TOPOLOGICAL QUANTUM DOUBLE

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Abstract : Following a preceding paper showing how the introduction of a t.v.s. topology on quantum groups leads to a remarkable *unification* and *rigidification* of the different definitions, we adapt here, in the same way, the definition of quantum double. This topological double is *dualizable* and *reflexive* (even for *infinite dimensional* algebras).

In a simple case we show, considering the double as the "zero class" of an extension theory, the uniqueness of the double structure as a quasi-Hopf algebra.

Résumé : A la suite d'un précédent article montrant comment l'introduction d'une topologie d'e.v.t. sur les groupes quantiques permet une *unification* et une *rigidification* remarquables des différentes définitions, on adapte ici de la même manière la définition du double quantique. Ce double topologique est alors *dualisable* et *reflexif* (même pour des algèbres de *dimension infinie*).

Dans un cas simple on montre, en considérant le double comme la "classe zéro" d'une théorie d'extensions, l'unicité de cette structure comme algèbre quasi-Hopf.

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0. INTRODUCTION.

The topological model of quantum groups (abbreviated QG) we introduced in [BFGP] is very useful: dualizing is straightforward and thus everything we show on it is valid for both the Drinfeld [D1] and Faddeev-Reshetikhin-Takhatajan [FRT] versions of QG. Moreover most of the works made in the generic case (not root of unity) seem to be easily adaptable to the new model.

In this paper we do this explicitely for the quantum double. Here again the notion of duality is primordial: the duality in the "QUE-dual" [D1] is very specific, and defines the double in a particular context. But the double can be defined more generally, for example in terms of the "double cross-product" of Majid [Ma1]. It is also of interest in the theory of monoidal categories. But in these works, because of the problem of dualizing an infinite dimensional Hopf algebra, it is supposed most of the time that the constructions are done for finite dimensional Hopf algebras (group algebras of finite groups for example).

Fréchet nuclear or dual of Fréchet nuclear topologies and strong duality get rid of these difficulties in the infinite dimensional case (see [BFGP]). Therefore, we can adapt the definition of Majid for finite groups to a quantum double structure on $\mathcal{H}_t(G)\bar{\otimes}\mathcal{A}_t(G)$ and $\mathcal{C}_t^\infty(G)\bar{\otimes}\mathcal{D}_t(G)$ either for $t=0$ or t generic parameter of the QG deformation (see [BFGP]) and for every compact Lie group G ($\mathcal{H}(G)$ is the coefficient algebra of G, $\mathcal{A}(G)$ its strong dual, $\mathcal{C}^\infty(G)$ the C^∞ complex functions on G and $\mathcal{D}(G)$ the distributions on G ; $\bar{\otimes}$ stands for the completed injective topological tensor product [Gr]).

We have, as topological vector spaces, $(\mathcal{H}_t(G)\bar{\otimes}\mathcal{A}_t(G))^* = \mathcal{A}_t(G)\hat{\otimes}\mathcal{H}_t(G)$ (projective topological tensor product) and thus, we can define a dual double structure. *The double defined here is, of course, reflexive.*

We were also interested in the notion of extensions for quantum groups. Two possibilities arise :

1) to adapt the Hochschild theory of associative algebras to the Hopf case and then the underlying vector space of the extension is a direct sum.

2) Since $\mathcal{U}(\mathcal{G}_1 \oplus \mathcal{G}_2) = \mathcal{U}\mathcal{G}_1 \otimes \mathcal{U}\mathcal{G}_2$ and $\mathbb{C}(G_1 \times G_2) = \mathbb{C}G_1 \otimes \mathbb{C}G_2$, for $\mathcal{G}_1, \mathcal{G}_2$ Lie algebras and G_1, G_2 groups, we have to find, if we want to pass from the level of Lie algebras or groups to that of their associated Hopf algebras and then to their QG deformations, an extension theory for Hopf algebras compatible with an underlying structure of tensor product of vector spaces.

This was done by Sweedler [Sw2] and Singer [Si] (with restrictions on the (co)-commutativity). We remark that the double, in the non deformed case of $\mathcal{A}(G)$ and $\mathcal{D}(G)$, is a smash-product (see [Sw1] or [A]), i.e., the "zero class" of the extension theory. Then, adapting the theory for quasi-Hopf algebras, we show with cohomological methods, that in the case of $\mathcal{D}(G)$ any quasi-Hopf extension of $\mathcal{D}(G)$ by $\mathcal{C}^\infty(G)$ is the double. This is a kind of uniqueness result.

We were also motivated by an article of Dijkgraaf-Pasquier-Roche [DPR] who find apparently non equivalent quasi-Hopf analogues of the double for finite groups. Here, with the extension theory we take, this is not possible.

The article is organized as follows :

We first recall what we need about the quantum double, then construct topological quantum doubles and study their properties.

Next, we recall the extension theory we use and the classifying cohomology. We adapt it for quasi-Hopf algebras in the way of [Bo] and prove the uniqueness property of $D(\mathcal{D}(G))$ by showing the nullity of a second cohomology.

Finally, we discuss what is left to be done about the double and more generally about the problem of extensions for QG.

1. BACKGROUND.

Let $(A, \mu, \Delta, \varepsilon, S)$ be a Hopf algebra, A^* a Hopf algebra "in duality" (to be defined) with A, and A^0 be A^* with the *opposite* coproduct. The antipode on A^0 is then ${}^tS'$, the transpose of the skew antipode of A (i.e. the antipode of the coalgebra A equipped with the opposite product as an algebra).

If we consider the vector space $A^* \otimes A$, Drinfeld [D1] defines the quantum double as follows :

i) $D(A) \simeq A^0 \otimes A$ as coalgebras,

ii) $(f \otimes Id_A).(Id_{A^0} \otimes b) = f \otimes b$,

iii) $(Id_{A^0} \otimes e_s).(e^t \otimes Id_A) = \Delta_s^{kjn}\ \mu_{plk}^t\ S_n'^p\ (e^l \otimes Id_A)\ (Id_{A^0} \otimes e_j)$, where $\{e_s\}$ is a basis of A and $\{e^t\}$ the dual basis.

