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# Postulates for Time-Evolution in Quantum Mechanics

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**Abstract**

A detailed list of postulates is formulated in an algebraic setting. These postulates are sufficient to entail the standard time evolution governed by the Schrödinger- or Dirac equation. They are also necessary in a strong sense: Dropping any one of the postulates allows for other types of time-evolution, as is demonstrated with examples. Some philosophical remarks shall hint on possible further investigations.

# 1 Introduction

Time has several appearances in physics:

A1: In general relativity it is itself an *object* of the theory. Unified with space it is in mutual interaction with matter.

A2: For smaller systems it is an external parameter, used to describe the *dynamics*, the *time-evolution*.

A3: There is a *symmetry* of all the physical laws, they are invariant under *translation* in time and, more generally, under transformations belonging to the Poincaré or Galilei-group.

A4: The basic laws are also invariant under *time-reflection*, if coupled with changes of charges and of parity.

A5: For large systems, there is the *arrow of time*, a breaking of time-reversal-symmetry, stated as the *second law of thermodynamics*.

A6: There is an *arrow of time* in *cosmology*.

A7: There is also an *arrow of time* in the *measurement* process. The "collapse of the wave packet" is irreversible.

This paper is mainly about A2, dynamics, in some part also linked with A3, time-translation-invariance. Up to now there exists a large amount of investigations about observables, states and measurement, but little can be found regarding the time-evolution [Pav]. It is to be expected that in future development all the different appearances of time will be recognized as different facets of one unified whole. This work cannot be more than a small brick in such a building, functioning as a link between general schemes in an algebraic setting, epistemological arguments and the well known Schrödinger equation.

Our basis consists of results of the axiomatic algebraic approach to quantum mechanics [Gil], [Gud], [Mac]. The structure of space is represented by the Weyl-operators of a system with a finite number of degrees of freedom [Thi]. They are considered as observable for all times $t \in \mathbf{R}$. The algebra they create is represented by bounded operators on a separable Hilbert space in an essentially unique way. Together with three postulates on the time evolution, postulating affinity, group property and continuity, this statement gives a short way, probably to be used in a course on the foundations of quantum mechanics, to imply the standard form of a time-evolution. In the sequel the continuity condition will be relaxed to demanding measurability, the basic statement and the group-property postulate will be dissected into several hypotheses and postulates, allowing for deeper analysis, epistemological discussions and probably for alternate theories of time-evolution.

# 2 Hypotheses and Postulates

Replacing Newton's hypotheses on absolute space and absolute time we can allow for some freedom to choose only finite time-intervals, discrete points of time ..

H1: Time is an external parameter, equipped with a pair-relation between different points in time: "$t$ is later than $s$", abbreviated as $t > s$.

H2: The set of observables at a fixed time $t$ forms a $C^*$-algebra $\mathcal{A}_t$, which has an irreducible representation in a Hilbert space $\mathcal{H}$. This representation is unique up to unitary equivalence. The possible physical states of the system are represented by the mathematical states of the algebra, $\mathcal{S}(\mathcal{A}_t)$.

H3: The Hilbert space $\mathcal{H}$ is separable (it has a countable basis).

H4: The set of observables is at each time the same: $\forall s, \forall t : \mathcal{A}_s = \mathcal{A}_t =: \mathcal{A}$.

H5: For the time $t$ later than $s$, there corresponds to each state $w_s \in \mathcal{S}(\mathcal{A}_s)$ one and only one state $w_t = T_{t,s} w_s \in \mathcal{S}(\mathcal{A}_t)$. The mapping $T_{t,s} : \mathcal{S}(\mathcal{A}_s) \to \mathcal{S}(\mathcal{A}_t)$ represents the time-evolution.

The last hypothesis is also the definition of "time-evolution", which will be specified in the following postulates. In P5 and P6 the preliminary freedom considering the structure of "time" in H1 will be removed.

P1: *Affinity (Superposition Principle):* The time-evolution is compatible with the mixing of states. $\forall \{v_s, w_s\} \subset \mathcal{A}_s, \forall \lambda \in [0,1] : T_{t,s}(\lambda v_s + (1-\lambda) w_s) = \lambda T_{t,s} v_s + (1-\lambda) T_{t,s} w_s$.

P2: *Partial Reversibility:* There exists not only a time-evolution to later times, into the "future", but there is also a law to find the "past", $T_{t,s}$ is *injective*:

$$\forall t, s, t > s : \exists T_{t,s}^{-1} : \{w_t = T_{t,s} w_s | w_s \in \mathcal{A}_s\} \to \{w_s\} = \mathcal{S}(\mathcal{A}_s)$$

$$T_{t,s}^{-1} T_{t,s} = \text{identity}|_{\mathcal{S}(\mathcal{A}_s)} .$$

P3: *Surjectivity:* Each state can be reached by the time-evolution; for $t$ later than $s$, each state at time $t$ does have a past history at time $s$, $T_{t,s}$ is surjective:

$$\{T_{t,s} w_s | w_s \in \mathcal{S}(\mathcal{A}_s)\} = \mathcal{S}(\mathcal{A}_t).$$

If both P2 and P3 hold, there holds *full reversibility:*

$$T_{t,s} T_{t,s}^{-1} = \text{identity}|_{\mathcal{S}(\mathcal{A}_t)} .$$

P4: *Divisibility:* Time-evolution can be split over times in between:

$$u > t > s \Rightarrow T_{u,s} = T_{u,t} T_{t,s}.$$

P5: *Invariance under time-translation:* The time differences $\{\tau =: t - s | t > s\}$ form a semigroup under addition, and the time-evolution is determined by time-differences only:

$$\forall s, \forall t : T_{s+\tau,s} = T_{t+\tau,t} =: T_\tau.$$

In P5 we need H4, which implies that the set of possible states is the same at each time: $\forall s, \forall t : \mathcal{S}(\mathcal{A}_s) = \mathcal{S}(\mathcal{A}_t)$.

