The (non–)Gibbsian nature of states invariant under stochastic transformations

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Abstract

We investigate the Gibbsian nature of systems that are invariant under some stochastic transformation. A prime example concerns stationary measures for interacting particle systems. We bring together various types of results concerning this question using entropy techniques and large deviation estimates.

Keywords: entropy inequalities, non–Gibbsianess, large deviations, stationary states, stochastic transformations, Toom model, restrictions of equilibrium measures.

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1 Introduction.

Equilibrium statistical mechanics has to do with the study of Gibbs states. They describe large systems in equilibrium for some given microscopic interaction potential. A lot is known about them, in particular for lattice spin systems, due to the rather explicit construction of the state. In this paper we ask for the Gibbsian nature of states on lattice spin configurations that are invariant under certain stochastic transformations. Understandably, invariance is a rather implicit way of defining the physical state and yet, this happens quite often to be the only information we have about it. Consider for example the problem of determining the character of the steady state in models of non–equilibrium statistical mechanics. We have in mind here the study of the stationary states of certain stochastic dynamics, such as interacting particle systems or probabilistic cellular automata. Given are the updating rules that define the continuous time or discrete time stochastic evolution and the question is whether a state that after updating is transformed back into itself, is a Gibbs state for some (and what) interaction. It should be clear in that respect that there is more to the question of "Gibbsianess" than being able to take the logarithm of the probabilities. One wants to see if in some *bulk* limit, the system can be consistently described in terms of a relative energy function. This could for example be related to the type of large deviations one sees. A detailed account of the relevance and the consequences of these type of considerations can be found in [1].

This paper grew out of the question how to unify various results in the literature concerning the nature of states that are not defined in terms of some explicit interaction potential. Crucial in these questions is the influence of applying stochastic transformations on the relative entropy density of two probability measures, defined on spin systems on a lattice. We deduce that for the type of transformations that we consider, the translation invariant stationary measures either are all Gibbsian with respect to the same interaction or either are non–Gibbsian. In the context of Markov spinflip processes on a lattice, a similar statement has been proven by Holley [2] and Künsch [3].
This result is important not only to get a characterization of so called *interacting particle systems*, but also, to learn more about the restrictions to a lower dimensional sublattice of certain Gibbs measures. Indeed, some equilibrium systems can be seen as Markov extensions of stationary measures for some transition operator (or, having a *global* Markov property) [4]. In fact, stationary measures of discrete time Markov processes can also be obtained from a projection of corresponding equilibrium systems on the space-time lattice [5]. A relevant question is, whether the above stationary measures and restrictions of Gibbs measures are again Gibbsian.
In the high–noise, high–temperature regime respectively, this is indeed the case. There we do not have long range order and the weak correlations make the probability of a configuration in a finite box to depend continuously on the configuration far away. For probabilistic cellular automata (PCA), discrete time Markov processes on a lattice with local and parallel updating, it was proven in [6] that for high enough noise, the unique

stationary measure is Gibbsian. A similar statement for continuous time spinflip processes was proven in [7]. The situation is also quite clear in the case when the *detailed balance condition* is satisfied ; see [8] for a discussion of the stochastic or kinetic Ising model. Very little is however known about the stationary measures of dynamics that are not reversible. A typical example is the Toom model [9], [6], on which we will come back later. As for the projections of Gibbs measures, it is also straightforward to check that these restrictions are Gibbsian at high temperature (see e.g. [10]). However, for the case of the two-dimensional ferromagnetic Ising model below the critical temperature, Schonmann [11] has proven that the restrictions of the $+$ and $-$ phase to a one-dimensional layer are both non–Gibbsian.
It would also be interesting to study fixed point measures of real space renormalization group transformations [1], but our methods are not adapted to treat these kind of transformations, because the rescaling of the lattice induces a rescaling in the entropy density as well, making the entropy density inequalities of Section 2 useless. We will come back to this at the end of the paper.
The paper is organized as follows:
In section 2 some definitions and notation is given. Thereafter the main result is presented. In Section 3 we discuss the situation for stochastic spinflip dynamics. Section 4 is devoted to a discussion of related problems.

