An inverse scattering result for several convex bodies

Latchezar Stoyanov

1 Introduction

Let K be a compact subset of $\mathbf{R}^n, n \geq 3, n$ odd, with C^∞ boundary ∂K such that

$$\Omega_K = \overline{\mathbf{R}^n \setminus K}$$

is connected. The *scattering operator* related to the wave equation in $\mathbf{R} \times \Omega$ with Dirichlet boundary condition on $\mathbf{R} \times \partial\Omega$ can be represented as a unitary operator

$$S : L^2(\mathbf{R} \times S^{n-1}) \longrightarrow L^2(\mathbf{R} \times S^{n-1})$$

(see [LP1]). The kernel of $S-$Id, which can be considered as a distribution

$$s_K(t, \theta, \omega) \in \mathcal{D}'(\mathbf{R} \times S^{n-1} \times S^{n-1}),$$

is called the *scattering kernel.*

It was shown by A.Majda [Ma] that for each obstacle K the convex hull of K can be recovered if one knows sing supp $s_K(t, -\omega, \omega)$ for a dense set of ω's in S^{n-1} (see also [LP2] for a similar result). Consequently, an obstacle K is completely determined by the singularities of the scattering kernel, provided we know in advance that K is convex. Moreover, one can distinguish between convex and strictly convex obstacles by the same data (see [So] and [Y]). A general problem arises: is an obstacle K completely determined if we know sing supp $s_K(t, \theta, \omega)$ for all $\theta, \omega \in S^{n-1}$?

In this paper we consider obstacles K of the form

$$K = \bigcup_{j=1}^{s} K_j \quad , \quad K_i \cap K_j = \emptyset \quad \text{for} \quad i \neq j, \qquad K_j \text{ convex for all } j = 1, \dots, s, \qquad (1)$$

which satisfy the following condition introduced by M.Ikawa [I]:

$$\text{(H)} \qquad (K \setminus (K_i \cup K_j)) \cap \text{ convex hull } (K_i \cup K_j) = \emptyset \quad \text{for all} \quad i, j = 1, \dots, s.$$

We show that among the obstacles with these properties, K can be completely recovered if one knows sing supp $s_K(t,\theta,\omega)$ for almost all $(\omega,\theta) \in S^{n-1} \times S^{n-1}$.

To state precisely the corresponding result, consider another obstacle L of the same type, namely

$$L = \bigcup_{j=1}^{r} L_j \quad , \quad L_i \cap L_j = \emptyset \quad \text{for} \quad i \neq j, \qquad L_j \text{ convex for all } j = 1, \ldots, r. \tag{2}$$

1.1. Theorem. *Let K and L have the form* (1) *and* (2)*, respectively, and let K and L satisfy the condition* (H). *Assume that there exists a subset $\mathcal{R}$ of full Lebesgue measure in $S^{n-1} \times S^{n-1}$, such that for all $(\omega,\theta) \in \mathcal{R}$ we have*

$$\text{sing supp} s_K(t,\theta,\omega) \subset \text{sing supp} s_L(t,\theta,\omega). \tag{3}$$

Then for every $i = 1, \ldots, s$ there exists $j_i = 1, \ldots, r$ with $K_i = L_{j_i}$. In particular, $K \subset L$.

As an immediate consequence one gets the following.

1.2. Corollary. *Let K and L have the form* (1) *and* (2)*, respectively, and let both K and L satisfy the condition* (H). *If there exists a subset $\mathcal{R}$ of full Lebesgue measure in $S^{n-1} \times S^{n-1}$ such that*

$$\text{sing supp } s_K(t,\theta,\omega) = \text{sing supp } s_L(t,\theta,\omega)$$

holds for all $(\omega,\theta) \in \mathcal{R}$, then $K = L$.

It is seen from the proof that the assertions of both Theorem 1.1 and Corollary 1.2 remain true if R is *residual* in $S^{n-1} \times S^{n-1}$ (instead of having full Lebesgue measure), i.e. if R contains a countable inersection of open dense subsets of $S^{n-1} \times S^{n-1}$.

Acknowledgement. This paper has been written while the author was visiting the University of Bordeaux I as chercheur associé of C.N.R.S.

2 Proof of Theorem 1.1

Throughout we assume that K and L satisfy the assumptions of Theorem 1.1. Let $\mathcal{R}$ be a subset of full Lebesgue measure of $S^{n-1} \times S^{n-1}$ such that (3) holds for all $(\omega,\theta) \in \mathcal{R}$. We are going to show that each convex component of K coincides with some convex component of L. To do this we shall use the connection between singularities of the scattering kernel and sojourn times of reflecting (ω,θ)-rays.

Let $\omega \in S^{n-1}, \theta \in S^{n-1}$. By a *reflecting* (ω,θ)*-ray* in Ω_K, we mean an infinite ray in Ω_K with incoming direction ω and outgoing direction θ making finitely many reflections at $\partial\Omega$ according to the law of the geometrical optics (see [PS1] or [PS2] for the precise definition). The *sojourn time* T_γ of γ is defined by

$$T_\gamma = \langle \omega, x_1 \rangle + \sum_{i=1}^{s-1} \|x_i - x_{i+1}\| - \langle \theta, x_s \rangle,$$

where $x_1, \dots, x_s$ are the successive reflection points of γ ([G]). If γ has no segments tangent to ∂K, then it is called an *ordinary* reflecting (ω,θ)-ray. By $\mathcal{L}_{\omega,\theta}(K)$ we denote the *set of all reflecting* (ω,θ)*-rays in* Ω_K.

