Localization at Weak Disorder: Some Elementary Bounds

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Dedicated to E.H. Lieb, in celebration of his sixtieth birthday.

Abstract

An elementary proof is given of localization for linear operators $H=H_o+\lambda V$, with H_o translation invariant, or periodic, and $V(\cdot)$ a random potential, in energy regimes which for weak disorder ($\lambda\to 0$) are close to the unperturbed spectrum $\sigma(H_o)$. The analysis is within the approach introduced in the recent study of localization at high disorder by Aizenman and Molchanov [AM]; the localization regimes discussed in the two works being supplementary. Included also are some general auxiliary results enhancing the method, which now yields uniform exponential decay for the matrix elements $<0|P_{[a,b]}e^{-itH}|x>$ of the spectrally filtered unitary time evolution operators, with [a,b] in the relevant energy range.

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1. Introduction

This work presents an elementary derivation of localization for time evolutions generated by linear operators consisting of a translation invariant, or periodic, part and an added random potential, at energy ranges which at weak disorder approach the unperturbed spectrum. Though the article is self contained, the method is a continuation of that presented in the recent work by Aizenman and Molchanov [AM]. The regimes considered in these two works are supplementary, as shown in figure 1.

The relevance of the existence, in the spectrum of operators with disorder, of spectral regimes in which the states are localized as well as regimes of extended states, and the nature of the transition between them, have been the subject of extensive analysis in physics literature (Anderson [A], Mott-Twose [MT], Thouless [T], Wegner [W], and various interesting recent works). Rigorous results on localization in dimensions not restricted to d=1 ([FMSS],[DLS],[SW]) have, before the appearance of ref. [AM], drawn on the multiscale analysis (MS) of Fröhlich and Spencer [FS] (d=1 results are discussed in [GMP],[CFKS],[CL],[FP]). The MS analysis is a powerful tool for the resolution of the small denominator problems, which are caused by the disorder and the fact that one is interested in the resolvent operators at energies in the spectrum. In terms of the general picture, the method yields very good qualitative results, which include the existence of localization at edges of the spectrum - at any disorder, and an explicit finite volume criterion for localization. However, these are not easily translated into effective quantitative bounds, and, though the method has been somewhat simplified ([S2,D,DK]), it still is not quite elementary. An elementary approach to localization was presented in ref. [AM]. It is used and further developed here. (A summary of the basic ideas can be found in Section 2.) The elementary approach has not yet reached all the known results, e.g., the two examples cited above, but it did yield some results which have previously resisted rigorous analysis ([AM]).

The main results of this work are split into two parts. The first concerns localization in the context stated above. The novelty is not in the existence of localization, which was known [though, according to the parallel work [FK], for some regions the results are being firmed up only now] - but mainly in the method and in the more explicit bounds, both on the localization regime (eq. (1.3)) and on the unitary evolution operator (eq. (1.6)). The second part (in the order of presentation only) consists of some general auxiliary results which, in addition to being used here, are of wider applicability. Their relation to past works is described below.

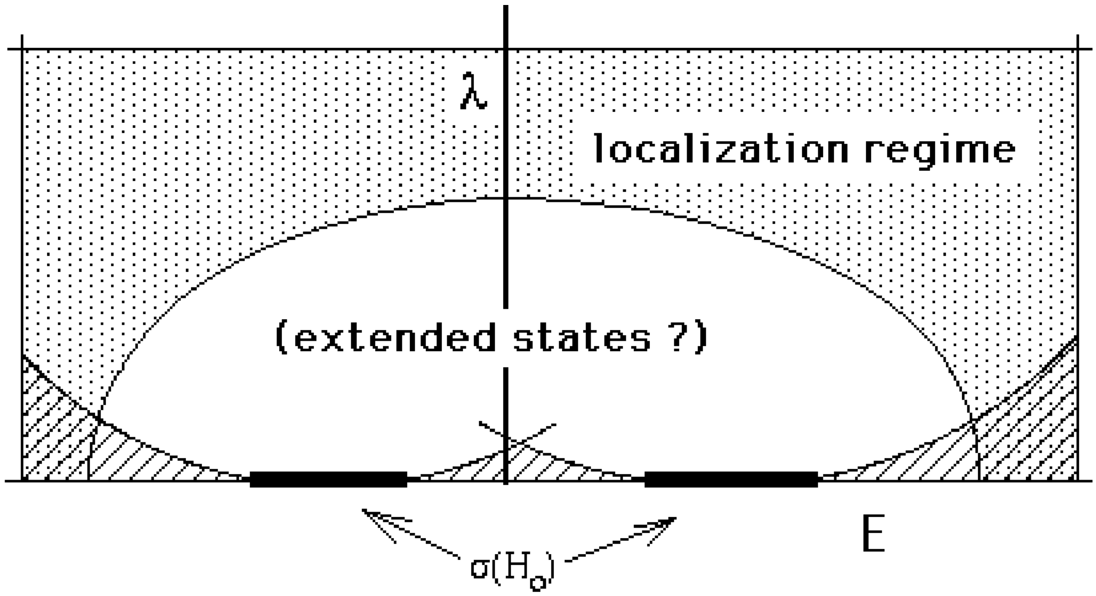

Fig. 1: A schematic depiction of the energy × disorder regimes covered by this work (stipps), and by ref. [AM] (dots).

1 a. Stability bounds for the localization regime

The first class of operators considered here act in the Hilbert space ℓ^2(\o(Z, Z)\s(d,)), and are of the form

$$H = T + U_o(x)+\lambda V(x) \ , \qquad (1.1)$$

where:

i. T a bounded self adjoint operator (not necessarilty real) whose matrix elements, $T_{x,y}$ = \o(T, ¯)$_{y,x}$, decay exponentially fast:

$$\left| T_{x\backslash,y} \right| \ ^2 \ B \exp(- m\ |x-y|\ ) \ , \qquad (1.2)$$

with some m > 0 and B < ∞. (The analysis applies also to operators with slower decay, down to the power law $|x-y|^{-d+\varepsilon}$ - see Section 4.) With no loss we set $T_{x,x} \equiv 0$.

ii. $H_o = T + U_o(x)$ is a periodic operator (i.e., it commutes with a d-dimensional subgroup of translations).

iii. $V(x)$ are random variables, i.e., $V(\cdot)$ is a random potential. Their joint probability distribution is required to satisfy some regularity conditions. These conditions are met, for example, for $\{V(x)\}$ independent identically distributed random variables with any of the following distributions: 1) the uniform distribution in [-1,1], 2) the standard Gaussian distribution, and even 3) the Cauchy distribution, for which \o(I, E)(|V|)=∞. The parameter η, referred to below, takes in those cases the optimal value $\eta=1$.

The symbol \o(I, E)(·) denotes the average over the realizations of the potential, i.e., the expectation value with respect to the corresponding probability measure.

