RIGOROUS THEORY OF THE BOLTZMANN EQUATION
IN THE LORENTZ GAS*

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Abstract: The Boltzmann limit conjecture of Grad is discussed in general and proved for the Lorentz gas case (where the Boltzmann equation is linear). This is a reprint of an unpublished preprint of 1972, with one footnote added, one postscript (to quote the Lanford theorem), and improved with language editing. I reprint it in this form to make it accessible, as it has been quoted by other authors in later papers. The original preprint was commissioned for a book that eventually was not published.

1. — Introduction

The Boltzmann equation is an approximation to the "true" evolution equation: this is due to the fact that in its derivation the following assumptions are made [1,2]:

1) only binary collisions are considered
2) "Molecular chaos" is assumed at all times: i.e. the high order correlation functions can be expressed in terms of the one–particle distribution as:

$$f(\underline{r}_1\underline{v}_1,\, \underline{r}_2\underline{v}_2, \ldots, \underline{r}_n\underline{v}_n \;;\; t) = \prod_{i=1}^{n} f(\underline{r}_1\underline{v}_1 \;;\; t) \tag{1.1}$$

3) in the computation of the collision term one disregards the fact that the molecules have a non vanishing dimension.

Having realized that the Boltzmann equation is only an approximation it becomes of interest to investigate if there are, at least, limiting cases in which it holds rigorously.

If n denotes the particle density and a the radius of interaction (i.e. a parameter proportional to the interaction range or to the square root of the scattering cross–section) then a critical examination of assumptions 1), 2), 3) suggest that the Boltzmann equation should hold rigorously in the limiting case $n \to \infty$, $a \to 0$ in such a way that [3]:

$$\text{i)} \quad na^2 \neq 0 \tag{1.2}$$

$$\text{ii)} \quad na^3 \to 0 \tag{1.3}$$

$$\text{iii)} \quad \text{“Molecular chaos” (1.1) is assumed at } t = 0 \tag{1.4}$$

In fact (1.2) says that the mean free path (m.f.p.) is finite (i.e. there are collisions); on the other hand (1.3) insures that, in the molecular scale, the gas is infinitely dilute (i.e. no particle can be found in a region of dimension a; hence the probability that a given particle collides with any other particle fixed vanishes *a priori* (although the particle will certainly suffer collisions because of (1.2)). The last facts prevent multiple collisions and the building up of correlations capable of destroying the molecularly chaotic character of the initial state.

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To discuss rigorous results we need, however, a more precise statement of the conjecture that in the above limiting case the Boltzmann equation is rigorously true. This is done in the following lines.

Consider a gas of particles described, at $t = 0$, by a molecularly chaotic state (i.e. by a state such that the n–point correlation function factorizes as in (1.1)) with a one particle distribution:

$$f(\underline{r}, \underline{v}\ ;\ 0) = \frac{1}{a^2} f_0(\underline{r}, \underline{v}) \tag{1.5}$$

where $f_0(\underline{r}, \underline{v})$ is a given (a–independent) function.

Suppose that the gas of particles just introduced evolves through the action of a pair potential $\varphi_a(\underline{r})$ having the form:

$$\varphi_a(\underline{r}) = \varphi\left(\frac{|\underline{r}|}{a}\right) \tag{1.6}$$

where φ is a short range force (without hard core to avoid inessential complications in the notations and definitions) so that the differential scattering cross section in the solid angle Ω is of the form

$$\sigma_a(\Omega) = a^2 \sigma(\Omega) \tag{1.7}$$

Let $f(\underline{r}_1, \underline{v}_1, \underline{r}_2, \underline{v}_2, \ldots, \underline{r}_n, \underline{v}_n\ ;\ t)$ be the m–particle correlation function describing the state into which the initial state evolves in time t.

This function is, of course, no longer “chaotic” (i.e. of the form (1.1)). The “Boltzmann limit conjecture” (BLC) can be now formulated as:

Conjecture BLC:
For all fixed $t > 0$ and under “mild assumption” on $f_0(\underline{r}, \underline{v})$, the following limit exists:

$$\tilde{f}(\underline{r}_1\underline{v}_1, \underline{r}_2\underline{v}_2, \ldots, \underline{r}_m\underline{v}_m\ ;\ t) = \lim_{a \to 0} a^{2m} f(\underline{r}_1\underline{v}_1, \ldots, \underline{r}_m\underline{v}_m\ ;\ t) \tag{1.8}$$

and is chaotic:

$$\tilde{f}(\underline{r}_1\underline{v}_1, \underline{r}_2\underline{v}_2, \ldots, \underline{r}_n\underline{v}_n\ ;\ t) = \prod_{i=1}^{n} f(\underline{r}_i\underline{v}_i\ ;\ t) \tag{1.9}$$

and, further, $f(\underline{r}, \underline{v}\ ;\ t)$ verifies the Boltzmann equation:

$$\begin{aligned}\frac{\partial \tilde{f}}{\partial r}(\underline{r}, \underline{v}, t) + \underline{v} \cdot \frac{\partial \tilde{f}}{\partial r}(\underline{r}, \underline{v}; t) = \int \mathrm{d}\,\underline{v}_1 \int \mathrm{d}\,\Omega \cdot \\ \cdot\, |\underline{v} - \underline{v}_1| \sigma(\Omega) (\tilde{f}(\underline{r}, \underline{v}'; t) \tilde{f}(\underline{r}_1, \underline{v}_1'; t) - \tilde{f}(\underline{r}, \underline{v}, t) \tilde{f}(\underline{r}, \underline{v}_1, t))\end{aligned} \tag{1.10}$$

with initial condition

$$\tilde{f}(\underline{r}, \underline{v}, 0) = f_0(\underline{r}, \underline{v}) \tag{1.11}$$

where $\underline{v}', \underline{v}_1'$ are functions of $\underline{v}$, $\underline{v}_1$, Ω in such a way to conserve kinetic energy, linear momentum and to have a relative direction parallel to Ω.

We refrain to state some examples of “mild assumptions” on $f_0(\underline{r}, \underline{v})$ since they should become clear in the course of a hypothetic proof of BLC.

We observe that, for the time being, it seems quite difficult to attack the problem of costructing a rigorous proof of the BLC. This is mainly due to the fact that the existence and stability theorems necessary for a proper mathematical definition of time evolutions of large systems are still far from being proved.

We stress that this is not a “technical point” but reflects our lack of understanding of some basic physical properties of the time evolution of large assemblies of particles (for an example of such problems see [5]; for an example of their applications see [6] and [7]).

In this paper we investigate the BLC in the case of simple models introduced by Lorentz [8] and used, for instance, to study the diffusion between gases of very different molecular weight [9] or, in its quantum version, to study the properties of a degenerate gas [10].

The Lorentz models are described in the next section and are such that the mathematical problems concerning existence and stability of the solutions of the microscopic equations of motion are very easily dealt with.

This mathematical simplicity, reflected also in the fact that the Boltzmann equation turns out to be linear, will enable us to push to the end the proof of the BLC at least in some cases.

2. — The Lorentz models

In the models there are two types of particles: the W–particles (wind–particles) and the T–particles (tree–particles).

The W–particles move through the space interacting only with the T–particles which, however, are supposed to be infinitely heavy compared to the W–particle and are supposed at rest and randomly distributed in space.

Each model is completely described by the $W-T$ interaction and by the T–particle distribution.

¿From now on we shall focus our interest to the case in which the T–particles are distributed as the space distribution of a perfect gas (Poisson distribution) with density n. We shall also assume that the T–particles are, with respect to the W–particles, hard spheres of radius a, reflecting the W–particles on their surface.

The assumed tree distribution is such that the probability for finding inside a given region Λ, with volume $V(\Lambda)$ exactly N tree particles, and for finding them in the infinitesimal cubes $d\underline{c}_1, \ldots, d\underline{c}_N$ around $\underline{c}_1, \ldots, \underline{c}_N$, is:

$$f_\Lambda(\underline{c}_1, \ldots, \underline{c}_N) d\underline{c}_1, \ldots, d\underline{c}_N = e^{-nV(\Lambda)} \frac{n^N}{N!} d\underline{c}_1, \ldots, d\underline{c}_N \tag{2.1}$$

Note that, since the T–particles are hard spheres only with respect to the W–particles but not with respect to the each other, there are configurations $\underline{c}_1, \ldots, \underline{c}_N$ of trees in which the hard spheres overlap, (for some comments on this point see §6).

If $x = (\underline{p}, \underline{q})$ is the W–particles phase space coordinate ($\underline{p}$= velocity, $\underline{q}$= position) the symbol:

$$S_t^{\underline{c}_1, \cdots \underline{c}_N} x \tag{2.2}$$

will denote the W–particle $x' = (\underline{p}', \underline{q}')$ into which x evolves in time t in the presence of N tree–particles located at $\underline{c}_1, \ldots, \underline{c}_N$.

The symbol $\omega(\underline{p})$ will denote the direction of $\underline{p}$. The symbol $\hat{x}$ will denote the pair $(\omega(\underline{p}), q)$ if $x = (\underline{p}, \underline{q})$.