It follows from Theorem 1.2 in [PS3] that there exists $\mathcal{R}' \subset S^{n-1} \times S^{n-1}$ having full Lebesgue measure in $S^{n-1} \times S^{n-1}$ (and residual at the same time) such that for $(\omega,\theta) \in \mathcal{R}'$ we have

$$\text{sing supp}\, s_K(t,\theta,\omega) = \{-T_\gamma : \gamma \in \mathcal{L}_{\omega,\theta}(K)\}$$

and

$$\text{sing supp}\, s_L(t,\theta,\omega) = \{-T_\gamma : \gamma \in \mathcal{L}_{\omega,\theta}(L)\}.$$

Then

$$\mathcal{R}'' = \mathcal{R} \cap \mathcal{R}'$$

is a subset of full Lebesgue measure in $S^{n-1} \times S^{n-1}$, and according to (3) and the above inclusions, one gets

$$\{-T_\gamma : \gamma \in \mathcal{L}_{\omega,\theta}(K)\} \subset \{-T_\gamma : \gamma \in \mathcal{L}_{\omega,\theta}(L)\} \quad , \quad (\omega,\theta) \in \mathcal{R}''. \tag{4}$$

Given $x \in \partial K$, we denote by $N_K(x)$ the *unit normal vector* to ∂K at x pointing into Ω_K. Thus

$$N_K : \partial K \longrightarrow S^{n-1}$$

is the *Gauss map* of ∂K.

Theorem 1.1 will be easily derived by the following lemma.

2.1. Lemma. *For every $x \in \partial K$ there exists $y \in \partial L$ such that*

$$N_L(y) = \pm N_K(x) \quad , \quad \langle x - y, N_K(x) \rangle = 0. \tag{5}$$

That is, the tangential hyperplane to K at x coincides with the tangential hyperplane to L at some $y \in \partial L$.

The proof of Lemma 2.1 is rather long and will be devided into several steps.

2.2. Lemma. *Let Y and Z be smooth convex (concave) hypersurfaces in $\mathbf{R}^n$ with $Y \cap Z = \emptyset$. Let $\Sigma(Y,Z)$ be the set of those $(y,z) \in Y \times Z$ such that the tangent* *hyperplanes to Y at y and Z at z coincide and the Gauss maps of Y at y and of Z at z are regular. Then $\Sigma(Y,Z)$ is a smooth $(n-2)$-dimensional submanifold of $Y \times Z$.*

Proof. Take smooth charts

$$\varphi : V \longrightarrow Y \quad , \quad \psi : W \longrightarrow Z,$$

$V, W \subset \mathbf{R}^{n-1}$, around some points $y_0 \in Y$ and $z_0 \in Z$ with $(y_0, z_0) \in \Sigma(Y,Z)$. For $v \in V$, denote by $N_Y(v)$ one of the unit normal vectors to Y at $\varphi(y)$ so that N_Y is continuos on V. In the same way we define N_Z on W. Since $(y_0, z_0) \in \Sigma(Y,Z)$, the normals to Y at y_0 and to Z at z_0 are collinear, and (choosing appropriately the normal fields N_Y and N_Z) we may assume that these normals coincide. We may also assume that N_Y is regular on V, N_Z is regular on W and

$$N_Y^{(n)}(v) \neq 0 \quad , \quad v \in V \quad , \quad N_Z^{(n)}(w) \neq 0 \quad , \quad w \in W.$$

Then $N_Y^{(n)}(v)$ and $N_Z^{(n)}(w)$ have one and the same sign for all $v \in V$ and $w \in W$.

Define the functions

$$f_i, g : V \times W \longrightarrow \mathbf{R}$$

by $f_i(v,w) = N_Y^{(i)}(v) - N_Z^{(i)}(w), i = 1, \ldots, n-1$ and $g(v,w) = \langle \varphi(v) - \psi(w), N_Y(v) \rangle$, and consider $F : V \times W \longrightarrow \mathbf{R}^n$ given by

$$F(v,w) = (f_1(v,w), \ldots, f_{n-1}(v,w), g(v,w)).$$

Then F is smooth and we have

$$\{(v,w) \in V \times W : (\varphi(v), \psi(w)) \in \Sigma(Y,Z)\} = F^{-1}(0).$$

Therefore the assertion will be proved if we show that $F^{-1}(0)$ is a smooth $(n-2)$-dimensional submanifold of $V \times W$. To do this it is sufficient to establish that F is submersion on $F^{-1}(0)$ (cf. [GG]).

