ONE DIMENSIONAL ANOMALY OF THE FERMI SURFACE

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Consider a N-fermions system with hamiltonian, see [BGPS]:

$$H=\sum_{i=1}^N \frac{-\Delta_{x_i}-p_F^2}{2m}+\lambda\sum_{i<j} v(x_i-x_j) \tag{1}$$

where m, p_F are two fixed parameters, λ is the coupling constant and v is the pair potential, supposed smooth and with short range p_0^{-1}.

The fermions will be supposed also to have spin 0 and to be enclosed in a periodic box of size L. Thus in the grand canonical ensemble the pair Schwinger function at $\xi=(x,t)\in R^2$ is defined by:

$$S(\xi)=\lim_{L\to\infty}\lim_{\beta\to\infty}\frac{\mathrm{Tr}\, e^{-(\beta-t)H}\varphi_x^- e^{-tH}\varphi_0^+}{\mathrm{Tr}\, e^{-\beta H}} \tag{2}$$

where $\varphi_x^\pm$ are the creation and annihilation operators.

The above system is particularly interesting in the light of recently proposed connections with the theory of 2–dimensional Fermi gases which, in some cases, might exhibit similar properties, see [A].

The free system, $\lambda=0$, is such that:

$$\begin{aligned} S(\xi)=&S^0_{m,p_F}(\xi)=\frac{1}{(2\pi)^2}\int dk_0 dk\,\frac{e^{-i(k_0t+kx)}}{-ik_0+(k^2-p_F^2)/2m}\simeq\\ \simeq_{\xi\to\infty}&-\frac{1}{\pi}\frac{\sin(p_Fx-\,\mathrm{arctg}\, v_Ft/x)}{(x^2+(v_Ft)^2)^{1/2}},\qquad\qquad v_F\equiv\frac{p_F}{m}\end{aligned} \tag{3}$$

if $\mathrm{arctg}\, v_Ft/x$ bounded away from $\pi/2$;

And $S(x,0^-)$ has a Fourier transform $1-\hat n(k)$ in x, with $\hat n(k)=\langle a_k^+a_k^-\rangle$ being the average occupation density of the state with momentum k. It is well known that $\hat n(k)=1$ if $|k|<p_F$ and $\hat n(k)=0$ otherwise.

We address the question: *which is the asymptotic behaviour of $S(\xi)$ when $\lambda\ne0$?* answering the question will give us a way to define the *Fermi momentum* and the *mass* of the particles in presence of interaction.

In fact we shall use the following definition:

Definition: *The hamiltonian (1) describes a system with particles of mass m' and Fermi momentum p'_F if there exist constants Z,η such that:*

$$S(\xi)=\frac{1}{Z}\frac{S^0_{m',p'_F}(\xi)}{|\xi|^{2\eta}} \tag{4}$$

where $|\xi|^2=p_0^2(x^2+{v'_F}^2t^2)$.

One says that the system (1) is a normal Fermi liquid *if $\eta=0$: in this case Z^{-1} is the jump of the Fermi distribution $\hat n(k)$ at $k=p'_F$. Otherwise the system is* anomalous *and the momentum distribution is less singular at $k=p'_F$, behaving proportionally to $||k|-p'_F|^{2\eta}$.*

Of course one expects that the quantities m', p'_F depend on λ and are different from the parameters m, p_F in (1). It is therefore convenient, if one wants to study the above problem by perturbation theory, to introduce a hamiltonian with more parameters giving us the possibility of fixing *a priori* the values m, p_F which describe the singularity of $S(\xi)$. Hence we shall consider the hamiltonian:

$$H=\sum_{i=1}^N\frac{-\Delta_{x_i}-p_F^2}{2m}+\lambda\sum_{i<j}v(x_i-x_j)+\sum_{i=1}^N\frac{-\alpha\Delta_{x_i}-\nu p_F^2}{2m} \tag{5}$$

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and call $m_0 = m/(1+\alpha)$ the *bare mass* and $\mu = -(1+\nu)p_F^2/2m$ the *chemical potential.* We shall then ask the question: is it possible to fix $\alpha = \alpha(\lambda), \nu = \nu(\lambda)$ in terms of λ so that (4) holds with $(m', p'_F) = (m, p_F)$? Or: can one fix the bare mass and bare chemical potential so that the interacting system has prescribed values of the mass and the Fermi momentum?

