Anderson Localization for the Almost Mathieu Equation, II: Point Spectrum for $\lambda > 2$

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Abstract

We prove that for any $\lambda > 2$ and a.e. ω, θ the point spectrum of the almost Mathieu operator $(H(\theta)\Psi)_n = \Psi_{n-1} + \Psi_{n+1} + \lambda \cos(2\pi(\theta + n\omega))\Psi_n$ contains the essential closure σ_{ess} of the spectrum. Corresponding eigenfunctions decay exponentially. The singular continuous component, if it exists, is concentrated on a zero measure set which is nowhere dense in σ_{ess}.

1. INTRODUCTION

This paper is another attack on the almost-Mathieu operator on $\ell^2(\mathbf{Z})$:

$$(H(\theta)\Psi)_n = \Psi_{n-1} + \Psi_{n+1} + \lambda \cos(2\pi(\theta + n\omega))\Psi_n$$

This simple-looking operator has been studied extensively for many years. We refer the reader to [1,2] for - still incomplete - list of references. The critical (and physical) value of the coupling constant λ is $\lambda = 2$ (from now on we assume without loss of generality that $\lambda \geq 0$); it is believed that at $\lambda = 2$ there occurs a transition from pure absolutely continuous to pure point spectrum. The ω here is supposed to be "irrational enough" since for rational ω the potential is periodic and the spectrum is absolutely continuous for all λ; for Liouville ω (abnormally well approximated by rationals) and $\lambda > 2$ the spectrum of $H(\theta)$ is purely singular continuous [3,4]. Up to recently the only rigorous reason for this belief was that for $\lambda > 2$ and irrational ω the Lyapunov exponents are positive; that proves the absence of the absolutely continuous part of the spectrum [5,6]. By Aubry duality there is no pure point

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spectrum for $\lambda < 2$ [7]. The latest development for any $\lambda < 2$ is the proof of existence of the absolutely continuous spectrum that was given by Last [8] for a.e. ω, θ and by Gesztesy and Simon [13] for all ω, θ; Last [8] also proved that for a.e. ω the absolutely continuous spectrum, σ_{ac}, coincides with the spectrum, σ, up to a set of zero Lebesque measure.

In the present paper we study λ's above the critical value. Localization, i.e., pure point spectrum with exponentially decaying eigenfunctions, was proved by Sinai [9] and Fröhlich, Spencer and Wittwer [10] in the perturbative regime: λ "big enough." In [11] we developed a nonperturbative way of proving localization, which worked for $\lambda \geq 15$. The method also allowed to prove localization in the middle of the spectrum for $\lambda \geq 5.4$. Those restrictions on λ were caused exclusively by the roughness of the a priory estimates from above on the growth of the formal solution Ψ_E, i.e., on the Lyapunov exponents. In this paper we notice that the results of [8,13] together with Aubry duality and certain regularity of the Lyapunov exponent [12] give a much nicer estimate for "most of" the Lyapunov exponents. We combine this soft argument with the method of [11] to prove the existence of "a lot of" pure point spectrum for a.e. ω. The result holds for any $\lambda > 2$ but we have to pay for that by not being able to rule out the possibility of some singular continuous spectrum.

We already denoted the spectrum of $H(\theta)$ by σ, and now denote the singular continuous part of the spectrum by σ_{sc}, and the pure point part of the spectrum by σ_{pp}. It is well known that for irrational ω the sets σ, σ_{sc} and σ_{pp} (understood as the closure of the set of eigenvalues) do not depend on the phase θ for a.e. θ. Though for $\lambda < 2$ the results on the absolutely continuous spectrum probably hold for every θ (see [13]), the pure point spectrum for $\lambda > 2$ should be an essentially "a.e." result, since for generic θ the spectrum is purely singular continuous in this case [14].
We put

$$\Theta = \{\theta : \text{for every } s > 1 \text{ the relation } |\sin(\theta + \frac{k}{2}\omega)| < k^{-s} \text{ holds for infinitely many } k\}.$$