Since the velocity $|\underline{p}|$ is conserved it is clear that $S_t^{\underline{c}_1, \ldots, \underline{c}_N} x$ depends only on the trees located within a distance $(|\underline{p}|t + a)$ from $\underline{q}$. The symbols:

$$\left(S_t^{\underline{c}_1, \ldots, \underline{c}_N} x\right)_1 \ , \ \left(S_t^{\underline{c}_1, \ldots, \underline{c}_N} x\right)_2 \ , \ \omega\left(\left(S_t^{\underline{c}_1, \ldots, \underline{c}_N} x\right)_1\right) \tag{2.3}$$

will, respectively, denote the velocity, position and momentum direction of (2.2). The following symbols will occur frequently:

$$S_t^{\underline{c}_1, \ldots, \underline{c}_N} \hat{x} = \left(\omega\left(\left(S_t^{\underline{c}_1, \ldots, \underline{c}_N} x\right)_1\right) \ , \ \left(S_t^{\underline{c}_1, \ldots, \underline{c}_N} x_m\right)\right) \tag{2.4}$$

Similarly we can give a natural meaning to the evolution of m W–particles:

$$S_t^{\underline{c}_1, \ldots, \underline{c}_N} (x_1, \ldots, x_m) = \left(S_t^{\underline{c}_1, \ldots, \underline{c}_N} x_1, \ldots, S_t^{\underline{c}_1, \ldots, \underline{c}_N} x_m\right) \tag{2.5}$$

which takes into account the fact that there are no $W-W$ interactions.

3. — The Boltzmann limit for the Lorentz gas

It is easy to derive the Boltzmann equation for W–particles in the case of the above described Lorentz gas. It is not difficult to realize that the assumptions to be made in order to derive the Boltzmann equation are essentially the same as conditions 1), 2), 3), of section 1. They are:

i) a W–particle never hits twice the same particle;

ii) molecular chaos is assumed;

iii) the size of the T–particles is negligible.

Here by "chaotic" W–particle state we again mean a state such that the W–particle correlation functions are a product of one W–particle distribution which are independent on the T–particle distribution: more precisely a chaotic state is such that the probability distribution for finding a certain configuration C of T–particles and a set of W–particles in $x_1, \ldots, x_m$ has the form: $p(C) \prod_{i=1}^m f(x_i)$, where $p(C)$ denotes the Poisson distribution (2.1) and this is interpreted as 0 if any wind particle is inside the hard cores of C.*

Clearly assumption i), ii), iii) can be only approximately true.

Let us formulate the BLC for the Lorentz gas. Assume that the initial W–particles state has the form:

$$f(x_1, \ldots, x_m \; ; \; 0) = \int_{C \text{ comp } (x_1, \ldots, x_m)} p(C) \prod_{i=1}^m f_0(x_i) \tag{3.1}$$

where $f_0(x)$ is a given function of x and the "integral" is the "sum" over all the T–particle configurations compatible with $x_1, \ldots, x_m$ (i.e. over the C's such that no W–particle is located inside the hard core of a T–particle).

Note that (3.1) is not a product state for the W–particles: this difference with respect to section 1 arises because here we have hard core interactions (which, for simplicity, were not considered in section 1).

Consider the state obtained by evolving the initial state (3.1):

$$f(x_1, \ldots, x_m \; ; \; t) = \int_{C \text{ comp } (x_1, \ldots, x_m)} p(C) \prod_{i=0}^m f_0(S^C_{-t} x_j) \tag{3.2}$$

and then let the T–particle density n tend to infinity and the hard core $W - T$ radius tend to zero in such a way that $na^3 \to 0$ but $na^2 \to l \neq 0, \infty$.

The BLC becomes:

Conjecture BLC:

If $t \geq 0$ and under "mild assumptions" on f_0, the following limit exists:

$$\lim_{\substack{na^3 \to 0 \\ na^2 \to \text{ const } \neq 0, \infty}} f(x_1, \ldots, x_m \; ; \; t) = \tilde{f}(x_1, \ldots, x_m \; ; \; t) \tag{3.3}$$

and:

$$\tilde{f}(x_1, \ldots, x_m \; ; \; t) = \prod_{i=1}^m \tilde{f}(x_i \; ; \; t) \tag{3.4}$$

and $f(x; t)$ verifies the Boltzmann equation:

$$\frac{\partial \tilde{f}}{\partial t}(x, t) + \underline{p} \cdot \frac{\partial \tilde{f}}{\partial \underline{q}}(x, t) = \lambda^{-1} |\underline{p}| \int (f(x', t) - f(x, t) \sigma(\Omega) \mathrm{d}\, \Omega \tag{3.5}$$

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* More explicitly this means the following. Let p be the probability of finding the W particles in a infinitesimal cube $dx_1 \ldots dx_M$ around the configuration $X = (x_1, \ldots, x_M)$ in the box Λ_0, and a tree configuration in the infinitesimal cube $d\underline{c}_1 \ldots d\underline{c}_M$ around $C = (\underline{c}_1, \ldots, \underline{c}_N)$ in the box Λ, assuming it a wider than Λ_0, at least. Here $x_i = (\underline{p}_i, \underline{q}_i)$. Then p is the product of (2.1) times $M!^{-1} (\prod_{i=1}^M f(x_i)\, dx_i)\, e^{-\int_C f(\underline{x}) d\underline{x}}$, where $\underline{x} = (\underline{p}, \underline{q})$ and $\int_C d\underline{x}$ means integration over $\underline{p}$ and over the $\underline{q} \in \Lambda_0$ which are outside the hard spheres centered on $C = (\underline{c}_1, \ldots, \underline{c}_N)$. In other words the W particles also have a Poisson distribution, in the region outside the T particles, with a density function f.