The above is not the only possible definition of mass and Fermi momentum: there are several others; for an example of an alternative one see [BGPS].

That the above analysis and definitions are non empty and have a non trivial meaning is shown by the main result of [BGPS].

Theorem: *Suppose that $a \equiv \hat v(0) - \hat v(2p_F) \neq 0$, and $|\lambda|$ small enough; then there exist functions $\alpha(\lambda), \nu(\lambda)$,-$\eta(\lambda)$ analytic in λ near $\lambda = 0$ and divisible by λ^2 such that, fixing $\alpha = \alpha(\lambda), \nu = \nu(\lambda)$ in (5) it is:*

$$S(\xi) = \frac{S^0_{m,p_F}(\xi)}{|\xi|^{2\eta}} + \frac{A_\xi(\lambda)}{|\xi|^{1+2\eta}} \tag{6}$$

with $\eta = \eta(\lambda)$ and $|A_\xi(\lambda)| \le C|\lambda|$ for a suitable constant C and all ξ. Furthermore $\eta(\lambda) \neq 0$.

In other words it is possible to fix $\alpha = \alpha(\lambda), \nu = \nu(\lambda)$ so that the system describes fermions with Fermi momentum p_F and mass m. However it will be, generically (*i.e.* if $a \neq 0$ at least), a system with anomalous Fermi surface.

The above theorem was discussed by Tomonaga, [T], who gave very strong arguments in favour of it: the arguments led to the formulation of the Luttinger model, which was conjectured, [L], to show the same behaviour, for the Schwinger functions, of the system (1) (hence to show an anomalous Fermi surface, by [T]), and to be exactly soluble. The model was indeed solved exactly by [ML] and gave the expected results (although quantitatively different from the exact conjecture in [L]).

The above theorem closes the missing gaps by proving that the "realistic" system (1) shows anomalous behaviour as predicted by [T].

Let me spend a few words about the strategy followed in our theory of (1).

Imagine to fix $L, \beta < \infty$. Then, $(d = 1)$:

1) the functions $S(\xi)$ are analytic in λ, α, ν near 0, with a radius of convergence at least of order $O(L^{-1})$, unformly in β.

2) the Schwinger functions are given, for small λ, α, ν by a functional integral:

$$S(\xi_1, \ldots, \xi_n) = \frac{\int P(d\psi)\, e^{-V(\psi)}\, \psi^-_{\xi_1} \ldots \psi^+_{\xi_n}}{\int P(d\psi) e^{-V(\psi)}} \tag{7}$$

where:

$$V(\psi) = \lambda \int v(x-y) \psi^+_{x,t}\psi^-_{x,t}\psi^+_{y,t}\psi^-_{y,t}\, dxdydt + \alpha \int \psi^+_{x,t} \frac{-\Delta}{2m} \psi^-_{x,t}\, dxdt + \nu \int \psi^+_{x,t}\psi^-_{x,t}\, dxdt \tag{8}$$

and $\psi^\pm_\xi$ are grassmanian variables on R^2 and $P(d\psi)$ is a grassmanian integrations with propagator:

$$g(\xi - \eta) = \langle \psi^-_\xi\, \psi^+_\eta \rangle \equiv \frac{1}{(2\pi)^2} \int dk_0 dk \frac{e^{-i\kappa(\xi-\eta)}}{-ik_0 + (k^2 - p_F^2)/2m} \tag{9}$$

where $\kappa = (k, k_0)$, $\int dk\, \cdot\, \equiv \frac{2\pi}{L}\sum \cdot$, $\int dk_0\, \cdot\, \equiv \frac{2\pi}{\beta}\sum \cdot$ with the sums running over the $k = \frac{2\pi}{L} n$ and $k_0 = \frac{2\pi}{\beta}(n_0 + \frac12)$ with n_0, n integers. [1]