2 Invariance for stochastic transformations.

2.1 Definitions and notation.

We will consider spin systems on the regular lattice ^{d} and on finite sublattices. For any subset $\Lambda \subset {}^d$ the configuration space is $\Omega_\Lambda = \{-1,+1\}^\Lambda$ and we write $\Omega \equiv \Omega_{d}$. The space Ω_Λ is equipped with the usual product topology and Borel σ-field $\mathcal{F}_\Lambda$. The set of probability measures on $(\Omega_\Lambda, \mathcal{F}_\Lambda)$ is denoted by $\mathcal{P}_\Lambda$ and we let $\mathcal{P} \equiv \mathcal{P}_{d}$ denote the set of probability measures on the full lattice. For σ, η in Ω, we say that $\sigma = \eta$ on Λ iff $\sigma(x) = \eta(x)$ for all x in Λ. Further we denote $\sigma(\Lambda) \equiv \prod_{x\in\Lambda}\sigma(x)$. For $x \in {}^d$ we define the translation $\tau_x : \Omega \to \Omega : \eta \mapsto \tau_x\eta$, where $\tau_x\eta(y) = \eta(y+x)$. τ_x acts on a probability measure $\mu \in \mathcal{P}$ as $\tau_x\mu(f) = \mu(f \circ \tau_{-x})$, for all measurable functions f on Ω. We call $\mu \in \mathcal{P}$ translation invariant if $\tau_x\mu = \mu$, for all $x \in {}^d$. We denote the set of translation invariant measures in $\mathcal{P}$ by $\mathcal{P}_{tr}$.

Definition 1 *For a finite set Λ and $\mu, \nu \in \mathcal{P}_\Lambda$ the relative entropy of μ w.r.t. ν is*

$$I(\mu,\nu) = \sum_{\eta\in\Omega_\Lambda} \mu(\eta) \log \frac{\mu(\eta)}{\nu(\eta)} \tag{2.1}$$

(with 0log0=0). It is not hard to verify that $I(\mu,\nu) \geq 0$ and $I(\mu,\nu) = 0$ iff $\mu = \nu$. For an infinite volume the relative entropy is almost always infinite, but for translation invariant measures we can work with the concept of relative entropy density. For $\mu \in \mathcal{P}$ and $\Lambda \subset {}^d$, let μ_Λ denote the restriction of μ on $(\Omega_\Lambda, \mathcal{F}_\Lambda)$. Further denote $I_\Lambda(\mu,\nu) = I(\mu_\Lambda, \nu_\Lambda)$. We make the following

Definition 2 *Let $\mu, \nu \in \mathcal{P}_{tr}$. Consider the sequence of boxes $\Lambda = [-L, L]^d$. The relative entropy density of μ w.r.t. ν is defined as*

$$i(\mu,\nu) = \lim_{L\uparrow\infty} \frac{1}{L^d} I_\Lambda(\mu,\nu), \tag{2.2}$$

provided this limit exists.

The existence of the limit (2.2) is a non–trivial problem. This limit is known to exist in case ν is a Gibbs measure for a translation invariant interaction [12]. We recall the definition of Gibbs measure for a translation invariant interaction.

Definition 3 *We call $\mu \in \mathcal{P}_{tr}$ a Gibbs measure w.r.t. the translation invariant interaction $\{J_V\}_{V\subset {}^d}$, if there exists a version of the local conditional probabilities of the form*

$$\mu[\sigma = \eta \textit{ on } \Lambda | \sigma = \omega \textit{ on } \Lambda^c] = Z_\Lambda^{-1}(\omega) \exp[\sum_{V\cap\Lambda\neq\emptyset} J_V\, \eta(\Lambda)\omega(V\backslash\Lambda)] \tag{2.3}$$

and $\sum_{V\ni o} |J_V| < \infty$.

With this definition, these conditional probabilities are continuous functions of $\omega \in \Omega$. Of course, there exist also other (non–continuous) versions, because these conditional probabilities are only defined μ-a.s.. We denote by $\mathcal{G}(J)$ the set of all translation invariant Gibbs measures with respect to $\{J_V\}_{V\subset {}^d}$. We will use the following results:

Lemma 1 [12] *Suppose ν is Gibbs for a translation invariant interaction, then μ is Gibbs for the same interaction iff $i(\mu,\nu) = 0$.*

Lemma 2 [13] *$\mu \in \mathcal{P}_{tr}$ is Gibbs for some translation invariant interaction iff the one point conditional probability $\mu[\sigma(o) = \eta(o) | \sigma = \omega$ on $\{o\}^c]$ is positive and has a version continuous in $\omega \in \Omega$.*

Often it is more convenient to work with relative energies instead of conditional probabilities.

Definition 4 *Let $\mu \in \mathcal{G}(J)$. The relative energy h_μ of μ for a spinflip in the origin is the continuous function defined by*

$$h_\mu(\eta) = \sum_V J_V \eta^o(V) - \sum_V J_V \eta(V) = -2 \sum_{V\ni o} J_V \eta(V), \tag{2.4}$$

where η^o is the configuration η flipped in the origin, so $\eta^o(x) = \eta(x)$ if $x \neq o$ and $\eta^o(o) = -\eta(o)$.

Combining the above definitions, we have the following relation between the relative energy and the one point conditional probabilities:

$$(1 + \exp h_\mu(\eta))^{-1} = \mu[\sigma(o) = \eta(o) | \sigma = \eta \text{ on } \{o\}^c]. \tag{2.5}$$

For μ not a Gibbs measure, a function h_μ can also be defined μ-a.s. via the relation (2.5), but it cannot be continuous.