Fix $(v,w) \in F^{-1}(0)$ and let

$$\sum_{i=1}^{n-1} A_i \mathrm{grad}_{v,w} f_i(v,w) + B \mathrm{grad}_{v,w} g(v,w) = 0 \tag{6}$$

for some reals A_i, B. Considering in (6) the derivatives with respect to w_j $(1 \leq j \leq n-1)$, we get

$$-\sum_{i=1}^{n-1} A_i \frac{\partial N_Z^{(i)}}{\partial w_j}(w) - B \langle \frac{\partial \psi}{\partial w_j}(w), N_Y(v) \rangle = 0. \tag{7}$$

Set $A_n = 0$ and $A = (A_1, \dots, A_n) \in \mathbf{R}^n$. Since $F(v,w) = 0$, we have $f_i(v,w) = 0$ for all $i = 1, \dots, n-1$, and therefore $N_Y(v) = N_Z(w)$. Therefore (7) implies

$$0 = \langle A, \frac{\partial N_Z}{\partial w_j}(w)\rangle + B\langle \frac{\partial \psi}{\partial w_j}(w), N_Z(w)\rangle = \langle A, \frac{\partial N_Z}{\partial w_j}(w)\rangle.$$

Since N_Z is regular at w, we have that $\{\frac{\partial N_Z}{\partial w_j}(w)\}_{j=1}^{n-1}$ is a linear basis in the tangent hyperplane $T_{\psi(w)}Z$. Hence the latter equality implies $A = \lambda N_Z(w)$ for some $\lambda \in \mathbf{R}$. Consequently, $0 = A_n = \lambda N_Z^{(n)}(w)$, which shows that $\lambda = 0$. Thus, $A = 0$.

Next, consider in (6) the derivatives with respect to v_j ($j = 1, \dots, n-1$). We get

$$B(\langle \frac{\partial \varphi}{\partial v_j}(v), N_Y(v)\rangle + \langle \varphi(v) - \psi(w), \frac{\partial N_Y}{\partial v_j}(v)\rangle) = 0,$$

and consequently

$$B\langle \varphi(v) - \psi(w), \frac{\partial N_Y}{\partial v_j}(v)\rangle = 0.$$

Since $F(v,w) = 0$, we have that $N_Y(v) = N_Z(w)$ and $\varphi(v) - \psi(w)$ is orthogonal to $N_Y(v)$. On the other hand, $\{\frac{\partial N_Y}{\partial v_j}(v)\}_{j=1}^{n-1}$ is a linear basis of $T_{\varphi(v)}Y$, so there exists $j = 1, \dots, n-1$ with

$$\langle \varphi(v) - \psi(w), \frac{\partial N_Y}{\partial v_j}(v)\rangle \neq 0.$$

This yields $B = 0$ which proves the assertion. ♠

Now we begin with the *proof of Lemma* 2.1.

For $i = 1, \dots, s$ denote by $\mathcal{N}_{K,i}$ the set of those $\omega \in S^{n-1}$ such that ω is a regular value of the restriction of N_K to ∂K_i. In the same way we indtroduce the sets $\mathcal{N}_{L,j}$ for $j = 1, \dots, r$. It follows by the Sard theorem that the sets $\mathcal{N}_{K,i}, \mathcal{N}_{L,j}$ have full Lebesgue measure in S^{n-1}. Moreover, they are open and dense in S^{n-1}, so

$$\mathcal{N} = (\bigcap_{i=1}^{s} \mathcal{N}_{K,i}) \bigcap (\bigcap_{j=1}^{r} \mathcal{N}_{L,j})$$

is an open and dense subset of S^{n-1} having full Lebesgue measure.

Note that to prove Lemma 2.1 it is sufficient to consider the case when $N_K(x) \in \mathcal{N}$. Indeed, assume that for every $x' \in \partial K$ with $N_K(x') \in \mathcal{N}$ there exists $y' \in \partial L$ with $N_L(y') = \pm N_K(x')$, $\langle x' - y', N_K(x')\rangle = 0$. Given an arbitrary $x \in \partial K$, there exists a sequence $\{x_m\} \subset \partial K$ with $N_K(x_m) \to N_K(x)$ and $N_K(x_m) \in \mathcal{N}$ for each m. Taking a subsequence, we may assume that all x_m belong to one and the same convex component K_i of K and $x_m \to x'' \in \partial K$. Then $N_K(x'') = N_K(x)$ and the convexity of K_i implies $x \in T_{x''}K_i$. Since $N_K(x_m) \in \mathcal{N}$, by our assumption there exists $y_m \in \partial L$ with $N_L(y_m) = \pm N_K(x_m)$, $\langle x_m - y_m, N_K(x_m)\rangle = 0$. Passing again to an appropriate subsequence, we may assume $y_m \to y \in \partial L$. Then we have $N_L(y) = \pm N_K(x'')$, $\langle x'' - y, N_K(x'')\rangle = 0$. Combining this with $N_K(x) = N_K(x'')$ and $x \in T_{x''}\partial K$, we see that (5) hold.