3) it is possible to *rearrange* the series in $\vec r_0 = (\lambda, \alpha, \nu)$ into power series in a new family of parameters $\vec r_0, \vec r_{-1}, \vec r_{-2}, \ldots$, with $\vec r_j \in R^3$, so that the disk of convergence for the series expressing the Schwinger functions in powers of such parameters is at least $|\vec r_h| \le \varepsilon_0$ with $\varepsilon_0 > 0$ *and β, L independent.* The new expansion parameters, *the running coupling constants*, are analytic functions of $\vec r_0$ with radius of analyticity of $O(L^{-1})$.

[1] this means that the numerator and denominators of (7) are defined by developing the exponential in powers and by computing the integrals via the Wick rule. It is well known since the work [C], see also [GK], [BGPS], that the resulting series are entire functions of λ, α, ν and, for λ, α, ν small enough, of order $O(L^{-1})$ at least, the denominator does not vanish and (7) holds.

Comment: in other words we can "push" the problem of computing the Schwinger functions into that of studying the true dependence of a certain sequence of scalar functions of $\vec{r}_0$, the running couplings. Of course the abstract formulation looks just a rephrasing of the original problem and no gain at all. Its interest depends on the actual definition of the resummation rule and of the running couplings.

4) the $\vec{r}_h$ verify, if $|\vec{r}_h| \le \varepsilon_0$, a recurrence relation:

$$\vec{r}_{h-1} = \Lambda\vec{r}_h + B(\vec{r}_h) + \mathcal{B}_h(\vec{r}_h, \vec{r}_h - \vec{r}_{h+1}, \vec{r}_h - \vec{r}_{h+2}, \ldots, \vec{r}_h - \vec{r}_0) \tag{10}$$

with Λ linear and diagonal, and $B, \mathcal{B}_j$ analytic in their arguments in the polydisk of radius ε_0. Furthermore $\mathcal{B}_j$ are functions with *short memory*; this means that there exist functions $D_j^{h,k}(\vec{x}_k, \vec{x}_{k+1}, \ldots, \vec{x}_{-1}, \vec{x}_0)$ such that for some $b, d > 0$:

$$\begin{aligned} &\mathcal{B}_j(\vec{x}_j, \vec{x}_{j+1}, \ldots, \vec{x}_{-1}, \vec{x}_0) = \sum_{k=j}^{-1} D_j^k(\vec{x}_j, \vec{x}_{k+1}, \ldots, \vec{x}_{-1}, \vec{x}_0) \\ &|D_j^k| \le 2^{-(k-j)b}\, |\vec{x}_j| \,\big(\sup_{k'>k} |\vec{x}_{k'}|\big)\, d \end{aligned} \tag{11}$$

Comment: this has the practical consequence that the dynamical system without memory defined by the map: $\vec{r}\,' = \Lambda\vec{r} + B(\vec{r})$ and the one in (10) behave essentially in the same way near $\vec{r}_h = \vec{0}$.

5) if $|\vec{r}_j| \le \varepsilon_0$ and $\vec{r}_j \xrightarrow[j\to-\infty]{} \vec{r}_{-\infty}$ then there exists an analytic function $\eta(\vec{r})$ of $\vec{r}$, defined for $|\vec{r}| \le \varepsilon_0$, and such that $\eta = \eta(\vec{r}_{-\infty})$ is the anomalous exponent for the Schwinger functions:

$$S(\xi) = \frac{S_{m,p_F}(\xi)}{|\xi|^{2\eta}} + \lambda O(|\xi|^{-1+2\eta}) \tag{12}$$

with $|\xi|^2 = p_0^2(x^2 v_F^2 t^2)$, $v_F = p_F/m$. And finally:
6) if $\vec{r} = (\lambda, \alpha, \nu)$ then the first two components of the *beta function* $B(\vec{r})$, *vanish identically* for $\vec{r} = (\lambda, \delta, 0)$

