Open spin chains and quadratic algebras

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Abstract

Algebraic framework for construction of a commuting set of operators that can be interpreted as integrals of motion of the open spin chain with boundary conditions and nearest neighbour interaction is investigated.

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In the end of the seventies quantum inverse scattering method was developed (for a review see e.g. [1]). One of the systems where it was applied was *periodic* spin chain with the nearest neighbour interaction. The algebra, from which the hamiltonian of this system as well as its integrals of motion were derived, is defined by the relations

$$R_{12}(u_1-u_2)L_1(u_1)L_2(u_2)=L_2(u_2)L_1(u_1)R_{12}(u_1-u_2) \tag{1}$$

where R is a matrix function $R: U \rightarrow End(V_0 \otimes V_0)$ satisfying the Yang–Baxter equation (YBE)

$$R_{23}(u_2-u_3)R_{13}(u_1-u_3)R_{12}(u_1-u_2)=R_{12}(u_1-u_2)R_{13}(u_1-u_3)R_{23}(u_2-u_3) \tag{2}$$

and the range U of the "spectral parameters" u is usually the field of complex numbers $\mathbf{C}$.

In 1988, Sklyanin proposed a method for constructing solvable models of quantum *open* (i.e. non–periodic) *spin chains* [2]. The method is based on reflection-type algebras given by the relations

$$R_{12}(u_1-u_2)M_1(u_1)R_{12}(u_1+u_2)M_2(u_2)=M_2(u_2)R_{12}(u_1+u_2)M_1(u_1)R_{12}(u_1-u_2)$$

$$\begin{aligned}R_{12}(u_2-u_1)K_1^{t_1}(u_1)R_{12}(-u_1-u_2-2\eta)K_2^{t_2}(u_2) &= \\ K_2^{t_2}(u_2)R_{12}(-u_1-u_2-2\eta)&K_1^{t_1}(u_1)R_{12}(u_2-u_1)\end{aligned}$$

The matrix R, beside the YBE, satisfies conditions

$$\begin{aligned}P_{12}R_{12}(u)P_{12}&=R_{12}(u),\\ R_{12}^{t_1}(u)&=R_{12}^{t_2}(u),\\ R_{12}(u)R_{12}(-u)&=\rho(u)\mathbf{1}_{12},\\ R_{12}^{t_1}(u)R_{12}^{t_1}(-u-2\eta)&=\tilde{\rho}(u)\mathbf{1}_{12},\end{aligned}$$

where t_1, t_2 mean the transposition in the first, respectively second pair of indices. Some of these conditions were later weakened [3] but stil they remained rather restrictive. The purpose of the present paper is to present a more general construction of open spin chains.

The starting point is associative algebra generated by elements $M_i^j(u), K_i^j(u)$, $i,j \in \{1,\ldots,d_0=dimV_0\}$ satisfying quadratic relations

$$A_{12}(u_1,u_2)M_1(u_1)B_{12}(u_1,u_2)M_2(u_2)=M_2(u_2)C_{12}(u_1,u_2)M_1(u_1)D_{12}(u_1,u_2), \tag{3}$$

$$\tilde{A}_{12}(u_1,u_2)K_1^{t_1}(u_1)\tilde{B}_{12}(u_1,u_2)K_2^{t_2}(u_2)=K_2^{t_2}(u_2)\tilde{C}_{12}(u_1,u_2)K_1^{t_1}(u_1)\tilde{D}_{12}(u_1,u_2). \tag{4}$$

where $A, B, \ldots, \tilde{D}$ are matrix functions $U \times U \rightarrow End(V_0 \times V_0)$ i.e. $A_{12}(u_1,u_2)=A_{i_1i_2}^{j_1j_2}(u_1,u_2)$, $i_1,i_2,j_1,j_2 \in \{1,\ldots,d_0=dimV_0\}$ and similarly for $B, C, \ldots, \tilde{D}$. Algebras of this type were investigated in another context in [4].

Our first task is to identify the conditions on the numerical matrices $A, B, \ldots, \tilde{D}$ that guarantee the existence of a commuting subalgebra that, when represented, will provide us with a commuting set of operators that eventually can be interpreted as the integrals of motion of a quantum system.