We will assume that the phase θ does not belong to this zero measure set. As we already mentioned, for $\lambda > 2$ the arithmetic nature of ω plays a major role. Let $\frac{p_n}{q_n}$ be the n^{th} continuous fractions approximant of ω. Throughout the rest of the paper we assume that ω is Diophantine, i.e., an irrational such that for some $r > 1$ and $C > 0$ we have $|q_n\omega - p_n| > Cq_n^{-r}$. In Theorem 1 we will also use another, rather technical, restriction on ω

$$\frac{C}{q_n^r} < |q_n\omega - p_n| = \frac{o(1)}{q_n} \text{ as } n \to \infty \tag{1}$$

The set of ω's described in (1) has full Lebesque measure [15].
For $\lambda > 2$ the spectrum σ is a set of positive Lebesque measure [13]:$|\sigma| \geq 2\lambda - 4$. We put

$$\sigma_{ess} = \{E \in \sigma : \text{ for any } \epsilon > 0 \ |(E - \epsilon, E + \epsilon) \cap \sigma| > 0\}.$$

We will prove

Theorem 1. *Suppose ω satisfies (1). Then for any $2 < \lambda < 15$ we have*

1. $|\sigma_{pp}| = |\sigma|$,
2. σ_{sc}, *if it exists, has measure zero and is nowhere dense in* σ_{ess} *(in the relative topology).*
3. *For* $\theta \notin \Theta$ *the set of eigenvalues of the operator* $H(\theta)$ *is dense in* σ_{pp} *and the corresponding eigenfunctions are exponentially decaying .*

The zero measure set of Diophantine ω's not satisfying (1) happens to include the golden mean - the most popular object for numerical studies. Without technical assumption (1) we can prove a slightly weaker result:

Theorem 1'. *For any Diophantine* ω *and any* $2 < \lambda < 15$ *we have*

1. $|\sigma_{pp}| \geq 2\lambda - 4$.
2. *There exists a closed set* $A \subset \sigma_{pp}$, $|A| \geq 2\lambda - 4$, *such that* σ_{sc} *is nowhere dense in* A.
3. *For* $\theta \notin \Theta$ *the set of eigenvalues of the operator* $H(\theta)$ *is dense in* A *and the corresponding eigenfunctions are exponentially decaying .*

Remarks.

1. As can be seen from the proof, in order to to prove the complete localization for $\lambda > 2$ it suffices to prove the continuity of the Lyapunov exponent $\gamma(E)$. For λ large enough the continuity of $\gamma(E)$ follows from the proof in [9]. This continuity would also be enough to rule out the singular continuous spectrum for $\lambda < 2$ [16].
2. An estimate on the rate of the exponential decay of the eigenfunctions can be easily obtained from the proof and is given by $|\Psi(x)| \leq const \left(\frac{\lambda}{2}\right)^{(-\frac{1}{4}+\epsilon)|x|}$ for any $\epsilon > 0$. This decay is slower than what is suggested by the Lyapunov exponent.
3. Theorems 1 and 1' can be proved with θ satisfying a weaker condition. Namely, let us fix $1 < \mu < \left(\frac{\lambda}{2}\right)^{\frac{1}{16}}$ and put $\Theta_\mu = \{\theta : |\sin(\theta + \frac{k}{2}\omega)| < \mu^{-k}$ holds for infinitely many $k\}$. Then for $\theta \notin \Theta_\mu$ the same result holds, but the estimate on the decay of the eigenfunctions will be: $|\Psi(x)| \leq const \left(\left(\frac{\lambda}{2}\right)^{-\frac{1}{4}} \mu^4\right)^{|x|}$. It is quite clear that the rate of decay should depend on μ. This result can be compared with the fact that for $\theta \in \Theta_\mu$, μ sufficiently large, the operator $H(\theta)$ has no pure point component in the spectrum [14].
4. The Diophantine property can be made weaker (see the comment in [11]) but not too weak since for Liouville ω the spectrum of $H(\theta)$ is purely singular continuous. The upper bound in (1) appears here only because of the same bound in [8] and is presumably an artifact of the Last's (and, of course, of the present) proof.