The last property will be the vault key of the whole theory: should it have failed it would have been easy to see that *no initial $\vec{r}_0 = (\lambda, \alpha, \nu)$ could be found generating, via (10), a sequence of running couplings $\vec{r}_h$ such that $|\vec{r}_h| \le \varepsilon_0$ for all $h \le 0$, and the whole discussion would be inconclusive.*

It remains to get down to work to explain the "resummation procedure" with the above remarkable properties. This is obviously a very essential key point: the renormalization group is a well known method for resummations, but there are "many" possible resummations and it is by no means clear that there is one with the above properties. It is in fact not difficult to exhibit many resummations with all the above properties except the last. They are non trivial, nevertheless they are useless, see [BGPS], —s4, for an example.
In order to understand the right resummation procedure or, in other words, *the right renormalization group transformation*, we must first introduce the notion of *quasi particles fields*. It is in fact by regarding the functional integrals in (7) as functional integrals over quasi particle fields that one can immediately recognize the analogy between the above Fermi surface problem and the well known theory of the anomalous dimension in $4-\varepsilon$ dimensions of [WF]. Once the analogy is recognized it is only a "technical problem" to work out the details to obtain the above propeties 1)%6).
We consider the grassmanian propagator and decompose it as follows; let $\varepsilon(k_0, k) \equiv -ik_0 + (k^2 - p_F^2)/2m$ and use the identity $1 \equiv \big(1 - e^{-\varepsilon(k_0,k)^2 p_0^{-2}}\big) + e^{-\varepsilon(k_0,k)^2 p_0^{-2}}$ where p_0 is a convenient momentum scale, equal to the inverse range of the potential; then:

$$g(\xi) = g^{(>0)}(\xi) + g^{(\le 0)}(\xi) \equiv \frac{1}{(2\pi)^2}\int \frac{dk dk_0}{\varepsilon(k_0,k)} \Big(\Big[1 - e^{-\varepsilon(k_0,k)^2 p_0^{-2}}\Big] + \Big[e^{-\varepsilon(k_0,k)^2 p_0^{-2}}\Big]\Big) \tag{13}$$

with the obvious identification of the two terms, that we call the *ultrviolet* and the *infrared* parts of the propagator g..
It is then possible to write the infrared part as:

$$g^{(\le 0)}(\xi) = \sum_{\omega=\pm 1} e^{-i\omega p_F x}\,\overline{g}^{(\le 0)}(\xi, \omega), \qquad \overline{g}^{(\le n)}(\xi) = \sum_{h=-\infty}^{n} 2^h \tilde{g}^{(h)}(2^h x, \omega) \tag{14}$$

with $\tilde g^{(h)}(\xi,\omega)$ essentially independent on h, for $-h$ large, see [BGPS].

This means that:

$$\psi^\pm_\xi\equiv\psi^{(>0)\pm}_\xi+\psi^{(\le0)\pm}_\xi\equiv\psi^{(>0)\pm}_\xi+\sum_{\omega=\pm1}e^{\pm i\omega p_F x}\psi^{(\le0)\pm}_{\xi,\omega}\tag{15}$$

where $\psi^{(>0)}_\xi$ and $\psi^{(\le0)}_{\xi,\omega}$, $\omega=\pm$, are *independent* grassmanian variables, (*i.e.* the crossed propagators vanish).

We call the $\psi^{(\le0)}_{\xi,\omega}$ the quasi particle fields and the integrals can be written $\int P(d\psi^{(>0)})P(d\psi^{(\le0)})e^{-V(\psi^{(\le0)})}$.

Forgetting the integral over $\psi^{(>0)}$ (see [BGPS]), I shall explain here the definition of the running couplings (*i.e.* the definition of the resummation procedure) for the easier problem of the evaluation of the partition function: [2]

$$\Xi=\int P(d\psi^{(\le0)})e^{-V(\psi)}\tag{16}$$

We already mentioned that we use the renormalization group methods, and that the proper formulation is to to think that interaction V and the integral in (16) are over quasi particle fields. The reason is that the quasi particle fields propagator $\overline g^{(\le0)}(\xi,\omega)$ have *no intrinsic length scale*: they have in fact a Fourier transform with singularity at the origin. *This is not the case for the particle fields* which contain, in the propagator, a fixed length scale p_F^{-1}, see (14).