Fix an arbitrary $x^{(0)} \in \partial K$ with $N_K(x^{(0)}) \in \mathcal{N}$. We want to find $y^{(0)} \in \partial L$ such that $T_{x^{(0)}}\partial K = T_{y^{(0)}}\partial L$. We may assume that $x^{(0)} \in \partial K_1$. It is sufficient to show that for each sufficiently small neighbourhood U of $x^{(0)}$ in ∂K_1 there exist $x \in U$ and $y \in \partial L$ such that $T_x\partial K = T_y\partial L$. If this is so, then the compactness of ∂L implies the existence of a point $y^{(0)} \in \partial L$ with $T_{x^{(0)}}\partial K = T_{y^{(0)}}\partial L$.

Let U_0 be a small neighbourhood of $x^{(0)}$ in ∂K_1 such that $N_K(x) \in \mathcal{N}$ for all $x \in U_0$. The first step in our proof will be to choose in a special way a vector $\omega_0 \in S^{n-1}$ tangent to ∂K_1 at some $x \in U_0$.

Given $i \neq j, i, j = 1, \dots, r$, denote by $\Sigma_{i,j}$ the set of those $\omega \in S^{n-1}$ such that $\omega = \frac{z-y}{\|z-y\|}$ for some $(y, z) \in \Sigma(\partial L_i, \partial L_j)$ (see the notation in Lemma 2.2). That is,

$$\Sigma_{i,j} = h_{i,j}(\Sigma(\partial L_i, \partial L_j)),$$

where

$$h_{i,j} : \partial L_i \times \partial L_j \longrightarrow S^{n-1}$$

is defined by

$$h_{i,j}(y, z) = \frac{z - y}{\|z - y\|}.$$

Since $h_{i,j}$ is smooth, it follows by the Sard theorem and Lemma 2.2 that $\Sigma_{i,j}$ is a finite union of compact subsets of measure zero in S^{n-1}. Therefore

$$\mathcal{S} = S^{n-1} \setminus (\bigcup_{i\neq j} \Sigma_{i,j})$$

is a subset of full Lebesque measure in S^{n-1}. In particular, $\mathcal{S}$ is dense in S^{n-1}.

Next, using the fact that K satisfies the condition (H), one can find a line l in $\mathbf{R}^n$ with $l \cap K = \{x^{(0)}\}$. Indeed, for each $i = 2, \dots, s$, $K_i' = K_i \cap T_{x^{(0)}}\partial K$ is a convex domain in $T_{x^{(0)}}\partial K$ with $x^{(0)} \notin K_i'$. Let

$$C_i = \{x^{(0)} + tz : t \geq 0, z \in K_i'\}.$$

Then C_i is a cone in $T_{x^{(0)}}\partial K$ with vertex $x^{(0)}$. The condition (H) implies $C_i \cap (\pm C_j) = \{x^{(0)}\}$ for all $i \neq j, i, j = 2, \dots, s$. It is then clear that there exists a line $l \subset T_{x^{(0)}}\partial K$ through $x^{(0)}$ with $l \cap C_i = \{x^{(0)}\}$ for all $i = 2, \dots, s$.

Consider an arbitrary line l with this property and denote by ξ one of the directions of l, $\xi \in S^{n-1}$. The compactness of $K \setminus K_1$ and the regularity of N_K at $x^{(0)}$ show that for all $\omega \in S^{n-1}$ sufficiently close to ξ there exists $x \in U_0$ (not unique) so that ω is tangent to ∂K_1 at x and the line $l(x, \omega)$, determined by x and

ω has no common points with $K \setminus K_1$. Since $\mathcal{S}$ is dense in S^{n-1}, there exist $\omega_0 \in \mathcal{S}$ and $x \in U_0$ with

$$l(x, \omega_0) \cap (K \setminus K_1) = \emptyset,$$

$l(x, \omega_0)$ being tangent to ∂K_1 at x. **Fix ω_0 and x with these properties**.

Since x will be fixed in the next arguments, without loss of generality we may **assume that** $x = 0$, i.e. x coincides with the initial point of the coordinate system in $\mathbf{R}^n$. We shall also assume that

$$N_K(0) = e = (0, \dots, 0, 1).$$

For each

$$\omega \in S^{n-1} \quad , \quad \langle \omega, e \rangle < 0, \tag{8}$$

define

$$\theta(\omega) = \omega - 2\langle \omega, e \rangle e \in S^{n-1}.$$

Then $\theta(\omega)$ is symmetric to ω with respect to $T_0 \partial K$. Consequently, for each ω with (8) there exists a reflecting $(\omega, \theta(\omega))$-ray $\gamma(\omega)$ in Ω_{K_1} with exactly one reflection point 0. It then follows by the choice of ω_0 that if $\delta > 0$ is small enough and

$$\mid \omega - \omega_0 \mid < \delta, \tag{9}$$

then $\gamma(\omega)$ is contained in Ω_K and moreover $\gamma(\omega)$ is an ordinary reflecting $(\omega, \theta(\omega))$-ray in Ω_K.

Fix $\delta > 0$ with the latter property. Note that

$$T_{\gamma(\omega)} = \langle \omega, x \rangle - \langle \theta, x \rangle = 0,$$

since $x = 0$ by our assumption.

2.3. Lemma. *For each ω with* (8) *and* (9) *there exists a reflecting* $(\omega, \theta(\omega))$*-ray $\gamma'(\omega)$ in Ω_L with* $T_{\gamma'(\omega)} = 0$.

