# Uniqueness of Ground State in Exactly Solvable Hubbard, Periodic Anderson, and Emery Models

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We study the exactly solvable strongly interacting electron models recently introduced by Brandt and Giesekus, and further generalized by other authors. For a very general class of models, including the Hubbard, the periodic Anderson, and the Emery models with certain hopping matrices and infinitely large on-site Coulomb repulsion on d-sites, we prove that the known exact ground sate is indeed the unique ground state for a certain electron number. The uniqueness guarantees that one can discuss physics of various strongly interacting electron systems by analyzing the exact ground states.

In spite of considerable interest, various aspects of strongly interacting electron systems remain to be understood. Recently Brandt and Giesekus [1] introduced models of tight binding electrons with infinitely large on-site Coulomb repulsion, in which they were able to write down the exact ground states. Some generalization of the models were found by Mielke [2], by Strack [3], and by Tasaki [4]. In particular the cell construction in [4], which we shall use in the present Letter as well, provides the most general treatment of the class of the solvable models [5]. The class of models now includes various versions of the Hubbard, the periodic Anderson, and the Emery models with specific hopping matrices and $U = \infty$ on d-sites.

Unlike in many solvable models, the ground states of Brandt and Giesekus have nontrivial structure, and are expected to contain rich physics. Although the solvable models are in some sense artificial, it is expected that the models provide typical examples which exhibit various interesting phenomena generated by interplay between strong Coulomb interaction and kinetic motion of electrons. In [4] it was pointed out that the exact ground states have the so-called RVB (resonating-valence-bond) structure, and was speculated that some of them exhibit superconductivity [6]. In [7], Bares and Lee performed a detailed analysis of the solvable Emery model in one dimension, and discussed its relevance to the physics of the Kondo insulator.

In [1, 2, 4], the exact ground state was speculated to be the unique ground state of each model, but no proof was given. This has been a serious disadvantage when one wishes to draw physical conclusions by analyzing the exact ground states. Recently Bares and Lee [7, 8] announced that they obtained a proof of uniqueness of the ground state for some one-dimensional models. In the present Letter, we prove that the exact ground state is nonvanishing and is indeed the unique ground state (in a finite volume) for a quite general class of models with the number of electrons fixed to the twice of the number of "cells" in the lattice. (Our argument is different from that of Bares and Lee [8].) The class includes *all* the concrete models (of Brandt-Giesekus type) considered in literature, among which are the two and three dimensional Hubbard and periodic Anderson models introduced in [1, 2], the two dimensional Emery [3] and Hubbard [4] models which mimic the $CuO_2$ structure, the one dimensional Emery and the periodic Anderson models [3, 7]. (See Fig.1.)

We believe that the present results provide a basis for future studies, in which one extracts various physics out of the exact ground states of the Brandt-Giesekus type.

*The models and main results:* We shall describe the solvable models in their most general forms. We first construct the lattice to work with. A cell $C$ is a finite set of sites, where each site $x \in C$ is classified either as a $U = \infty$ site (or a d-site) which can carry at most one electron, or a $U = 0$ site (or a p-site) which can carry at most two electrons. The lattice $\Lambda_N$ is constructed by starting from the empty set $\Lambda_0 = \phi$, and successively adding cells $C_1, C_2, \ldots, C_N$, where the cells need not be identical. When adding a new cell $C_i$ to the lattice $\Lambda_{i-1}$ (which consists of $C_1, \ldots, C_{i-1}$), we identify some (including none) of the sites in $C_i$ with sites in $\Lambda_{i-1}$ in a one-to-one fashion, noting that sites of the same type should be identified. (See Fig.1.) The only nontrivial requirement (which is introduced in the present Letter) in the construction is the following "three electrons condition". *When a new cell $C_i$ is added to $\Lambda_{i-1}$ to form $\Lambda_i$, we have either; 1) the sites in $C_i$ which are not identified with sites in $\Lambda_{i-1}$ can carry at least three electrons, or 2) the unidentified site in $C_i$ is a single p-site.* This is quite a reasonable requirement [9], which is satisfied in all the concrete models (of Brandt-Giesekus type) studied in literature [10].