P6: *Measurability:* The time differences form an interval of real numbers. The expectation values form measurable functions:

$$\forall \tau, \forall a \in \mathcal{A}, \forall w \in \mathcal{S}(\mathcal{A}) : \tau \mapsto (T_\tau w)(a) \qquad \text{is measurable.}$$

The observables can be represented by operators on the Hilbert space $\mathcal{H}$, each state is represented by a density matrix. Since the representation of the algebra $\mathcal{A}$ is irreducible, the density matrix $\rho_w$ associated to the state $w$ is unique. Expectation values are equal to the trace:

$$\forall\, w \in \mathcal{S}(\mathcal{A}) \exists\, \rho_w\ \forall\, a \in \mathcal{A} : w(a) = \mathrm{Tr}\ \rho_w a.$$

The postulates are necessary and sufficient to imply the standard form of time-evolution: There is a self-adjoint operator $H$, the Hamiltonian, unique except for an additive constant, generating a unitary group $U_\tau := \exp(-iH\tau/\hbar)$, which represents the time-evolution:

$$\forall\, t, \tau, w, a : w_{t+\tau}(a) = \mathrm{Tr}\ e^{-iH\tau/\hbar} \rho_w e^{iH\tau/\hbar} a.$$

## 3 Sufficiency

There is no postulate on "quantum determinism" (that pure states remain pure), no postulate on $w^*$-continuity (that closely neighbouring states remain neighbouring), and no postulate on continuity of evolution (that $t \to w_t$ is a continuous function). All these properties of the Schrödinger equation are consequences of the stated postulates.

The proof has *three steps*. The *first one* is a special case of the first half of a theorem of Kadison [Kad], which is closely related to a theorem of Wigner (appendix to chapter 20 of [Wig]). Here the Kadison theorem is not needed in its most general form in the algebraic setting; because of H2 we are interested in the standard Hilbert space setting: For this case there is a more elementary proof by Hunziker [Hun]. Only the postulates P1, P2, P3 are used, since we analyze the "symmetry transformation" $T_{t,s}$ for fixed $s$ and $t$.

Proof of "quantum determinism": $T_{t,s}^{-1}$, the inverse of the affine mapping $T_{t,s}$, is affine. If $w_s$ is pure, $w_t = T_{t,s} w_s$, $w_t = \lambda u_t + (1-\lambda) v_t$, $\lambda \in (0,1)$, then (using P1 and P3)

$$w_s = T_{t,s}^{-1} w_t = \lambda T_{t,s}^{-1} u_t + (1-\lambda) T_{t,s}^{-1} v_t$$

implies $T_{t,s}^{-1} u_t = T_{t,s}^{-1} v_t$, since $w_s$ is pure. Applying $T_{t,s}$ gives $u_t = v_t = w_t$, so $w_t$ is a pure state, what was to be proven.

Now see [Hun] for the proof that $T_{t,s}$ can be represented by an operator $U : \mathcal{H} \to \mathcal{H}$, which is either an isometry or an antiisometry and unique, except for a phase-factor:

$$(T_{t,s} w_s)(a) = \mathrm{Tr}\ U \rho_{w_s} U^* a.$$

The $w^*$-continuity is a consequence of this formula, as is the superposition principle for Hilbert space-vectors: If the pure states represented by the vectors $|\psi\rangle$ and $|\varphi\rangle$ evolve to pure states represented by $U|\psi\rangle$ and $U|\varphi\rangle$, then the pure state represented by $\alpha|\psi\rangle + \beta|\varphi\rangle$ evolves into a pure state represented by $\alpha U|\psi\rangle + \beta U|\varphi\rangle$.

Hunziker's proof makes use of affinity P1 and quantum determinism, which implies partial reversibility P2 but may hold without full reversibility. Using now again both P2 and P3, full reversibility, we infer that $U$ has to be invertible, hence either unitary or antiunitary.

In the *second step* of the sufficiency proof we need the group-property. Divisibility and time translation invariance, P4 and P5, give a semigroup property: $T_\sigma T_\tau = T_{\sigma+\tau}$. We define $T_0 =$ identity, and so P2 and P3 allow for completion of the group property, with

$$T_{-\tau} := T_\tau^{-1}.$$

The representation of the states as density matrices and observables as operators enables the extension of taking expectation values to all of $\mathcal{B}(\mathcal{H})$, especially to Hilbert-Schmidt operators. The property of weak measurability, stated in P6, is transfered to all $a$ in the Hilbert-Schmidt class $\mathcal{HS}$, since the limits of measurable functions are measurable. On the space $\mathcal{HS}$ the $T_\tau$ act as a group of unitaries, and a theorem of von Neumann ([vNeu], theorem VIII.9 in [RSi]) states, that the $T_\tau$ act as a strongly continuous group. (Here we need the separability, H3.) The details are given in Appendix A.