The quasi particles fields, therefore, arise naturally to recover the scale invariance that is lost by the presence of the Fermi momentum.

We can write the interaction V as an interaction between quasi particles and define the relevant part and the irrelevant parts of it *via the same prescription that would be used in the renormalization group approach to scalar fields*, (see [WF],[G]). By a power counting analysis one identifies the relevant and marginal operators and then one defines the relevant part of the interaction to be $\mathcal{L}V$ where $\mathcal{L}$ is a *localization operator* defined by:

$$\begin{aligned}\mathcal{L}\psi^+_{\xi_1,\omega_1}\psi^+_{\xi_2,\omega_2}\psi^-_{\xi_3,\omega_3}\psi^-_{\xi_4,\omega_4}&=\psi^+_{\xi_1,\omega_1}\psi^+_{\xi_1,\omega_2}\psi^-_{\xi_1,\omega_3}\psi^-_{\xi_1,\omega_4}\\ \mathcal{L}\psi^+_{\xi_1,\omega_1}\psi^-_{\xi_2,\omega_2}&=\psi^+_{\xi_1,\omega_1}\Big(\psi^-_{\xi_1,\omega_2}+(x_2-x_1)\partial\psi^-_{x_1,\omega_2}\Big)\end{aligned}\tag{17}$$

as it would be done in quantum field theory of the Gross–Neveu model, (see [GK], and [BG]).

The *effective potential* will the be defined recursively as follows. Let $Z_0=1$ and denote $P^0_{Z_0}$ the integration with propagator $Z_0^{-1}g^{(\le0)}(\xi)$. We shall write such functional integration formally as:

$$P^0_{Z_0}(d\psi)\propto e^{-Z_0(\psi^+,[(-\Delta-p_F^2)/2m]Q^0\psi^-)}\,d\psi\tag{18}$$

where $Q^0(\partial)$ has Fourier transform $Q^0(k)=e^{\varepsilon(k)^2p_0^{-2}}$, where again $\varepsilon(k)^2=k_0^2+\big((k^2-p_F^2)/2m\big)^2$. For later use we define also $Q^h(\partial)$ as $Q^h(k)=e^{2^{-2h}\varepsilon(k)^2p_0^{-2}}$.

Then, setting $V\equiv V^{(0)}$:

$$\begin{aligned}\Xi=&\int e^{-V^{(0)}(\sqrt{Z_0}\psi^{(\le0)})}P^0_{Z_0}(d\psi^{(\le0)})\equiv\int e^{-V^{(0)}(\sqrt{Z_0}(\psi^{(<0)}+\psi^{(0)})}\hat P_{Z_0}(d\psi^{(0)})P^0_{Z_0}(d\psi^{(\le-1)})=\\ =&\int P_{Z_0}(d\psi^{(\le-1)})e^{-\tilde V(\sqrt{Z_0}\psi^{(\le-1)})}\end{aligned}\tag{19}$$

where the propagator of $\psi^{(\le-1)}$ is $Z_0^{-1}g^{(\le-1)}(\xi)$ and that of $\psi^{(0)}$ is $Z_0^{-1}\tilde g^{(0)}(\xi)$, see (14).

The integration over $\psi^{(0)}$ is performed by thinking $\psi^{(0)}$ and $\psi^{(\le-1)}$ as a sum of quasi particle fields. The result of the integration will be expressed in terms of the function $\tilde V(\sqrt{Z_0}\psi^{(\le-1)})$ *which can be regarded either as a function of the particle field* $\psi^{(\le-1)}_\xi$ *or of the quasi particle field* $\psi^{(\le-1)}_{\xi,\omega}$.