We shall consider an electron system on the resulting lattice [11] $\Lambda_N = C_1 \cup \cdots \cup C_N$. For a site $x \in \Lambda_N$, $c_{x\sigma}$, $c^\dagger_{x\sigma}$, and $n_{x\sigma} = c^\dagger_{x\sigma} c_{x\sigma}$ denote the annihilation, the creation, and the number operators, respectively, of an electron at site $x$ with spin $\sigma = \uparrow, \downarrow$. The states are constructed by operating $c^\dagger_{x\sigma}$ with various $x$ and $\sigma$ to the vacuum state $\Phi_0$, but we only allow the states which satisfy $n_{x\uparrow} n_{x\downarrow} \Phi = 0$ for any $U = \infty$ site $x$. This restriction effectively takes into account infinitely large on-site Coulomb repulsion on d-sites.

With a cell $C$, we associate the Hamiltonian

$$H(C) = \sum_{\sigma=\uparrow,\downarrow} \alpha_\sigma(C) \mathcal{P}(C) \alpha^\dagger_\sigma(C), \tag{1}$$

with

$$\alpha_\sigma(C) = \sum_{x \in C} \lambda_x^{(C)} c_{x\sigma}, \tag{2}$$

where $\lambda_x^{(C)}$ are nonvanishing real coefficients [12]. The projection operator onto the space of the allowed sates is

$$\mathcal{P}(C) = \prod_{x \in C_{U=\infty}} (1 - n_{x\uparrow} n_{x\downarrow}), \tag{3}$$

where $C_{U=\infty}$ is the set of $U = \infty$ sites in $C$.

Then the Hamiltonian for the whole lattice $\Lambda_N$ is

$$H_N = \sum_{i=1}^{N} H(C_i), \tag{4}$$

where the coefficients $\lambda_x^{(C_i)}$ are chosen and fixed independently in each cell.

Before discussing the ground state of the model, we shall rewrite the Hamiltonian (4) into the "standard form" (6). (This rewriting is not necessary for the uniqueness proof.) Note that there are operator identities [13] $c_{y\sigma} \mathcal{P}(C) c^\dagger_{x\sigma} = -\mathcal{P}(C) c^\dagger_{x\sigma} c_{y\sigma} \mathcal{P}(C)$ for

$x \neq y \in C$, $c_{x\sigma}\mathcal{P}(C)c^\dagger_{x\sigma} = \mathcal{P}(C)(1-n_{x\uparrow}-n_{x\downarrow})\mathcal{P}(C)$ for $x \in C_{U=\infty}$, and (trivially) $c_{x\sigma}\mathcal{P}(C)c^\dagger_{x\sigma} = \mathcal{P}(C)(1-n_{x\sigma})\mathcal{P}(C)$ for $x \notin C_{U=\infty}$. By using these identities we find

$$H_C = \sum_{\sigma=\uparrow,\downarrow}\sum_{x,y\in C}\lambda^{(C)}_x\lambda^{(C)}_y c_{y\sigma}\mathcal{P}(C)c^\dagger_{x\sigma} = \mathcal{P}(C)\left\{\epsilon(C) - \sum_{\sigma=\uparrow,\downarrow}\sum_{x,y\in C} t^{(C)}_{xy}c^\dagger_{x\sigma}c_{y\sigma}\right\}\mathcal{P}(C), \quad (5)$$

where $t^{(C)}_{xy} = \lambda^{(C)}_x\lambda^{(C)}_y$ for $x \neq y$, $t^{(C)}_{xx} = 2(\lambda^{(C)}_x)^2$ if $x \in C_{U=\infty}$, $t^{(C)}_{xx} = (\lambda^{(C)}_x)^2$ if $x \notin C_{U=\infty}$, and $\epsilon(C) = \sum_{x\in C}(\lambda^{(C)}_x)^2$. By summing up this expression, we get