Our main result provides a bound on the rate at which the continuous spectrum of H expands when the random part of the potential is turned on. The range of energies in which there can be only localized states is described in terms of the following notation. For each $\lambda > 0$, and $0 < \eta$ $({}^{2}1)$,

$$R_\eta(\lambda) = \underset{s\in(0,\eta)}{\mathrm{Y}} \left\{ E \in \mathbb{R} \setminus \sigma(H_o) \,\middle|\, \lambda\, K(s) \sup_x \left(\sum_y \left| < x \,|\, \frac{1}{(H_o - E)} \,|\, y > \right|^s \right)^{-s} < 1 \right\} \quad (1.3)$$

i.e., $R_\eta(\lambda)$ is the range of energies, *off the spectrum of H_o*, where the condition seen in the brackets holds *for some* $s \in (0,\eta)$.

(We employ Dirac's notation for the Hilbert space and its dual, $<x|A|y> \equiv (x, Ay)$ and denote by $|x>$ the (Kronecker -) δ function state corresponding to the site x∈ ∴o(Z, Z)\s(d,).)

For the study of the inverse operators (resolvents) $[H-E]^{-1}$ at energies in the spectrum, it is convenient to employ some cutoff. We do that by considering the approximating operators

$$H^{(L)} = H_0 + \lambda\, V(x)\, I[|x| \leq L] \quad (1.4)$$

for which the random potential is 'turned on' only in the finite volumes $\Lambda_L = [-L,L]^d$ [though H_o is *not* restricted to a finite region].

<u>Theorem 1.1:</u> Let a random operator have the form (1.1) with the off-diagonal elements obeying the exponential bound (1.2) and the random potential satisfying the regularity assumptions $\mathcal{A}$ and $\mathcal{B}$, spelled below, at some $\eta \le 1$. Then, for each $\lambda>0$, at energies $E \in R_\eta(\lambda)$

$$\mathbb{E}\left(\left|\langle x| \frac{1}{H^{(L)}-E} |y\rangle\right|^{s} \right) \le A_s \exp[- \mu_s |x-y|] \quad , \tag{1.5}$$

for $s < \eta$, with $\mu_s>0$ and $A_s<\infty$ independent of L.

Furthermore, for each $[a,b]\in R_\eta(\lambda)$, the unitary evolution operator filtered with the corresponding spectral projection $P_{(a,b)}$, satisfies

$$\mathbb{E}\left(\sup_{t} | \langle x| P_{(a,b)}\, e^{-itH} |y\rangle | \right) \le \tilde{A} \exp[- \tilde{\mu} |x-y|] \tag{1.6}$$

with some $\tilde{A} < \infty$ and $\tilde{\mu} > 0$.

<u>Remarks:</u> i) By the general analysis of Kunz and Souillard [KS] (and additional comments found in [CFKS]), (1.6) implies localization in the spectral sense, which means that for each $\lambda>0$ the operator H has, with probability one, only pure point spectrum in $R(\lambda)$ and eigenfunctions with energies within that range decay exponentially:

$$|\varphi_n(x)| \le A_n(\omega) \exp[- \mu(E) |x-x_n(\omega)|] \quad , \tag{1.7}$$

with $A_n(\omega)$ almost surely finite. That implies also zero conductivity.

ii) The existing criteria of Delyon Levy and Souillard [DLS] and Simon and Wolff [SW] permit to deduce the spectral localization (1.7) directly from (1.5). However, they do not provide the implication (1.5) $\Rightarrow$ (1.6).

Our derivation of (1.6) proceeds via (1.5). It is based on a new criterion, which is presented here as a separate result (Theorem 1.2).

iii) Let us note that for each $s>0$, the behavior of the quantity

$$g_s(E) = \sup_{x} \left(\sum_{y} \left| \langle x | \frac{1}{(H_o - E)} | y \rangle \right|^{s} \right)^{-s} \tag{1.8}$$

is described by the asymptotic bounds (see Appendix II, equations (II.3) and (II.4)):

$g_s(E)$ ² \b\lc\{(\a\co2\vs20\hs7(Const. $s^{-d/s}\,\Delta^{-(1+d/s)}$, for $\Delta\to 0$,Δ^{-1} [1+o(1)] , for $\Delta\to\infty$)) (1.9)

with Δ= dist{E,$\sigma(H_o)$}. Hence, for small λ the boundary of R(λ) separates rather fast from $\sigma(H_o)$ - as a fractional power of λ (though not faster than $\lambda^{1/(1+d+\varepsilon)}$). However, for larger λ the increase is at the linear rate λ [K(s) - o]. Therefore, condition (1.11) and the fact that σ(H) expands at the linear rate $\lambda\|V\|_\infty$, assure that even when V is uniformly bounded R(λ) is not void of spectrum for λ large enough. Of course, for unbounded potentials R(λ)$\cap\sigma$(H)_\o(o,/) as soon as λ_0.

1 b. The regularity assumptions

In the analysis we assume some uniform regularity of the conditional probability distributions of V(x) conditioned on the values of the potential elsewhere, i.e., on V({x}^{c}) - with {x}$^c \equiv$ \o(Z, Z)\s(d,)\{x}). The main assumption is:

A. For all s small enough, 0<s<η with η²1, the conditional averages over V(x) - conditioned on V({x}^{c}) satisfy uniformly a bound of the form:

\o(I, E)$_x$(\f(|V|^s,|V-α|^s) | ...) / \o(I, E)$_x$(\f(1,|V-α|^s) | ...) ² K(s)^s (1.10)

with

K(s) < \o(sup, $x\in$\o(Z, Z)\s(d)) ess.sup |V(x)| $\equiv \|V\|_\infty$, (1.11)

(otherwise, (1.10) is a trivial statement). We use here the abbreviated notation: \o(I, E)$_x$(g(V)|...) ≡ \o(I, E)(g(V(x)) | V({x}^{c})).

This condition allows for singular measures, though not with point δ functions. (Localization for δ potentials has been successfully studied in [CKM,DK]).

For the most complete result we also require the conditional probability distributions to be absolutely continuous with respect to the Lebesgue measure, which means that they are of the form $P(V(x) \mid V_{\{x\}^c})\, dV(x)$ and, furthermore, to satisfy

$$\mathcal{B}. \qquad \int \left(P(V(x)|V_{\{x\}^c})^{1+q} \, dV(x) \right) \le Q \tag{1.12}$$

for some $q > 0$ and a uniform constant $Q < \infty$.

Under $\mathcal{B}$, the condition $\mathcal{A}$ is implied by the following.

$\mathcal{C}$. For some $\tau \le 1$, the conditional τ-moments of $V(x)$ are uniformly bounded:

$$\mathbb{E}\left(|V(x)|^\tau \mid V_{\{x\}^c} \right) \equiv \int \left(|V(x)|^\tau P(V(x)|V_{\{x\}^c})\, dV(x) \right) \le B_\tau \ (< \infty) \ . \tag{1.13}$$

The implication $\mathcal{B}$+$\mathcal{C}$ => $\mathcal{A}$ is presented in Appendix I. Other scenarios under which $\mathcal{A}$ holds were discussed in ref.[AM].

Not much effort is made here to minimize the regularity assumptions, but let us note that their role in the argument is incremental:

- Only $\mathcal{A}$ is required in the proof of the exponential decay of the resolvent, i.e., (1.5), for energies in $R_\eta(\lambda)$. (Derived in Section 2).

