Random Sets and Spectra of Unbounded Linear Random Operators

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1 Introduction and Main Results

The purpose of this note is to present a possible framework in which to study unbounded linear random operators. A. V. Skorohod defines a notion of random linear operators in [7], but it seems to be unsuitable for the purpose of a corresponding spectral theory, because the realizations are not linear operators. Spectral theory of random Schrödinger operators is the subject in [1] and [6]. A strict generalization of some of the basic facts in this theory will be presented in the following. Our results complete the results in [8]. Application of the theory of random sets as found in [5] and [2] turns out to be a promising approach.

Definition 1.1 (Random Set) *A random set Σ in a topological space X is a mapping from a probability space Ω into the set of subsets of X such that*

$$\Sigma_U := \{\omega \in \Omega \mid \Sigma(\omega) \bigcap U \neq \emptyset\} \tag{1}$$

is an event when U is open.

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces. An operator T from $\mathcal{H}_1$ into $\mathcal{H}_2$ is a linear mapping from a subspace $D(T)$ of $\mathcal{H}_1$ into $\mathcal{H}_2$ [9, p.50-]. The domain $D(T)$ is not required to be dense and the operator is not required to be a continuous mapping. Motivated by the gap topology for closed operators [3, p.197-207] we use the graph of the operator to avoid difficulties with domains:

Definition 1.2 (Random Operator) *A random operator T from $\mathcal{H}_1$ into $\mathcal{H}_2$ is a mapping $\omega \mapsto T(\omega)$ from a probability space into the set of operators from $\mathcal{H}_1$ into $\mathcal{H}_2$ such that the graph $G(T)$ is a random set in $\mathcal{H}_1 \oplus \mathcal{H}_2$.*

Our first aim is to give important examples of random operators.

Theorem 1.1 (Bounded Random Operators) *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be separable Hilbert spaces. A mapping T from a probability space into the set of bounded operators defined on $\mathcal{H}_1$ with values in $\mathcal{H}_2$ is a random operator if and only if the inner product $\langle f, Tg\rangle$ is a random number for all g in $\mathcal{H}_1$ and all f in $\mathcal{H}_2$.*

The previous theorem is the most important characterization theorem. It ensures that the canonical definition of a bounded random operator gives a special class of

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random operators. In order to generalize this we turn to matrix operators. We will use the results in [9, p.149-153] without further comment. Let $\{e_m\}$ and $\{\tilde{e}_n\}$ be orthonormal systems, and let (a_{nm}) be a matrix of complex numbers. The domain $D(A)$ of the induced maximal matrix operator A is given by the set of vectors f which satisfy: (i) $g_n := \sum_{m\geq 1} a_{nm} \langle e_m, f\rangle$ exists for all n, and (ii) $\sum_{n\geq 1} |g_n|^2 < \infty$. The action is given by $Af := \sum_n g_n \tilde{e}_n$. Observe that e_m is in $D(A)$ if and only if $\sum_n |a_{nm}|^2 < \infty$, and that (g_n) in (i) is defined by pointwise convergence and not convergence in l^2. The minimal operator A_0 is the restriction of A by restricting the domain to the linear span of $\{e_n\}$.
A random vector f in a separable Hilbert space is a mapping $\omega \mapsto f(\omega)$ from the probability space into the Hilbert space such that the inner product $\langle g, f\rangle$ is a random number for any fixed vector g.

Theorem 1.2 (Random Matrices) *Let $\{e_m\}$ and $\{\tilde{e}_n\}$ be random orthonormal systems in the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively, and let (a_{nm}) be a matrix of random numbers. The induced minimal operator A_0 is a random operator. Assume further that $\{e_m\}$ is complete in $\mathcal{H}_1$ and that $\mathcal{H}_2$ is separable. If $P\{\sum_{m\geq 1} |a_{nm}|^2 < \infty\} = 1$, then the maximal matrix operator A is a closed, densely defined, random operator.*
If T is a closed random operator from a separable Hilbert space into a separable Hilbert space, then T is a restriction of a random matrix operator.

The previous theorem shows in particular that we have obtained a generalization of the definition in [6, p.13, p.37-40]. A discussion on how to define unbounded random operators is found in [6, p.35-36], containing also some further references. That our definition is a strict generalization of the definition in [1, p.241-] is a consequence of the following characterization theorem.

Theorem 1.3 (Normal Random Operators) *If T is a mapping from a probability space into the set of normal operators defined in a separable Hilbert space, then it is a random operator if and only if all spectral measures $E \mapsto \langle f, 1_E(T) g\rangle$ are random measures.*

Observe that the topological support of a finite random measure is a random set: $\Sigma := \operatorname{supp} \rho := \{z \mid \rho(V) > 0,\ \forall^{open} V \ni z\}$, so $\Sigma_U = \{\omega \mid \rho_\omega(U) > 0\}$. Since the Lebesgue components of a finite random measure are also random measures [10], this gives randomness of corresponding spectra of a normal random operator. Randomness of spectra holds also in the general case:

Theorem 1.4 (Random Spectra) *Let T be a closed random operator in a separable Hilbert space, and assume the probability space to be complete. The set of eigenvalues $\sigma_p(T)$, the core spectrum $\tilde{\sigma}(T)$, and the spectrum $\sigma(T)$ are random sets.*

The above theorem seems to be a novelty also in the case of bounded operators.