In the *third step* of our proof the continuity property of $T_\tau$ is transfered together with the group property to the operators $U_\tau$ acting on $\mathcal{H}$. Since $U_{2\tau} = U_\tau^2$ and since the product of two antiunitaries is unitary (as is the product of unitaries), $U_{2\tau}$ is a unitary operator, whenever $\tau$ appears as a time difference.

For a discrete set of time differences $\{\tau = n\tau_0, n \in \mathbf{Z}\}$ there is no problem in transfering the group property. Choose any phase for the representative $U_{\tau_0}$ and define $U_{n\tau_0} = U_{\tau_0}^n$. For continuously varying $\tau \in \mathbf{R}$ one may proceed step by step to smaller time-differences by choosing one of the two possible representatives $U_{\tau/2}$ with the property $U_{\tau/2}^2 = U_\tau$. With the right choice of the phases it is then possible to extend the group property from the dense set $\{b\tau_0$, $b$ a binary digit with finite length$\}$ continuously to all $\tau \in \mathbf{R}$. The details are given in Appendix B.

## 4 Necessity

The usual "necessity" is the implication that a time evolution governed by the Schrödinger equation really does have the properties P1 – P6. This is standard: P1 is linearity, P2 – P5 is the group property, and P6 is a consequence of the strong continuity of $U_\tau = \exp(-i\tau H/\hbar)$. But we will show that each postulate is also necessary in another sense: dropping it from the list would destroy the sufficiency of the rest. This is demonstrated by examples of toy-models, which violate just one of the postulates but not the others, and which are not representable by a continuous unitary group:

¬P1: A non-linear time-evolution: $\mathcal{H} = \mathbf{C}_2$, states are represented by density matrices

$$\rho = \begin{pmatrix} r & z \\ z^* & 1-r \end{pmatrix}, \qquad r \in [0,1], \qquad |z|^2 \le r(1-r).$$

The time-evolution is

$$T_\tau : \rho \mapsto \rho_\tau = \begin{pmatrix} r & ze^{ir\tau} \\ z^*e^{-ir\tau} & 1-r \end{pmatrix}.$$

¬P2: Destruction in a localized trap:

$$\mathcal{H} = \mathcal{L}^2(\mathbf{R}_-, dx) \oplus \mathbf{C}, \qquad \begin{pmatrix} \psi(x) \\ c \end{pmatrix} \in \mathcal{H} : \left\| \begin{pmatrix} \psi \\ c \end{pmatrix} \right\|^2 = \int_{-\infty}^0 |\psi(x)|^2 dx + |c|^2.$$

A vector in Hilbert space is defined by a wave function on the negative half line and in general with an admixture of "vacuum" vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. The time-evolution is

$$T_\tau : \rho \mapsto \rho_\tau = S_\tau \rho S_\tau^* + \Pi_\tau(\rho)\widehat{\rho},$$

where

$$S_\tau : \begin{pmatrix} \psi(x) \\ c \end{pmatrix} \mapsto \begin{pmatrix} \psi(x-\tau) \\ c \end{pmatrix},$$

(a right-shift of the wave function),

$$\Pi_\tau \left( \left| \begin{matrix} \psi(x) \\ c \end{matrix} \right\rangle \left\langle \begin{matrix} \varphi(x) \\ d \end{matrix} \right| \right) := \int_{-\tau}^{0} \varphi^*(x)\psi(x)dx,$$

(a partial trace of the component lost by the shift),

$$\widehat{\rho} := \left| \begin{matrix} 0 \\ 1 \end{matrix} \right\rangle \left\langle \begin{matrix} 0 \\ 1 \end{matrix} \right|,$$

(the vacuum state). Time-evolution shifts the particle on the negative half line to the right. If it encounters the boundary at $x = 0$ it is destructed and the lost part is replaced by the vacuum. This time-evolution is surjective but not injective.

$\neg$P3: Approach to equilibrium:

$$\dim \mathcal{H} \geq 2, \qquad \frac{d}{dt}\rho_t = \widehat{\rho} - \rho_t, \qquad T_\tau \rho = e^{-\tau}\rho + (1 - e^{-\tau})\widehat{\rho}.$$

This time-evolution is injective but not surjective. It can be inverted by

$$T_\tau^{-1}\rho = e^{\tau}\rho + (1 - e^{\tau})\widehat{\rho},$$

but this mapping leads out of the set of positive density matrices, unless $\rho$ is already in the range of $T_\tau$ (we always consider positive time differences). This is an example of "partial reversibility" in the absence of full reversibility.

Another example violating surjectivity is:

$$\mathcal{H} = \mathcal{L}^2(\mathbf{R}_+, dx), \qquad T_\tau : \rho \mapsto U_\tau \rho U_\tau^*,$$

where $U_\tau$ is only a partial isometry:

$$(U_\tau \psi)(x) = \psi(x - \tau).$$

The second example is a case obeying quantum determinism, the first one is not.