5. All the results certainly hold for $\lambda \geq 15$ as well [11].

In Section 2 we describe the method for proving localization which is a certain modification of the method of [17] (see also [10], [18]). In Section 3 we formulate our main technical result, Theorem 2, which is very similar to Theorem 2 in [11], and present the soft arguments which prove Theorems 1 and 1' from Theorem 2. In Section 4 we prove Theorem 2.

2. THE GENERAL SETUP.

We start with some definitions

Definition. A formal solution $\Psi_E(x)$ of the equation $H(\theta)\Psi_E = E\Psi_E$ will be called **a generalized eigenfunction** if $\Psi_E(x) \leq C(1+|x|)$ for some $C = C(\Psi_E) < \infty$. The energy E for which such a solution exists will be called **a generalized eigenvalue**.

It is well known that to prove pure point spectrum one only needs to prove that generalized eigenfunctions belong to ℓ^2 (see [17]).
We denote the Green's function $(H-E)^{-1}$ of the operator $H(\theta)$ restricted to the interval $[x_1, x_2]$ with zero boundary conditions at $x_1 - 1$ and $x_2 + 1$ by $G_{[x_1,x_2]}(E)$.
Let us fix a number $m < 1$.

Definition. A point $y \in \mathbf{Z}$ will be called (m,k)-regular if <u>there exists</u> an interval $[x_1, x_2]$ containing y such that

$$|G_{[x_1,x_2]}(y, x_i)| < m^k, \text{ and } dist(y, x_i) \leq k; \; i = 1, 2.$$

Otherwise y will be called (m,k)-singular.

Let $E(\theta)$ be a generalized eigenvalue of H_θ, $\Psi(x)$ the corresponding generalized eigenfunction.

Lemma 1. *For every $x \in \mathbf{Z}$ such that $\Psi(x) \neq 0$ there exists $k_0 = k_0(x, m, \theta, E)$ such that for $k > k_0$ the point x is (m,k)-singular.*

Lemma 1 is the same kind of statement as Lemma 3.1 in [10] and so is the proof. □
Suppose one can prove that (m,k)-singular points are "far apart" which we formulate as the following

Quasilemma. *Suppose the points x_1, x_2 are (m,k)-singular, k is large enough and $dist(x_1, x_2) > \frac{k}{2}$, then $dist(x_1, x_2) > k$.*

Then the rest of the proof can be organized as follows. Assume without loss of generality that $\Psi(0) \neq 0$. Let $|x|$ be bigger than $k_0(0, m, \theta, E)$ and sufficiently large so that we can apply Quasilemma with $k = |x|$. Suppose x is $(m, |x|)$-singular. Since 0 is $(m, |x|)$-singular, the Quasilemma asserts that $dist(0, x) = |x| > |x|$. The contradiction implies that x is $(m, |x|)$-regular. Thus we have that there exists an interval $[x_1, x_2]$ containing x such that

$$|x_i - x| \leq |x|, \quad |G_{[x_1,x_2]}(x, x_i)| \leq m^{|x|}, \; i = 1, 2.$$

We now can use the formula

$$\Psi(x) = G_{[x_1,x_2]}(x, x_1)\Psi(x_1 - 1) + G_{[x_1,x_2]}(x, x_2)\Psi(x_2 + 1)$$

to obtain the estimate:

$$|\Psi(x)| \leq 2C(1 + 2|x|)m^{|x|}.$$

This argument shows that the localization follows immediately if we make a Lemma out of the Quasilemma, i.e. provide it with hypotheses and, of course, with a proof.