This can be formulated in an other way ([Ma1]).

DEFINITION 1. *Let $(A, \mu, \Delta, \varepsilon, S)$ be a Hopf algebra, A^* its dual for a given duality notion. We denote by A^0 the Hopf algebra A^* with the opposite coproduct, ${}^t\mu'$, S' the skew antipode of A and $< ., . >$ the duality pairing.*

We call **quantum double of** A, $D(A)$, *the Hopf algebra $(A^0 \otimes A, \mu_D, \Delta_D =^t \mu' \otimes \Delta, S_D =^t S' \otimes S)$ with*

$$\mu_D\ ((f \otimes a) \otimes (g \otimes b)) = \sum_{(a)} f\ \ < g, S'(a_{(3)})\ ?\ a_{(1)} > \otimes\ a_{(2)}\ b$$

where ? is a variable of A, using the notation of Sweedler [Sw1] for the coproduct. Equivalently,

$$\mu_D((f \otimes a) \otimes (g \otimes b)) = \sum_{(a)(g)} < g_{(1)}, a_{(1)} >< S^0(g_{(3)}), a_{(3)} >\ fg_{(2)} \otimes a_{(2)}b$$

LEMMA 1. *If A is cocommutative,*

$$\mu_D((f \otimes a) \otimes (f \otimes b)) = \sum_{(a)} f(a_{(1)} \bullet g) \otimes a_{(2)}b$$

with $x \bullet h$ the coadjoint action of A on A^0 (i.e. $x \bullet h(y) = \sum_{(x)}\ < h, S(x_{(1)})y\ x_{(2)} >$)

Proof :

(i) $S = S'$

(ii) we have $\sum_{(a)}\ f(a_{(1)} \bullet g) \otimes a_{(2)}b = \sum_{(a)(a_{(1)})}\ f < g, S(a_{(1)})\ ?\ a_{(3)} > \otimes a_{(2)}b$

$= \sum_{(a)}\ f < g, S(a_{(1)})\ ?\ a_{(2)} > \otimes a_{(3)}b$ (coassociativity)

$= \sum_{(a)}\ f < g, S(a_{(3)})\ ?\ a_{(1)} > \otimes a_{(2)}b$ (cocommutativity)

∎

The product in this cocommutative case is known as the *smash product* of A^0 by A via the coadjoint action [Sw2].

In the general case, we have the "double cross product" structure of Majid [Ma1] : the product of the double can be written as follows :

$(f \otimes a).(g \otimes b) = \sum_{(a)(g)} f\ \alpha(a_{(1)} \otimes g_{(1)}) \otimes \beta(a_{(2)} \otimes g_{(2)})b$ where α defines an A-module structure on A^0 by

$$\alpha(c \otimes h) = \sum_{(c)}\ < h, S'(c_{(2)})\ ?\ c_{(1)} >$$

and β defines an A^0-module structure on A by

$$\beta(c \otimes h) = \sum_{(c)} c_{(2)}\ \ < h, S'(c_{(3)})\ c_{(1)} > .$$

2. TOPOLOGICAL QUANTUM DOUBLE.

As in [BFGP] we would like, with the help of topology, to find a notion of quantum double which is reflexive and more natural (in the definition of duality for example).

I think that we can find general enough conditions on a topological Hopf algebra A so that the construction described in the following is always possible. This would certainly require quite technical notions of topological vector spaces. For example, the dual of the topological tensor product of a Fréchet space with its dual, is, in the general case, especially pathological (see [Gr, Chap. II, paragraph 4]). And this would lead us too far from our main subject : the quantum groups.

Therefore, I shall restrict myself here to the two cases in which we were interested in [BFGP], when G is a compact, semi-simple, connected Lie group :

1) $\mathcal{A}(G) = \Pi_{\pi\in\hat{G}}\, End(V_\pi)$ $\quad (\mathcal{A}(G)^* = \mathcal{H}(G) = \oplus_{\pi\in\hat{G}}\, \mathcal{C}_\pi)$
with $\hat{G} = \{$ non equivalent irreducible finite dimensional (f.d.) representations of $G\}$
$\mathcal{C}_\pi = \{$ coefficients of the f.d. representation $(\pi, V_\pi)\}$
and their "quantum" deformations.

2) $\mathcal{D}(G) = \{$ distributions on $G\}$ $\quad (\mathcal{D}(G)^* = C^\infty(G) = \{C^\infty$ complex functions on $G\})$
and their "quantum deformations" (for details see [BFGP]).

DEFINITION 2. *We call* **topological Hopf algebra,** *a Hopf algebra* $(A, \mu, \Delta, \varepsilon, S)$ *such that*
$\mu : A\tilde{\otimes}A \to A$
$\Delta : A \to A\tilde{\otimes}A$ $\qquad$ *are continuous*
$\varepsilon : A \to \mathbb{C}$
$S : A \to A$
where $\tilde{\otimes}$ *stands either for the completed projective topological tensor product (noted* $\hat{\otimes}$*), or for the injective one (noted* $\bar{\otimes}$*) (see* [Gr]*).*

Now we have

PROPOSITION 1. *Let* A *be a topological Hopf algebra (for* $\bar{\otimes}$*) and* A^* *its strong dual.*

Then Definition 1 gives a topological Hopf algebra structure (for $\bar{\otimes}$*) on* $A^0\bar{\otimes}A$*.*

The proof is straightforward :

the elements $f \otimes a$, $f \in A^0$, $a \in A$, generate topologically $A^0\bar{\otimes}A$, Δ_A, S'_A, μ_A are (by definition) continuous and the duality pairing is a separately continuous form on $A^0 \times A$, thus a continuous linear form on $A^0\bar{\otimes}A$.

∎

Therefore, we can define the double for every (injective) topological Hopf algebra. However the problems mentioned above occur for the dualization. That is why, from now on, *we shall restrict ourselves to the cases* $\mathcal{A} = \mathcal{A}(G)$ *and* $\mathcal{D} = \mathcal{D}(G)$ $(\mathcal{A}^* = \mathcal{H}$ *and* $\mathcal{D}^* = \mathcal{C})$, *and to their QG deformations* (which have the same vector space structure).