$\neg$P4: A hidden driving force:

$$\dim \mathcal{H} \geq 2, \qquad H = H^* \neq const.\mathbf{1}, \qquad \hbar = 1,$$

$$T_\tau : \rho \mapsto U_\tau \rho U_\tau^*, \qquad U_\tau := e^{-iH\sin\tau}.$$

These unitaries do not form a group.

¬P5: A time dependent Hamiltonian $H(t)$:

$$T_{t,s}\rho_s = U_{t,s}\rho_s U_{t,s}^*, \qquad \hbar\frac{d}{dt}U_{t,s} = iH(t)U_{t,s}, \qquad U_{s,s} = \mathbf{1}.$$

To avoid mathematical subtleties in integrating the differential equation, we consider an example, where all $H(t)$ commute:

$$H(t) = A + 2Bt, \qquad [A, B] = \emptyset.$$

Then

$$U_{t,s} = \exp[i(A(t-s) + B(t^2 - s^2))/\hbar].$$

¬P6: Non-measurable dynamics: By referring to the axiom of choice, we can state that there exists a set of real numbers $\{e_\alpha\}$ which are mutually irrational ($\alpha \neq \beta \Rightarrow e_\alpha/e_\beta$ is not a rational number), such that each real number $\tau$ can uniquely be represented as

$$\tau = \sum_{\alpha \text{ finite}} r_\alpha e_\alpha, \qquad r_\alpha \text{ rational.}$$

Now we take for each $\alpha$ some Hamiltonian $H_\alpha$, such that all $H_\alpha$ commute (f.e. only one $H_\alpha \neq 0$), and define

$$U_{r_\alpha e_\alpha} := e^{-ir_\alpha e_\alpha H_\alpha/\hbar}, \qquad U_\tau := \prod_\alpha U_{r_\alpha e_\alpha}.$$

These unitary operators form a discontinuous group, the affine mappings

$$T_\tau : \rho \mapsto U_\tau \rho U_\tau^*$$

also form a discontinuous group, and $s + \tau = t \mapsto \mathrm{Tr}\ U_\tau \rho_s U_\tau^* a$ is in general not measurable. In section 6 there will be a special epistemological remark on this example.

## 5 The Heisenberg Scheme

Here we look at linear mappings $S$ of the algebras of observables. Such mappings can be transposed to mappings of the dual spaces, the spaces $\mathcal{L}(\mathcal{A})$ of linear functionals, which contain the states $\mathcal{S}(\mathcal{A})$

$$\begin{array}{ccccc}
 & \mathcal{A}_t & & & \\
 & & \ell_t \in \mathcal{L}(\mathcal{A}_t) & & \\
S_{s,t} & & S_{s,t}^\dagger & & \mathbf{C} \\
 & & \ell_s \in \mathcal{L}(\mathcal{A}_s) & & \\
 & \mathcal{A}_s & & &
\end{array}$$

$\ell_t = S_{s,t}^\dagger \ell_s$ is defined by $(S_{s,t}^\dagger \ell_s)(a_t) := \ell_s(S_{s,t}a_t)$. In order to map the states $\mathcal{S}(\mathcal{A}_s) \subset \mathcal{L}(\mathcal{A}_s)$ to states $\mathcal{S}(\mathcal{A}_t)$, it is necessary that $S_{s,t}$ maps positive observables to positive observables and the unity to the unity.

Now there is a feature which is not so obvious in the standard treatment of quantum mechanics: in order to predict expectation values at later times, the mapping of the algebra has to act *backwards* in time! If I know the state $w_s$ of the system at time $s$ and want to predict the expectation value of an observable $a$ at a later time $t$, I have either to map the state $w_s$ onto a state at time $t$, or I have to map the observable $a \in \mathcal{A}_t$ to an observable of the algebra $\mathcal{A}_s$ by $S_{s,t}$; then I can calculate $w_s(S_{s,t}a)$. If one wants to drop the hypothesis H4 (as may probably be the case in systems whose space changes in time because of a moving horizon) this fact is not to be neglected. In standard quantum mechanics it is somehow disguised, because the set of observables and the Hilbert space of a system remain unchanged in time. It only surfaces in systems with time-dependent Hamiltonians or in the interaction representation, since

$$\frac{d}{dt}a(t) = \frac{d}{dt}U^*_{t,s}aU_{t,s} = \frac{i}{\hbar}U^*_{t,s}[H(t), a]U_{t,s},$$

which means that an additional infinitesimal shift has to act on the observable *before* it is translated back into the past.

One problem in the Heisenberg scheme is the right choice of the algebra. The $C^*$-algebra generated by the Weyl operators is too small for Hamiltonians containing not only linear and quadratic terms of coordinate and momenta [FVe]. In ordinary quantum mechanics this problem can easily be solved. It makes sense to declare each operator which can be approximated by operators of the original algebra as observable. So one enlarges the algebra to its weak closure in the (unique) irreducible representation, which equals $\mathcal{B}(\mathcal{H})$, to make sure that the dynamics can be defined as a mapping of observables.