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2 Random Operators

In this section the stated characterizations of random operators will be proven using the measurable selection theorem [2, p.65-66].

Theorem 2.1 (Measurable Selections) *If Σ is a complete, non empty, random set in a separable metric space, then there exists a sequence (x_i) of random variables such that $\Sigma = \overline{\{x_i\}}$.*

The following theorem gives an alternative characterization of random operators, since the graph of a linear operator is a linear space.

Theorem 2.2 (Random Vector Spaces) *If E is a random subspace in a separable Hilbert space, then any inner product $\langle f, Pg \rangle$ is a random number, where P is the orthogonal projection onto $\overline{E}$. Randomness of these inner products are also sufficient for the randomness of E.*

Proof. Let (x_i) be a sequence of random vectors such that $\overline{E} = \overline{\{x_i\}}$. We define a random orthonormal system from the Gram-Schmidt procedure by induction as in [6, p.16-17]. Define the normalized version e_1 of x_1 from $e_1(\omega) := 0$ on $\{\omega \mid x_1(\omega) = 0\}$ and $e_1(\omega) := x_1(\omega) / \|x_1(\omega)\|$ otherwise. Assume that $e_1, \ldots, e_n$ have been defined. Set $y_{n+1} := x_{n+1} - \sum_{i \leq n} e_i \langle e_i, x_{n+1} \rangle$ and define e_{n+1} to be the normalized version of y_{n+1}. The first part of the theorem follows from $\langle f, Pg \rangle = \sum_i \langle f, e_i \rangle \langle e_i, g \rangle$.
Let (y_i) be a dense set of vectors in the Hilbert space and let U be an open set in the Hilbert space. The second claim in the theorem follows from $E_U = \bigcup_i \{\omega \mid P(\omega) y_i \in U\}$, which is measurable. □

Let T be an operator acting between Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. The adjoint operator T^* has domain consisting of all vectors g in $\mathcal{H}_2$ such that $f \mapsto \langle g, Tf \rangle$ is continuous. The action of T^* is uniqly determined from the requirements $T^* g \in \overline{D(T)}$ and $\langle g, Tf \rangle = \langle T^* g, f \rangle$ for all f in $D(T)$. This extends the usual definition of the adjoint operation, since the domain $D(T)$ is not required to be dense in $\mathcal{H}_1$.

Theorem 2.3 (Adjoint Operator) *If T is a random operator between separable Hilbert spaces, then the adjoint operator T^* is a closed random operator.*

Proof. Set $A_{inv} := \{(f, g) \mid (g, f) \in A\}$. The graph of the adjoint is given by $G(T^*) = [G(-T)_{inv}]^{\perp}$, where the orthogonal complement is taken in $\mathcal{H}_2 \oplus \overline{D(T)}$. The adjoint is then a closed operator. The domain of T is a random set, because $D(T)_U = \{\omega \mid \exists f \in D(T) \bigcap U,\ Tf \in \mathcal{H}_2\}$. Consequently $\mathcal{H}_2 \oplus \overline{D(T)}$ is a random vector space. Randomness of the adjoint follows now from the characterization of random vector spaces in terms of orthogonal projections. □

Proof. **(Theorems 1.2 and 1.1)** Let T be a closed random operator from a separable Hilbert space into a separable Hilbert space. Take a measurable selection (x_i, Tx_i) for the graph of T. A random orthonormal system $\{e_i\}$ contained in $D(T)$ follows from the Gram-Schmidt procedure used on $\{x_i\}$. The adjoint T^* is a closed densely defined random operator, so the same procedure gives a random orthonormal basis $(\tilde{e}_i)$ contained in $D(T^*)$. Observe that Te_i is a random variable, since it is a linear combination of the random variables $Tx_1, \ldots, Tx_i$. Set $a_{ij} := \langle \tilde{e}_i, Te_j \rangle$, which is then a random complex number. The operator T is a restriction of the maximal matrix operator defined from (a_{ij}) and the orthonormal systems [9, p.150-151].
Randomness of $\langle f, Tg \rangle$ in the case of a bounded operator defined on $\mathcal{H}_1$ follows from expansion in the constructed random orthonormal bases, since $D(T) = \mathcal{H}_1$ and $D(T^*) = \mathcal{H}_2$.
Now we turn to the converse statements in Theorem 1.2. Let Q be the set of finite sequences of complex numbers, where each component are rational. The domain of A_0 is the linear span of $\{e_i \mid \sum_j |a_{ji}|^2 < \infty\}$. Since $\{\omega \mid \exists f \in D(A_0(\omega)) \bigcap U, A_0(\omega) f \in V\}$ for open sets U, V equals $\bigcup_{q \in Q} \{\omega \mid \sum_j q_j e_j \in U, c_j := \sum_i a_{ji} q_i,\ \sum_j c_j \tilde{e}_j \in V\}$, so A_0 is a random operator.
The matrix operator A is a random operator, since it is the adjoint of the minimal matrix operator from the adjoint matrix.
Randomness of a bounded operator from randomness of the inner products $\langle f, Tg \rangle$ is a special case. □

The following is a generalization of parts of the result for bounded random operators, and will be used in the characterization proof for normal random operators.