PROPOSITION 2. *As vector spaces, we have* $(\mathcal{A}\hat{\otimes}\mathcal{H})^* = \mathcal{H}\bar{\otimes}\mathcal{A}$ *and* $(\mathcal{H}\bar{\otimes}\mathcal{A})^* = \mathcal{A}\hat{\otimes}\mathcal{H}$, *since* $A^* = \mathcal{H}$ *and* $A^{**} = A$, *and the same for* $\mathcal{D}$ *and* $\mathcal{C}$.

Proof :

1) *for* $\mathcal{A}$ *:*

$$
\begin{aligned}
(\mathcal{A}\hat{\otimes}\mathcal{A}^*)^* &= \left[\left(\prod_{\pi\in\hat{G}} \mathcal{L}(V_\pi)\right)\hat{\otimes}\left(\bigoplus_{\pi'\in\hat{G}} \mathcal{L}(V_{\pi'})\right)\right]^* \\
&\simeq \left[\prod_{\pi\in\hat{G}} \left(\mathcal{L}(V_\pi)\hat{\otimes}(\bigoplus_{\pi'\in\hat{G}} \mathcal{L}(V_{\pi'}))\right)\right]^* \\
&\simeq \bigoplus_{\pi\in\hat{G}} \left[\mathcal{L}(V_\pi)\hat{\otimes}(\bigoplus_{\pi'\in\hat{G}} \mathcal{L}(V_{\pi'}))\right]^* \\
&\simeq \bigoplus_{\pi\in\hat{G}} \left[\bigoplus_{\pi'\in\hat{G}} [\mathcal{L}(V_\pi)\otimes\mathcal{L}(V_{\pi'})]\right]^* \quad (\mathcal{L}(V_\pi) \text{ normable }) \\
&\simeq \bigoplus_{\pi\in\hat{G}} \prod_{\pi'\in\hat{G}} [\mathcal{L}(V_\pi)\otimes\mathcal{L}(V_{\pi'})]^* \\
&\simeq \bigoplus_{\pi\in\hat{G}} \prod_{\pi'\in\hat{G}} (\mathcal{L}(V_\pi)\otimes\mathcal{L}(V_{\pi'})) \\
&\simeq \bigoplus_{\pi\in\hat{G}} \left[\mathcal{L}(V_\pi)\bar{\otimes}(\prod_{\pi'\in\hat{G}} \mathcal{L}(V_{\pi'}))\right] \quad (\mathcal{L}(V_\pi) \text{ Fréchet }) \\
&\simeq \left(\bigoplus_{\pi\in\hat{G}} \mathcal{L}(V_\pi)\right)\bar{\otimes}\left(\prod_{\pi'\in\hat{G}} \mathcal{L}(V_{\pi'})\right) \simeq \mathcal{A}^*\bar{\otimes}\mathcal{A}
\end{aligned}
$$

Thus we obtain $(\mathcal{A}\hat{\otimes}\mathcal{A}^*)^* \simeq \mathcal{A}^*\bar{\otimes}\mathcal{A}$ and by the same arguments we have $(\mathcal{A}^*\bar{\otimes}\mathcal{A})^* = \mathcal{A}\hat{\otimes}\mathcal{A}^*$.

2) *for* $\mathcal{D}(G)$ *:* in [Gr], chap. II, we see p. 129 that $\mathcal{D}(G)\hat{\otimes}\mathcal{C}^\infty(G)$ is bornological, thus (theorem 14.1) its strong dual is complete and therefore (corollary of lemma 9) $(\mathcal{D}(G)\hat{\otimes}\mathcal{C}^\infty(G))^* \simeq \mathcal{D}(G)^*\bar{\otimes}\mathcal{C}^\infty(G)^*$ $(= C^\infty(G)\bar{\otimes}\mathcal{D}(G)$) and $\mathcal{D}(G)\hat{\otimes}\mathcal{C}^\infty(G)$ is reflexive.

■

We now have to dualize the Hopf algebra structure : for the coproduct, this is straightforward.

For the product we consider the "double cross-product" form :

$$(f\otimes a).(g\otimes b) = \sum_{(a)(g)} f\ \alpha(a_{(1)}\otimes g_{(1)})\otimes\beta(a_{(2)}\otimes g_{(2)})b$$

with $\alpha : A\bar{\otimes}A^* \to A^*$ and $\beta : A\bar{\otimes}A^* \to A$ continuous (see definition of α and β).

An equivalent form (without Sweedler notation, denoting (34) the permutation of two middle factors), is

$$\mu_D = (\mu_*\otimes\mu)\ (Id_*\otimes\alpha\otimes\beta\otimes Id)\ (34)\ (Id_*\otimes\Delta_*\otimes\Delta\otimes Id)$$

which shows by transposition that ${}^t\mu_D$ define a continuous coproduct on $(A\bar{\otimes}A^*)^* \simeq A^*\hat{\otimes}A$ and thus make the dual of the topological quantum double of A a topological Hopf algebra (for $\hat{\otimes}$). Because of the reflexivity of the vector spaces involved, we have in the same way $D(A)^{**} = D(A)$.

All these arguments work for A considered as a topological $\mathbb{C}[[t]]$-Hopf algebra thanks to the material developped in [BFGP]. Therefore we can construct and dualize the double of the "topological quantum groups" studied in [BFGP] ($\mathcal{A}(G)$ and $\mathcal{D}(G)$ with deformed coproduct on $\mathbb{C}[[t]]$).

We can formulate all of these into a theorem :

THEOREM 1. *Let* A *be either* $\mathcal{A}(G)$, $\mathcal{D}(G)$, *or their deformed versions,* H *be* $\mathcal{H}(G)$, $\mathcal{C}^\infty(G)$ *or their deformed versions. Then :*
$D(A) = A^*\bar{\otimes}A = H\bar{\otimes}A$ *with the classical Hopf structure of double*
$D(A)^* = A\hat{\otimes}A^* = A\hat{\otimes}H$ *with the dualized structure and* $D(A)^{**} = D(A)$.