- The implication that the spectrum is of pure point type with exponentially localized eigenfunctions requires only the absolute continuity of the conditional measures. (For ergodic random operators it suffices to require even less: that all the conditional distributions have some absolutely continuous component.) The argument can be based on the criteria of DLS and SW instead of the one presented in Theorem 1.2, below.

- The extra assumption $\mathcal{B}$ permits us to derive the explicit and convenient statement (1.6). A key role is played by the general auxiliary result stated in Theorem 1.2, below.

<u>Organization of the proof of Theorem 1.1</u>: The proof is split into two parts. The first consists of the derivation of the resolvent bound (1.5) for the regime described in the statement of Theorem 1.1. The analysis is presented in Section 2, with the result summarized in Lemma 2.1. The fact that under the stated assumptions (1.5) $\Rightarrow$ (1.6) is the content of the localization criterion presented below in Theorem 1.2. Other than this explanation, no statement formally entitled "<u>Proof of Theorem 1.1</u>" is offered here.

In Sections 4 and 5 we discuss extensions of the main result to operators for which the matrix elements of T decay by only a power law, and to Bethe lattices - where exponential decay does not guaranty summability. In both cases, the regime provided by the condition (1.9) does not shrink, as $\lambda \to 0$, to $\sigma(H_o)$. [Instead, it converges to the L^1 spectrum of H]. As mentioned in Section 5, the earlier work of Abou-Chacra and Thouless [AT] indicates that for Bethe lattices the discrepancy is a real effect, and not a limitation of the method.

1 c. A localization criterion for e^{-itH}

In addition to the result presened above, we derive in this work some estimates which are of independent interest, and in particular the following criterion. It permits to deduce the explicit bound (1.6) also for the regimes of high disorder or extreme energies discussed in reference [AM], directly from the results which were obtained there.

Theorem 1.2: Let H be given by (1.1), with both $\sum_y |T_{x,y}|^2$ and U(x) bounded, uniformly in x, and $V(\cdot)$ having a probability distribution satisfying the regularity conditions $\mathcal{B}$ and $\mathcal{C}$. Then, a sufficient condition for an exponential bound of the form (1.6) is that the following bound holds

$$\int_a^b \mathbb{E}\left(\left|\langle x| \frac{1}{H^{(L)}-E} |y\rangle\right|^s \right) dE \le A' \exp[-\mu' |x-y|] \qquad (1.14)$$

with some L-independent constants $0 < s < 1$, $\mu' > 0$ and $A' < \infty$, for some approximating sequence of operators $H^{(L)}$ such that: 1) $H^{(L)} \to H$ in the strong resolvent sense, and 2) each $H^{(L)}$ has only discrete spectrum in some neighborhood of [a,b]. (The implied value for $\tilde{\mu}$ is $\tilde{\mu} = \mu'/[1+q(2-s)]$).

The two listed conditions on $H^{(L)}$ are easily seen to be satisfied by the choice presented in (1.4), for any $[a,b] \subset \mathbb{R} \setminus \sigma(H_o)$. The proof of Theorem 1.2 is presented in Section 3, along with some other statements of general interest.

Theorem 1.2 is related to the localization criteria of Delyon Levy and Souillard [DLS] and Simon and Wolff [SW] (which incorporate a key observation of Kotani [K], mentioned here below (3.8)). The DLS and SW criteria allow to deduce from (1.14) the localization statement (1.7) under the weaker assumption of just the absolute continuity of the conditional distributions of V(x). Theorem 1.2 requires slightly more, but it yields the more explicit statement (1.6). Such a result has previously been proven only for one dimension [DKS] (see also [CFKS]), where other methods apply.

2. Exponential decay of the resolvent

At the heart of the proof of localization is the exponential decay of the resolvent, in the sense of (1.14). In this section we derive such a decay for the regime described in Theorem 1.1. The only regularity assumption used here is $\mathcal{A}$.

We start by noting that, by the Combes-Thomas [CT] estimate (reproduced here in Appendix II), for energies off the unperturbed spectrum $\sigma(H_o)$

$$\left|\langle x| \frac{1}{H_o-E} |y\rangle\right| \le \frac{2}{\Delta} \exp\left(- \Delta \frac{a}{b+\Delta} |x-y| \right) \qquad (2.1)$$

with $\Delta \equiv \mathrm{dist}\{E,\sigma(H_o)\}$. The constants $0<a,b<\infty$ depend only on the dimension and the parameters $\{B,m\}$ of eq. (1.2). (For large Δ there is another, more suitable, bound.) This information will be used to estimate of the effect of the random potential, which is treated as a perturbation.

We denote now

$$G^{(L)}_E(x,y) = \langle x| \frac{1}{H^{(L)}-E} |y\rangle , \quad \text{and} \quad G_{E,o}(x,y) = \langle x| \frac{1}{H_o-E} |y\rangle, \qquad (2.2)$$

with $H^{(L)}$ defined by (1.4). The main result of this section is:

<u>Lemma 2.1</u> Assuming the exponential decay (1.2) and the regularity condition $\mathcal{A}$ (of Section 1), for each $\lambda>0$ and $E\notin\sigma(H_o)$ in the regime $R_\eta(\lambda)$ [defined by (1.3)], there are constants s ($< \eta \le 1$), A, and μ [$\approx \Delta \frac{a}{b+\Delta}$], with which

$$\mathbb{E}\left(|G^{(L)}_E(x,y)|^s \right) \le A \exp[- \mu |x-y|] \quad . \qquad (2.3)$$

For random potentials the perturbation theory is complicated by the fact that in any realization there will be regions in which the random terms are unusually large, or create conditions of resonance. That may cause both difficulties, e.g., small denominators which call for delicate estimates, and possibly some real effects (such as the Griffith singularities in random coupling Ising models, and anomalous decay rates in various other situation). The difficulties are resolved, or bypassed, here by taking an approach similar to that presented in ref. [AM], where it was used in a different regime. The key points are:

1) Focus on suitable moments of the matrix elements of the resolvent, instead of keeping track of some more complete information on its behavior for typical individual realizations of the potential. The resolvent is studied via some of the equation which it obeys. In ref. [AM] a key role was played by [H-E] $G(x,y)=\delta_{x,y}$, here the starting point is the resolvent equation (2.5).

2) Identify the range of moments (small enough) for which the expectation value is not dominated by the divergence caused by the resonances. The choice depends on the regularity conditions, but in all cases the power is $s<1$.

3) Factorize (with some added constants) the expectation values of products of V(u) with G(x,y) by making use of: i) the simple dependence of the resolvent on any particular potential (at fixed values of all the other terms), ii) the regularity of the conditional distribution of V(u).

4) Extract exponential decay from the resulting 'closed' inequalities.

In the third step, a key role is played by the following observation.

Lemma 2.2: In its dependence on V(u), each of the quantities G\s((L),E)(u,y) is of the form

G\s((L),E)(u,y) = \f(D,V(u)–F) (2.4)

where D and F depend on the values of the potential elsewhere - but not on V(u).

Proof For finite matrices (2.4) is explained by Cramer's formula, which expresses the matrix elements of the inverse as ratios of two determinants. The determinant in the numerator is free of V(u) (since it is erased along with the entire u - row) and the one in the denominator, which is det(H\s((L),)- E), is linear in V(u). For infinite dimensional operators the statement is reduced to the above case by a finite rank perturbation argument (Krein formula) found, e.g., in an Appendix to ref. [AM] ■

We now turn to the main analysis.