The hypothesis H5, defining the concept of time-evolution, has in the Heisenberg scheme to be replaced by:

H5$_{\mathrm{H}}$: For the time $t$ later than $s$, there corresponds to each observable $a_t \in \mathcal{A}_t$ one and only one observable $a_s = S_{s,t}a_t \in \mathcal{A}_s$. The mapping $S_{s,t} : \mathcal{A}_t \to \mathcal{A}_s$ represents the time-evolution. It is linear:

$$S_{s,t}(a + \alpha b) = S_{s,t}a + \alpha S_{s,t}b,$$

it maps positive observables to positive observables and the unity to the unity.

These properties are stated in the hypothesis (which is a kind of definition) since a violation would contradict the fundamental meaning of unity, positivity, or summation.

The postulate P1 is fulfilled by the transposed mapping automatically. P2 and P3, which together imply the full reversibility of $S^\dagger_{s,t}$ have to be replaced by

P2$_{\mathrm{H}}$: *Surjectivity:* Each element $a_s \in \mathcal{A}_s$ is the image of some $a_t \in \mathcal{A}_t$ under the mapping $S_{s,t}$.

P3$_{\mathrm{H}}$: *Homomorphicity:* The mapping $S_{s,t}$ is an algebraic *-homomorphism:

$$S_{s,t}(ab) = (S_{s,t}a)(S_{s,t}b)$$

$$S_{s,t}(a^*) = (S_{s,t}a)^*.$$

Together with the uniqueness of the representation of $\mathcal{A}_t$, stated in H2, the postulate P3$_{\mathrm{H}}$ implies the injectivity of $S_{s,t}$: Any state $w_s \in \mathcal{A}_s$ corresponds to a state $w_t \in \mathcal{A}_t$ by

$$w_t(a) := w_s(S_{s,t}a),$$

with the property
$$w_t(dac) = w_s((S_{s,t}d)(S_{s,t}a)(S_{s,t}c)).$$

If $a \neq b$, there must exist $d$ and $c$, such that $w_t(dac) \neq w_t(dbc)$, because of the uniqueness of the irreducible components of each representation, which implies that $w_t$ may be represented by a density matrix in a Hilbert space $\mathcal{H}$, where the representation of the algebra $\mathcal{A}$ is dense in $\mathcal{B}(\mathcal{H})$. Injectivity of $S_{s,t}$ is the consequence of the property stated above: $S_{s,t}(a) \neq S_{s,t}(b)$. Considering the transpose $S^{\dagger}_{s,t} =: T_{t,s}$, $\mathrm{P2_H}$ implies the injectivity P2 and $\mathrm{P3_H}$ via injectivity of $S_{s,t}$ the surjectivity of $T_{t,s}$, P3. Since P1 holds, we can apply the Kadison theorem: $T_{t,s}$ and hence $S_{s,t}$ can be represented by unitary or antiunitary transformations $U_{t,s}$:

$$S_{s,t}(a) = U^*_{t,s} a U_{t,s}.$$

The three remaining postulates P4, P5, P6 can be reformulated as follows:

$\mathrm{P4_H}$: *Divisibility:* $u > t > s \Rightarrow S_{s,u} = S_{s,t}S_{t,u}$.

$\mathrm{P5_H}$: *Invariance under time-translation:* The time differences form a semigroup under addition and
$$\forall\, s, \forall\, t, \forall\, \tau : S_{s,s+\tau} = S_{t,t+\tau} =: S_\tau.$$

$\mathrm{P6_H}$: *Measurability:* The time differences form an interval of real numbers, and
$$\forall\, \tau, \forall\, a \in \mathcal{A}, \forall\, w \in \mathcal{S}(\mathcal{A}) : \tau \mapsto w(S_\tau a) \qquad \text{is a measurable function.}$$

## 6 Philosophical Remarks

Having established the hypotheses and postulates as the mathematical foundation of the Schrödinger equation, one may ask for further epistemological underpinnings. This does not exclude violations of the postulates in limiting cases, as ¬P1 in soliton theory, ¬P2 and ¬P3 in non-equilibrium thermodynamics, ¬P4 at non-Markowian diffusion equations, ¬P5 in the presence of time-dependent external fields. These are limiting situations of systems in interaction with some other systems. Here we are interested in small isolated systems for short times, making an idealization in ignoring A1 and not investigating A7.

In classical mechanics it has been considered as possible to make each measurement with any desired sharpness, without disturbing the object. Determinism then guaranteed, that the object will be and has been in pure states at all times. In the early days of quantum mechanics this view has been transfered to the new physics. There was little doubt that each system is "in reality" in a pure state, described by a wave function. Mixed states were used to describe the limited knowledge of larger systems. This was probably the unspoken reason to believe in "quantum determinism".

Being acquainted with the mathematics of quantum mechanics, with the EPR paradox, with Bell's theorem and its experimental verifications, we have to realize that often a subsystem of a larger one, even if the total is in a pure state, will in reality be in a mixed state. So the preference of pure states as a reason for quantum determinism is not unquestionable, we should search for other underpinnings. The following remarks to the postulates might give some hints:

RP1: A proposal of St. Weinberg to make precision measurements of possible nonlinearities in the Schrödinger equation [Wei], initiated a closer theoretical analysis of nonlinearities, leading to the conclusion that any nonlinearity would turn the EPR paradox to a real violation of relativistic locality [Pol]. It seems hard to fight for a nonlinear Schrödinger equation against this argument.