$$H_N = -E_0 - \mathcal{P}_N\sum_{\sigma=\uparrow,\downarrow}\sum_{x,y\in\Lambda_N} t_{xy}c^\dagger_{x\sigma}c_{y\sigma}, \quad (6)$$

where $t_{xy} = \sum_{i=1}^N t^{(C_i)}_{xy}$, $E_0 = -\sum_{i=1}^N \epsilon(C_i)$, and

$$\mathcal{P}_N = \prod_{i=1}^N \mathcal{P}(C_i). \quad (7)$$

To derive (6), we have used that fact that $H_N$ operates only on the allowed states, which are now characterized as $\mathcal{P}_N\Phi = \Phi$.

The main result of the present Letter is the following.

**Theorem:** Consider the model on $\Lambda_N$. Then we have;
i) Let $N_\uparrow, N_\downarrow$ be nonnegative integers with $N_\uparrow < N$ or $N_\downarrow < N$. Then there is no state $\Phi$ with $N_\uparrow$ up electrons and $N_\downarrow$ down electrons that satisfies $H_N\Phi = 0$.
ii) There exists a unique state $\Phi$ with $N$ up electrons and $N$ down electrons that satisfies $H_N\Phi = 0$. It is written as

$$\Phi = \mathcal{P}_N A^\dagger_N \Phi_0, \quad (8)$$

where

$$A^\dagger_N = \prod_{\sigma=\uparrow,\downarrow}\prod_{i=1}^N \alpha^\dagger_\sigma(C_i). \quad (9)$$

¿From the simple operator identity $\mathcal{P}(C_i)\alpha^\dagger_\sigma(C_i)\mathcal{P}_N\alpha^\dagger_\sigma(C_i) = \mathcal{P}_N(\alpha^\dagger_\sigma)^2 = 0$, and the fact that the Hamiltonian (4) is nonnegative, it obviously follows that the state (8), if nonvanishing, is an exact ground sate of the model. More delicate issues, which are solved in the present Letter, have been to show the state (8) is nonvanishing and is the unique ground state.

By using the standard argument based on the $SU(2)$ invariance of the model, the above theorem implies the following.

**Corollary:** Consider the model on $\Lambda_N$. In the sector with the total electron number $2N$, the ground state of the Hamiltonian (4) is unique, and is given by (8).

For the electron number strictly greater than $2N$, one can easily see (by construction) that the ground sates are degenerate [1, 2]. Although we do not study such cases in detail, we remark that i) of the above theorem has an immediate consequence that the ground states for $2N+n$ electrons has total spin not greater than $n/2$.

*Proof:* We shall prove the theorem by induction. We first set $N = 0$ and $\Lambda_0 = \phi$. Then both i) and ii) are trivial, where in the latter we let $A_0^\dagger = 1$.

We assume the statements of the theorem for $\Lambda_N = C_1 \cup \cdots \cup C_N$, and prove them for $\Lambda_{N+1} = C_1 \cup \cdots \cup C_{N+1} = \Lambda_N \cup C_{N+1}$. In what follows, we abbreviate $\Lambda_N$ and $C_N$ as $\Lambda$ and $C$, respectively.

We start from ii). Let $\Phi$ be a state on $\Lambda_{N+1} = \Lambda \cup C$ with $(N+1)$ up electrons and $(N+1)$ down electrons, and assume that

$$H_{N+1}\Phi = 0. \tag{10}$$

Since $H_{N+1}$ is a sum of nonnegative operators, we find that (10) is equivalent to the condition that both

$$H_N\Phi = 0, \tag{11}$$

and

$$\mathcal{P}(C)\alpha_\sigma^\dagger(C)\Phi = 0, \tag{12}$$

for $\sigma =\uparrow,\downarrow$ hold.