2.2 Main result.

Finite volume inequalities Fix the finite volumes Λ, Λ' and suppose we have a pointwise strictly positive stochastic matrix P, defined by its transition probabilities

$$\begin{aligned} & p(\cdot|\cdot) : \Omega_\Lambda \times \Omega_{\Lambda'} \to [0,1] \\ & \sum_{\sigma \in \Omega_\Lambda} p(\sigma|\cdot) = 1 \\ & p(\cdot|\cdot) > 0 \end{aligned} \tag{2.6}$$

P acts on a probability measure $\mu \in \mathcal{P}_{\Lambda'}$ as

$$\mu P(\sigma) = \sum_{\eta \in \Omega_{\Lambda'}} p(\sigma|\eta)\mu(\eta), \qquad \sigma \in \Omega_\Lambda. \tag{2.7}$$

It is also convenient to define the measure $\mu \times P$ on $\Omega_\Lambda \times \Omega_{\Lambda'}$ by

$$\mu \times P(\sigma, \eta) = \mu(\eta) p(\sigma|\eta) \tag{2.8}$$

and let $\mu \times P[\sigma]$ denote the conditional probability of $\mu \times P$ given σ on Ω_Λ, i.e.

$$\mu \times P[\sigma](\eta) = \frac{\mu \times P(\sigma, \eta)}{\sum_{\eta'} \mu \times P(\sigma, \eta')}.$$

We remind the reader of the following

Lemma 3 *Let $\mu, \nu \in \mathcal{P}_{\Lambda'}$, P as in (2.6), then*

$$I(\mu, \nu) = I(\mu P, \nu P) + \sum_{\sigma \in \Omega_\Lambda} \mu P(\sigma)\, I(\mu \times P[\sigma], \nu \times P[\sigma]). \tag{2.9}$$

Proof:

In case μ is not absolutely continuous w.r.t. ν, both sides equal $+\infty$. Otherwise let $f(\eta) = \mu(\eta)/\nu(\eta)$ and $g(\sigma) = \mu P(\sigma)/\nu P(\sigma)$ for $\eta \in \Omega_{\Lambda'}, \sigma \in \Omega_\Lambda$. Then the identity follows from the observation that
$\mu \times P[\sigma](\eta)/\nu \times P[\sigma](\eta) = f(\eta)/g(\sigma)$.

$\square$

Local stochastic operators In this paragraph we deal with stochastic operators P on the full lattice ^{d}. They are defined by their kernel $P(\cdot|\cdot)$. That is, $P(\cdot|\eta) \equiv P^\eta(\cdot) \in \mathcal{P}$ for every $\eta \in \Omega$ and $P(A|\cdot) \in \mathcal{F}$ for every $A \in \mathcal{F}$. Let Λ be any translation of $[-L, L]^d$, i.e. $\Lambda = [-L, L]^d + x$ for some $x \in {}^d$. We suppose that P satisfies the following conditions:

1. $P_\Lambda(\eta|\omega) \equiv P(\sigma = \eta \text{ on } \Lambda|\omega)$ depends on ω only through its coordinates in $\Lambda' = [-L-R, L+R]^d + x$, where $R > 0$ is a constant. We can thus look upon P_Λ as a transition kernel $P_\Lambda : \Omega_\Lambda \times \Omega_{\Lambda'} \to [0, 1]$.

2. P_Λ satisfies all of (2.6).

3. $$\lim_{L \uparrow \infty} \frac{P_\Lambda(\eta|\omega^o)}{P_\Lambda(\eta|\omega)} \text{ is continuous in } \omega \;\text{ for every } \eta \in \Omega. \tag{2.10}$$

We have the following result.

Theorem 1 *Suppose we have a stochastic operator P satisfying the above conditions 1,2 and 3. If P has an invariant (or stationary) measure ν (i.e. $\nu P = \nu$), which is Gibbs for some translation invariant interaction, then any other translation invariant stationary measure is also Gibbs with respect to the same interaction.*

Corollary 1 *The translation invariant stationary measures of P are either all Gibbsian with respect to the same potential or either they are all non–Gibbsian. Moreover, if two of them have non–vanishing relative entropy density, then they are all non–Gibbsian.*

<u>Remarks:</u>

- Theorem 1 can easily be extended to stochastic operators P that do not satisfy condition 1 of (2.10), but can be suitably approximated by a sequence of operators satisfying this condition. (The reason is that (2.14) below is uniform in R.)

- For the same reason, Theorem 1 can be applied to continuous time spinflip processes with positive and local rates. Such a process can be approximated by a sequence of discrete time Markov processes with parallel updating satisfying all of (2.10) (see e.g. [14], [15]). Also positive, continuous rate functions that are not strictly local pose no problem because of our first remark.