3. PROOF OF THEOREM 1.

We define

$$B(\theta, E, \lambda) = \begin{pmatrix} E - \lambda \cos 2\pi\theta & -1 \\ 1 & 0 \end{pmatrix}, B_k(\theta, E, \lambda) = B(\theta + k\omega, E, \lambda),$$

$$M_k(\theta, E, \lambda) = B_k(\theta, E, \lambda) ... B_0(\theta, E, \lambda).$$

The Lyapunov exponent $\gamma(E, \lambda)$ is defined by $\gamma(E, \lambda) = \inf_k \int_0^{2\pi} |k|^{-1} \ln ||M_k(\theta, E, \lambda)|| d\theta$. Theorem 1 will follow from

Theorem 2. *Let θ, ω be as in Theorem 1 and suppose E, λ are such that $\frac{\ln \frac{\lambda}{2}}{\gamma(E,\lambda)} > \frac{3}{4}$ and E is a generalized eigenvalue of $H(\theta)$. Then the corresponding generalized eigenfunction $\Psi(x)$ is exponentially decaying.*

Proof of Theorem 1. Take any $\lambda > 2$. The Aubry duality and the Thouless formula yield the famous relation [19,6,7]

$$\gamma(E, \lambda) = \ln \frac{\lambda}{2} + \gamma\left(\frac{2E}{\lambda}, \frac{4}{\lambda}\right) \tag{2}$$

Since $\frac{4}{\lambda} < 2$, we can usethe result of Last [8] that for a.e. $E \in \sigma(H(\theta, \frac{4}{\lambda}))$ the Lyapunov exponent $\gamma(E, \frac{4}{\lambda})$ is equal to 0.
Craig and Simon have proven [12] that $\gamma(E)$ is continuous at points with $\gamma(E) = 0$. We set

$$G = \{E : \gamma\left(E, \frac{4}{\lambda}\right) < \frac{1}{3} \ln \frac{\lambda}{2}\}.$$

The set $2\lambda^{-1}G \cap \sigma$ contains an open (in σ_{ess}) dense (in σ_{ess}) set of measure $|\sigma|$, thus the set $\sigma \backslash 2\lambda^{-1}G$ is a nowhere dense in σ_{ess} set of zero measure. For $E \in 2\lambda^{-1}G$ relation (2) implies that $\gamma(E,\lambda) < \frac{4}{3}\ln\frac{\lambda}{2}$ and we can apply Theorem 2 to obtain that $\sigma_c \cap 2\lambda^{-1}G = \emptyset$ which gives the statement of the theorem. $\square$

Proof of Theorem 1'. Gesztesy and Simon [13] have proven that for any irrational ω and any λ the inequality $|\sigma_{ac}| \geq 4 - 2\lambda$ holds. This plus Ishii-Pastur-Kotani theorem (see [1]) and the same argument as above implies the result. $\square$

4. PROOF OF THEOREM 2.

As we learned from Section 2 it suffices to prove the Quasilemma under the conditions of Theorem 2.
Let us put

$$P_k(\theta, E) = \det\left[(H(\theta) - E)\Big|_{[0,k-1]} \right].$$

$P_k(\theta, E)$ is an even function of the argument $\theta + \frac{k-1}{2}\omega$ and can be written as a polynomial of the degree k in $\cos(2\pi(\theta + \frac{k-1}{2}\omega))$:

$$P_k(\theta, E) = \sum_{j=0}^{k} b_j(E)\cos^j(2\pi(\theta + \frac{k-1}{2}\omega))$$

To simplify the notation we will sometimes omit the dependence on E.