where $x' = (\underline{p}\,;\ \underline{q})$ and $\underline{p}'$ is a vector with the same length as $\underline{p}$ but forming with it an angle Ω; $a^2\sigma(\Omega) = \frac{a^2}{4\pi}$ is the scattering cross section of a hard sphere with radius a and $\lambda^{-1} = \pi\ na^2$.

A similar conjecture can be formulated in a two–dimensional model; here the solid angle Ω has to be replaced by the deflection angle ψ and $\sigma(\Omega)$ by $\sigma(\psi) = \frac{1}{4}\sin\frac{\psi}{4}$ and $\lambda^{-1} = 2an$. Of course the Boltzmann limit will be, in this case, $na^2 \to 0$, $2na \to \lambda^{-1} \neq 0, \infty$.

In the next sections we construct a proof of the above conjecture in the two–dimensional case. The three–dimensional case could be treated along the same lines as it will become apparent from the proofs.

4. — Results on the BLC

Assume the spatial dimension to be two. The direction $\omega(\underline{p})$ will be in this case the angle θ between $\underline{p}$ and a fixed axis.

The function $f_0(x)$ will be thought as $f_0(|\underline{p}|, \omega(\underline{p}), \underline{q})$ if $x = (\underline{p}, \underline{q})$ and we can write:

$$f_0(|\underline{p}|, \omega(\underline{p}), \underline{q}) = \int \mathrm{d}\,\underline{q}'\mathrm{d}\,\omega' f_0(|\underline{p}|, \omega', \underline{q}')\delta(\underline{q} - \underline{q}') \cdot \delta(\omega(\underline{p}) - \omega') \tag{4.1}$$

we shall shorten (ω', q') as ξ, $\mathrm{d}\,\underline{q}'\mathrm{d}\,\omega'$ as $\mathrm{d}\,\xi$, $\delta(\underline{q} - q')\delta(\omega(\underline{p}) - \omega')$ as $\delta(x - \xi)$. Hence, by using definition (2.4), the (3.2) becomes for $m = 1$:

$$f(x;t) = \int \mathrm{d}\,\xi f_0(|\underline{p}|, \xi) \int_{C\ \mathrm{comp}\ x} \delta(S^C_{-t}\hat{x} - \xi) p(C) \tag{4.2}$$

It is therefore useful to consider the Green's function:

$$g(\xi; x; t) = e^{\pi n a^2} \int_{C\ \mathrm{comp}\ x} p(C)\delta(S^C_{-t}\hat{x} - \xi) \tag{4.3}$$

where the factor $e^{\pi na^2}$ has been introduced for normalization purposes (note that it tends to 1, in the Boltzmann limit).

It is easily checked that:

$$g(\xi; x; 0) = \delta(\hat{x} - \xi), \quad \int g(\xi; x; t)\mathrm{d}\,\xi \equiv 1, \quad f(x,t) = \int \mathrm{d}\,\xi f_0(|\underline{p}|, \xi) g(\xi; x; t) \tag{4.4}$$

we shall show that as $na^2 \to 0$, $2na \to \lambda^{-1} \neq 0, \infty$ the function $g(\xi; x; t)$ will tend to a limit $\tilde{g}(\xi; x; t)$ which verifies the two dimensional analogue of equation (3.5) with initial condition $\tilde{g}(\xi; x; 0) = \delta(\hat{x} - \xi)$ and $|\underline{p}|$ fixed.

The linearity of (3.5), and of the third (4.4), will imply, under suitable assumptions on f_0, that also $\tilde{f}(x,t)$ verifies (3.5).

We will not insist in discussing in which sense $g(\xi; x; t)$ converges to $\tilde{g}(\xi; x; t)$.

It will appear from the proofs below that at least $g(\xi; x; t)$ converges to $\tilde{g}(\xi; x; t)$ pointwise for $t \neq 0$, and in the sense of the distributions for all $t \geq 0$.

However a close examination of the proof will provide evidence against any uniformity of the convergence in t, unless t is restricted to a bounded interval (for further remarks on this point see section 6).

Under the above convergence conditions the "mild assumptions" in BLC could, for instance, be the continuity and boundedness of f_0.

5. — Proof

The proof is based on several simple changes of variables in (4.3).