- Condition 2 avoids degeneracies and is certainly necessary for the statements above.

- Condition 3 is very similar to condition 1 but is independent of it : in a sense, it requires good locality properties of the reversed transformation. It implies that if you condition a Gibbs measure on what state you get after transforming it via P, then this conditioned measure is still Gibbsian.

Proof of Theorem 1:

Take $\mu, \nu \in \mathcal{P}_{tr}$. Suppose that $\nu P = \nu$, $\nu \in \mathcal{G}(J)$. For $\eta \in \Omega_\Lambda$, we have that

$$(\mu P)_\Lambda(\eta) = \mu_{\Lambda'} P_\Lambda(\eta). \tag{2.11}$$

This just means that

$$\int P(\sigma = \eta \text{ on } \Lambda|\omega)\,\mu(d\omega) = \sum_{\omega \in \Omega_{\Lambda'}} P_\Lambda(\eta|\omega)\,\mu_{\Lambda'}(\omega) \tag{2.12}$$

For $\sigma_\Lambda \in \Omega_\Lambda$, define

$$(\mu \times P[\sigma_\Lambda])(d\omega) = \mathcal{N}(\sigma_\Lambda)\mu(d\omega)P_\Lambda(\sigma_\Lambda|\omega)$$

where $\mathcal{N}(\sigma_\Lambda)$ is a normalizing factor. Then, for $\omega \in \Omega_{\Lambda'}$ we have

$$(\mu \times P[\sigma_\Lambda])_{\Lambda'}(\omega) = \mu_{\Lambda'} \times P_\Lambda[\sigma_\Lambda](\omega).$$

For each L we can apply the identity (2.9) to the stochastic matrix P_Λ:

$$\begin{aligned} \frac{1}{L^d} I_{\Lambda'}(\mu, \nu) = \frac{1}{L^d} I_\Lambda(\mu P, \nu P) + \\ \frac{1}{L^d} \sum_{\sigma_\Lambda \in \Omega_\Lambda} (\mu P)_\Lambda(\sigma_\Lambda) I_{\Lambda'}(\mu \times P[\sigma_\Lambda], \nu \times P[\sigma_\Lambda]). \end{aligned} \tag{2.13}$$

Since R is fixed, it will not matter if we write L^d instead of $(L+R)^d$ when we let $L \uparrow \infty$. Now note that P_Λ obeys condition 3 above. Therefore $\nu \times P[\sigma_\Lambda]$ is Gibbs because ν is. Moreover,

$$h_{\nu \times P[\sigma_\Lambda]}(\eta) = h_\nu(\eta) + \log \frac{P_\Lambda(\sigma_\Lambda|\eta^o)}{P_\Lambda(\sigma_\Lambda|\eta)}, \qquad \eta \in \Omega. \tag{2.14}$$

Suppose now that μ is not Gibbs with respect to the same interaction as ν, then $i(\mu, \nu) > 0$ (see Lemma 2) or equivalently $I_\Lambda(\mu, \nu) = \mathcal{O}(L^d)$ (this means that for any strictly positive real numbers c_1, c_2, d_1, d_2 with $d_1 < d < d_2$, there exists L_0, such that for $L > L_0$ $c_1 L^{d_1} < I_\Lambda(\mu, \nu) < c_2 L^{d_2}$).
We want to show that also $I_{\Lambda'}(\mu \times P[\sigma_\Lambda], \nu \times P[\sigma_\Lambda]) = \mathcal{O}(L^d)$ for every $\sigma_\Lambda \in \Omega_\Lambda$.
Since ν is Gibbs, we take h_ν continuous. Because μ is not Gibbs for the same interaction as ν, there exists a set $\mathcal{A}$ of μ-measure > 0 such that $h_\mu(\eta) \neq h_\nu(\eta)$, for $\eta \in \mathcal{A}$. Using (2.13), we get that μ-a.s.

$$h_{\mu \times P[\sigma_\Lambda]} - h_{\nu \times P[\sigma_\Lambda]} = h_\mu - h_\nu, \tag{2.15}$$

because the P_Λ just drops out.
But then, $\mu \times P[\sigma_\Lambda]$ cannot be Gibbs for the same interaction as $\nu \times P[\sigma_\Lambda]$.
So, $I_{\Lambda'}(\mu \times P[\sigma_\Lambda], \nu \times P[\sigma_\Lambda]) = \mathcal{O}(L^d)$. Taking $L \uparrow \infty$, we obtain now that

$$i(\mu, \nu) > i(\mu P, \nu) \tag{2.16}$$

and μ cannot be stationary for P.