Theorem 1: Let $\mathcal{A}$ is the associative algebra generated by elements $M_i^j(u)$, $K_i^j(u)$, relations (3), (4) and

$$M_1(u_1)K_2(u_2) = K_2(u_2)M(u_1). \tag{5}$$

If the matrices $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}$, are related to A, B, C, D by

$$\tilde{A}_{12}(u_1,u_2) = (A_{12}^{t_1 t_2}(u_1,u_2))^{-1}, \quad \tilde{D}_{12}(u_1,u_2) = (D_{12}^{t_1 t_2}(u_1,u_2))^{-1}, \tag{6}$$

$$\tilde{B}_{12}(u_1,u_2) = ((B_{12}^{t_1}(u_1,u_2))^{-1})^{t_2}, \quad \tilde{C}_{12}(u_1,u_2) = ((C_{12}^{t_2}(u_1,u_2))^{-1})^{t_1}, \tag{7}$$

then the elements $t(u) = K_i^j(u)M_j^i(u) = tr[K(u)M(u)]$ form a commutative subalgebra i.e.

$$[t(u_1), t(u_2)] = 0.$$

Remark: Note that there are no restrictions on A, B, C, D.

Proof: Repeats the steps of [2]. Denoting $K_1 \equiv K_1(u_1)$, $K_2 \equiv K_2(u_2)$, $M_1 \equiv M_1(u_1)$, $M_2 \equiv M_2(u_2)$, $A_{12} \equiv A_{12}(u_1,u_2), \ldots, D_{12} \equiv D_{12}(u_1,u_2)$ and using the properties of the trace

$$trX^tY^t = trXY, \quad tr(XY^t) = tr(X^tY)$$

and relations (3)– (7) we get

$$t(u_1)t(u_2) = tr_1(K_1M_1)tr_2(K_2M_2) = \ldots = tr_{12}(K_1^{t_1}K_2M_1^{t_1}M_2)$$

$$= tr_{12}(K_1^{t_1}K_2\tilde{B}_{12}^{t_2}B_{12}^{t_1}M_1^{t_1}M_2) = tr_{12}[(K_1^{t_1}\tilde{B}_{12}K_2^{t_2})^{t_2}(M_1B_{12}M_2)^{t_1}] =$$

$$tr_{12}[(K_1^{t_1}\tilde{B}_{12}K_2^{t_2})^{t_1t_2}\tilde{A}_{12}^{t_1t_2}A_{12}(M_1B_{12}M_2)] = tr_{12}[(\tilde{A}_{12}K_1^{t_1}\tilde{B}_{12}K_2^{t_2})^{t_1t_2}(A_{12}M_1B_{12}M_2)]$$

$$= tr_{12}[(K_2^{t_2}\tilde{C}_{12}K_1^{t_1}\tilde{D}_{12})(M_2C_{12}M_1D_{12})^{t_1t_2}] = tr_{12}[(K_2^{t_2}\tilde{C}_{12}K_1^{t_1})^{t_1}(M_2C_{12}M_1)^{t_2}]$$

$$= tr_{12}[K_2^{t_2}K_1M_2^{t_2}M_1] = tr_{12}[K_2^{t_2}M_2^{t_2}K_1M_1] = t(u_2)t(u_1)$$

Q.E.D

The fundamental property that enabled to construct the operators describing integrals of motion of periodic spin chains was the possibility to define a coproduct in the algebra (1) because the commuting operators then could be expressed in the form

$$t(u) = tr[L_{(N)}(u)L_{(N-1)}(u)\ldots L_{(1)}(u)]$$

where $L_{(j)}$ were matrices of operators acting nontrivially only in the space of the j–th spin. However, it seems that it is not posible (in the unbraided categories) to define a coproduct in the algebra $\mathcal{A}$. Nevertheless, we can use the algebra for the construction of spin chain operators due to the following covariance property.

Theorem 2: Let $\mathcal{B}$ is the algebra generated by $L(u) = L_i^j(u), N(u) = N_i^j(u)$, $i, j \in \{1, \ldots, d_0 = dim V_0\}$ and relations

$$A_{12}(u_1, u_2)L_1(u_1)L_2(u_2) = L_2(u_2)L_1(u_1)A_{12}(u_1, u_2) \tag{8}$$

$$D_{12}(u_1, u_2)N_1(u_1)N_2(u_2) = N_2(u_2)N_1(u_1)D_{12}(u_1, u_2) \tag{9}$$

$$N_1(u_1)B_{12}(u_1, u_2)L_2(u_2) = L_2(u_2)B_{12}(u_1, u_2)N_1(u_1) \tag{10}$$

$$L_1(u_1)C_{12}(u_1, u_2)N_2(u_2) = N_2(u_2)C_{12}(u_1, u_2)L_1(u_1). \tag{11}$$