RP2: I don't know of any separate argument in favor of partial reversibility, except that by experience we know that the past of a small system may in principle be calculated with similar accuracy as the future. But unification of P2 and P3 gives full reversibility, and this might have its origin in the CPT invariance, A4.

RP3: There also seems to be no separate argument in favor of surjectivity. There are actually proposals for a time-evolution violating P3, [GRW], aiming to change the puzzling theories of measurement, A7. But, as stated above, this might contradict A4. I propose to investigate this problem more closely.

RP4: If a violation of divisibility should occur, it could probably be removed by enlarging the algebra of observables, preferably by measuring the time-derivatives, or some hitherto unnoticed degrees of freedom. In this way classical mechanics lifts the Zeno-paradox of the arrow which cannot move, because in every moment it is resting at some fixed point. Zeno considered only the observable location $x$, Galilei and Newton added the velocity $\dot{x}$.

RP5: Time translation invariance is a statement about the flow of time in the universe. It must have its origin in A1.

RP6: The measurability postulate might be dropped if one is willing to adopt alternate foundations of measure theory [Sol]. In constructive mathematics there exists no proof for the existence of a function which is not measurable, whereas the important tools of Banach space and Hilbert space theory are ready for use [Bis]. The example $\neg$P6 is nonconstructive, its presumed "existence" relies on the axiom of choice, which is independent of the other axioms of set theory, as is the continuum hypothesis [Coh63]. We may also keep in mind that one should consider only physical theories about measurable quantities. And there is probably a relation between mathematical and experimental measurability. I do not know about a deeper investigation, but there is an intuitive feeling on both sides for a link between the foundations of mathematics and the foundations of physics [Coh71], [Whe].

We might also drop the assumption about $t$ varying continuously. The constructions in Appendix B tell us what would happen. Instead of an energy we would have a quasienergy, defined only modulo $h/\Delta t$ ($\Delta t$ the shortest time difference, $h = 2\pi\hbar$). The situation would be analogous to the quasimomenta $q$ of phonons in solids, defined only modulo $2\pi/a$ ($a$ a lattice constant): at low temperatures, when only phonons with small $q$ appear, the quasimomenta behave like ordinary momenta. Only when two or more phonons with large $q$ collide, such that the sum exceeds $2\pi/a$, an Umklapp process may happen, changing the total quasimomentum by $2\pi/a$. In particle physics, the analogue of this effect would happen, if particles collide, with a sum of energies exceeding $h/\Delta t$. If $\Delta t$ is on the scale of the Planck-time $(\hbar G/c^5)^{1/2}$, then this would afford energies exceeding $10^{28}$ eV. At the moment we are still far away from this regime, and are unable to decide, whether time is continuous or discrete.

## Appendix A: A Variation of von Neumann's Theorem

**Theorem:** Let $\mathcal{A}$ be a $C^*$-algebra with a unique irreducible representation in a separable Hilbert space $\mathcal{H}$. Let $\{T_\tau | \tau \in \mathbf{R}\}$ be a one-parameter group of affine invertible mappings $\mathcal{S}(\mathcal{A}) \to \mathcal{S}(\mathcal{A})$, such that $\forall\, w \in \mathcal{S}(\mathcal{A})$, $\forall\, a \in \mathcal{A}$ the function $\tau \mapsto (T_\tau w)(a)$ is measurable, then $\tau \mapsto T_\tau w$ is strongly continuous.

**Proof:**

(a) The first part of Kadison's theorem implies that there is a transposed mapping $S_\tau : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$, namely $S_\tau : a \mapsto U_\tau^* a U_\tau$, $U_{-\tau} = U_\tau^* = U_\tau^{-1}$, such that $(T_\tau w)(a) = w(S_\tau a)$. Considering the space of operators of the Hilbert-Schmidt class $\mathcal{HS}$ as a Hilbert space, with the scalar product
$$\langle b|a\rangle_{\mathcal{HS}} := \mathrm{Tr}\ b^* a,$$
the mapping $S_\tau$ acts as a unitary operator in $\mathcal{HS}$, since
$$\langle b|S_\tau a\rangle_{\mathcal{HS}} = \mathrm{Tr}\ U_\tau b^* U_\tau^* a = \langle S_{-\tau} b|a\rangle_{\mathcal{HS}}$$
and $S_{-\tau} = S_\tau^{-1}$. Acting on density matrices, which represent the states and form a subset of the Hilbert-Schmidt operators, $S_{-\tau}$ is a representation of $T_\tau$:
$$v = T_\tau w \Rightarrow S_{-\tau}\rho_w = \rho_v.$$
The measurability of the function $\tau \mapsto (T_\tau w)a$ extends to the limits $a_n \to a \in \mathcal{HS}$ and $\rho_w \to b \in \mathcal{HS}$. Since also $\mathcal{HS}$ is separable (if the $|\psi_n\rangle$ are a countable basis of $\mathcal{H}$, then the $a_{mn} = |\psi_m\rangle\langle\psi_n|$ are a countable basis of $\mathcal{HS}$), von Neumann's theorem [vNeu] holds: $\tau \mapsto S_\tau a$ is strongly continuous in the $\mathcal{HS}$ norm.