$\square$

Proof of Corollary 1 :

Suppose P has a translation invariant stationary Gibbs measure. Then Theorem 1 together with Lemma 1 give that for any two translation invariant stationary measures μ, ν, $i(\mu, \nu) = 0$. If, on the contrary, there exist two translation invariant stationary measures with non–zero relative entropy density, it follows that all translation invariant stationary measures are non–Gibbsian.

$\square$

Theorem 1 provides a common framework to understand various results in the literature having to do with the question Gibbs versus non–Gibbs. In the following Sections we illustrate this point more clearly.

3 Spinflip dynamics.

By a spinflip dynamics we mean a stochastic time evolution for infinite configurations of spins on a lattice in which every individual spin flips with a certain rate or a certain probability according to the configuration of spins nearby. Such continuous time (sequential) interacting spin systems are thus determined via a spinflip rate $c(x, \eta)$ giving the probability per unit time that the spin at site x flips if the present configuration is η. A well known example is the so called Glauber dynamics, where the rates $c(x, \eta) \sim \exp[-\beta h_x(\eta)]$ are (essentially) proportional to the Boltzmann factor of the relative energy $h_x(\eta) = H(\eta^x) - H(\eta)$ for some given Hamiltonian given of the spin system. More generally, such dynamics that are governed by an energy function and that satisfy the so called detailed balance condition, go under the name of stochastic Ising models. Quite a lot is known about their stationary states. In particular, the Gibbs states $\mu \sim e^{-\beta H}$ are time invariant and the dynamics is reversible w.r.t. μ. However, there are many other physically relevant examples of spinflip dynamics, which do not satisfy the detailed balance condition and for which in general very litlle is known about their stationary states. Could it be that there is still some (effective) Hamiltonian out there for which the stationary states are Gibbsian? We discuss this question first by commenting on some work of Künsch ([3]) and a paper of Garrido and Marro ([16]). We end this section by a discussion of the situation for discrete time spinflip dynamics, examples of probabilistic cellular automata. We must however start with a disclaimer. We know that in general the answer to the above question is no. (Non–trivial) counter examples are provided by voter models in 3 or more dimension [17].

Künsch (1984) is studying the local conditional distributions of a stationary measure for a spinflip process with strictly positive and continuous flip rates. He shows among other things that either all translation invariant stationary measures are Gibbsian with the same potential or no translation invariant stationary Gibbs measure exists. This has to be compared with our Theorem 1 (Remarks). Moreover, he constructs an example with infinitely many stationary measures (all of which are Gibbsian), but no reversible measures. In fact, the general tone of the paper is that, again for strictly positive and continuous rates, reversible or not, all stationary measures should be Gibbsian. In particular, this would mean that the local conditional distributions of a single spin given all other spins are the same for all stationary measures. Although this last statement is formulated as a conjecture in the paper, we wish to point out here that local conditional distributions as such, carry very little information unless assumed to have continuous versions. More precisely, if μ and ν are both stationary measures, we can always find a function h on Ω for which

$$\mu[\sigma(o) = \eta(o) | \sigma(x) = \eta(x), x \neq o] = h(\eta) \qquad \mu - \text{a.s.}$$
$$\nu[\sigma(o) = \eta(o) | \sigma(x) = \eta(x), x \neq o] = h(\eta) \qquad \nu - \text{a.s.}$$

and μ and ν need not be Gibbsian.

Effective Hamiltonian description? It has been suggested in Garrido and Marro (1989) that strictly positive and local flip rates for the dynamics always give rise to Gibbs states as stationary states. In fact they give an explicit form of the interaction potential (they call it the effective Hamiltonian). It basically amounts to writing down the equation ($c(x, \eta)$ can be for example a sum of Glauber rates with different temperatures)

$$\exp[-2\beta \sum_{A \ni x} J_A \eta(A)] = \frac{c(x, \eta)}{c(x, \eta^x)} \tag{3.1}$$

for the coupling constants $\{J_A\}$ which would appear in the (effective) Hamiltonian

$$\text{“}H(\eta) = -\sum_A J_A \eta(A)\text{”}. \tag{3.2}$$

Then, the Gibbs measures with respect to this Hamiltonian are of course invariant under the dynamics. There is however only a consistent solution to (3.1) if the rates satisfy the condition

$$\sum_\eta \eta(A) \log \frac{c(x, \eta)}{c(x, \eta^x)} \frac{c(y, \eta)}{c(y, \eta^y)} = 0 \tag{3.3}$$

for every finite set A and all $x, y \in A$. It is easy to see that this condition implies in fact that we are really dealing with a stochastic Ising model and (3.3) can never be satisfied by a dynamics which is not reversible.