The $\tilde V$ will be written as a sum $\sum_{\omega_1,\ldots}\int w(\xi_1,\ldots,\omega_1,\ldots)\psi^{+(\le-1)}_{\xi_1\omega_1}\ldots d\xi_1\ldots$ of terms and the calculation of the coefficients w of its expansion in (even) field monomials $\psi^{+(\le-1)}\ldots$, is of course a delicate technical problem. It is obtained "trivially" by a power series expansion, which is shown to be convergent by the essential use of the Hadamard inequality on determinants following the work [GK] (and the idea of [C]).

Regarding $\tilde V$ as a function of the quasi particle fields we can apply to it the localization operator $\mathcal{L}$ thus isolating the relevant and marginal parts from the irrelevant ones (which, of course, are the really interesting

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[2] for the treatement of the ultraviolet problem and for the (non trivial) discussion of how to connect the theory of the partition function with that of the asymptotic behaviour of the Schwinger functions see [BGPS].

ones for the purpose of calculating the Schwinger functions and the physical properties of the system). It will result, see (17):

$$\begin{aligned}\tilde V(\sqrt{Z_0}\psi^{(\le -1)}) = &\overline V(\sqrt{Z_0}\psi^{(\le -1)}) + Z_0^2\lambda_0\int \psi^{(\le-1)+}_{\xi,+}\psi^{(\le-1)+}_{\xi,-}\psi^{(\le-1)-}_{\xi,+}\psi^{(\le-1)-}_{\xi,-}\,d\xi+\\ &+Z_0\int\Big(\delta_0\psi^{(\le-1)+}_{\xi,\omega}\,\omega'\partial_x\psi^{(\le-1)-}_{\xi,\omega'}+\zeta_0\psi^{(\le-1)+}_{\xi,\omega}\,(\partial_t+i\omega'\partial_x)\psi^{(\le-1)-}_{\xi,\omega'}\Big)e^{ip_F(\omega-\omega')x}\,d\xi+\\ &+Z_0 2^0\nu_0\int\psi^{(\le-1)+}_{\xi,\omega}\psi^{(\le-1)-}_{\xi,\omega'}e^{ip_F(\omega-\omega')x}\,d\xi\end{aligned}\tag{20}$$

where $\overline V\equiv(1-\mathcal L)\tilde V$ and the other terms are the result of the application of $\mathcal L$ to $\tilde V$, *i.e. they are a definition of the constants* $\lambda_0,\delta_0,\zeta_0,\nu_0$. The factor $2^0=1$ is so written because it will be generalized, in the iterative construction of the running couplings, into a factor 2^h at the step $-h$.

Looking at the part of $\tilde V$ of second degree in the fields ψ (or, more generally, of any fixed degree) we realize that the dependence of the kernels in the integrals over the ξ variables and the sums over the quasi particle labels ω must be special so that it must be possible to write them as functionals of the particle fields: this explains why, in (20), the coefficients δ_0,ν_0,ζ_0 are just constants and not functions of ω,ω', as *a priori*, one would expect. This is a very important property called in [BG] "gauge invariance".

Hence by adding and subtracting to the r.h.s. of (20) the second degree part $W=(\zeta_0-\delta_0)Z_0\int\psi^{(\le-1)+}_{\xi,\omega}\cdot\frac{\Delta_\xi}{2m}\psi^{(\le-1)-}_{\xi,\omega'}e^{ip_F(\omega-\omega')x}d\xi$ we see, with an easy calculation, that $\tilde V$ can be written as a sum of an "irrelevant part" $\overline V-W$ (in the sense of power counting) plus a "relevant part" given, at second degree in the fields by:

$$Z_0\int d\xi\,\psi^{(\le-1)+}_\xi\Big(\delta_0\frac{-\Delta_\xi-p_F^2}{2m}\psi^{(\le-1)-}_\xi+\zeta_0\,(\partial_t+\frac{-\Delta_\xi-p_F^2}{2m})\psi^{(\le-1)-}_\xi+2^0\nu_0\psi^{(\le-1)-}_\xi\Big)\tag{21}$$