Proof of Lemma 2.1: Let [a,b] ⊂ \o(I, R)\σ(H_o) be an interval of energies separated from the spectrum of H_o. Since H\s((L)) (defined by (1.4)) is a finite rank perturbation of H, its spectrum in [a,b] consists of only a finite number of proper eigenvalues (not more than $|\Lambda_L|$). For all other energies $E \in [a,b]$ the resolvent formula yields:

$$\frac{1}{H^{(L)} - E} = \frac{1}{H_0 - E} - \frac{1}{H_0 - E}\lambda V^{(L)} \frac{1}{H^{(L)} - E} \tag{2.5}$$

where V\s((L))(x) = V(x) I[|x|^{2}L]. In terms of the matrix elements equation (2.5) reads:

G\s((L),E)(x,y) = G\s(,E\,o)(x,y) - λ \i\su(u$\in\Lambda_L$,,)G\s(,E\,o)(x,u) V(u) G\s((L),E)(u,y) , (2.6)

where the sum is over the finite region Λ_L={x$\in$\o(Z, Z)\s(d,) | |x|^{2}L }.

Following the path taken in ref.[AM], where a different equation was studied, we use equation (2.6) for estimating |G\s((L),E)(x,y)| raised to a power s < 1. As explained in [AM], and as also follows by a much more general argument from Lemma 3.2 (i), that restriction resolves the divergence caused by the resonances. Since for 0<s<1
|\i\su(,, A_n) |^s ^{2} \i\su(,, |A_n|^s), we get

\o(I, E)(|G\s((L),E)(x,y)|^s) ^{2} |G\s(,E\,o)(x,u)|^s + λ^s \i\su(u$\in\Lambda_L$,,)|G\s(,E\,o)(x,u)|^s \o(I, E)(|V(u)|^s |G\s((L),E)(u,y)|^s)
(2.7)

The expectation on the right can be performed by first averaging |V(u)|^s |G\s((L),E)(u,y)|^s with respect to the conditional probability distribution \o(I, E)(- - | $V_{\{u\}c}$), i.e., taking the conditional average over V(u) - conditioned on the values of V($\cdot$) elsewhere. The representation (2.4) permits us to handle the conditional average of the product. With the aid of the the regularity condition iii. a. we obtain, for s < η (^{2}1):

\o(I, E)(|G\s((L),E)(x,y)|^s) ^{2} |G\s(,E\,o)(x,y)|^s + λ^s K(s)^s \i\su(u$\in\Lambda_L$,,)|G\s(,E\,o)(x,u)|^s \o(I, E)(|G\s((L),E)(u,y)|^s) . (2.8)

Unlike (2.7), the inequality (2.8) is a "closed" statement on \o(I, E)(|G\s((L),E)($\cdot$,$\cdot$)|^s). In order to see its implications, it is useful to consider the weighted sums:

χ\s((L),E)(s,μ) = \o(sup, $_{x\in\Lambda_L}$) \i\su(y$\in\Lambda_L$,,)\o(I, E)(|G\s((L),E)(x,y)|^s) exp(+μ|x-y|) (2.9)

and
$$\chi_{E,o}(s,\mu) = \sup_{x\in\Lambda_L} \sum_{y\in\Lambda_L} \left|G_{E,o}(x,y)\right|^s \exp(+\mu|x-y|) \tag{2.10}$$

[Note that a uniform bound on $\chi_{L,E}$ is tantamount to exponential decay for $\overline{E}(|G_{L,E}(x,y)|^s)$]. With the help of the triangle inequality, $|x-y| \le |x-u|+|u-y|$, (2.8) yields:

$$\chi_{L,E}(s,\mu) \le \chi_{E,o}(s,\mu) + \lambda^s K(s)^s \chi_{E,o}(s,\mu)\, \chi_{L,E}(s,\mu) \ . \tag{2.11}$$

It is now relevant to note that $\chi_{E,o}(s,\mu)$ is a continuous function of s and μ, for $E\notin\sigma(H_o)$ and $\mu < \Delta \frac{a}{s(b+\Delta)}$ (by (2.1)). Thus, for each $\lambda < 1/[K(s)\, \chi_{E,o}(s,0)^{1/s}]$, at small enough μ:

$$\lambda^s K(s)^s \chi_{E,o}(s,\mu) < 1 \ . \tag{2.12}$$

Under the above condition, (2.12) yields

$$\chi_{L,E}(s,\mu) \le \frac{\chi_{E,o}(s,\mu)}{1-\lambda^s K(s)^s \chi_{E,o}(s,\mu)} := A(s,\mu)\ , \tag{2.13}$$

which implies the uniform exponential bound:

$$\overline{E}\left(\left|G_{L,E}(x,y)\right|^s\right) \le A(s,\mu) \exp(-\mu\, |x-y|) \tag{2.14}$$

which is the assertion (2.3). ■

In the next section we turn to the derivation of an explicit localization statement from (2.14).

3. Localization criterion

The goal of this section is to derive the localization criterion stated in Theorem 1.2. The following auxiliary result plays a key role in relating moments of the resolvent with the time evolution. In this lemma, |x> and |y> are any two unit vectors in the Hilbert space in which the operator M acts. The notation suggests, however, the way the result is applied in our more specific context.

Lemma 3.1: Let $M_\upsilon = M_o + \upsilon\, |x><x|$ be a self adjoint operator incorporating a rank one term (with || |x> || = 1) whose coefficient is a random variable, and [a,b] an interval [an energy range] with a neighborhood in which the spectral projection of M is of finite rank, i.e., dim Ran $P_{[a-\varepsilon,b+\varepsilon]}(M) < \infty$, for some ε>, $P_{(\cdot,\cdot)}$ being the spectral projection of M_υ. If:

i) the probability distribution of υ is of the form ρ(dυ) = f(υ) dυ, with the density satisfying \i([illegible],, $f(\upsilon)^{1+q}$ dυ) = Q < ∞, for some q > 0,

ii) <x| M_o | x>|, <x| M^2 |y>, and \o(I, E)($|\upsilon|^r$) are finite, for some r >0,

then, for any unit vector |y>, and any 0 < s ² 2:

\o(I, E)(\o(sup, t) |<x| $P_{(a,b)}\, e^{-itM_\upsilon}$ |y>|) ²

² 2 $Q^{1/[1+q(2-s)]}$ \b\bc(\i(a,b, |\f(<x|$[M_o-E]^{-1}$|y>,<x|$[M_o-E]^{-1}$|x>)$|^s$ dE))$^{1/[1+q(2-s)]}$ (3.1a)

² Const. |\o(I, E)(\i(a,b, |<x|$[M_\upsilon-E]^{-1}$|y>$|^{2s}$ dE))$|^{1/[2+2q(2-s)]}$ (3.1b)

where \o(I, E)(·) represents the average over υ, and in the second bound 2s ² r and the constant is an explicit function of the numerical quantities listed in the statement.