Discrete time In this paragraph, we deal with probabilistic cellular automata (PCA). They can be viewed as discrete time versions of spinflip dynamics. PCA are discrete time

Markov processes on ^{d}, with local and parallel updating of Ising type spins. The finite volume transition probabilities can be written as

$$P_\Lambda(\sigma|\eta) = \prod_{x\in\Lambda} P_{\{x\}}(\sigma(x)|\eta) \tag{3.4}$$

and we take the $P_{\{x\}}(\sigma(x)|\eta)$ positive and local in η (cf. conditions 1 and 2). It is easy to check that also our condition 3 is verified. Theorem 1 can thus be applied to them. In the reversible case, where the PCA satisfies the detailed balance condition (see [6]) with respect to some Gibbs measure for some translation invariant interaction $\{J(V)\}_{V\subset^d}$, it is a consequence of Theorem 1 that the translation invariant stationary measures are precisely all Gibbs measures for $\{J(V)\}_{V\subset^d}$.
At high noise (i.e. when the dependence for a spin on the value of its neighboring spin on the previous timestep is small) it is known that there is a unique stationary Gibbs measure [6].
In the non–reversible case, practically nothing is known about the nature of the stationary measures in the regime where there is more than one stationary measure. Perhaps it is possible that these stationary measures are non–Gibbsian. To prove such a statement, it would be sufficient (because of Theorem 1) to prove that the relative entropy density between two different stationary measures is nonzero. A typical example of a non–reversible PCA (for which it is conjectured that the stationary measures at low noise are non–Gibbsian [1]) is the Toom model. The simplest version of the Toom model is a PCA which can be seen as a North–East–Center majority vote model on the square lattice. That is, in (3.19) one takes

$$P_{\{x\}}(\sigma(x)|\eta) = 1/2(1-(1-2\epsilon)\sigma(x)\mathrm{sgn}[\eta(x)+\eta(x')+\eta(x'')]) \tag{3.5}$$

with $x' = x+(1,0)$ and $x'' = x+(0,1)$ the "eastern" and "northern" nearest neighbor of x on ^{2}. Equivalently, we can define the Toom model on the triangular lattice, but arrange it so that, for the space-time lattice, each site is the center of a triangle on the previous layer. More precisely, if we denote by $\mathcal{L}_n$ the layer at time $t=-n$, then under projection perpendicular to the lattice plane, the sites of $\mathcal{L}_{n-1}$ coincide with the centers of alternate triangles in $\mathcal{L}_n$, i.e. those with some fixed orientation. The sites of $\mathcal{L}_{n-2}$ coincide with the centers of the similarly oriented triangles in $\mathcal{L}_{n-1}$ and finally $\mathcal{L}_{n-3}$ coincides with $\mathcal{L}_n$.
The deterministic Toom rule ($\epsilon = 0$) assigns to the spin at site x the majority of the spins at the corners of the triangle directly above. The evolution in the stochastic Toom model follows the deterministic rule with probability (at least) $1-\epsilon$ and does something else with probability (at most) ϵ.
The deterministic dynamics has $\sigma(x)\equiv +1$ and $\sigma(x)\equiv -1$ as stationary states. Moreover, these states are stable against finite excitations of the opposte sign. In fact, such excitations shrink at a constant rate and hence disappear in a finite time. This is the so called eroder property. For ϵ small but different from zero, there are still (at least) two stationary measures, in one of which most spins are $+1$, in the other of which most are

-1, obtained from $+$ and $-$ begin conditions respectively (they will be called ρ_+ and ρ_- in the following). For more information, see [6]. We take as basis vectors to generate our three dimensional space-time lattice, the vectors e_1, e_2, e_3 connecting the origin o with the corners of the triangle directly above o. The time coordinate $t(x)$ of a space–time point x is given by $t(x) = x.(e_1 + e_2 + e_3)$.
As mentioned before, to prove that ρ_+, ρ_- are non–Gibbsian, one should show that $i(\rho_+, \rho_-) > 0$. It is equivalent to show the probability for ρ_+ of having a large cluster of minusses is (at least) exponentially small in the volume of the cluster. (see for example [1]). This would be true if the probability to find $\sigma_x \equiv -1$ in the triangle $\Delta \equiv \{x|t(x) = -K; x.e_1, x.e_2, x.e_3 \leq K\}$, starting from the all $+$ configuration at time $t = -N$, is smaller than *cte* $\epsilon^{|\Delta|}$, for all $K > 0$ and for all $N > K$; ($|\Delta|$ is the cardinality of the set Δ).
In [9], it is shown that

$$\text{Prob}[\sigma(o) = -1] \leq 2\epsilon \tag{3.6}$$

by making an expansion in graphs for this probability. However, this expansion cannot be used to prove our statement. In fact, there is a model of Domany (see [6]) for which Toom's expansion can be used to prove non–ergodicity, but for which the stationary measures are Gibbsian. The transition function for this model is

$$P_{\{x\}}(\sigma(x)|\eta) = 1/2[1 + \sigma(x) \tanh \beta(\eta(x) + \eta(x') + \eta(x''))]. \tag{3.7}$$