Proof of Lemma 2.3. Fix ω with (8) and (9). Since $\mathcal{R}''$ is dense in $S^{n-1} \times S^{n-1}$, $(\omega, \theta(\omega))$ can be approximated by elements of $\mathcal{R}''$. Let $\{(\omega_m, \theta_m)\} \subset \mathcal{R}''$ be a sequence with $\omega_m \to \omega$, $\theta_m \to \theta(\omega)$. It is easy to see that for sufficiently large m, there exists a (unique) reflecting (ω_m, θ_m)-ray γ_m in Ω_K with one reflection point $x_m \in \partial K_1$. We may assume that there exists $\lim x_m = y \in \partial K_1$. Since

$$N_K(y) = \lim N_K(x_m) = \lim \frac{\theta_m - \omega_m}{\|\theta_m - \omega_m\|} = \frac{\theta(\omega) - \omega}{\|\theta(\omega) - \omega\|} = N_K(0),$$

and N_K is regular at $0 = x \in U_0$, we find that $y = 0$, i.e. $x_m \to 0$. So we may also assume that $\|x_m\| \leq 1/2$ for all m.

It follows from $(\omega_m, \theta_m) \in \mathcal{R}''$ and (4) that there exists a reflecting (ω_m, θ_m)-ray γ'_m in Ω_L with

$$T_{\gamma'_m} = T_{\gamma_m} = \langle \omega_m, x_m \rangle - \langle \theta_m, x_m \rangle = \langle \omega_m - \theta_m, x_m \rangle. \tag{10}$$

In particular, $T_{\gamma'_m} \leq 1$ for all m. This implies that the reflecting rays γ'_m have bounded number of reflection points. Indeed, let d be the *minimal distance* between convex components of L and let D be the *radius of a ball with center* 0 *containing* L. If k_m is the number of reflection points of γ'_m, then we clearly have

$$1 \geq T_{\gamma'_m} \geq (k_m - 1)d - 2D,$$

and so $k_m \leq \frac{1+2D}{d} + 1$.

Now passing to an appropriate subsequence, we may assume that each γ'_m has exactly k reflection points (k does not depend on m). Moreover, we may assume that for each m and each $i = 1, \ldots, k$ the i-th reflection point $x_i^{(m)}$ of γ'_m belongs to one and the same convex component L_{j_i} of L and there exists $\lim_{m\to\infty} x_i^{(m)} = x_i \in \partial L_{j_i}$. It is then easy to see that $x_1, \ldots, x_k$ are the successive reflection points of a reflecting (not necessarily ordinary) $(\omega, \theta(\omega))$-ray $\gamma'(\omega)$ in Ω_L with (cf (10))

$$T_{\gamma'(\omega)} = \lim_{m\to\infty} T_{\gamma'_m} = 0.$$

This proves the assertion. ♠

Denote by $k(\omega)$ the number of reflection points of the reflecting $(\omega, \theta(\omega))$-ray $\gamma'(\omega)$ from Lemma 2.3. The next step will be to show that $k(\omega) \leq 2$ for all ω with (8) and (9), provided δ is chosen small enough. To do this we use the fact that L satisfies the condition (H).

2.4. Lemma. *There exists $\varphi_0 \in (0, \frac{\pi}{2})$ such that if X, Y, Z are three convex components of L with $X \neq Y, Y \neq Z$ and $\xi \in X, \eta \in Y, \zeta \in Z$, then the angle between the vectors $\eta - \xi$ and $\zeta - \eta$ is not less than φ_0.*

The proof of this assertion is an easy consequence of the condition (H) and the compactness of L. We omit the details. ♠.

We continue with the proof of Lemma 2.1. Assume that

$$\delta < \frac{d}{D} \sin^2 \frac{\varphi_0}{4}, \tag{11}$$

and fix ω with (8) and (9). Let $y_1, \ldots, y_k$ ($k = k(\omega)$) be the successive reflection points of the reflecting $(\omega, \theta(\omega))$-ray $\gamma'(\omega)$ in Ω_L (see Lemma 2.3). Then we have

$$0 = T_{\gamma'(\omega)} \quad = \quad \langle \omega, y_1 \rangle + \sum_{i=1}^{k-1} \|y_i - y_{i+1}\| - \langle \theta(\omega), y_k \rangle$$

$$= \quad \langle \omega - \theta(\omega), y_k\rangle + \sum_{i=1}^{k-1} [\|y_{i+1} - y_i\| - \langle \omega, y_{i+1} - y_i\rangle].$$

Since $| \langle \omega - \theta(\omega), y_k\rangle | \leq 2D\delta$ and

$$\|y_{i+1} - y_i\| - \langle \omega, y_{i+1} - y_i\rangle = (1 - \cos\varphi_i)\|y_{i+1} - y_i\|,$$

$\varphi_i \in [0, \frac{\pi}{2}]$ being the angle between the vectors ω and $y_{i+1} - y_i$, we find

$$2D\delta \geq 2\sum_{i=1}^{k-1} \|y_{i+1} - y_i\| \sin^2 \frac{\varphi_i}{2} \geq 2d \sum_{i=1}^{k-1} \sin^2 \frac{\varphi_i}{2}.$$

Combining this with (11), one obtains

$$\sin^2 \frac{\varphi_i}{2} \leq \frac{D}{d}\delta < \sin^2 \frac{\varphi_0}{4}$$

and therefore $\varphi_i < \frac{\varphi_0}{2}$ for each $i = 1, \ldots, k-1$.