Remark A point worth noting is the lack of dependence on υ (!) of the vector

$|\varphi_E>$ = \f($[M_\upsilon-E]^{-1}$ |x>,<x|$[M_\upsilon-E]^{-1}$|x>) = \f($[M_o-E]^{-1}$ |x>,<x|$[M_o-E]^{-1}$|x>) (3.2) which is initially defined only for υ such that $E \notin \sigma(M_\upsilon)$. (A simple explanation is provided by Cramer's formula for the inverse of a matrix. More generally, one may derive this fact via a resolvent identity.) Since the ratio appearing in (3.1a) is the scalar product of $|y\rangle$ with $|\varphi_E\rangle$, M_o can be replaced there by M_υ.

Proof: For the study of the quantity appearing in (3.1a) it may be assumed that $|x\rangle$ is a cyclic vector for M_o , since otherwise $\langle x|\, P_{(a,b)} \exp(-itM_v)\, |y\rangle$ can be evaluating by first projecting $|y\rangle$ into the subspace for which that is true.

Let now $\Gamma(E) = -1/\langle x|[M_o-E]^{-1}|x\rangle$. One easily derives the equation:

$$[M_v - E]\, |\varphi_E\rangle = [v - \Gamma(E)]\, |\varphi_E\rangle \ . \tag{3.3}$$

Elementary considerations show that within the interval (a,b):

i) The spectrum of M_υ consists of precisely those energies $E \in (a,b)$ for which

$$\Gamma(E) = v \tag{3.9}$$

ii) The eigenfunctions are given by $|\varphi_E\rangle$ - at those values of E at which (3.9) holds. Notice, however, that $|\varphi_E\rangle$ is not normalized, satisfying instead the condition

$$\langle x|\varphi_E\rangle = 1 \quad (\text{in general } \langle\varphi_E|\varphi_E\rangle \_1). \tag{3.5}$$

iii) The spectral measure of M_υ associated with the state $|x\rangle$,

$\mu_v(dE)$ I[a<E<b] = dE \i\su($E_n \in \sigma(M_\upsilon) \cap (a,b)$, , $\delta(E-E_n)$) \f($|\langle x|\varphi_{E_n}\rangle|^2$,$\langle\varphi_{E_n}|\varphi_{E_n}\rangle$) , (3.6)

can be written in the form:

$$\mu_v(dE) = dE\ \delta(\Gamma(E) - v)\ I[a<E<b] \ . \tag{3.7}$$

The representation (3.7) of the spectral measure immediately implies the important observation of Delyon Lévy and Souillard [DLS] and of Simon and Wolff [SW] (where the statement is in a slightly modified form) that:

\i(\o(I,R),, μ_v(--) dv) = $\mathcal{L}$(--) (3.8) where $\mathcal{L}$ is the Lebesgue measure on $\mathbb{R}$. (The approach of [SW] yields yet another derivation of (3.8).) This relation permitted DLS and SW to make explicit the fundamental observation of Kotani that sets of zero Lebesgue carry also zero spectral measure for almost every realization of the potential.

Now, for each v,

\o(sup,) |<x| P(a,b) e-itMυ |y>| ² \i\su(En∈σ(Mυ)∩(a,b), ,)
t
\f(|<x|φEn><φEn|y>|,<φEn| φEn>)

= \i(a,b, \b\bc\|(\f(<φE|y>,<φE|x>))) μv(dE) = \i(a,b, |<φE|y>|) μv(dE) (3.9)

(since <φE|x> = 1).

Due to the orthogonality, and the completeness of the eigenfunctions,

\i((a,b),, |<φE|y>|2 μv(dE)) = <y| P(a,b) |y> = 1. (3.10)

Thus, for any any s∈(0,1) we may estimate the quantity on the right side of (3.9) with the aid of the Hölder inequality\o(I, E)(fλa+(1-λ)b) ²\o(I, E)(fa)λ \o(I, E)(fb)(1-λ), with the power 1 expressed as an interpolation between s and 2: 1 = λ s + (1-λ) 2, at λ=1/(2-s). That yields

\o(sup,) |<x| P(a,b) e-itMυ |y>| ² \b\bc\((\i(a,b, \b\bc\|(<φE|y>))s μv(dE))1/(2-s). (
t

Averaging this inequality over v, with a repeated use of the Hölder inequality (in a form valid for a probability average - \o(I, E)(g1/(2-s)) ² \o(I, E)(g)1/(2-s)) we obtain:

\i(,, \o(sup,) |<x| P(a,b) e-itMυ |y>| ρ(dv)) ² \b\bc\((\i(,,)\i(a,b,) \b\bc\|(<φE|y>)s
t
1/(2-s)
μv(dE) ρ(dv))

² \b\bc\((||f||∞ \i(a,b, |<φE|y>|s dE))1/(2-s), (3.12)

with $\|f\|_\infty = \sup\{f(v)\}$. The last step is based on the key fact (3.8).

To handle the case of unbounded f(v), we split the integration over v into two regions, according to whether f(·) ² κ holds or not - with κ a parameter to be optimized. Using (3.12) for the integral over the first regime, and the unitarity based bound |<x| $P_{(a,b)}$ $e^{-itM_{\upsilon}}$ |y>| ² 1 for the second (where 1²$[f(v)/\kappa]^q$) we get

\i(,, \o(sup, t) |<x| $P_{(a,b)}$ $e^{-itM_{\upsilon}}$ |y>| ρ(dv)) ² \b\bc\((κ \i(a,b, $|<\varphi_E|y>|^s$ dE))$^{1/(2-s)}$ + \f(\i(,,f(υ)$^{1+q}$dυ),κ^q)

(3.13)

The choice of κ which equalizes the two terms yields

\i(,, \o(sup, t) |<x| $P_{(a,b)}$ $e^{-itM_{\upsilon}}$ |y>| ρ(dv)) ²

² 2 \b\bc\((\i(,,f(υ)$^{1+q}$ dυ))$^{1/[1+q(2-s)]}$ \b\bc\((\i(a,b, $|<\varphi_E|y>|^s$ dE))$^{1/[1+q(2-s)]}$. (3.14)

Which proves (3.1a). (Under the optimal choice for κ, the factor of 2 is replaced by $\alpha(q,s) = [q/(2-s)]^{1/[1+q(2-s)]}+[(2-s)/q]^{q(2-s)/[1+q(2-s)]}$.)

To derive the bound (3.1b) which is the more relevant for our problem, we first replace M_o in (3.1a) by M_v (as justified by the Remark there), and average the integral over v - ignoring the fact that at this stage there is no real v dependence. Next, we apply the Schwarz inequality and bound the term contributed by the denominator with the help of Lemma 3.2 (ii) stated below. ■

At the end of the last proof we use the second of the two related statements in the following lemma. The first one is also of interest (as mentioned in the proof of Lemma 2.1).