The "free" integration $P_{Z_0}(d\psi^{(\le-1)})$ will be written formally as proportional to $d\psi\,\exp-(\psi^{(\le-1)+},(\partial_t+\omega\partial_x)Q^0(\partial)\psi^{(\le-1)-})$ so that the term with coefficient ζ_0 can be put togheter with the free integration and one gets, setting $Z_{-1}=Z_0(1+\zeta_0)$:

$$\Xi=\int P^{-1}_{Z_{-1}}(d\psi^{(<-1)})\hat P_{Z_{-1}}(d\psi^{(-1)})e^{-V^{(-1)}(\sqrt{Z_{-1}}(\psi^{(\le-2)}+\psi^{(-1)})}\tag{22}$$

where the propagator of the integration $\hat P_{Z_{-1}}(d\psi^{(-1)})$ is *defined*, by difference, so that the following quadratic forms are identical:

$$\begin{aligned}&Z_0(f,[\partial_t+(-\Delta-p_F^2)/2m]Q^{(-1)}f)+Z_0\zeta_0(f,[\partial_t+(-\Delta-p_F^2)/2m]f)=\\ &=Z_{-1}(f,[\partial_t+(-\Delta-p_F^2)/2m]Q^{(-1)}f)+Z_{-1}(f,\tilde Qf)\end{aligned}\tag{23}$$

Hence, we finally get:

$$\begin{aligned}\Xi=&\int\hat P_{Z_0}(d\psi^{(0)})P^0_{Z_0}(d\psi^{(<0)})e^{-V^{(0)}(\sqrt{Z_0}(\psi^{(<0)})+\psi^{(0)}))}=\\ =&\int\hat P_{Z_{-1}}(d\psi^{(-1)})P^{-1}_{Z_{-1}}(d\psi^{(<-1)})e^{-V^{(-1)}(\sqrt{Z_{-1}}(\psi^{(<-1)})+\psi^{(-1)}))}\end{aligned}\tag{24}$$

One then iterates the above construction: the key remark is that at the step $-h$ the propagator of $\hat P_{Z_h}(d\psi^{(h)})$, regarded as a quasi particles field propagator, written as $2^h\hat g^{(h)}(2^h\xi,\omega)$ is essentially scale invariant (this is not completely obvious and it requires a calculation; which shows that $\hat g^{(h)}\ne\tilde g^{(h)}$ for $h<0$, but that nevertheless $\hat g$ has the same approximate scale invariance as $\tilde g^{(h)}$). The propagator $\hat g^{(h)}(\xi,\omega)$ results approximately scale independent (*i.e.* h–independent) and decaying exponentially fast both in position and in momentum space.

The above analysis describes the definition of the running couplings; the approximate scale invariance of the propagators $\hat g^{(h)}$ is the reason for the corresponding scale invariance of the beta function, *i.e.* of the relation (10) and the h–independence of the "main term" $B(\vec r)$. The convergence and analyticity properties of the beta function are a simple consequence of the techniques developed in [GK], see [BGPS].

The final property, item 6) above, *i.e.* the vanishing of the first two components of $B(\lambda,\delta,0)$, is based on the comparison of the above theory with the analogous theory of the Luttinger model, [BGM], [GS].

One shows, see [BG], that the function $B(\vec{r})$ (*i.e.* the leading part of the recurrence (10)) is the same for the model (5) and for the Luttinger model. And one infers that the first two components of $B(\lambda, \delta, 0)$ must vanish by an "indirect argument", namely by showing that otherwise the exact solution of the Luttinger model would be different from what it is, see [BGPS]. There are also arguments, see [DM], suggesting that a "direct" check of the vanishing of the appropriate part of the beta function might actually be possible.
Finally the property in item 5) is based on a careful analysis of the series (now known to be convergent) for the Schwinger functions in terms of the running couplings. The analysis is somewhat delicate: see [BGPS]: note however that heuristically the result should be expected from the closeness between the above approach and the work [WF].

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