Let $\tilde{C}$ be the set of sites in $C$ which are not identified with sites in $\Lambda$ when one constructs $\Lambda_{N+1} = \Lambda \cup C$. (We then have $\Lambda \cap \tilde{C} = \phi$ and $\Lambda \cup \tilde{C} = \Lambda_{N+1}$, where $\cup$ means the simple union.) We can decompose $\Phi$ according to the numbers of up and down electrons (denoted as $n_\uparrow$ and $n_\downarrow$, respectively) contained in $\tilde{C}$ as

$$\Phi = \sum_{n_\uparrow=0,1,2,\ldots}\ \sum_{n_\downarrow=0,1,2,\ldots} \Phi_{n_\uparrow,n_\downarrow}. \tag{13}$$

In the state $\Phi_{n_\uparrow,n_\downarrow}$ with $n_\sigma > 1$ for $\sigma =\uparrow$ or $\downarrow$, the number of spin-$\sigma$ electrons in $\Lambda$ is strictly less than $N$. Then (11) and the assumed i) for $\Lambda$ imply $\Phi_{n_\uparrow,n_\downarrow} = 0$. Thus the decomposition (13) becomes

$$\Phi = \Phi_{1,1} + \Phi_{1,0} + \Phi_{0,1} + \Psi_{0,0}, \tag{14}$$

with

$$\Phi_{1,1} = \sum_{x,y\in\tilde{C}} c_{x\uparrow}^\dagger c_{y\downarrow}^\dagger \Psi_{1,1}(x,y), \tag{15}$$

and

$$\Phi_{1,0} = \sum_{x\in\tilde{C}} c_{x\uparrow}^\dagger \Psi_{1,0}(x), \quad \Phi_{0,1} = \sum_{y\in\tilde{C}} c_{y\downarrow}^\dagger \Psi_{0,1}(y), \tag{16}$$

where various $\Psi$ denote states in which electrons live only on $\Lambda$.

Since $H_N$ acts only on $\Lambda$, the condition (11) implies (among other relations) $H_N\Phi_{1,1} = 0$, which further reduces to $H_N\Psi_{1,1}(x,y) = 0$ for each $x, y \in \tilde{C}$. Noting that the state $\Psi_{1,1}(x,y)$ has $N$ up electrons and $N$ down electrons, the assumed ii) for $\Lambda$ implies

$$\Psi_{1,1}(x,y) = \psi_{x,y}\mathcal{P}_N A_N^\dagger \Phi_0, \tag{17}$$

where $\psi_{x,y}$ are undetermined coefficients.

Next we examine the condition (12), which, with the decomposition (14), now reads

$$\mathcal{P}(C)\sum_{z\in C}\lambda_z^{(C)} c_{z\sigma}^\dagger(\Phi_{1,1} + \Phi_{1,0} + \Phi_{0,1} + \Psi_{0,0}) = 0, \tag{18}$$

for $\sigma =\uparrow,\downarrow$, where we used the definition (2) of $\alpha_\sigma(C)$. We shall again decompose the left-hand side of (18) according to the numbers of up and down electrons in $\tilde{C}$. Clearly each state in the decomposition must vanish independently.

Let us suppose that the cell $C$ satisfies 1) of the "three electrons condition", and start from the sector with three electrons in $\tilde{C}$. The contribution comes only from $\Phi_{1,1}$, and we get from (18) that

$$\begin{aligned} 0 &= \mathcal{P}(C) \sum_{z\in\tilde{C}} \lambda_z^{(C)} c_{z\sigma}^\dagger \sum_{x,y\in\tilde{C}} \psi_{x,y} c_{x\uparrow}^\dagger c_{y\downarrow}^\dagger \mathcal{P}_N A_N^\dagger \Phi_0 \\ &= \sum_{x,y,z\in\tilde{C}} \psi_{x,y} \lambda_z^{(C)} \mathcal{P}(C) c_{z\sigma}^\dagger c_{x\uparrow}^\dagger c_{y\downarrow}^\dagger \mathcal{P}_N A_N^\dagger \Phi_0, \end{aligned} \tag{19}$$