<u>Lemma 3.2:</u> Let H be a self adjoint operator and [a,b] an interval with a neighborhood where the spectral projection of H is of finite rank. Then for any $0 \le r < 1$:

$$\text{i)}\quad \int_a^b \left|\langle x|\frac{1}{H-E}|x\rangle\right|^r dE \;\le\; \frac{1+r}{1-r} \tag{3.15}$$

and ii)

$$\int_a^b \left|\langle x|\frac{1}{H-E}|x\rangle\right|^{-r} dE \;\le \tag{3.16}$$

$$\le\; |b-a| \left(\max\{|b|^r,|a|^r\} + |\langle x|H|x\rangle|^r\right) + (1-r)^{-1}\, 2^r\, |b-a|^{1-r} \sum_{y\,(\perp x)} |\langle x|H|y\rangle|^2$$

where $|x\rangle$ is a unit vector, and $|y\rangle$ ranges over a complete set of orthonormal vectors orthogonal to $|x\rangle$.

A key fact, used in the derivation of Lemma 3.2, is that for any function of the form

$$g(E) = \sum_n q_n \frac{1}{u_n - E} \quad , \tag{3.17}$$

with u_n real, $q_n \ge 0$ and $\Sigma\, q_n < \infty$, the Lebesgue measure of the level sets of $g(\cdot)$ satisfies the identity (!)

$$\mathcal{L}\left(\left\{ E\in \mathbb{R} \;\middle|\; g(E) \substack{\ge t \\ (\le -t)} \right\}\right) = \frac{\Sigma\, q_n}{t}\,, \qquad \text{for any } t>0. \tag{3.18}$$

Equation (3.18) has an easy proof, although the statement itself is quite remarkable. In the nineteenth century it was reported by Boole [B]. In this century it was rediscovered by Stein and Weiss [SW] (who discuss it in a more general form, and also relate the associated bound to general properties of singular integrals). The author became aware of the equality due to its even more recent rediscovery by E. Lieb.

<u>Proof of Lemma 3.2:</u> i) The spectral representation of $\langle x|[H-E]^{-1}|x\rangle$ is:

$$\langle x|\frac{1}{H-E}|x\rangle = \sum_{u_n\in[a,b]} b_n \frac{1}{u_n - E} + \int_{\sigma(H)\in[a-\varepsilon,b+\varepsilon]} \rho(du)\frac{1}{u-E} \tag{3.19}$$

where the sum is over a finite number of terms with $b_n>0$, and $\rho(du)$ is a positive measure (which may include additional discrete terms) satisfying

$$\sum_n b_n + \rho(\{\sigma(H)\setminus[a,b]\}) = 1 \;. \tag{3.20}$$

Since dist{[a,b], $\sigma(H)$ } > 0, for $E\in[a,b]$ the second term is uniformly approximated by discrete sums, of the type seen in (3.17). By a simple continuity argument, (3.18) yields:

$\mathcal{L}$({E∈[a,b] | |<x|$[H\text{-}E]^{-1}$|x>| $\geq$ t } $\leq$ \f(2,t) . (3.21)

The statement (3.15) follows now by elementary integration.

ii) The function $\Gamma(E)$ = - 1/<x|$[H\text{-}E]^{-1}$|x>, with E∈ ^, maps the upper half plane into itself. Its structure and the asymptotic behavior (for $E\to\infty$), easily imply that the Herglotz representation of $\Gamma(E)$ (defined for such functions) takes the form

$$\Gamma(E) = \langle x|\, [E - H]\, |x\rangle \ + g(E) \tag{3.22}$$

with
$$g(E) = \sum_{v_n\in[a,b]} q_n \frac{1}{v_n - E} + \int_{\mathrm{IR}\setminus[a-\varepsilon,b+\varepsilon]} \zeta(dv)\frac{1}{v-E} \tag{3.23}$$

where the sum is over a finite number of terms with $q_n \geq 0$ (corresponding to the zeros of <x|$[H\text{-}E]^{-1}$|x> in [a,b]), and $\zeta(du)$ is a positive measure whose support is separated from [a,b] with the total mass satisfying:

\i\su(n,, b_n) + ζ(\o(I, R)) = <x| M^2 |x> - <x| M |x>^{2} = \i\su(y (_x),, |<x|M|y>|^2) . (3.24)

If $\zeta(du)$ is in addition a discrete measure, the identity (3.18) easily leads to the bound (3.16). Since the support of $\zeta(du)$ is separated from [a,b] - which is the range of integration in (3.16), a simple continuity argument shows that the bound is valid whether the measure be discrete or not. ■

We are now ready to reach for the main result of this section.

<u>Proof of Theorem 1.2:</u> For a fixed energy range (a,b), and a pair of sites x,y∈\o(Z, Z)\s(d,) let

$$h(t) = \langle x|\, P_{(a,b)}(H)\, \exp(-itH)|y\rangle$$

and
$$h_L(t) = \langle x|\, P_{(a,b)}(H^{(L)})\, \exp(-itH^{(L)})|y\rangle \ . \tag{3.25}$$

For the conditional average of $h_L(t)$ - at fixed values of the potential at sites other than x, we are in the situation covered by Lemma 3.1. Using the bound (3.1b), and then the Hölder inequality, we learn that if (1.14) holds with L independent constants, and the regularity conditions iii.b. and iii.c. are satisfied, then the desired bound:

$$\backslash o(I, E)(\backslash o(\sup_t,)\, |h_L(t)|\,)\ ^2\ A \exp[- \mu\, |x-y|\,] \tag{3.26}$$

holds, uniformly in L (!), for the approximating sequence. It now remains to be demonstrated that (3.26) survives the limit $L\to\infty$.

The strong resolvent convergence, $H^{(L)}\to H$, is equivalent to:

$$<x|\, \exp(-itH_L)\, |y> \backslash o(\to_{L\to\infty},)\ <x|\, \exp(-itH)\, |y>\ , \tag{3.27}$$

for any t∈ ∴o(I, R) and {x, y}. [For our choice of $H^{(L)}$ that continuity statement can be established using the Dyson series expansion for e^{-itH} in which T is treated as the perturbation. Under the condition (1.2), T is a bounded operator and (3.27) holds regardless of the potential (!).]

By implication, also

$$h(t) = \backslash o(\lim_{L\to\infty},)\ h_L(t) \tag{3.28}$$

whenever a and b are not eigenfunctions of H ([RS]). The caveat may be ignored here since the set of energies E for which Prob({E is a (proper) eigenvalue of H}) _ 0 is at most countable and h(t) (defined with the projection on an open interval) is lower - semi continuous in a and b. [We could even make the stronger statement, that in our case the above set of energies is empty, and hence the conclusion holds regardless of the boundary choice]. Applying two standard limiting principles we get:

$$\begin{aligned} \mathbb{E}(\sup_t |h(t)|\ ) &\le \mathbb{E}(\lim_{L\to\infty}\ \sup_t\ |h_L(t)|\ ) \\ &\le \lim_{L\to\infty}\ \mathbb{E}(\sup_t\ |h_L(t)|\ ) \le\ A\ \exp(-\mu\,|x-y|) \quad . \quad \blacksquare \end{aligned} \tag{3.29}$$

4. Operators with slowly decaying off diagonal terms

Except for exponential decay, the above discussion applies also to operators in $\ell^2(\mathbb{Z}^d)$ based on T for which (1.2) is not valid, but is replaced by the power law decay:

$$\left|T_{x,y}\right| \le \frac{B}{|x-y|^{\alpha}} \quad . \tag{4.1}$$

Our method shows that for

$$\alpha > 2d \tag{4.2}$$

a modified version of Theorem 1.1 holds, with equation (1.6) replaced by

$$\mathbb{E}\left(\sup_t \left| \langle x| P_{(a,b)}\, e^{-itH} |y\rangle \right| \right) \le \frac{A}{(1+|x-y|)^{\mu}} \quad . \tag{4.3}$$

A device which permits to minimize the necessary changes is the replacement in the analysis of the Euclidean distance $|x-y|$ by the metric:

$$\delta(x,y) = \log(1+|x-y|). \tag{4.4}$$

(This device is imported from statistical mechanics context, where it was introduced by L. Gross [G], and used in a somewhat analogous way in ref. [Ai1]). Notice that (4.1) can be expressed as exponential decay in $\delta(\cdot,\cdot)$, i.e., in a form similar to (1.2). However, in this metric exponential decay implies summability only if the decay rate is large enough.