Remark : $D(\mathcal{D}(G))^* = \mathcal{D}(G)\hat{\otimes}\mathcal{C}^\infty(G) \simeq \mathcal{C}^\infty(G, \mathcal{D}(G)).$
$\simeq \mathcal{D}(G, C^\infty(G))$
This can lead to explicit and interesting expressions for the coproduct on the dual double [P].

3. THE DOUBLE AS TRIVIAL EXTENSION OF BIALGEBRAS.

In this section *we show a uniqueness property of the topological double* in the non-deformed case of the double, $D(\mathcal{D}(G))$, of $\mathcal{D}(G)$.

But first we have to fix the framework of this result :

Notations. Let $(A, \Delta_A, \varepsilon_A)$ be a coalgebra, $(\Lambda, \mu_\Lambda, \eta_\Lambda)$ an algebra :
On $Hom_{\mathbb{C}}(A, \Lambda)$ we put the algebra structure defined by the product (often called "convolution")

$$f * g = \mu_\Lambda \circ (f \otimes g)\Delta_A$$

The unit of $Hom_{\mathbb{C}}(A, \Lambda)$ for $*$ is $\eta_\Lambda \circ \varepsilon_A$.
If A is cocommutative and Λ commutative, $*$ is commutative.
We note $Reg(A, \Lambda)$ the invertible elements of $Hom_{\mathbb{C}}(A, \Lambda)$ for $*$, and f^{-1} the inverse of $f \in Reg(A, \Lambda)$.
For the notions of module algebra, comodule algebra,...we refer to the book of Abe [A].
For Ψ a (right) A-comodule structure on X, for X a coalgebra, we note $\Psi(x) = \sum_{(x)} x_X \otimes x_A$.

For A a Hopf algebra, ε_A its counit, we note $A^+ = Ker\ \varepsilon_A$.

Extensions :

DEFINITION 3 [Sw2]. *Let A be a cocommutative Hopf algebra, Λ a commutative A-module algebra.*
We call (B, Ψ) an **algebra tensorial extension of Λ by A** *if B is an algebra containing Λ and $\Psi : B \to B \otimes \Lambda$ is an algebra homomorphism verifying*
(i) $A = \Psi^{-1}(B \otimes 1) = \{b \in B | \Psi(b) = b \otimes 1\}$.
(ii) Ψ gives to B the structure of a right A-comodule (with respect to the underlying coalgebra structure of A).
(iii) $b\lambda = \sum_{(b)} (b_A.\lambda)\ b_B \qquad Jb \in B,\ \lambda \in \Lambda.$
We say that the extension (Ψ, B) is **cleft** *if there is a comodule morphism in $Reg(A, B)$.*

PROPOSITION 3 [By]. *Let A be a cocommutative Hopf algebra, Λ a commutative one. Then (i, B, ρ) with B a Hopf algebra, i an injective Hopf morphism from Λ to B, ρ a surjective one from B to A such that*
1) $Ker\ \rho = \Lambda^+ B$ (ideal generated by Λ^+ in B)
2) $\exists\ q : B \to \Lambda$ left Λ-modules map with $(i \circ q) * (j \circ \rho) = 1$
3) $\exists\ j : A \to B$ right A-comodules map with $q \circ j = y \circ \varepsilon : A \to \Lambda$
defines a cleft algebra tensorial extension (with $\psi = (1 \otimes \rho) \circ \Delta$).

Proposition 3 can be taken as an alternative (and more natural) definition of cleft algebra extension. In the terminology of Byott, (i, B, ρ) and (i', B', ρ') are *equivalent* if $\exists f : B \to B'$ Hopf algebras isomorphism *s.t.* $\quad f \circ i = i \quad$ and $\quad \rho \circ f = \rho$. This notion of extension is classified by a cohomological theory that I shall briefly recall now, *the Sweedler cohomology* ([Sw2]) :

For A a cocommutative Hopf algebra, Λ a commutative A-module algebra, $Reg(A^{\otimes n}, \Lambda)$ are n-cochains.
Let $D^n : Reg(A^{\otimes n}, \Lambda) \to Reg(A^{\otimes n+1}, \Lambda)$

$$\begin{aligned} D^n f = [\tau \circ (Id_A \otimes f)] &* [f^{-1} \circ (m_A \otimes Id_A \otimes ... \otimes Id_A)] \\ &* [f \circ (Id_A \otimes m_A \otimes Id_A \otimes ... \otimes Id_A] \\ &* ... \\ &* [f^{\pm 1} \circ (Id_A \otimes ... \otimes Id_A \otimes m_A)] * [f^{\mp 1} \otimes \varepsilon_A] \end{aligned}$$

where m_A and ε_A are the product and counit on A and $\tau : A \otimes \Lambda \to \Lambda$ defines the $A-$module structure on Λ.

Normalized n-cochains :
$Reg_+(A^{\otimes n},\Lambda)=\{f\in Reg(A^{\otimes n},\Lambda)|f(a_1\otimes ...\otimes a_n)=\varepsilon_\Lambda(a_1)...\varepsilon_\Lambda(a_2)$ if $a_i\in \mathbb{C}$, for at least one $i\}$.

As always, we define $Z^n_{sw}(A,\Lambda)$, $B^n_{sw}(A,\Lambda)$ and $H^n_{sw}(A,\Lambda)$ for n-cocycles, n-coboundaries, and n-cohomology of the Sweedler complex $\{Reg^n(A,\Lambda)=Reg(A^{\otimes n},\Lambda)\ ;\ D^n\}$.