Assume that $k > 2$ and denote by φ the angle between the vectors $y_2 - y_1$ and $y_3 - y_2$. Since $\varphi \leq \varphi_1 + \varphi_2$, we get $\varphi < \varphi_0$, which is a contradiction with the choice of φ_0 (see Lemma 2.4). Thus, $k = k(\omega) \leq 2$.

In this way we have shown that, if δ satisfies (11), then $k(\omega) \leq 2$ for each ω with (8) and (9).

Next, we assume that δ satisfies (11). Given ω with (8) and (9), we have $\omega \neq \theta(\omega)$, and so $\gamma'(\omega)$ has at least one *transversal reflection point* y, that is $\gamma'(\omega)$ is not tangent to ∂L at y. The final step in the proof of Lemma 2.1 will be to show that there exists ω with (8) and (9) so that $\gamma'(\omega)$ has exactly one transversal reflection point. This will be derived by the following lemma.

2.5. Lemma. *Let $P \neq Q$ be two convex components of L and let $\mathcal{T}$ be the set of those ω with* (8) *and* (9) *such that $\gamma'(\omega)$ has two transversal reflection points, the first of which belongs to P and the second to Q. Assume that ω_0 is a point of Lebesgue measure density of $\mathcal{T}$ in S^{n-1}. Then there exist $y \in P, z \in Q$ such that*

$$N_L(y) = \pm N_L(z) = \pm e \quad , \quad \frac{z - y}{\|z - y\|} = \omega_0.$$

Proof of Lemma 2.5. Recall that by our assumptions we have $x = 0$ and $N_K(0) = e = (0, \ldots, 0, 1)$. Consequently, $\omega_0 = (u_0; 0)$ for some $u_0 \in \mathbf{R}^{n-1}$ with $\|u_0\| = 1$. Set

$$B_0 = \{u \in \mathbf{R}^{n-1} : \|u\| < 1\}.$$

For $u = (u_1, \ldots, u_{n-1}) \in B_0$ define

$$\omega(u) = (u; -\sqrt{1 - \|u\|^2}) \ , \ \theta(u) = \theta(\omega(u)) = (u; \sqrt{1 - \|u\|^2}).$$

Clearly, if u is sufficiently close to u_0, then $\omega(u)$ satisfies (8) and (9). Therefore u_0 is a point of positive Lebesgue measure density of

$$U' = \{u \in B_0 : \omega(u) \in \mathcal{T}\}$$

in $\mathbf{R}^{n-1}$.

Let $u' \in U'$ be sufficiently close to u_0. Then there exist $y(u') \in P$ and $z(u') \in Q$ which are the successive reflection points of a reflecting $(\omega(u), \theta(u))$-ray in Ω_L. This is in fact the ray $\gamma'(\omega(u))$, and so $T_{\gamma'(\omega(u))} = 0$.

Set $L' = P \cup Q$. It is clear that there exists an open U with $U' \subset U \subset B_0$ such that for each $u \in U$ there exist unique $y(u) \in \partial P, z(u) \in \partial Q$ which are the successive reflection points of a reflecting $(\omega(u), \theta(u))$-ray $\gamma'(u)$ in $\Omega_{L'}$. However, the sojourn time $T(u) = T_{\gamma'(u)}$ might be non-zero. It is easy to see that $y(u)$ and $z(u)$ are smooth maps of $u \in U$ (cf. for example Appendix b in [Sj] or Chapter 10 in [PS2]). Hence

$$T(u) = \langle \omega(u), y(u)\rangle + \|y(u) - z(u)\| - \langle \theta(u), z(u)\rangle$$

is a smooth function of u. Let us calculate $\frac{\partial T}{\partial u_i}$ for $u \in U$ and $i = 1, \ldots, n-1$. We have

$$\begin{aligned}\frac{\partial T}{\partial u_i}(u) \;&=\; \langle \frac{\partial \omega}{\partial u_i}(u), y(u)\rangle + \langle \omega(u), \frac{\partial y}{\partial u_i}(u)\rangle + \langle \frac{z(u) - y(u)}{\|z(u) - y(u)\|}, \frac{\partial z}{\partial u_i}(u) - \frac{\partial y}{\partial u_i}(u)\rangle \\ &\quad -\langle \frac{\partial \theta}{\partial u_i}(u), z(u)\rangle - \langle \theta(u), \frac{\partial z}{\partial u_i}(u)\rangle.\end{aligned}$$

Since $\frac{\partial y}{\partial u_i}(u)$ is tangent to ∂P at $y(u)$, it follows by the reflection law at $y(u)$ that

$$\langle \frac{z(u) - y(u)}{\|z(u) - y(u)\|}, \frac{\partial y}{\partial u_i}(u)\rangle = \langle \omega(u), \frac{\partial y}{\partial u_i}(u)\rangle.$$