Then

1) the algebra $\mathcal{B}$ can be turned into bialgebra by coproduct

$$\Delta(L_i^j(u)) = L_i^k(u) \otimes L_k^j(u), \quad \Delta(N_i^j(u)) = N_k^j(u) \otimes N_i^k(u) \tag{12}$$

and counit

$$\epsilon(L_i^j(u)) = \delta_i^j, \epsilon(N_i^j(u)) = \delta_i^j. \tag{13}$$

2) the algebra $\mathcal{M}$ generated by the $M_i^j(u)$ and relations (3) is $\mathcal{B}$–comodule algebra. The coaction on $\mathcal{M}$ is given by

$$\beta : \mathcal{M} \to \mathcal{M} \otimes \mathcal{B}, \quad \beta(M_i^j(u)) = M_k^l(u) \otimes L_i^k(u)N_l^j(u) \tag{14}$$

that with a slight abuse of notation can be written as $\beta(M) = \tilde{M} = LMN$.

Remark: A similar covariance algebra can be defined for the algebra $\mathcal{K}$ generated by $K_i^j(u)$.

Proof: It is straightforward to check that the relations (8-11) are invariant under (12,13). Proof of invariance of (3) under (14) is

$$A_{12}\tilde{M}_1 B_{12}\tilde{M}_2 = A_{12}L_1M_1(N_1B_{12}L_2)M_2N_2 = (A_{12}L_1L_2)M_1B_{12}N_1M_2N_2 =$$

$$L_2L_1(A_{12}M_1B_{12}M_2)N_1N_2 = L_2L_1M_2C_{12}M_1(D_{12}N_1N_2) =$$

$$L_2M_2(L_1C_{12}N_2)M_1N_1D_{12} = (L_2M_2N_2)C_{12}(L_1M_1N_1)D_{12} = \tilde{M}_2C_{12}\tilde{M}_1D_{12}.$$

Q.E.D. (We have again deleted the $(u_1, u_2)-$dependence in the above formulae.)

Importance of the Theorem 2 is in the fact that it gives a possibility to define a set of commuting operators composed from the operators acting nontrivially only in spaces V_i of single spin states. Indeed, if ρ_i are representation of $\mathcal{B}$ on spaces V_i, $i = 1, \ldots, N$, then

$$\hat{L}(u) := (\rho_1 \otimes \rho_2 \otimes \ldots \otimes \rho_N) \circ (\Delta^{N-1})(L(u)),$$

$$\hat{N}(u) := (\rho_1 \otimes \rho_2 \otimes \ldots \otimes \rho_N) \circ (\Delta^{N-1})(N(u))$$

are operators that represent the algebra $\mathcal{B}$ on $\mathcal{H} \equiv V_1 \otimes V_2 \otimes \ldots \otimes V_N$ that is the Hilbert space of the system of N spins and the operators $\hat{L}(u), \hat{N}(u)$ can be written as

$$\hat{L}(u) = \hat{L}_{(N)}(u)\hat{L}_{(N-1)}(u) \ldots \hat{L}_{(1)}(u) \tag{15}$$

$$\hat{N}(u) = \hat{N}_{(1)}(u)\hat{N}_{(2)}(u) \ldots \hat{N}_{(N)}(u) \tag{16}$$

where

$$\hat{L}_{(j)}(u) = \mathbf{1} \otimes \ldots \otimes \mathbf{1} \otimes \rho_j(L(u)) \otimes \mathbf{1} \otimes \ldots \otimes \mathbf{1}$$

$$\hat{N}_{(j)}(u) = \mathbf{1} \otimes \ldots \otimes \mathbf{1} \otimes \rho_j(N(u)) \otimes \mathbf{1} \otimes \ldots \otimes \mathbf{1}$$

.

The representations of $\mathcal{B}$ on V_i such that $dimV_i = dimV_0$ follow from

Theorem 3: Let there are $\alpha, \delta \in U$ such that the Matrices A, B, C, D satisfy the equations

$$A_{12}(u_1, u_2)A_{13}(u_1, \alpha)A_{23}(u_2, \alpha) = A_{23}(u_2, \alpha)A_{13}(u_1, \alpha)A_{12}(u_1, u_2) \tag{17}$$

$$D_{12}(u_1, u_2)D_{13}(u_1, \delta)D_{23}(u_2, \delta) = D_{23}(u_2, \delta)D_{13}(u_1, \delta)D_{12}(u_1, u_2) \tag{18}$$

$$D_{13}(u_1, \delta)B_{12}(u_1, u_2)A_{23}(u_2, \alpha) = A_{23}(u_2, \alpha)B_{12}(u_1, u_2)D_{13}(u_1, \delta) \tag{19}$$

$$A_{13}(u_1, \alpha)C_{12}(u_1, u_2)D_{23}(u_2, \delta) = D_{23}(u_2, \delta)C_{12}(u_1, u_2)A_{13}(u_1, \alpha) \tag{20}$$

for all $u_1, u_2 \in U$. (note the unusual order of indices in (19,20)).