where we used explicit form of $\Psi_{1,1}(x,y)$ in (17). The equation is easy to analyze since the right-hand side factorizes into the states on $\tilde{C}$ and on $\Lambda$. It is also essential that $\mathcal{P}_N A_N^\dagger \Phi_0$ is nonvanishing from the assumed ii) for $\Lambda$. By setting $\sigma =\uparrow$, (19) yields

$$\psi_{x,y} \lambda_z^{(C)} = \psi_{z,y} \lambda_x^{(C)}, \tag{20}$$

for the compatible combinations of $x,y,z \in \tilde{C}$, i.e., those satisfy $x \neq y$, and $z \neq x,y$ if $z \in C_{U=\infty}$. By setting $\sigma =\downarrow$ in (19), we get

$$\psi_{x,y} \lambda_z^{(C)} = \psi_{x,z} \lambda_y^{(C)}, \tag{21}$$

again for $x,y,z \in \tilde{C}$ with similar compatible conditions. Because of 1) of the "three electrons condition", we see that the sets of compatible $(x,y,z)$ in (20), (21) are not empty. Then the equations (20), (21) are easily found to possess the unique (apart from multiplication by a constant) solution

$$\psi_{x,y} = \lambda_x^{(C)} \lambda_y^{(C)}. \tag{22}$$

When the cell $C$ satisfies 2) of the "three electrons condition", the analysis is trivial. Since we must have $x = y$ in (15), we can set $\psi_{xx} = (\lambda_x^{(C)})^2$ to be consistent with (22). In the following, we do not have to distinguish between the cases 1) and 2).

Next we set $\sigma =\downarrow$ in (18) and consider the sector with one up electron and one down electron in $\tilde{C}$. Both $\Phi_{1,1}$ and $\Phi_{1,0}$ contribute, and we get

$$\sum_{x,y\in\tilde{C}} \sum_{z\in\delta C} \lambda_x^{(C)} \lambda_y^{(C)} \lambda_z^{(C)} \mathcal{P}(C) c_{z\downarrow}^\dagger c_{x\uparrow}^\dagger c_{y\downarrow}^\dagger \mathcal{P}_N A_N^\dagger \Phi_0 + \sum_{x,z\in\tilde{C}} \lambda_z^{(C)} \mathcal{P}(C) c_{z\downarrow}^\dagger c_{x\uparrow}^\dagger \Psi_{1,0}(x) = 0, \tag{23}$$

where $\delta C = C\backslash\tilde{C}$ (i.e., the set of sites in $C$ identified with those in $\Lambda$). We also used the solution (22). By using the operator identities in [13], we find that $\mathcal{P}(C) c_{x\uparrow}^\dagger c_{y\downarrow}^\dagger c_{z\downarrow}^\dagger \mathcal{P}_N = \mathcal{P}(C) \mathcal{P}_z c_{x\uparrow}^\dagger c_{y\downarrow}^\dagger c_{z\downarrow}^\dagger \mathcal{P}_z \mathcal{P}(\Lambda\backslash\{z\}) = \mathcal{P}(C) c_{x\uparrow}^\dagger c_{y\downarrow}^\dagger \mathcal{P}_z c_{z\downarrow}^\dagger \mathcal{P}_z \mathcal{P}(\Lambda\backslash\{z\}) = \mathcal{P}(C) c_{x\uparrow}^\dagger c_{y\downarrow}^\dagger \mathcal{P}_N c_{z\downarrow}^\dagger$, where $\mathcal{P}(\Lambda\backslash\{z\}) = \prod_{x\in(\Lambda\backslash\{z\})_{U=\infty}} (1 - n_{x\uparrow} n_{x\downarrow})$ commutes with $c_{z\downarrow}^\dagger$. Then the equation (23) can be rewritten as

$$\sum_{x,y\in\tilde{C}} \lambda_x^{(C)} \lambda_y^{(C)} \mathcal{P}(C) c_{x\uparrow}^\dagger c_{y\downarrow}^\dagger \{\mathcal{P}_N \sum_{z\in\delta C} \lambda_z^{(C)} c_{z\downarrow}^\dagger A_N^\dagger \Phi_0 - (\lambda_x^{(C)})^{-1} \Psi_{1,0}(x)\} = 0, \tag{24}$$

which is again factorized, and yields

$$\Psi_{1,0}(x) = \lambda_x^{(C)} \mathcal{P}_N \sum_{y \in \delta C} \lambda_y^{(C)} c_{y\downarrow}^\dagger A_N^\dagger \Phi_0. \tag{25}$$