Theorem 2.4 *Let T be a closed random operator and let f be a random vector such that $f \in D(T)$. It follows that Tf is a random vector.*

Proof. Since T^* is densely defined, there exists a random orthonormal basis $\{e_i\}$ contained in $D(T^*)$ with T^*e_i random. This gives that $\langle g, Tf\rangle$ equals $\sum_i \langle g, e_i\rangle \langle T^*e_i, f\rangle$, so Tf is a random variable. $\square$

Proof. **(Theorem 1.3)** Let T be a normal random operator, so $D(T) = D(T^*)$, and $\|Tf\| = \|T^*f\|$. The real part $X := (T + T^*)/2$ of T is a selfadjoint operator. A measurable selection (f_i, Tf_i) for T gives a measurable selection (f_i, Xf_i) for X, since T^*f_i is a random variable and $\|Xf\| \leq \|Tf\|$. Randomness of the spectral measures of X follows from equality of $-\pi \langle f, 1_{(-\infty,t]}(X)g\rangle$ and $\lim_{\delta\to 0+} \lim_{\epsilon\to 0+} \int_{-\infty}^{t+\delta} \mathrm{Im}\, \langle f, (s + i\epsilon - X)^{-1}g\rangle \; ds$. This argument is also valid for the imaginary part of T. The spectral family of T is then random, because it is given by the product of the spectral families from the real and imaginary parts of T.
The proof in the opposite direction follows likewise, by consideration of the resolvents of the real and imaginary parts of T. $\square$

3 Random Spectra

In this section randomness of the spectra will be proven using the measurable projection theorem [2, p.75].

Theorem 3.1 (Measurable Projections) *Let Ω be a complete probability space, and let X be a complete separable metric space, equipped with the Borel field. If $E \subset \Omega \times X$ is product measurable, then the projection $\{\omega \mid \exists (\omega, x) \in E\}$ is an event.*

In the following we assume the probability space to be complete. Particular consequences for a closed random set Σ in a separable Hilbert space are that the graph $\{(\omega, x) \mid x \in \Sigma(\omega)\}$ is a measurable set, and that Σ_E is an event for any Borel set E. We mention as a curiosity that completeness also was assumed in the first proof of randomness of the Lebesgue parts of the spectrum of a selfadjoint random operator [4].

Proof. **(Random point spectrum)** Let $U \subset \mathbb{C}$ be open and let T be a closed random operator. Then $\{\omega \mid \sigma_p(T(\omega)) \bigcap U \neq \emptyset\}$ equals $G(T)_E$ with $E = \{(f, zf) \mid \|f\| = 1, z \in U\}$, which proves the claim since E is a Borel set in $\mathcal{H}\times\mathcal{H}$: $E = A \bigcap B$, where $A = \{(f, g) \mid |\langle f, g\rangle| = \|f\| \, \|g\|\}$, and $B = \{(f, g) \mid \|f\| = 1, \langle f, g\rangle \in U\}$. $\square$

Proof. **(Random core spectrum)** The core spectrum is given by approximate eigenvectors, so $\tilde{\sigma}(T) = \{z \in \mathbb{C} \mid \forall \epsilon > 0,\ \exists f \in D(T) \quad \|(z - T)f\| < \epsilon \|f\|\}$. For each $n \in \mathbb{N}$ let $\{I^n_m\}, m \in \mathbb{N}$ be a partition of the complex plane into disjoint squares with sides $1/n$, and choose $z^n_m \in I^n_m$. It follows that $\{(\omega, z) \mid z \in \tilde{\sigma}(T(\omega))\}$ equals $\bigcap_{N\geq 1} B_N = \bigcap_{N\geq 1} A_N$, where $B_N := \{(\omega, z) \mid \exists f \in D(T(\omega)), \|f\| > 1,\ \|(z - T(\omega))f\| < 1/N\}$ and $A_N := \bigcup_{M\geq 1} \bigcap_{m\geq M} \bigcup_{n\geq 1} \{(\omega, z) \mid z \in I^m_n,\ \exists f \in D(T), \|f\| > 1,\ \|(z^m_n - T)f\| < 1/N\}$. The first equality is straightforward, and the second follows from $B_{N+1} \subset A_{N+1} \subset B_N$. This gives randomness of the core spectrum. $\square$

Proof. **(Random spectrum)** The union of random sets $A_1, A_2, \ldots$ is a random set, since $\{\omega \mid (\bigcup_i A_i(\omega)) \bigcap U \neq \emptyset\}$ equals $\bigcup_i \{\omega \mid A_i(\omega) \bigcap U \neq \emptyset\}$. Randomness of the spectrum follows from $\sigma(T) = \tilde{\sigma}(T) \bigcup \tilde{\sigma}(T^*)^*$. $\square$

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