Then the map $\rho_{\alpha\delta} : End(V_0) \otimes \mathcal{B} \to End(V_0 \otimes V_0)$

$$[\rho_{\alpha\delta}(L^j_k(u))]^n_m = A^{jn}_{km}(u, \alpha) \tag{21}$$

$$[\rho_{\alpha\delta}(N^j_k(u))]^n_m = D^{jn}_{km}(u, \delta) \tag{22}$$

is a representation of the algebra $\mathcal{B}$ on V_i such that $dimV_i = dimV_0$.

Proof: Direct check of relations (8)–(11) by means of (17)–(18).

Remark : Note that full Yang–Baxter type equations are not required in the theorem. It is suficient if they are satisfied for single $(\alpha, \delta) \in U \times U$.

If we find a representation σ of $\mathcal{A}$ on $\mathcal{H}$, then due to the Theorems 1,2 we get the set of commuting operators on $\mathcal{H}$

$$\hat{t}(u) = Tr[\sigma(K(u))\hat{L}(u)\sigma(M(u))\hat{N}(u)]$$

Assuming that there are numerical matrices $m(u), k(u) \in End(V_0)$ that satisfy the relations (3), (4), we can choose

$$\sigma(M^j_i(u)) = m^j_i(u)\mathbf{1}_{\mathcal{H}}, \; \sigma(K^j_i(u)) = k^j_i(u)\mathbf{1}_{\mathcal{H}}$$

and then

$$\hat{t}(u) = tr[k(u)\hat{L}_{(N)}(u)\dots\hat{L}_{(1)}(u)m(u)\hat{N}_{(1)}(u)\dots\hat{N}_{(N)}(u)] \tag{23}$$

where the operator matrices $L_{(k)}$ and $N_{(k)}$ act nontrivailly only in the k-th factor of the space $\mathcal{H} = V_0 \otimes V_0 \otimes \dots \otimes V_0$.

The last goal we want to achieve is finding the hamiltonian H of the open chain system with the nearest neighbour interaction and boundary terms.

Theorem 4: Let there is one–dimensional representation of the algebra $\mathcal{A}$ by numerical matrices $m(u)$, $k(u)$ and the representation of $\mathcal{B}$ on V_i is $\rho_i = \rho_{\alpha,\delta}$ for all $i \in \{1,\dots,N\}$.

If there is $u_0 \in U$ such that

$$A_{12}(u_0,\alpha) = \kappa P_{12}, \quad D_{12}(u_0,\delta) = \lambda P_{12}, \quad m(u_0) = \mu\mathbf{1} \tag{24}$$

where κ, λ, μ are constants and P is the permutation matrix, then

$$\hat{t}(u_0) = \mu(\kappa\lambda)^N tr[k(u_0)] \tag{25}$$

and

$$\frac{d\hat{t}}{du}(u_0) = (\kappa\lambda)^N \mu\{Htr[k(u_0)] + tr[\frac{dk}{du}(u_0)]\} \tag{26}$$

$$H = \sum_{n=1}^{N-1} H_{n,n+1} + \mu^{-1}\frac{dm_{(1)}}{du}(u_0) + [tr\ k(u_0)]^{-1} tr_0[k_0(u_0)H_{N,0}] \tag{27}$$

$$H_{n,n+1} = \lambda^{-1}\frac{dD_{n,n+1}}{du}(u_0,\delta)P_{n,n+1} + \kappa^{-1}P_{n,n+1}\frac{dA_{n,n+1}}{du}(u_0,\alpha) \tag{28}$$

Proof: From (15), (16) and (21)–(24) we get

$$\hat{L}(u) = \kappa^N P_{0N}P_{0n-1},\dots,P_{01}, \quad \hat{N}(u) = \lambda^N P_{01}P_{02},\dots,P_{0N} \tag{29}$$

wherefrom (25) immediately follows. Similarly, (26)–(28) is obtained by differentiating (23) and using (29) and the identity on $End(V_0^{\otimes N+1})$

$$P_{0,n+1}X_{0n} = X_{n+1,n}P_{0,n+1}.$$

In the conclusion, we have shown that the *open spin chains with the nearest neighbour interaction can be constructed from rather general quadratic algebras* defined by matrix functions A, B, C, D that

1) satisfy the equations (17,18,19,20),

2) admit numerical matrices $m(u), k(u)$ that satisfy (3,4,6,7).

3) together with $m(u)$ satisfy the regularity condition (24).

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