It is not hard to see that, if one sets $1-2\epsilon = 1/2(\tanh 3\beta + \tanh \beta)$, this transition function appoaches that of the Toom model (3.5) if $\beta \uparrow \infty, \epsilon \downarrow 0$. So, it is not clear that one should expect the stationary measures of the Toom model at low noise to be non–Gibbsian.
For this reason we are trying to extract the necessary information from simulations on the Toom model. However, so far this has not given conclusive evidence. Although we are inclined to interpret the simulation data as showing Gibbsian behavior for the stationary states, a further and more systematic search needs to be done before drawing any conclusions.

One can also see from here that the question of Gibbsianess is more than an academic question. Of course it is an important characterization of the stationary states but more than that, here we see that the question is equivalent to an estimate of the efficiency of the repeated transformation in removing mistakes : saying that the relative entropy density of the two stationary states is non–zero means that you need to create a volume fraction of "mistakes" to see in one stationary state, an island of the opposite stationary state — for equilibrium dynamics relaxing towards one of the extremal stationary Gibbs states, it suffices to essentially introduce a boundary of the "wrong" phase for the dynamics to automatically fill up the enclosed volume with this (for example, opposite) phase. It may also happen that you need less than a boundary (such as in the voter model, see [17]) and then the stationary states may again be non–Gibbsian even though their relative entropy densites are zero. The picture that emerges is therefore that the stationary states are Gibbsian whenever, when the dynamics is started from a particular stationary state, by introducing *on the boundary* of a volume a configuration that is "typical" of another stationary state, after a time of the order of the diameter of that volume, one sees this "wrong" state *in the interior* of the volume.

4 Relation with other problems.

4.1 Restriction of + and − phases of the Ising model.

Consider the standard ferromagnetic Ising model in d dimensions. In finite boxes $\Lambda = [-L, L]^d$ we consider + and − boundary conditions : the corresponding Gibbs measures $\mu_\Lambda^\pm \sim \exp(-\beta H_\Lambda^\pm)$ converge as $L \uparrow \infty$ to Gibbs measures $\mu^\pm$. At low temperatures ($\beta \uparrow \infty$) $\mu^+ \neq \mu^-$ are two extremal, translation invariant Gibbs measures. Both are satisfying the global Markov property [4]. For both measures, we can consider the restrictions $\nu^\pm$ to a $(d-1)$-dimensional hypersurface ($\cong {}^{d-1}$). Schonmann [11] proved that in $d = 2$ dimensions and below the critical temperature, both these measures $\nu^\pm$ are non–Gibbsian (i.e. (2.3) is never satisfied). We wish to add here two remarks:

1. The higher dimensional situation ($d > 2$) (open question 6 in [1]): Schonmann's analysis consists of two main steps: a large deviation estimate and the study of a wetting problem. The large deviation estimate boils down to the statement that the relative entropy density of the projected phases is non–zero: $i(\nu^+, \nu^-) > 0$. If we allow to look at very small temperatures, this result is valid in any dimension $d \geq 2$, from standard arguments [18]. As for the wetting problem, one has to prove that for each n, there exists an $N(n)$ such that

   $$\nu^+[\cdot|\tilde{\sigma} = -1 \text{ on } A_{n,N}] \to \nu^- \text{ as } N \uparrow \infty,$$

   where $\tilde{\sigma}$ is a configuration on the $(d-1)$-dimensional annulus

   $$A_{n,N} = \{x = (i, 0), i \in {}^{d-1} : n \leq |i_k| \leq N, k = 1, \ldots, d-1\}.$$

   The proof of this statement is a consequence of the recent work of Holický and Zahradník [19] on the problem of entropic repulsion in low temperature Ising models. Therefore, for any dimension ($d \geq 2$) the projection of the Ising phases on a hypersurface is non–Gibbsian at low temperatures.

2. The transfer matrix
   A well known tool in the analysis of the Ising model is the machinery of the transfer matrix formalism. In some sense one could say that the projections on hypersurfaces are invariant measures for this transfer matrix. Therefore, Corrolary 1 (with P= transfer matrix) very much resembles the statement of Proposition 2 in [11] that if

ν^+ is Gibbsian, then ν^- is Gibbsian for the same potential. We show however that there exists no stochastic operator P, satisfying the conditions of (2.10), for which both ν^+ and ν^- are invariant.
Consider a box $B = [-K, K] \times V_L; V_L = [-L, L]^{d-1}$. Let the (finite volume) transfer matrix a_L be defined by its matrix elements $a_L(\tau, \tau'), \tau, \tau' \in \Omega_{V_L}$:

$$a_L(\tau, \tau') = \exp[\beta/2(\sum_{<xy> \subset V_L} (\tau(x)\tau(y) + \tau'(x)\tau'(y)) + \beta \sum_{x \in V_L} \tau(x)\tau'(x)]. \qquad (4.1)$$