It is easy to see that $b_k = 2\lambda^k$. We now fix $E \in R$; $1 < m_1 < \frac{\lambda}{2}$. Given $k > 0$ we set

$$A_k = \{x : |P_k(\theta + x\omega)| > m_1^k\}$$

For any $x_1,\ x_2 = x_1 + k - 1,\ x_1 \leq y \leq x_2$ we have

$$|G_{[x_1,x_2]}(x_1, y)| = \left| \frac{P_{x_2-y}(\theta + (y+x_1)\omega)}{P_k(\theta + x_1\omega)} \right|$$

$$|G_{[x_1,x_2]}(y, x_2)| = \left| \frac{P_{y-x_1}(\theta + x_1\omega)}{P_k(\theta + x_1\omega)} \right| \tag{3}$$

Since $P_k(\theta, E)$ is one of the entries of the matrix $M_k(\theta, E)$ we have an evident upper bound $P_k(\theta, E) \leq ||M_k(\theta, E)||$. It is proved in [12] that for **all** θ, E, λ the inequality

$$\limsup |k|^{-1} \ln ||M_k(\theta, E, \lambda)|| \leq \gamma(E, \lambda)$$

holds.

Thus for any $\epsilon > 1$ there exists $k(\epsilon, \theta, E)$ such that for $k > k(\epsilon, \theta, E)$ we have

$$|P_k(\theta, E)| < (\epsilon\gamma(E, \lambda))^k. \tag{4}$$

It follows from (3),(4) that for $x_1 \in A_k,\ x_2 = x_1 + k - 1,\ k > k(\epsilon, \theta, E),\ \frac{1}{m_1} < m < 1$ and $y \in [x_1, x_2]$ such that

$$k\left(1 - \frac{\ln(m_1 m)}{\ln(\epsilon\gamma(E, \lambda))}\right) < y - x_1 < k\left(\frac{\ln(m_1 m)}{\ln(\epsilon\gamma(E, \lambda))}\right)$$

we have

$$|G_{[x_1,x_2]}(y, x_i)| < m^k, \quad i = 1, 2. \tag{5}$$

We set $c_{\lambda,\epsilon} = \frac{\ln(m_1 m)}{\ln(\epsilon\gamma(E,\lambda))}$.

Proposition 1. *Suppose $y \in \mathbf{Z}$ is (m, k)-singular. Then for any x such that $k(1 - c_{\lambda,\epsilon}) \leq y - x \leq kc_{\lambda,\epsilon}$ we have that x does not belong to A_k.*

Proposition 1 follows immediately from (5). □

The statement of the Quasilemma can now be obtained from the following lemma:

Lemma 2.[11] *Suppose ω is Diophantine. For $\theta \notin \Theta$ there exists $k_1(\theta, E)$ such that for $k > k_1(\theta, E)$ if the points x_1, x_2 satisfy*
1) $x_i, x_i + 1, ..., x_i + [\frac{k+1}{2}] \notin A_k$, $i = 1, 2$
2) $dist(x_1, x_2) > [\frac{k+1}{2}]$
then

$$dist(x_1, x_2) > \alpha^k$$

with $\alpha = \alpha(m_1, m, \lambda, E, s, r) > 1$.

Indeed, let us suppose that $\frac{\ln\frac{\lambda}{2}}{\gamma(E,\lambda)} > \frac{3}{4}$. Then there exist $1 < m_1 < \frac{\lambda}{2}$, $m < 1$ and $\epsilon > 1$ such that $2c_{\lambda,\epsilon} - 1 > \frac{1}{2}$. Let k be bigger than $\max[k(\epsilon, \theta, E), k_1(\theta, E)]$. Suppose x_1', x_2' are (m, k)-singular and $dist(x_1', x_2') > [\frac{k+1}{2}]$. Since $2c_{\lambda,\epsilon} - 1 > \frac{1}{2}$, we obtain using Proposition 1 that the points $x_1 = x_1' - c_{\lambda,\epsilon}k$ and $x_2 = x_2' - c_{\lambda,\epsilon}k$ satisfy the conditions of Lemma 2. Applying Lemma 2 we get that $dist(x_1', x_2') > \alpha^k > k$ for large k which proves the statement of the Quasilemma. □

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