(b) In Appendix B this continuity will be used, with $a = |\psi\rangle\langle\psi|$, to infer the possibility of constructing the representatives $U_\tau$ as a strongly continuous unitary group. This implies strong continuity of $T_\tau$: consider a pure state $w$, represented by the Hilbert space vector $|\varphi\rangle$
$$\|w - T_\tau w\| = \sup_{a\in\mathcal{A}} \frac{|w(a) - (T_\tau w)(a)|}{\|a\|} = \sup_{a} \frac{\langle\varphi|a|\varphi\rangle - \langle\varphi U_\tau^* a U_\tau \varphi\rangle|}{\|a\|} \le 2\|\varphi - U_\tau\varphi\| \overset{\tau\to 0}{\longrightarrow} 0.$$
So $T_\tau$ acts continuously on pure states, hence on finite superpositions of pure states, which are dense in $\mathcal{S}(\mathcal{A})$, and the continuity extends to all states.

## Appendix B

With standard Hilbert space methods we prove the second half of Kadison's theorem for the special case, where the algebra of bounded operators on a Hilbert space is involved:

**Theorem:** Given a one parameter group of mappings of the states of $\mathcal{B}(\mathcal{H})$, $\mathcal{H}$ a Hilbert space,

$$T_\tau : w \mapsto \mathcal{T}_\tau w, \qquad T_0 = \text{identity}, \qquad \forall \{\sigma, \tau\} \subset \mathbf{R} : T_\sigma T_\tau = T_{\sigma+\tau},$$

where each mapping can be represented by a unitary transformation,

$$\forall \tau \in \mathbf{R}\ \exists U_\tau = U^*_{-\tau} = U^{-1}_{-\tau} \quad \forall a \subset \mathcal{B}(\mathcal{H}) : (T_\tau w)(a) = w(U_{-\tau} a U_\tau),$$

and which acts weakly continuous on pure states applied to finite rank operators:

$$\text{for } a \text{ of finite rank, } w \text{ pure, } \tau \mapsto (T_\tau w)(a) \text{ is a continuous function,}$$

then the unitary operators $U_\tau$ can be chosen in such a way that they form a strongly continuous group,

$$U_0 = \mathbf{1}, \quad \forall \{\sigma, \tau\} \subset \mathbf{R} : U_\sigma U_\tau = U_{\sigma+\tau}, \quad \forall \psi \in \mathcal{H} : \tau \mapsto \|U_\tau \psi\| \text{ is a continuous function.}$$

**Proof:**

(a) We choose some reference vector $|\psi\rangle \in \mathcal{H}$, fixed throughout the proof. Since for any operator $a = |\psi\rangle\langle\varphi|$ and all pure states represented by the vectors $|\psi + i^n \eta\rangle$ the expectation values of $U_{-\tau} a U_\tau$ are continuous, we know that

$$\frac{1}{4} \sum_{n=0}^{3} i^{-n} \langle \psi + i^n \eta | U_{-\tau} a U_\tau | \psi + i^n \eta \rangle = \langle \psi | U_{-\tau} | \psi \rangle \langle \varphi | U_\tau | \eta \rangle$$

is a continuous function. If we have constructed a group $U_\tau$ such that $\tau \mapsto \langle \psi | U_\tau | \psi \rangle$ is continuous (that will be achieved in (b) to (g)), then we know that there is some interval around $\tau = 0$, where $\langle \psi | U_\tau | \psi \rangle \neq 0$, and therefore also $\langle \varphi | U_\tau | \eta \rangle$ is continuous. This property holds for all vectors $|\varphi\rangle$, $|\eta\rangle$, extends to all $\tau \in \mathbf{R}$ by the group property and implies the strong continuity.

(b) Setting $|\varphi\rangle = |\eta\rangle = |\psi\rangle$ in the formula in (a), we see that $|\langle \psi | U_\tau | \psi \rangle|^2$ is a continuous function and is independent of the chosen phases for $U_\tau$. For some $\varepsilon$, which shall be smaller than $2^{-10}$, there exists a $\tau_0 > 0$, such that

$$\forall \{\sigma, \tau\} \subset [-\tau_0, \tau_0] : ||\langle \psi | U_\sigma | \psi \rangle| - |\langle \psi | U_\tau | \psi \rangle|| \leq \varepsilon < 1/2^{10}.$$

(c) $U_{\tau_0}$ considered as fixed, there is a phase $\alpha \in [-\pi, \pi]$, such that for

$$V_0 := e^{-i\alpha} U_{\tau_0}$$

the expectation value $\langle \psi | V_0 | \psi \rangle$ is real and positive. Two of the representatives for $T_{\tau_0/2}$ are square roots of $V_0$. We choose the one with non-negative real part of the expectation value and denote it $V_1$. In this way we go on and construct a family $\{V_n\}$:

$$V_{n+1}^2 = V_n, \qquad \text{Re } \langle \psi | V_n | \psi \rangle \geq 0.$$

Since $w(V_n^{-1} a V_n) = (T_\tau w) a$ for $\tau = 2^{-n} \tau_0$, we have constructed a group of unitaries $U_\tau$, representing $T_\tau$ for all time differences $\tau = b\tau_0$, where $b$ is a binary digit with finite length:

$$b = b_I + \sum_{n=1}^{N} \delta_n 2^{-n}, \qquad b_I \in \mathbf{Z}, \qquad \delta_n \in \{0, 1\},$$

$$U_{b\tau_0} := e^{ib\alpha} V_N^{(2^N b)}.$$

(d) Next we have to show that the continuity of the absolute value of $\langle\psi|U_\tau|\psi\rangle$ implies a spectral concentration of $|\psi\rangle$. We represent each $V_N$ as the exponential of a quasi-Hamiltonian $H_N$ (here we assume units, where $\tau_0/\hbar = 1$):

$$V_N = \exp(-i2^{-N}H_N), \qquad -2^N\pi < H_N \le 2^N\pi, \qquad \text{for } n \le N : V_n = \exp(-i2^{-n}H_N).$$