Similarly, one finds

$$\langle \frac{z(u) - y(u)}{\|z(u) - y(u)\|}, \frac{\partial z}{\partial u_i}(u)\rangle = \langle \theta(u), \frac{\partial z}{\partial u_i}(u)\rangle.$$

Using the latter two equalities, we deduce from above that

$$\frac{\partial T}{\partial u_i}(u) = \langle \frac{\partial \omega}{\partial u_i}(u), y(u)\rangle - \langle \frac{\partial \theta}{\partial u_i}(u), z(u)\rangle. \tag{12}$$

Fix $\epsilon \in (0, 1)$ such that $u \in U, \|u - u_0\| < \epsilon$ imply $\|\omega(u) - \omega_0\| < \delta$. Clearly, there exist points $u \in U$ with $\|u - u_0\| < \epsilon$ which are points of positive Lebesgue measure density of U' (otherwise U' would have measure zero). Fix a point $\tilde{u}$ with these properties. We claim that

$$\frac{\partial T}{\partial u_i}(\tilde{u}) = 0 \quad , \quad i = 1, \ldots, n-1. \tag{13}$$

Consider an arbitrary $i = 1, \ldots, n-1$. It is sufficient to show that each neighbourhood V of $\tilde{u}$ in $\mathbf{R}^{n-1}$ contains a point v with $\frac{\partial T}{\partial u_i}(v) = 0$. Fix a cubic neighbourhood V of $\tilde{u}$ and let $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ be the i-th basis vector in $\mathbf{R}^{n-1}$. Since $V \cap U'$ has positive measure in $\mathbf{R}^{n-1}$, there exists $v = (v_1, \ldots, v_{n-1}) \in V$ such that $v + se_i \in V \cap U'$ for at least two different values of $s \in \mathbf{R}$. For such values of s the definition of U' implies $T(v + se_i) = 0$. Therefore there exists $v' = v + s'e_i \in V$ with $\frac{\partial T}{\partial u_i}(v') = 0$. This proves (13).

Now it follows from (12) and (13) that

$$\langle \frac{\partial \omega}{\partial u_i}(\tilde{u}), y(\tilde{u}) \rangle - \langle \frac{\partial \theta}{\partial u_i}(\tilde{u}), z(\tilde{u}) \rangle = 0 \quad , \quad i = 1, \ldots, n-1.$$

On the other hand,

$$\frac{\partial \omega}{\partial u_i}(\tilde{u}) = e_i + \frac{\tilde{u}_i}{\sqrt{1 - \|\tilde{u}\|^2}} e \quad , \frac{\partial \theta}{\partial u_i}(\tilde{u}) = e_i - \frac{\tilde{u}_i}{\sqrt{1 - \|\tilde{u}\|^2}} e,$$

therefore

$$y_i(\tilde{u}) + \frac{\tilde{u}_i}{\sqrt{1 - \|\tilde{u}\|^2}} y_n(\tilde{u}) - z_i(\tilde{u}) + \frac{\tilde{u}_i}{\sqrt{1 - \|\tilde{u}\|^2}} z_n(\tilde{u}) = 0.$$

That is

$$z_i(\tilde{u}) - y_i(\tilde{u}) = \lambda \tilde{u}_i \quad , \quad i = 1, \ldots, n-1$$

with $\lambda = \frac{z_n(\tilde{u}) + y_n(\tilde{u})}{\sqrt{1 - \|\tilde{u}\|^2}}$. Consequently, there exists $\mu \in \mathbf{R}$ with

$$z(\tilde{u}) - y(\tilde{u}) = \lambda \tilde{u} + \mu e.$$

In particular, we see that the vector $z(\tilde{u}) - y(\tilde{u})$ is collinear with the two-dimensional plane $\Pi(\tilde{u})$, determined by the point $y(\tilde{u})$ and the vectors $\tilde{u}$ and e. Since the vector $\omega(\tilde{u})$ is collinear with this plane, the law of reflection at $y(\tilde{u})$ yields that $N_L(y(\tilde{u}))$ is also collinear with $\Pi(\tilde{u})$. Using the same argument for $\theta(\tilde{u})$ and the law of reflection at $z(\tilde{u})$, we find that $N_L(z(\tilde{u}))$ is collinear with $\Pi(\tilde{u})$, too. In this way we have shown that the vectors $\tilde{u}, \omega(\tilde{u}), \theta(\tilde{u}), N_L(y(\tilde{u})), N_L(z(\tilde{u}))$ are collinear.