Let $x = (\underline{p}, \underline{q})$ and let $R(x,t)$ be the sphere with center $\underline{q}$ and radius $(|p|t + a)$; then $S^{c}_{-t}x$ depends only on the T–particles in c contained in $R(x,t)$.

Hence the integral (4.3) can be explicitly written as:

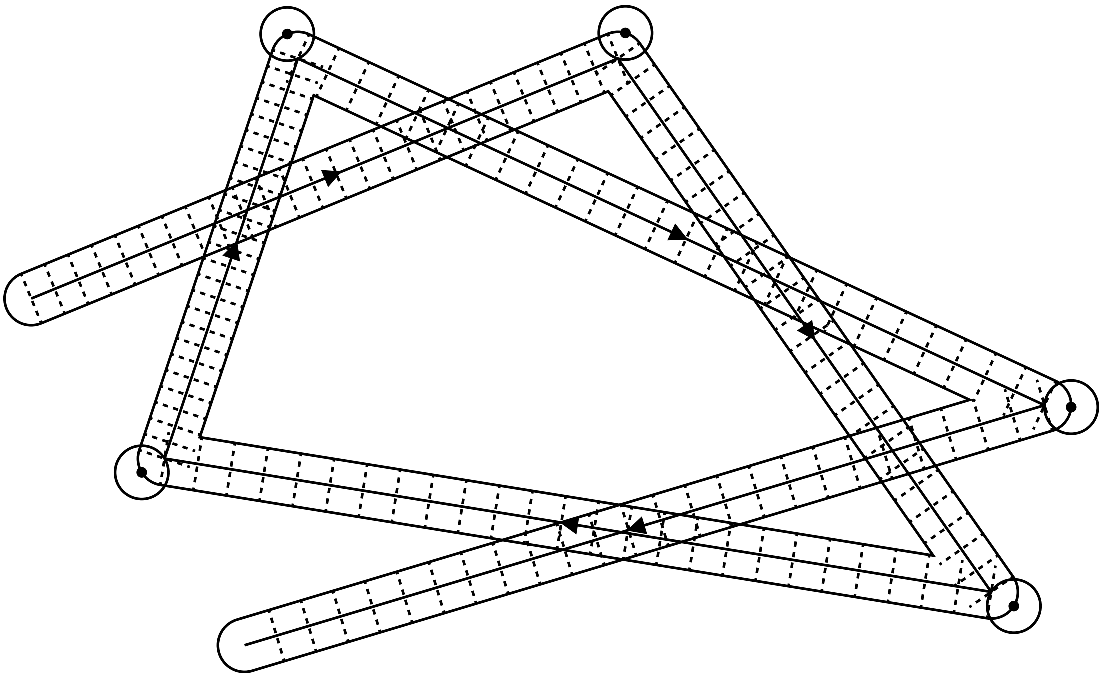
$$g(\xi;x;t) = e^{\pi n a^2} \sum_{M=0}^{\infty} \int_{R(x,t)^M} e^{-nV(R(x,t))} \frac{n^M}{M!} \delta\left(S_{-t}^{\underline{c}_1,\ldots,\underline{c}_M}\hat{x} - \xi\right) \mathrm{d}\,\underline{c}_1 \ldots \mathrm{d}\,\underline{c}_M \tag{5.1}$$

where $V(R(x,t))$= area of $R(x,t)$ and where use has been made of the assumed Poisson distribution of the T–particles (2.1).

Note that, in general, not all the T–particles $\underline{c}_1, \ldots, \underline{c}_M$ in (5.1) will be hit by the trajectory $S_{-t}^{\underline{c}_1,\ldots,\underline{c}_M}x$ $0 \le \tau \le t$. Let $A_{x,t,N}$ denote the set of configurations $\underline{c}_1, \ldots, \underline{c}_N$ of N T–particles such that a W–particle with initial coordinate x hits, in the time t, all the N particles in $\underline{c}_1, \ldots \underline{c}_N$ at least once. We deduce from (5.1):

$$\begin{aligned} g(\xi;x;t) = e^{\pi n a^2} \sum_{N=0}^{\infty} \int_{A_{x,t,N}} n^N \frac{\mathrm{d}\,\underline{c}_1, \ldots, \mathrm{d}\,\underline{c}_N}{N!} \chi_{\underline{c}_1,\ldots,\underline{c}_N}(x) \cdot \delta\left(S_{-t}^{\underline{c}_1,\ldots\underline{c}_N}\hat{x} - \xi\right) \cdot \\ \cdot \left[\sum_{M=N}^{\infty} \int_{R(x,t)^{M-N};\ \underline{c}'_1,\ldots,\underline{c}'_{M-N} \in P(t;\underline{c}_1,\ldots,\underline{c}_N)} \frac{\mathrm{d}\,\underline{c}'_1, \ldots, \mathrm{d}\,\underline{c}'_{M-N}}{(M-N)!} n^{M-N} e^{-nV(R(x,t))} \right] \end{aligned} \tag{5.2}$$

where $\chi_{\underline{c}_1,\ldots,\underline{c}_N}(x)$ is one if w is compatible with the hard cores of $\underline{c}_1, \ldots, \underline{c}_N$ and zero otherwise: the region $P(t;\underline{c}_1, \ldots, \underline{c}_N)$ is the tube like region (see fig. (5.3)) swept by an ideal T–particle when its center is moved along the path $S_{-\tau}^{\underline{c}_1,\ldots,\underline{c}_N}x$, $0 \le \tau \le t$.