We conclude from (b) and the definition of $V_0$ that $\langle\psi|V_0|\psi\rangle \ge 1-\varepsilon$, and, using the spectral representation of $H_N$,

$$\langle\psi|f(H_N)|\psi\rangle = \int f(E)d\mu_N(E),$$

that

$$1-\varepsilon \le \langle\psi|V_0|\psi\rangle = \langle\psi|\cos H_N|\psi\rangle \le \sum_{n,|n|\le 2^{N-1}} a_n + (1-\sum_n a_n)\cos\vartheta,$$

where

$$a_n := \int_{2\pi n-\vartheta}^{2\pi n+\vartheta} d\mu_N(E).$$

So $\quad 1 \ge \sum_n a_n \ge 1-\varepsilon/(1-\cos\vartheta) > 1-\sqrt{\varepsilon} \qquad$ for $\vartheta = 2\varepsilon^{1/4}$.

(e) We specify the spectral concentration further, using $\varepsilon \ge 1-\langle\psi|V_N^{-\nu}|\psi\rangle\langle\psi|V_N^{\nu}|\psi\rangle$ for all $\nu \le 2^N-1$:

$$\begin{aligned}
1-\varepsilon &\le \frac{1}{2^N}\sum_{\nu=0}^{2^N-1}\langle\psi|\exp(i\nu 2^{-N}H_N)|\psi\rangle\langle\psi|\exp(-i\nu 2^{-N}|\psi\rangle = \\
&= \frac{1}{2^N}\sum_\nu \iint \exp(i\nu 2^{-N}(E-E')d\mu_N(E)d\mu_N(E') = \\
&= \iint \frac{1}{2^N}\sum_\nu \cos(\nu 2^{-N}(E-E'))d\mu_N(E)d\mu_N(E') = \\
&= \iint \cos\left((1-2^{-N})\frac{E-E'}{2}\right)\frac{\sin\frac{E-E'}{2}}{2^N\sin 2^{-N}\frac{E-E'}{2}}d\mu_N(E)d\mu_N(E') \le \\
&\le \sum_n a_n^2 + \frac{\vartheta}{\pi}\sum_{n\ne n'} a_n a_n' + \varepsilon \le a_{\max}^2 + (1-a_{\max})^2 + \varepsilon^{1/4} + \varepsilon.
\end{aligned}$$

The last two inequalities are consequences of (d) and the convexity of the function $a \mapsto a^2$. Consequently, for the maximum of the concentration weights $a_n$:

$$(1-a_{\max})^2 \le \varepsilon^{1/4}$$

$$a_{\max} \ge 1-\varepsilon^{1/8} > 3/4.$$

With this inequality, the definition of the $V_n$, and their representations as functions of $H_N$, we conclude that $a_{\max} = a_0$.

(f) We consider the phases:

$$\forall n < 2^N : |\mathrm{Im}\,\langle\psi|V_N^n|\psi\rangle| \le 2\varepsilon^{1/4} + \varepsilon^{1/8} \le 2\varepsilon^{1/8}.$$

Combined with the inequality for the absolute value in (b):

$$\|\psi - V_N^n\psi\|^2 \le (\varepsilon^2 + 4\varepsilon^{1/4})^{1/2} \le 5\varepsilon^{1/4}.$$

$N$ was finite, but arbitrary, so

$$|b| \le 1 \Longrightarrow \|\psi - U_{b\tau_0\psi}\| \le |b\alpha| + 3\varepsilon^{1/8}.$$

By unitary equivalence

$$|b_1 - b_2| \le 1 \Longrightarrow \|U_{b_1\tau_0}\psi - U_{b_2\tau_0}\psi\| \le |(b_1 - b_2)\alpha| + 3\varepsilon^{1/8},$$

and, since $|\alpha| \le \pi$,

$$\|U_{b_1\tau_0}\psi - U_{b_2\tau_0}\psi\| \le 4\varepsilon^{1/8} \qquad \text{if } |b_1\tau_0 - b_2\tau_0| \le \varepsilon\tau_0(\varepsilon).$$

(g) Now for any smaller $\varepsilon$ we can do the same argument starting with some $\tau_0(\varepsilon) = b\tau_0$ instead of $\tau_0$. Because of the inequalities already proven for the first chosen $\varepsilon$, the $U_{b\tau_0}$ will not change in the construction (c). So we have proven that $\tau \mapsto \langle\psi|U_\tau|\psi\rangle$ is equicontinuous for $\{\tau = b\tau_0\}$ (which is dense in $\mathbf{R}$), and can thus be extended to a continuous function for all real $\tau$.

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