In particular, the modified Combes-Thomas bound, presented in Appendix II, requires the condition (4.2) and yields only a power law decay. Likewise, the weighted sums χ introduced in (2.9) and (2.10) acquire the form:

$$\chi^{(L)}_{E}(s,\mu) = \sup_{x\in\Lambda_L} \sum_{y\in\Lambda_L} \mathbb{E}\left(\left|G^{(L)}_{E}(x,y)\right|^{s}\right) (1+|x-y|)^{\mu} \quad , \tag{4.5}$$

and equation (2.13) holds in the modified sense.

A point which was not worth noticing before, but now is significant is that Theorem 1.1 does not generally prove that the localization regime, in the energy-disorder space, connects (as $\lambda\to 0$) with the unperturbed spectrum $\sigma(H_o)$. Instead, under the optimal regularity assumptions (i.e., η=1), it implies only that the localized regime connects to the edge of the range of energies for which G\s(,E\,o)(x,y) has better-than-summable decay, and not just square summable. (In case of the regular exponential decay the two points coincide, by the Combes-Thomas bound). More is said on this point in the next section.

5. Bethe Lattices

Our analysis applies also to operators associated with the Bethe lattice, but one should notice a significant difference in its implication: unlike the case for \o(Z, Z)\s(d,), the localization regime described by the set $R_\eta(\lambda)$ does not converge to the unperturbed spectrum $\sigma(H_o)$.

The regular Bethe lattice of coordination number K is the graph (Γ_K) with no loops in which each site has K+1 neighbors. As with the power law decay on the regular lattice, on the Bethe lattice the ordinary exponential decay (1.2) does not guarantee summability. It is necessary, therefore, to pay attention to the point discussed at the end of Section 4.

In particular, the random Schrödinger operator $H=T+\lambda V$, with $T_{x,y}=\delta_{|x-y|,1}$ (and $U_o(\cdot)\equiv 0$) was discussed for Bethe lattices in references [ANT,AT,KS,MA]. The (L^2) spectrum of H_o is $\sigma(H_o)$ = [−2\r(K), 2\r(K)] (less than what a naive guess may suggest). For energies |E| > 2\r(K) the resolvent decays exponentially at a square summable rate (faster that $K^{-|x|/2}$). However, the decay rate reaches the summable rate $(K^{-|x|})$ only at |E| = K+1. Thus, Theorem 1.1 (whose proof applies verbatim also to the Bethe lattice) implies localization at weak λ only for energies beyond [−K,K].

Naturally, one may want to know whether the existence of a gap between the unperturbed spectrum and the edge of regime for which localization is established at weak disorder is a limitation of the method, or whether it reflects a real effect. The much earlier work of Abou-Chacra and Thouless indicates that the latter is the case. This topic will be discussed more extensively in ref. [Ai2] (which includes a proof of the existence of absolutely continuous spectrum for $V(\cdot)$ generated with the Cauchy distribution).

Appendix I. The regularity condition

The most important regularity condition for this work is $\mathcal{A}$, of Section 1. Since the relevant bound, (1.10), is not stated explicitly in terms of the probability distribution, we offer here an easily approachable sufficiency condition. In the terminology of Section 1, what is shown is $\mathcal{B}+\mathcal{C}\Rightarrow\mathcal{A}$.

Lemma A.1: Let $\mathbb{E}(\cdot)$ represent an average over $V\in\mathbb{R}$ with respect to a probability measure $f(V)\, dV$, with

i) $\int_{\mathbb{R}} f^{1+q}(V)\, dv \le Q \ (< \infty)$ (I.1)

ii) $\mathbb{E}(|V|^{\tau}) \le B \ (< \infty)$ (I.2)

for some $q > 0$, and $\tau \le 1$. Then, for any

$$s < \left[1+ \frac{2}{\tau} + \frac{1}{q}\right]^{-1} \qquad \text{(I.3)}$$

there is a finite constant $K=K(s,q,Q,\tau,B)$ with which

$$\text{for all } \alpha \in \mathbb{R}: \quad \mathbb{E}\left(\frac{|V|^s}{|V-\alpha|^s}\right) \le K\, \mathbb{E}\left(\frac{1}{|V-\alpha|^s}\right) \quad . \qquad ($$

Furthermore, if V is essentially bounded then $K < \|V\|_{\infty}$.

Proof: First, we show that (I.2) implies integrability of $|V-\alpha|^{-s}$, for all $s <1/[1+ 1/q]$. The preliminary bound is by the Hölder inequality:

$$\int_{[a,b]} f(V)\, dV \le \left(\int f^{1+q}(V)\, dV\right)^{\frac{1}{1+q}} \left(\int_{[a,b]} dV\right)^{\frac{q}{1+q}} = Q^{\frac{1}{1+q}}\, |b-a|^{\frac{q}{1+q}} \quad . \qquad \text{(I.5)}$$

Hence

$$\mathbb{E}\left(\frac{1}{|V-\alpha|^s}\right) \le 1 + \int_1^{\infty} \mathrm{Prob}\left(\frac{1}{|V-\alpha|^s} \ge t\right) dt \le 1 + \frac{s(2^q Q)^{1/(1+q)}}{q/(1+q)-s} \equiv C_s \quad , \qquad \text{(I.6)}$$

where the integral is estimated with the aid of (I.5).

Estimating separately the contribution from the two regimes $|V| \le (>) |\alpha|/2$ we have:

$$\mathbb{E}\left(\frac{|V|^s}{|V-\alpha|^s}\right) \le \left|\frac{2}{\alpha}\right|^s \mathbb{E}(|V|^s) + \left|\frac{2}{\alpha}\right|^s \mathbb{E}\left(\frac{|V|^{2s}}{|V-\alpha|^s}\right) \le$$

$$\le \left|\frac{2}{\alpha}\right|^s \left[B + B^{2s/\tau} C_p{}^{s/p}\right] \qquad (I.7)$$

provided:

\f(2s,τ) + \f(s,p) = 1, and p < \f(1,1+1/q) (I.8)

where the equality is the Hölder condition, and the inequality is needed for $C_p < \infty$. In terms of s, the conditions (I.8) are, of couse, just (I.3).