(5.3)

Fig. (5.3): The set $P(t;\underline{c}_1, \ldots, \underline{c}_N)$ is the dashed region. The circles represent trees $\underline{c}_1, \ldots, \underline{c}_N$, $(N = 5)$ and the length of the trajectory in the dashed region is $|\underline{p}|t$.

The sum within square brackets in (5.2) can be performed (since the integrals are trivials) and yields:

$$e^{-nV(P(t;\underline{c}_1,\ldots,\underline{c}_N))} \tag{5.4}$$

so:

$$g(\xi;x;t)=e^{n\pi a^2}\sum_{N=0}^{\infty}\int_{A_{x,t,N}} n^N e^{-nV(P(t;\underline{c}_1,\dots,\underline{c}_N))}\frac{\delta\underline{c}_1\dots\mathrm{d}\,\underline{c}_N}{N!}\cdot\delta\left(S_{-t}^{\underline{c}_1,\dots,\underline{c}_N}\hat{x}-\xi\right)\ . \tag{5.5}$$

The reader should remark the very simple probabilistic meaning of this equation which makes it almost self–evident [11].

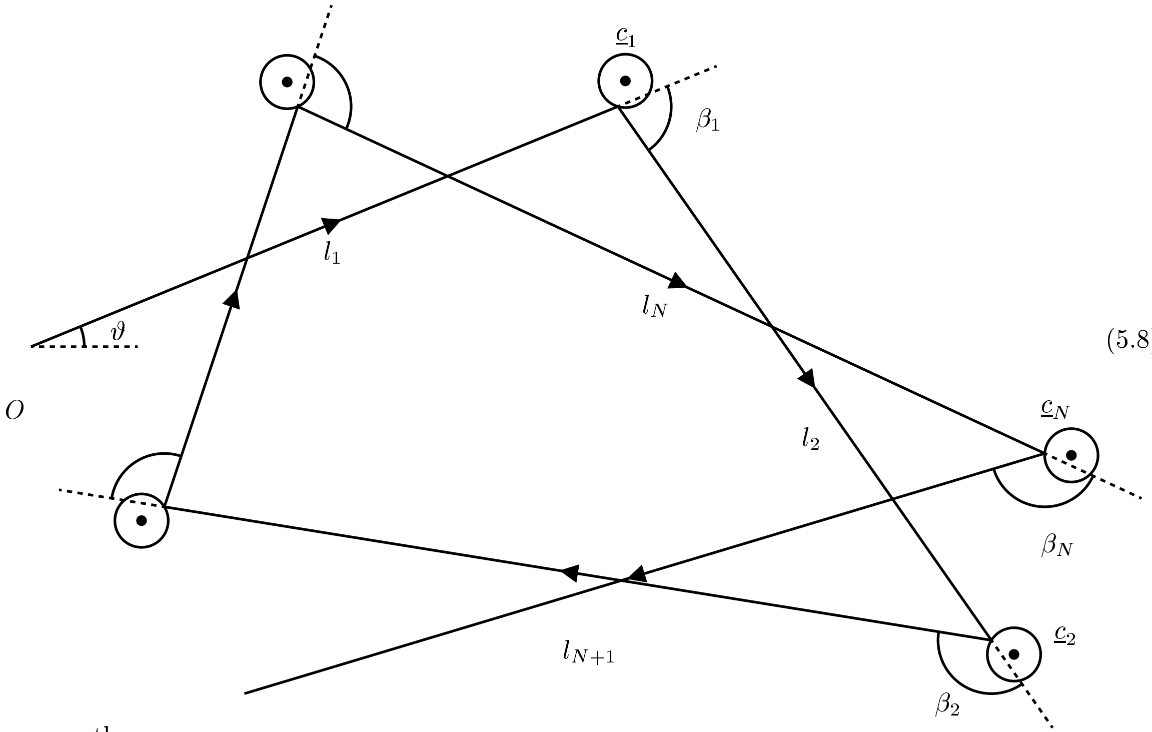
The T–particles in $A^1_{x,t,N}$ can be hit more than once in the time t. Divide $A_{x,t,N}$ as $A^1_{x,t,N}\cup A'_{x,t,N}$ where $A^1_{x,t,N}$ is the set of T–configurations in $A_{x,t,N}$ such that all their T–particles are hit just once by the trajectory $S_{-\tau}^{\underline{c}_1,\dots,\underline{c}_N}x$ $0\le\tau\le t$.