By setting $\sigma =\downarrow$ in (18) and looking at the same sector, we get

$$\Psi_{0,1}(y) = -\lambda_y^{(C)} \mathcal{P}_N \sum_{x \in \delta C} \lambda_x^{(C)} c_{x\uparrow}^\dagger A_N^\dagger \Phi_0. \tag{26}$$

Finally we set $\sigma =\uparrow$ in (18) and consider the sector with only one up electron in $\tilde{C}$. We see that $\Phi_{1,0}$ and $\Psi_{0,0}$ contribute, and by using (25), we get

$$-\sum_{x \in \tilde{C}} \sum_{y,z \in \delta C} \lambda_x^{(C)} \lambda_y^{(C)} \lambda_z^{(C)} c_{x\uparrow}^\dagger \mathcal{P}_N c_{z\uparrow}^\dagger c_{y\downarrow}^\dagger A_N^\dagger \Phi_0 + \sum_{z \in \tilde{C}} \lambda_z^{(C)} c_{z\uparrow}^\dagger \Psi_{0,0} = 0, \tag{27}$$

which imply

$$\Psi_{0,0} = \mathcal{P}_N \sum_{x,y \in \delta C} \lambda_x^{(C)} \lambda_y^{(C)} c_{x\uparrow}^\dagger c_{y\downarrow}^\dagger A_N^\dagger \Phi_0. \tag{28}$$

By combining the solutions (17), (22), (25), (26), and (28) with the decomposition (14), (15), and (16), and noting that $\mathcal{P}(C)\mathcal{P}_N = \mathcal{P}_{N+1}$, we find that, only by using necessary conditions for (10), the state $\Phi$ has been uniquely determined to have the desired form

$$\Phi = \mathcal{P}_{N+1} \left( \prod_{\sigma=\uparrow,\downarrow} \alpha_\sigma^\dagger(C) \right) A_N^\dagger \Phi_0 = \mathcal{P}_{N+1} A_{N+1}^\dagger \Phi_0, \tag{29}$$

which clearly satisfies (10).

It only remains to show $\Phi$ is nonvanishing. ¿From (29), we find

$$\Phi_{1,1} = \left( \mathcal{P}(C) \sum_{x,y \in \tilde{C}} \lambda_x^{(C)} \lambda_y^{(C)} c_{x\uparrow}^\dagger c_{y\downarrow}^\dagger \right) \mathcal{P}_N A_N^\dagger \Phi_0. \tag{30}$$

Noting that $\mathcal{P}(C) \sum_{x,y \in \tilde{C}} \lambda_x^{(C)} \lambda_y^{(C)} c_{x\uparrow}^\dagger c_{y\downarrow}^\dagger \Phi_0$ is nonvanishing due to the "three electrons condition", and $\mathcal{P}_N A_N^\dagger \Phi_0$ is nonvanishing due to the assumed ii) for $\Lambda$, we find that $\Phi_{1,1}$ (which may be regarded as the direct product of these sates) is nonvanishing as well. Since the states in the decomposition (14) are mutually orthogonal, we have shown that $\Phi$ is nonvanishing. The desired ii) in the theorem has been proved.