For a lower bound, let us note that (I.2) implies Prob($|V|^\tau > (2B)$) $\ge 1/2$. Taking just the contribution from the regime $|\alpha| > (2B)^{1/\tau}$ we have

$$E\left(\frac{1}{|V-\alpha|^s}\right) \ge \frac{1}{2}\, \frac{1}{||\alpha| + (2B)^{1/\tau}|^s} \quad . \qquad (I.9)$$

The inequalities (I.7) and (I.9) yield a value of K with which the desired bound (I.4) holds for $\alpha \ge (B)^{1/\tau}$. The extension (with a modified K) to small α (e.g., $\alpha < (B)^{1/\tau}$) is elementary.

For the case $\|V\|_\infty < \infty$, the strict inequality $K < \|V\|_\infty$ follows from an estimate on the relative contribution to \o(I, E)($1/|V-\alpha|^s$) from small enough intervals at the edges of $[-\|V\|_\infty, \|V\|_\infty]$. An explicit bound can be easily obtained using (I.5) as in (I.6). ■

Appendix II. A generalized Combes - Thomas estimate

For completeness, enclosed here is the proof of a somewhat generalized version of the Combes-Thomas estimate. The basic argument is well known (see [S1]), although the extension to the power-law decay discussed in Section 4 may not be standard.

Lemma II.1: (Based on Combes-Thomas [CT]) Let M=T+U be a self adjoint operator on $\ell^2(\Gamma)$, Γ a countable set of sites, whose off diagonal part T (U is an arbitrary diagonal operator) satisfies a bound of the form

$$\left|T_{x,y}\right| \leq B \exp[- m\, \delta(x,y)] \ , \tag{II.1}$$

where $\delta(\cdot,\cdot)$ is a metric on Γ for which, at some $\alpha > 0$:

$$\sup_{x\in\Gamma} \sum_{y\in\Gamma} \exp[-(m-\alpha)\delta(x,y)] = S < \infty. \tag{II.2}$$

Then, for energies not in the spectrum of M, with dist$\{E,\sigma(M)\}=\Delta > 0$,

$$\left|< x \,|\frac{1}{M-E}|\, y >\right| \leq \frac{2}{\Delta} \exp[-\frac{\Delta\,\alpha}{2BS+\Delta}\ \delta(x,y)\ ] \tag{II.3}$$

Furthermore, for $\Delta > 3BS$, the off diagonal terms obey also:

$$\left|< x \,|\frac{1}{M-E}|\, y >\right| \leq \left|\frac{2BS}{\Delta - BS}\right|^2 \exp[-\alpha\ \delta(x,y)\ ] \ , \text{ for all } x\neq y\ . \tag{II.4}$$

Proof: 1) The CT argument is based on the observation that

$$<x|\frac{1}{M-E}|y> \exp[\mu\delta_L(x,y)] = <x|\, R^{-1} \frac{1}{M-E} R\, |y> = <x|\frac{1}{R^{-1}(M-E)R}|y> \tag{II.5}$$

where R is the multiplication operator $R\,|y> = \exp[\mu\delta(x,y)]\ |y>$. The right side is then estimated by the operator norm:

$$\left\| \frac{1}{R^{-1}(M-E)R} \right\| \leq \left| \Delta - \|R^{-1}MR-M\| \right|_{+}^{-1} = \left| \Delta - \|R^{-1}TR-T\| \right|_{+}^{-1} \tag{II.6}$$

To justify these statements it is convenient to first carry the argument for R defined with $\delta_L(x,y)=\max\{L,\delta(x,y)\}$, which makes R a bounded operator, and remove the cutoff only at the end.

Now, using a standard estimate for the norm of an operator, and the triangle inequality, $|\delta(x\,y)-\delta(x,z)| \le \delta(z\,y)$, we have

$$\|R^{-1}TR-T\| \le \ \backslash o(\sup_{z\in\Gamma},\ ) \ \backslash i\backslash su(y\in\Gamma, ,\ \left|T_{z\,y}\left(\exp[\mu\delta(z\,y)] - 1\right)\right|)$$

$$\le BS \ \backslash f(\mu,\alpha-\mu) \ = \ \backslash f(\Delta,2) \qquad , \qquad \text{for } \ \mu = \ \therefore\phi(\Delta\ \alpha,2BS+\Delta) \ . \qquad \text{(II.7)}$$

(under the assumption (II.2)). That yields the bound (II.3).

2) For large Δ we repeat the above argument with R redefined as: $R\ |y\rangle = C^{I[x\_y]} \exp[\alpha\delta_L(x,y)]\ \ |y\rangle$, with $C=\backslash f(\Delta-BS,2BS)$. Under the condition $\Delta>3BS$, the multiplier is the exponential of a metric (since $\log C > 0$), and the argument used in the first step of (II.7) applies. Hence

$$\|R^{-1}TR-T\| \le (C+1)BS = (\Delta+BS)/2$$

with the net result stated in (II.4). The point to be appreciated is that there are now two factors of the order of Δ – one due to the term $C^{I[x\_y]}$ in the definition of R, and the other due to the norm estimate. ■

In this work this statement is used with two different metrics: $\delta^{(1)}(x\,y)=|x-y|$ (in Sections 1) and 2), and $\delta^{(2)}(x\,y)=\log(1+|x-y|)$ in Section 4). When used with the latter, the implication is a mild power-law decay of the resolvent, at energies off the spectrum - for operators whose off-diagonal matrix elements, $T_{x,y}$, are not assumed to decay exponentially in the Euclidean metric.

In the discussion of the localization regime $R(\lambda)$, in Section 1 (below equation (1.9)), we invoke both (II.3) and (II.4) - for large Δ.

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Appendix 1: Alternative derivation of (3.8) - Not included

Lemma 2.1: Let μ_υ(dz) be the spectral probability measure (for normalized |x>) which is explicitly defined by the Herglotz spectral representation:

<x| \f(1,H_υ-E-iε)|x> = \i(,, \f(1,z-E-iε) μ_υ(dz)) , for all E real and ε > 0.

Then the Lebesgue integral of μ_υ(dz) over υ reproduces the Lebesgue measure, i.e.

\i(,,dυ μ_υ(--)) = ℒ(--)

Proof: By a standard rank-one perturbation formula:

<x| \f(1,H_υ-E-iε)|x> = \f(1,υ-Γ_o(E+iε))

with Γ_o(E+iε) defined by this relation at υ=0. Hence for all E∈ % and ε>0

\i(,, \f(1,z-(E+iε)) μ_υ(dz)) = \f(1,υ - Γ_o(E+iε))

The imaginary parts of the above expressions are positive, and thus (by Fubini) their integrals over dυ are unambiguous. The result is:

\i(,,dυ μ_υ(dz)) \f(ε,$(z-E)^2 + ε^2$) = π

for all real E, and ε > 0. That condition is known to imply that \i(,,dυ μ_υ(dz)) is the Lebesgue measure. (A proof may proceed by showing that the integral of the indicator function I[z∈[0,t]] is the limit of the integral of the function \i(o,t,ε$\pi^{-1}[(z-E)^2+ε^2]^{-1}$ dE), with ε→0). ■

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([B] : in Levinson's book: (Density ..& gap thorems) SM 8153 .58L)