To this decomposition of $A^1_{x,t,N}$ corresponds a decomposition $g(\xi;x;t)=g_1(\xi;x;t)+g'(\xi;x;t)$ with:

$$g_1(\xi;x;t)=e^{\pi na^2}\sum_{N=0}^{\infty}\int_{A^1_{x,t,N}} n^N\frac{\mathrm{d}\,\underline{c}_1,\dots,\mathrm{d}\,\underline{c}_N}{N!}\chi_{\underline{c}_1,\dots,\underline{c}_N}(x)\cdot\delta\left(S_{-t}^{\underline{c}_1,\dots,\underline{c}_N}\hat{x}-\xi\right)e^{-nV(P(t;\underline{c}_1,\dots,\underline{c}_N))} \tag{5.6}$$

We now perform the change of variables, illustrated in fig. (5.8), from the $2N$ variables $\underline{c}_1,\dots,\underline{c}_N$ to the new $2N+1$ variables $l_1,\dots,l_{N+1}$, $\beta_1,\dots,\beta_N$; we get:

$$\frac{\mathrm{d}\,\underline{c}_1,\dots,\mathrm{d}\,\underline{c}_N}{N!}=a^N\delta\left(\sum_{i=1}^{N+1}l_i-|\underline{p}|t\right)\left(\prod_{i=1}^{N+1}\mathrm{d}\,l_1\right)\prod_{j=1}^{N}\left(\frac{\mathrm{d}\,\beta_j}{2}\sin\frac{\beta_j}{2}\right) \tag{5.7}$$



(5.8)

Hence the N^{th} order contribution to (5.6) is given by (if $x=(p,q)=(|p|,\omega(\underline{p}),\underline{q})$, $\xi=(\theta',q')$:

$$* \ e^{\pi na^2}(2na)^N\int_0^{\infty}\prod_{i=1}^{N+1}\mathrm{d}\,l_1\int_0^{2\pi}\prod_{i=1}^{N}\left(\sin\frac{\beta_i}{2}\frac{\mathrm{d}\,\beta_i}{4}\right)\delta\left(\sum_{i=1}^{N+1}l_i-|\underline{p}|t\right)\cdot$$

$$\cdot\,\delta\left(\sum_{i=1}^{N+1}\underline{l}_i-(\underline{q}'-\underline{q})\right)\delta\left(\sum_{i=1}^{N}\beta_i-(\theta'-\omega(\underline{p}))\right)e^{-nV(P(t;\underline{c}_1,\dots,\underline{c}_N))} \tag{5.9}$$

where $\underline{l}_i$ are the vectors represented by arrows in fig. (5.3) ($|\underline{l}_1| = l_i$); the * in (5.9) means that there is an extra condition on the integration region. It is the condition that none of the spheres of radius a around $\underline{c}_1, \ldots, \underline{c}_N$ has intersection with the straight segments of the broken line representing the trajectory in fig. (5.8) (i.e. this is the condition that $\underline{c}_1, \ldots, \underline{c}_N$ really belongs to $A^1_{x,t,N}$). Of course in (5.9), $\delta\left(\sum_{i=1}^N \beta_i - (\theta - \omega(\underline{p}))\right)$ means $\sum_{h=-\infty}^{+\infty} \delta(\sum_i \beta_i - (\theta - \omega(\underline{p})) - 2\pi h)$.

In the limit $na^2 \to 0$, $2na \to \lambda^{-1} \neq 0, \infty$ the restrictions indicated by the * in (5.9) become unimportant and $nV(P(t; \underline{c}_1, \ldots, \underline{c}_N))$ simplifies enormously:

$$nV(P(t; \underline{c}_1, \ldots, \underline{c}_N)) \to 2na \sum_{j=1}^{N+1} l_j = \lambda^{-1}|p|t \ . \tag{5.10}$$

Hence:

$$\tilde{g}(\xi; x; t) = \lim_{\substack{na^2 \to 0 \\ 2na \to \lambda^{-1}}} g_1(\xi; x; t) = \sum_{N=0}^{\infty} \lambda^{-N} \int_0^{\infty} \left(\prod_{i=1}^N \mathrm{d}\, l_i \right) \int_0^{2\pi} \prod_{j=1}^N \left(\sin \frac{\beta_j}{2} \frac{\mathrm{d}\,\beta_j}{4} \right) \cdot$$
$$\cdot\, \delta\left(\sum_i \underline{l}_i - (\underline{q}' - \underline{q}) \right) \cdot \delta\left(\sum_i \beta_i - (\theta' - \omega(\underline{p})) \right) \cdot e^{-\lambda^{-1}|p|t} \tag{5.11}$$

In the derivation of equation (5.11) we have systematically disregarded convergence problems connected with the summation over N, M etc. since they are trivial as a consequence of the presence of the factorials and of the boundedness of the integration regions.