We have $Reg^0(A,\Lambda)=A^{inv}$ and $H^0_{sw}(A,\Lambda)=\{\lambda\in\Lambda^{inv}/a\bullet\lambda=\varepsilon(a)\lambda \quad \forall a\in A\}$

$$\begin{aligned}
f\in Z^1_{sw}(A,\Lambda) &\Leftrightarrow D^1 f=[\tau\circ(Id\otimes f)]*[f^{-1}\circ m_A]*[f\otimes\varepsilon_A]=\eta_\Lambda\circ(\varepsilon_A\otimes\varepsilon_A)\\
&\Leftrightarrow f(ab)=\sum_{(a)}(a_{(1)}\bullet f(b))\ f(a_{(2)}),\qquad \forall a,b\in A\\
f\in B^1_{sw}(A,\Lambda) &\Leftrightarrow \exists\lambda\in\Lambda^{inv}\text{ s.t. } f=D^0\,\lambda\qquad (\Leftrightarrow f(a)=(a\bullet\lambda)\lambda\ )
\end{aligned}$$

Then we have :

PROPOSITION 4. [Sw2]. *If B is a cleft algebra extension of Λ by A, there exists $\tau\in Z^2_{sw}(A,\Lambda)$ s.t. the product on $B\simeq\Lambda\otimes A$ (as vector spaces) is*

$$(\lambda\sharp_\tau a)(\gamma\sharp_\tau x)=\sum_{(a)(x)}\lambda(a_{(1)}\bullet\gamma)\ \tau(a_{(2)}\otimes x_{(1)})\ \sharp_\tau\ a_{(3)}\ x_{(2)}.$$

If $\tau\in B^2_{sw}(A,\Lambda)$, then the product on B is equivalent to the smash-product :

$$(\lambda\sharp a).(\gamma\sharp x)=\sum_{(a)}\lambda(a_{(1)}\bullet\gamma)\ \sharp\ a_{(2)}x.$$

As wanted $H^2_{sw}(A,\Lambda)$ classifies the extensions. Of course, we can dualize Sweedler cohomology : this is the cohomology the complex $\{Reg(A,\Lambda^{\otimes m}),\ D^m_*\}$ with

$$\begin{aligned}
D^m_* f=[(Id_\Lambda\otimes f)\circ\varphi] &*[(\Delta_\Lambda\otimes Id_\Lambda\otimes ...\otimes Id_\Lambda)\circ f^{-1}]\\
&*[(Id_\Lambda\otimes\Delta_\Lambda\otimes Id_\Lambda\otimes ...\otimes Id_\Lambda)\circ f]\\
&*...\\
&*[(Id_\Lambda\otimes ...\otimes Id_\Lambda\otimes\Delta_\Lambda)\circ f^{\pm1}]*[f^{\mp1}\otimes\eta_\Lambda]
\end{aligned}$$

for Λ a commutative Hopf algebra and A a cocommutative Λ-comodule coalgebra with φ the structure map.

We will note, "canonically", $Z^n_{cosw}(A,\Lambda)$, $B^n_{cosw}(A,\Lambda)$ and $H^n_{cosw}(A,\Lambda)$ for the Sweedler complex in the coalgebra case. As before, we can define *cleft coalgebra tensorial extension of A by Λ* and prove that the underlying vector space is always $\Lambda\otimes A$ and that for each of them, there exists $\theta\in Z^2_{cosw}(A,\Lambda)$ s.t. the coproduct on the extension is

$$\Delta_\theta(\lambda\sharp^\theta a)=\sum_{(\lambda)(a)(a_{(1)})}(\lambda_{(1)}\ a^I_{(1)}\ \sharp^\theta\ a_{(2)A})\otimes(\lambda_{(2)}\ a^{II}_{(1)}\ a_{(2)\Lambda}\ \sharp^\theta\ a_{(3)})$$

with $\theta(a)=\sum_{(a)} a^I\otimes a^{II}$.

The structures of algebra tensorial extension and coalgebra tensorial extension (from now on we omit "cleft" but it will always be implicit) can be glued together if the Λ-comodule Hopf algebra A and the A-module Hopf algebra Λ verify some conditions called "matched pair" conditions (for details see [Si]). Then the compatibility condition between μ_σ and Δ_θ is expressed $D^{12}\theta=D^{21}_*\sigma$.

We call these extensions : *Hopf algebra tensorial extensions.* They are classified by the 2-cohomology of the following bicomplex [Si] :

$$\begin{array}{ccccccc}
Reg(A,\Lambda) & \stackrel{D^{11}}{\longrightarrow} & Reg(A^{\otimes 2},\Lambda) & \stackrel{D^{21}}{\longrightarrow} & Reg(A^{\otimes 3},\Lambda) & \stackrel{D^{31}}{\longrightarrow} ... \\
\downarrow D_*^{11} & & \downarrow D_*^{21} & & \downarrow D_*^{31} & \\
Reg(A,\Lambda^{\otimes 2}) & \stackrel{D^{12}}{\longrightarrow} & Reg(A^{\otimes 2},\Lambda^{\otimes 2}) & \stackrel{D^{22}}{\longrightarrow} & Reg(A^{\otimes 3},\Lambda^{\otimes 2}) & \stackrel{D^{32}}{\longrightarrow} ... \\
\downarrow D_*^{12} & & \downarrow D_*^{22} & & \downarrow D_*^{32} & \\
Reg(A,\Lambda^{\otimes 3}) & \stackrel{D^{13}}{\longrightarrow} & Reg(A^{\otimes 2},\Lambda^{\otimes 3}) & \stackrel{D^{23}}{\longrightarrow} & Reg(A^{\otimes 3},\Lambda^{\otimes 3}) & \stackrel{D^{33}}{\longrightarrow} ... \\
\downarrow D_*^{13} & & \downarrow D_*^{23} & & \downarrow D_*^{33} & \\
... & & ... & & ... &
\end{array}$$

with cochains $C^n(A,\Lambda) = \bigoplus_{p+q=n+2,\ p\geq 1, q\geq 1} Reg(A^{\otimes p},\Lambda^{\otimes q})$.

Remark : During the colloquium "Enveloping algebras and quantum groups" in Paris, July 92, M. Rosso exposed the same kind of extension theory about *quasi-Hopf algebras* (see [D2] for definition) and claimed that it is classified by the same bicomplex connected with the complex $\{Reg(\mathbb{C},\Lambda^{\otimes n}) \simeq \Lambda^{\otimes n},\ n\geq 1\}$ as first column, the cochains becoming $\bigoplus_{p+q=n+1,\ p\geq 0, q\geq 1} Reg(A^{\otimes p},\Lambda^{\otimes q})$.