Since u_0 can be approximated by vectors $\tilde{u} \in U$ having the latter property, using the compactness of ∂P and ∂Q, we deduce that there exist $y \in \partial P$ and

$z \in \partial Q$ such that $\frac{z(u)-y(u)}{\|z(u)-y(u)\|} = \omega_0$, $\langle z - y, N_L(y)\rangle = 0$, $\langle z - y, N_L(z)\rangle = 0$ and the vectors $N_L(y)$ and $N_L(z)$ are collinear with the two-dimensional plane, determined by ω_0 and e. This clearly implies that both $N_L(y)$ and $N_L(z)$ are collinear with e, i.e. $N_L(y) = \pm N_L(z) = \pm e$. ♠

Now assume that for each ω with (8) and (9), $\gamma'(\omega)$ has two transversal reflection points. For $i, j = 1, \ldots, r$ denote by $\mathcal{T}_{i,j}$ the set of those ω with (8) and (9) such that the first reflection point of $\gamma'(\omega)$ belongs to L_i and the second one to L_j. Since $\cup_{i,j=1}^r \mathcal{T}_{i,j}$ is an open subset of S^{n-1} (so it has positive measure) and ω_0 belongs to its closure, there exist $i \neq j$ such that ω_0 is a point of positive Lebesgue measure density of $\mathcal{T}_{i,j}$. Fix i and j with this property and apply Lemma 2.5 to $P = L_i$, $Q = L_j$ and $\mathcal{T} = \mathcal{T}_{i,j}$. We get that there exist $y \in \partial L_i$ and $z \in \partial L_j$ such that $N_L(y) = \pm N_L(z) = \pm e$ and $\omega_0 = \frac{z-y}{\|z-y\|}$. Since $e = N_K(0) \in \mathcal{N}$, it follows by the definition of $\mathcal{N}$ that N_L is regular at both y and z. Consequently, $(y, z) \in \Sigma(\partial L_i, \partial L_j)$ and $\omega_0 = h_{i,j}(y, z) \in \Sigma_{i,j}$ in contradiction with $\omega_0 \in \mathcal{S}$.

Therefore there exists ω with (8) and (9) such that $\gamma'(\omega)$ has exactly one transversal reflection point $y(\omega)$. Fix ω with this property and set $y = y(\omega)$. Then $\omega - \theta(\omega)$ is a non-zero vector collinear with both $N_K(0) = N_K(x)$ (recall that $x = 0$ by our assumption) and $N_L(y)$. Hence $N_K(x) = \pm N_L(y)$. On the other hand,

$$0 = T_{\gamma'(\omega)} = \langle \omega - \theta(\omega), y\rangle$$

shows that $\langle x - y, N_K(x)\rangle = 0$ which implies that $T_x\partial K = T_y\partial L$. This completes the proof of Lemma 2.1. ♠

Proof of Theorem 1.1. It is sufficient to show that K_1 coincides with some convex component of L.

Fix an element x of ∂K_1. It follows from Lemma 2.1 that there exists $y \in \partial L$ with $T_x\partial K = T_y\partial L$. Fix y with this property and let $y \in L_i$ for some $i = 1, \ldots, r$. We will show that $K_1 = L_i$. According to the condition (H) and Lemma 2.1, one gets easily that the set W of those $x' \in \partial K_1$ for which there exists $y' \in \partial L_i$ with $T_{x'}\partial K = T_{y'}\partial L$ is open in ∂K_1. On the other hand, W is also closed in ∂K_1, and since $W \neq \emptyset$ and ∂K_1 is connected, we find $W = \partial K_1$. Consequently, any supporting hyperplane to K_1 is a supporting hyperplane to L_i as well. Now using the convexity of K_1 and L_i we conclude that $K_1 = L_i$.

References

[GG] M. Golubitskii and V. Guillemin, *Stable Mappings and their Singularities*, Berlin, Springer, 1973.

[G] V. Guillemin, *Sojourn time and asymptotic properties of the scattering matrix*, Publ. RIMS Kyoto Univ. **12** (1977), 69-88.

[I] M.Ikawa, *Decay of solutions of the wave equation in the exterior of several strictly convex bodies*, Ann. Inst. Fourier **38** (1988), 113-146.

[LP1] P.Lax and R.Phillips, *Scattering Theory*, Academic Press, New York, 1967.

[LP2] P.Lax and R.Phillips, *The scattering of sound waves by an obstacle*, Comm. Pure Appl. Math. **30** (1977), 195-233.

[Ma] A.Majda, *A representation formula for the scattering operator and the inverse problem for arbitrary bodies*, Comm. Pure Appl. Math. **30** (1977), 165-194.

[PS1] V.Petkov and L.Stoyanov, *Singularities of the scattering kernel and scattering invariants for several strictly convex obstacles*, Trans. Amer. Math. Soc. **312** (1989), 203-235.

[PS2] V.Petkov and L.Stoyanov, *Geometry of Reflecting Rays and Inverse Spectral Problems*, John Wiley and Sons, Chichester, 1992.

[PS3] V.Petkov and L.Stoyanov, *Sojourn times of trapping rays and the behaviour of the modified resolvent of the Laplacian*, University of Bordeaux I - CNRS, Preprint n° 9209, 1992.

[Sj] J.Sjöstrand, *Geometric bounds on the density of resonances for semiclassical problems*, Duke Math. J. **60** (1990), 1-57.

[So] H. Soga, *Singularities of the scattering kernel for convex obstacles*, J. Math. Kyoto Univ. **22** (1983), 729-765.

[Y] K. Yamamoto, *Characterization of a convex obstacle by singularities of the scattering kernel*, J. Diff. Equations **64** (1986), 283-293.

Department of Mathematics, University of Western Australia, Nedlands, Perth 6009, Western Australia; stoyanov@maths.uwa.edu.au