The limit (5.11) is pointwise for $t \neq 0$ and it could be checked that it holds also in the sense of the distributions for $t \geq 0$.

Furthermore it could be checked that for $t > 0$ the function $g(\xi, x, t) \geq g_1(\xi, x, t)$ is bounded above by a $L_1(d\xi)$ function; hence the limit (5.11) holds also in the $L_1(d\xi)$ sense. Finally, by direct computation, it follows from (5.11) that:

$$\int \tilde{g}(\xi; x; t) \mathrm{d}\,\xi \equiv 1 \tag{5.12}$$

and this fact, together with the above convergence properties and (4.4), implies the validity of the limit relation: $\lim_{\substack{na^2 \to 0 \\ 2na \to \lambda^{-1}}} g(\xi; x; t) = \tilde{g}(\xi; x; t)$ in $L_1(\mathrm{d}\,\xi)$ for $t > 0$; furthermore it could be proved that this limit holds, for $t \geq 0$, in the sense of the distributions.

It is known [12] that (5.11) is a solution of the Boltzmann equation (and this can be checked directly by substituting $\tilde{g}$ into (3.5)), with initial condition $\tilde{g}(\xi; x; 0) = \delta(\hat{x} - \xi)$ and $|p|$ fixed.

To complete the proof of the BLC it remains to deal with the m–particle distributions. However we skip this point since it involves straightforward calculations based on changes of variable of the type illustrated in fig. (5.8).

6. — Concluding remarks

In the preceding sections we have described a proof of the Boltzmann limit conjecture in the case of a two–dimensional Lorentz gas with hard core $W - T$ interactions and free gas distribution of the T–particles. The generalization to three dimensions would be trivial.

A less trivial generalization would be obtained by keeping the hard core $W - T$ interaction but assuming that the T–particles are spatially distributed as if they were a gas of hard spheres with hard core size being proportional to the the $W - T$ radius. Other generalizations are conceivable in the direction of allowing soft

$W - T$ particle interactions and more general T–particle distributions. So far none of these generalizations have been attempted.

Much more difficult and interesting would be the treatment of the *Knudsen model*, in which the T–particles are allowed to move without suffering changes in their momentum in the collisions in the W–particles.

Another remark is that had we done the calculations associated with the proof of equation (3.4); we would have also found evidence of a lack of uniformity of the Boltzmann limit in the number m of W–particles even at fixed t: the larger m is the closer one has to get near the Boltzmann limit in order to see factorization of the W–particle correlations.

We also wish to remark that even when the Boltzmann limit conjecture is true, one cannot expect that the function $\tilde{f}(\underline{r}, \underline{v}, t)/a^2$ (see section 1) is a good approximation to $f(\underline{r}, \underline{v}, t)$ for large t: in fact one intuitively expects that for times of the order of $tm\underline{r}\underline{p}/na^3$ some non trivial correlations will start building up thus destroying the molecular chaos and spoiling the validity of the Boltzmann equation.

This last remark is quite deceiving since it tells us that we cannot use, *without further assumptions*, the Boltzmann equation to investigate the long time behaviour and, in particular, to compute the transport coefficients. From a rigorous point of view we cannot even be sure that the lowest order in na of the transport coefficients is correctly given by the value obtained in the Boltzmann limit. However it seems reasonable that this is, indeed, the case at least if the dimension of the space is larger than two (in one–dimension a simple counter example can be found by using soluble models [4]; in this case, however, the Boltzmann equation is a priori expected not to be a good approximation).

For further readings on the Lorentz gas see ref. [13].

The idea of the Boltzmann limit is clearly stated in [3]; the present proof in the case of the Lorentz gas is done in ref. [12] (for the case of $\tilde{g}(\underline{x}; x; t)$ only) and was inspired by discussions and suggestions from J.L. Lebowitz.

Postscript:

*The Boltzmann limit conjecture has been proved a few years after the present paper was written. It is due to O. Lanford, [14] (1974), and it holds under some very reasonable restrictions on the initial data and for a hard spheres system, but with a still standing limitation on the time interval of validity. The time interval is strictly positive, but it is a small fraction ε of the mean free flight time t_0 (which is defined as the ratio of the mean free path over a mean velocity computed in the initial state): $\varepsilon \sim \frac{1}{5}$. This clearly did put an end (or at least it should have) to the diatribes on whether the irreversibility can or cannot be deduced from a microscopically reversible mechanical model (*Boltzmann vindicatus est*); but it left open the question of the mathematical justification of the validity of the Boltzmann equation over the time scales on which it is usually used, going beyond the mean free time by several orders of magnitude. There has been, since, one case in which the proof of the BLC has been pushed to infinite time, [15].*

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