What I shall expose now should be part of the proof of this result, but since I never saw it anywhere and the point of view and the concepts here are a bit different, I shall do it in details. The idea is to adapt the Hopf algebra case to the bialgebra (not necessarily coassociative) one, exactly in the same way that I did in [Bo].

DEFINITION 4. *Let A be a cocommutative Hopf algebra, Λ a commutative one. (A,Λ) form a matched pair* [Si]. *We call* **bialgebra tensorial extension of Λ and A** *a cleft algebra extension of Λ by A (defined by $\sigma \in Z^2_{sw}(A,\Lambda)$) with an extension coproduct defined by a compatible $\theta \in Z^2_{cosw}(A,\Lambda)$.*

We say that two extensions $(B,\mu_\sigma,\Delta_\theta)$ and $(B',\mu_{\sigma'},\Delta_{\theta'})$ are equivalent if $(B,\mu_\sigma) \stackrel{alg.}{\simeq} (B',\mu_{\sigma'})$ and $\exists\, P_\Lambda \in (\Lambda\otimes\Lambda)^{inv}$ s.t. $\Delta_\theta = \tilde{P}_\Lambda\ \Delta_{\theta'}\ \tilde{P}_\Lambda^{-1}$ where $\tilde{P}_\Lambda = (23)\ (P_\Lambda \otimes 1_{A^{\otimes 2}})$.

THEOREM 2. *The 2-cohomology of the complex*

$$\begin{array}{ccccccccc}
\Lambda^{inv} \simeq Reg\ (\mathbb{C},\Lambda) & \rightarrow & Reg(A,\Lambda) & \rightarrow & Reg(A^{\otimes 2},\Lambda) & \rightarrow & Reg(A^{\otimes 3},\Lambda) & \rightarrow ... \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
(\Lambda\otimes\Lambda)^{inv} \simeq Reg\ (\mathbb{C},\Lambda^{\otimes 2}) & \rightarrow & Reg(A,\Lambda^{\otimes 2}) & \rightarrow & Reg(A^{\otimes 2},\Lambda^{\otimes 2}) & \rightarrow & Reg(A^{\otimes 3},\Lambda^{\otimes 2}) & \rightarrow ...
\end{array}$$

with "Sweedler" bialgebra cochains $C^n_{bis\omega}(A,\Lambda) = \bigoplus_{p+q=n+1,\ p\geq 0, q=1,2} Reg(A^{\otimes p},\Lambda^{\otimes q})$ classifies the bialgebra tensorial extensions for the trivial Λ-comodule structure on A (ie : $a \mapsto 1_\Lambda \otimes a$).

We note $Z^n_{bis\omega}(A,\Lambda)$, $B^n_{bis\omega}(A,\Lambda)$ and $H^n_{bis\omega}(A,\Lambda)$ the n-cocycles, n-coboundaries and n-cohomology of the above complex.

Proof : The "matched pair" property is reduced, for a trivial φ, to the fact that A must be a Λ-comodule Hopf algebra, which follows from [Ma1]. For the product see [Sw2]. For the compatibility between product and coproduct in the case of "an abelian matched pair" see [Si].

We consider the extension defined by $(\sigma,\theta)\in Z^2_{bisw}(A,\Lambda)$.

We have $(\lambda\sharp^\theta_\sigma a).{}_\sigma(\gamma\sharp^\theta_\sigma b)=\sum_{(a)}\lambda(a_{(1)}\bullet\gamma)\,\sigma(a_{(2)}\otimes b_{(1)})\ \sharp^\theta_\sigma\ a_{(3)}b_{(2)}$

$$\Delta_\theta(\lambda\sharp^\theta_\sigma a)=\sum_{(\lambda)(a)(a_{(1)})}(\lambda_{(1)}\ a^I_{(1)}\ \sharp^\theta_\sigma\ a_{(2)})\otimes(\lambda_{(2)}\ a^{II}_{(1)}\ \sharp^\theta_\sigma\ a_{(3)})$$

We have to show : $\Delta_\omega=\tilde P_\Lambda\ \cdot_\sigma\ \Delta_\theta\ \cdot_\sigma\ \tilde P_\Lambda^{-1}$ iff $\theta^{-1}*\omega$ is a coboundary with $\tilde P_\Lambda=\sum_{(P_\Lambda)}(p^{(1)}\ \sharp^\theta_\sigma\ 1_A)\otimes(p^{(2)}\ \sharp^\theta_\sigma\ 1_A)$.

To simplify notations, we shall write now $\sharp$ for $\sharp^\theta_\sigma$, . for $\cdot_\sigma$ and omit the "$\sum$".

$$\begin{aligned}\tilde P_\Lambda\ .\ \Delta_\theta(\lambda\sharp a)&=[(p^{(1)}\sharp 1_A)\otimes(p^{(2)}\sharp 1_A)].[(\lambda_{(1)}\ a^I_{(1)}\ \sharp\ a_{(2)})\otimes(\lambda_{(2)}\ a^{II}_{(1)}\ \sharp\ a_{(3)})]\\&=[(p^{(1)}\sharp 1_A).(\lambda_{(1)}\ a^I_{(1)}\ \sharp\ a_{(2)})]\otimes[(p^{(2)}\sharp 1_A).(\lambda_{(2)}\ a^{II}_{(1)}\ \sharp\ a_{(3)})]\\&=[p^{(1)}(1_A\bullet(\lambda,a^I_{(1)}))\ \tau(1_A\otimes a_{(2)(1)})\ \sharp\ a_{(2)(2)}]\\&\qquad\otimes[p^{(2)}(1_A\bullet(\lambda_{(2)}\ a^{II}_{(1)}))\ \tau(1_A\otimes a_{(3)(1)})\ \sharp\ a_{(3)(2)}]\end{aligned}$$

We use *normalized* cochains then $\tau(a\otimes 1_A)=\tau(1_A\otimes a)=\varepsilon(a),\ \forall a\in A$.

So : $\tilde P_\Lambda\ .\ \Delta_\theta(\lambda\sharp a)=(p^{(1)}\ \lambda_{(1)}\ a^I_{(1)}\ \sharp\ a_{(2)})\otimes(p^{(2)}\ \lambda_{(2)}\ a^{II}_{(1)}\ \sharp\ a_{(3)})$ applying the classical properties of counit.