To prove i), we assume that $\Phi$ is a state on $\Lambda_{N+1} = \Lambda \cup C$ with the number of up or down electrons strictly less than $N+1$, and that $\Phi$ satisfies (10). Again we decompose $\Phi$ as in (13). The condition (12) implies that there is at least one combination $(n_\uparrow, n_\downarrow)$ with $n_\uparrow \geq 1$, $n_\downarrow \geq 1$ such that $\Phi_{n_\uparrow,n_\downarrow} \neq 0$. This, however, contradicts with the conclusion from (11) and the assumed i) for $\Lambda$ that $\Phi_{n_\uparrow,n_\downarrow} = 0$ whenever $n_\uparrow \geq 1$, $n_\downarrow \geq 1$.

I wish to thank Pere-Anton Bares and Patrick Lee for stimulating correspondence.

## References

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[3] R. Strack, Phys. Rev. Lett. **70**, 833 (1993).

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[5] In [4], models only with d-sites (see below) was considered. By relaxing this condition (as we do here), one gets literally the most general description of the solvable models.

[6] In [4] it was claimed, directly below eq. (8), that a loop configuration with nonvanishing contribution must contain a loop which has the two source bonds in it. This claim is incorrect in general since there is a contribution from a configuration with a chain of "connected" loops containing the source bonds. The calculation in [4] for the tree models is not affected by this error.

[7] P.-A. Bares and P.A. Lee, preprint.

[8] P.-A. Bares and P.A. Lee, private communication.

[9] In some models, we have to impose open boundary conditions to fulfill the "three electrons condition". Although one can construct models in which the "three electrons condition" is violated, no such model of physical interest has been proposed. We note, in particular, that if the condition is violated for all $i = 1, \ldots, N$, then the resulting lattice can contain at most $2N$ electrons. Therefore the only possibilities are that there is no state with $2N$ electrons, or that the lattice is completely filled by electrons and the ground states are degenerate. (Note that we always have some d-sites, on which we can put either up or down electron.)

[10] In a model of "electrons" with spin $S$ as in [3], we replace "three" in 1) by $2(S+1)$. Then the uniqueness theorem can be extended in a trivial manner.

[11] Throughout the present Letter, the symbol $\cup$ implicitly means that proper identifications of sites are made.

[12] By letting the coefficients $\lambda_x^{(C)}$ complex, we get models which cannot be mapped to the models with real $\lambda_x^{(C)}$ by gauge transformations. Here we do not go into details of such models with complex hoppings.

[13] Let $\mathcal{P}_x = 1 - n_{x\uparrow}n_{x\downarrow}$. It is easy to verify that $\mathcal{P}_x c_{x\sigma}^\dagger \mathcal{P}_x = \mathcal{P}_x c_{x\sigma}^\dagger$, and $[\mathcal{P}_x, c_{y\sigma}^\dagger] = 0$, $[\mathcal{P}_x, \mathcal{P}_y] = 0$ for $x \neq y$. Then $c_{y\sigma}\mathcal{P}_y\mathcal{P}_x c_{x\sigma}^\dagger = \mathcal{P}_y c_{y\sigma}\mathcal{P}_y\mathcal{P}_x c_{x\sigma}^\dagger \mathcal{P}_x = \mathcal{P}_x\mathcal{P}_y c_{y\sigma} c_{x\sigma}^\dagger \mathcal{P}_x\mathcal{P}_y$, which proves the first identity. The second identity follows from $n_{x\sigma}c_{x\sigma}^\dagger = c_{x\sigma}^\dagger$ and $(n_{x\sigma})^2 = n_{x\sigma}$.

**Fig. 1.** Examples of models (lattice structures) to which our uniqueness theorem apply. A d-site with $U = \infty$ is denoted by $\bullet$, and a p-site with $U = 0$ by $\circ$. On the left of each lattice is the corresponding unit cell. The models are; (a) the one-dimensional periodic Anderson model [3], (b) the two-dimensional $CuO_2$-like model with extra hopping between O-sites [1], and (c) the two-dimensional extended Emery model [3]. These models become solvable by choosing appropriate hopping matrices and filling factor. We fix the electron number equal to the twice of the number of cells in the lattice. The "three electrons condition" is satisfied in the models (a) and (b) with open boundary conditions, and in the model (c) with open or periodic boundary conditions.