We write $\tilde P_\Lambda^{-1}=\sum_{(P_\Lambda^{-1})}(\bar p^{(1)}\sharp 1_A)\otimes(\bar p^{(2)}\sharp 1_A)$

$$\begin{aligned}&\tilde P_\Lambda\ .\ \Delta_\theta(\lambda\sharp^\theta_\sigma a)\ .\ \tilde P_\Lambda^{-1}\\&=[p^{(1)}\ \lambda_{(1)}\ a^I_{(1)}(a_{(2)(1)}\bullet\bar p^{(1)})\ \tau(a_{(2)(2)}\otimes 1_A)\ \sharp\ a_{(2)(3)}]\\&\qquad\otimes[p^{(2)}\ \lambda_{(2)}\ a^{II}_{(1)}\ (a_{(3)(1)}\bullet\bar p^{(2)})\ \tau(a_{(3)(2)}\otimes 1_A)\ \sharp\ a_{(3)(3)}]\\&=[p^{(1)}\ \lambda_{(1)}\ a^I_{(1)}(a_{(2)(1)}\bullet\bar p^{(1)})\ \sharp\ a_{(2)(2)}]\otimes[p^{(2)}\ \lambda_{(2)}\ a^{II}_{(1)}\ (a_{(3)(1)}\bullet\bar p^{(2)})\ \sharp\ a_{(3)(2)}]\\&=[p^{(1)}\ \lambda_{(1)}\ a^I_{(1)}(a_{(2)(1)}\bullet\bar p^{(1)})\ \sharp\ a_{(3)}]\otimes[p^{(2)}\ \lambda_{(2)}\ a^I_{(1)}\ (a_{(1)}\bullet\bar p^{(2)})\ \sharp\ a_{(3)}]\end{aligned}$$

from coassociativity we have

$$\begin{aligned}\sum_{(a)}a_{(1)}\otimes a_{(2)}\otimes a_{(3)}\otimes a_{(4)}\otimes a_{(5)}&=\sum_{(a)}a_{(1)(1)}\otimes a_{(1)(2)(1)}\otimes a_{(1)(2)(2)}\otimes a_{(2)}\otimes a_{(3)}\\&=\sum_{(a)}a_{(1)(1)}\otimes a_{(1)(2)(1)}\otimes a_{(2)}\otimes a_{(1)(2)(2)}\otimes a_{(3)}\ \text{(by cocommutativity), so}\end{aligned}$$

$$\tilde P_\Lambda\cdot_\sigma\ \Delta_\theta(\lambda\sharp^\theta_\sigma a)\cdot_\sigma\tilde P_\Lambda^{-1}=\sum_{(a)(a_{(1)})(\lambda)}[p^{(1)}\lambda_{(1)}a^I_{(1)(2)}(a_{(1)(2)(1)}\bullet\bar p^{(1)})\ \sharp\ a_{(2)}]\otimes[p^{(2)}\lambda_{(2)}a^I_{(1)(1)}(a_{(1)(2)(2)}\bullet\bar p^{(2)})\ \sharp\ a_{(3)}]$$

if we write $\Delta_\omega(\lambda\sharp^\omega_\sigma a)=\sum_{(a)(a_{(1)})(\lambda)}[\lambda_{(1)}\ a^{\tilde I}_{(1)}\sharp a_{(2)}]\otimes[\lambda_{(2)}\ a^{\tilde II}_{(1)}\sharp a_{(3)}]$ where $\omega(a)=\sum_{(a)}a^{\tilde I}\otimes a^{\tilde II}$

we get $\Delta_\omega(\lambda\sharp^\omega_\sigma a)=\tilde P_\Lambda\ \cdot_\sigma\ \Delta_\theta(\lambda\sharp^\theta_\sigma a)\ \cdot_\sigma\ \tilde P_\Lambda^{-1}$ if and only if $\theta^{-1}*\omega=D^{02}\ (P_\Lambda^{-1})$.

Now, as $\Delta_\omega=(\phi\otimes\phi)\circ\Delta_\theta\circ\phi^{-1}\quad\Leftrightarrow\quad\theta^{-1}*\omega=D^{11}_*\ \xi_\phi$ [Si] for $\phi:B_\theta\to B_\omega$ isomorphism and $\xi_\phi\in Reg(A,\Lambda)$ attached to ϕ, we have the result.

■

Application to $D(\mathcal{D}(G))$: Let A be either $\mathcal{A}(G)$ or $\mathcal{D}(G)$, Λ be $\mathcal{H}(G)$ or $\mathcal{C}^\infty(G)$. Theorem 2 could have been written with continuous cochains. Then the continuous bialgebra tensorial extensions of $\Lambda (= A^*)$ and A, via

$$\begin{array}{lcl} \varphi : \Lambda \to \Lambda \bar{\otimes} A & \text{and} & \tau : A \bar{\otimes} \Lambda \to \Lambda \\ \quad \lambda \mapsto \lambda \otimes 1_A & & \quad a \otimes \lambda \mapsto \lambda \circ ad(a) \end{array},$$

are classified by the continuous version of the "$bis\omega$" 2-cohomology *with respect to the injective topological tensor product.* $D(A)$ is the trivial extension of Λ and A in this case. (For details about topologization of algebraic cohomological theory see [Ta]).

THEOREM 3. *Let G a simply connected, connected, compact Lie group. Then the "biSweedler" continuous 2-cohomology vanishes :*

$$H^2_{bis\omega,c}\left(\mathcal{D}(G), \mathcal{C}^\infty(G)\right) = \{0\}$$

If we extend the name "double" to all the extensions for which the "classical" quantum double is the zero class, then we show that $D(\mathcal{D}(G))$ is the only bialgebra and thus the only quasi-Hopf double structure on $\mathcal{D}(G)$. It is interesting (and is one of the motivations of this work) to see in [DPR] that for finite groups they find quasi-Hopf double structures apparently not equivalent to the classical one. Their notion of extension is however apparently somewhat different

Proof :

LEMMA 2. *We have the following long exact sequence*

$$\ldots \to H^{n-1}_{sw,c}(A, \Lambda^{\otimes 2}) \to H^n_{bis\omega,c}(A, \Lambda) \to H^n_{sw,c}(A, \Lambda) \to H^n_{sw,c}(A, \Lambda^{\otimes 2}) \to \ldots$$

The proof is exactly the same as for the analoguous result in [Bo], and we have the straightforward corollary.

COROLLARY 1. *If* $H^1_{sw,c}(A, \Lambda^{\otimes 2}) = \{0\} = H^2_{sw,c}(A, \Lambda)$ *then* $H^2_{bis\omega,c}(A, \Lambda) = \{0\}$.

LEMMA 3. $H^n_{sw,c}(\mathcal{D}(G), \Lambda) \simeq H^n_{gr,c}(G, \Lambda^{inv})$, *the continuous group cohomology.*

Proof : Sweedler shows in [Sw2] that $H^n_{sw}(\mathbb{C}G, \Lambda) = H^n_{gr}(G, \Lambda^{inv})$. We can adapt the proof to have

$$H^n_{sw,c}(\mathbb{C}G, \Lambda) = H^n_{gr,c}(G, \Lambda^{inv}) \qquad (1)$$

Indeed, if $f \in Reg_c(\mathbb{C}G^{\otimes n}, \Lambda)$ is a continuous invertible element, its restriction to G^n takes values in Λ^{inv} and is continuous. f^{-1} (on G) maps g into $[f(g)]^{-1}$ and is continuous.

Conversely if f is a continuous map : $G \to \Lambda^{inv}$, one can extend it by linearity to an element of $Reg_c(\mathbb{C}G^{\otimes n}, \Lambda)$. Since G^n is a basis of $\mathbb{C}G^{\otimes n}$ we obtain an isomorphism of complexes.

Now, as f^{-1} (for $f \in Reg_c(\mathbb{C}G^{\otimes n}, \Lambda)$) is continous, we can extend f and f^{-1} to $\overline{\mathbb{C}G^{\otimes n}} = \mathcal{D}(G)^{\otimes n}$ and they stay inverse to each other for $*$. Thus

$$Reg_c(\mathbb{C}G^{\otimes n}, \Lambda) \simeq Reg_c(\mathcal{D}^{\otimes n}, \Lambda) \qquad (2)$$

and this is an isomorphism of complexes. By (1) and (2) we have the result.

∎

Finally, we have to calculate $H^2_{gr,c}(G, \mathcal{C}^\infty(G)^{inv})$ and $H^1_{gr,c}(G, [\mathcal{C}^\infty(G)^{\otimes 2}]^{inv})$. We have the following exact sequence :

$$0 \to \mathbf{Z} \to \mathcal{C}^\infty(G) \xrightarrow{exp} \mathcal{C}^\infty(G)^{inv} \to 0$$

where *exp* is the exponential. Then by the classical long exact sequence of cohomology and the fact that $H^2_{gr,c}(G, \mathbf{Z}) = \{0\} = H^2_{gr,c}(G, \mathcal{C}^\infty(G))$ for G compact, we have $H^2_{gr,c}(G, \mathcal{C}^\infty(G)^{inv}) = \{0\}$.

In the same way, knowing that $\mathcal{C}^\infty(G)\hat{\otimes}\mathcal{C}^\infty(G) = \mathcal{C}^\infty(G)\bar{\otimes}\mathcal{C}^\infty(G) = \mathcal{C}^\infty(G\times G)$ we have $0 \to \mathbf{Z}\times\mathbf{Z} \to \mathcal{C}^\infty(G\times G) \to \mathcal{C}^\infty(G\times G)^{inv} \to 0$ and therefore, $H^1_{gr,c}(G, [\mathcal{C}^\infty(G)\bar{\otimes}\mathcal{C}^\infty(G)]^{inv}) = \{0\}$.

Theorem 3 then follows from Lemma 3 and Corollary 1.

∎

4. CONCLUSION.

The paper can be completed in several directions. First it is interesting to look more in details at the explicit structure of the dual of the double (namely at its coproduct). Using $\mathcal{D}(G)\hat{\otimes}\mathcal{C}^\infty(G) \simeq \mathcal{C}^\infty(G,\mathcal{D}(G)) \simeq \mathcal{D}(G,\mathcal{C}^\infty(G))$, this should be achieved by an adaptation of the formulas given by Majid [Ma1] for finite groups.

Secondly, it should be possible to take a general coaction ψ and not just the trivial one in Theorem 2.

Third, Theorem 3 should also be true for $\mathcal{A}(G)$. For this one needs to prove

$$H^1_{gr,c}(G, [\mathcal{H}(G)\bar{\otimes}\mathcal{H}(G)]^{inv}) = \{0\} = H^2_{gr,c}(G, \mathcal{H}(G)^{inv}).$$

Finally this extension theory should be developed in the quantum groups domain. Indeed paragraph 3 is just a first step, an example of the techniques and the concepts to use.

But to look at extensions of a Hopf algebra A and an arbitrary A-module algebra M we must have a theory which works with a non commutative M and a non-cocommutative A (or to begin with a, may be, easier problem, extensions of deformations of a cocommutative A and a M ; this will be enough for the quantum group cases of $\mathcal{A}_t(G)$ or $\mathcal{D}_t(G)$). This seems not straightforward but there are some interesting examples which might be useful : Majid gives some in [Ma2] and Truini and Varadarajan try in [TV], to define a "quantum extension" of two algebras of coefficients of (compact) semi-simple groups. This is very interesting because extending two semi-simple Lie algebras, one by the other, is always trivial in the category of Lie algebras. But is it the same in the category of Hopf algebras, or bialgebras (non necessarily coassociative) ? And for two "quantum groups" ?

The [TV] example is not trivial in the sense that it is not a smash-product. Moreover, this extension seems not to be one in the sense of Molnar [Mo], who gives a very natural and interesting concept of extensions of Hopf algebras since it is valid in every case, even for non (co)-commutative ones. But he does not give a cohomological classification for these. It is the difficult point because we shall certainly have to go to the domain of non commutative cohomology theory.

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