(here we sum over nearest neighbor pairs)
¿From the Perron–Frobenius theorem, we have that the largest eigenvalue λ_L of a_L is non–degenerate. Let us call the corresponding normalized eigenvector ψ_L and its components $\psi_L(\tau), \tau \in \Omega_{V_L}$. The matrix a_L can be normalized to become a stochastic matrix π_L:

$$\pi_L(\sigma|\eta) = \frac{a_L(\sigma, \eta)\psi_L(\sigma)}{\lambda_L \psi_L(\eta)} \qquad \sigma, \eta \in \Omega_{V_L}. \qquad (4.2)$$

Note that the combination

$$\pi_L(\sigma|\eta)\psi_L^2(\eta)$$

is symmetric in (η, σ) which is the analogue of the detailed balance condition for PCA.
Consider now $\mu_{K,L}$, the Gibbs state on the finite box B, corresponding to the Hamiltonian H_B with free boundary conditions:

$$H_B(\sigma) = - \sum_{<xy> \subset B} \sigma(x)\sigma(y). \qquad (4.3)$$

We have that

$$\mu_{K,L}[\sigma \text{ on } \{0\} \times V_L | \eta \text{ on } \{-1\} \times V_L] = a_L(\sigma, \eta) \frac{\sum_\zeta a_L^K(\zeta, \sigma)}{\sum_\zeta a_L^{K+1}(\zeta, \eta)} \qquad (4.4)$$

and thus

$$\lim_{K \uparrow \infty} \mu_{K,L}[\sigma \text{ on } \{0\} \times V_L | \eta \text{ on } \{-1\} \times V_L] = \pi_L(\sigma|\eta). \qquad (4.5)$$

Hence,

$$\mu[\cdot \text{ on } \{0\} \times {}^{d-1} | \eta \text{ on } \{-1\} \times {}^{d-1}] = \lim_{L \uparrow \infty} \pi_L(\cdot|\eta) \qquad \text{for } \mu\text{-a.e. } \eta. \qquad (4.6)$$

The left hand side is the probability on the zero'th layer (the hypersurface passing through the origin) in the measure $\mu = 1/2(\mu^+ + \mu^-)$ conditioned on the "previous" layer. Any P that leaves ν^+ and ν^- invariant also leaves $\nu = 1/2(\nu^+ + \nu^-)$ invariant. Therefore its kernel should satisfy

$$P[\cdot|\eta] = \mu[\cdot \text{ on } \{0\} \times {}^{d-1} | \eta \text{ on } \{-1\} \times {}^{d-1}] \qquad (4.7)$$

for μ-a.e. η.
We claim that the right-hand side (and so the left-hand side) of (4.7) cannot have any version satisfying condition 3 of (2.10). We apply the following Lemma (the proof follows from [1], Prop. 4.12)

Lemma 4 *Suppose that $\nu = 1/2(\nu^+ + \nu^-)$ is a Gibbs measure for some interaction, then ν^+ and ν^- are also Gibbs with respect to the same interaction and so $i(\nu^+, \nu^-) = 0$.*

Since $i(\nu^+, \nu^-) > 0$, we conclude that the measure ν is non–Gibbsian. In particular (see Lemma 1), $\nu[\sigma(o) = \eta(o)|\sigma(x) = \eta(x), \quad x \neq o]$ is an essential discontinuous function of η, i.e. no modification on a set of ν-measure zero can change it into a continuous function. Using (4.3), we calculate that for ν-a.e. ω

$$\frac{P_{V_L}[\eta|\omega]}{P_{V_L}[\eta|\omega^o]} = \lim_{N\uparrow\infty}\{\frac{\psi_N(\omega^o)}{\psi_N(\omega)}\}\times\exp[\beta\sum_{x:<ox>}(\eta(x)\eta(o)+\omega(x)\omega(o))+2\beta\eta(o)\omega(o)]. \quad (4.8)$$

Since $\psi_N^2(\cdot)$ gives, for every N, the probabilities in the finite volume measure ν_{V_N}, with $\nu_N \to \nu$, and ν is not–Gibbsian, it is clear that the right–hand side of (4.8) cannot have a continuous version. The first factor of that right-hand side is essentially discontinuous because of the non–Gibbsianess of ν.

4.2 Difference with renormalization transformations.

Theorem 1 is not adapted to treat real space renormalization transformations, since they necessarily involve a rescaling of the lattice at each application of the transformation. Thus, these transformations don't satisfy property 1 of (2.10), because for every t, the spins on a line segment of length L at time $t+1$ depend on the spins on a line segment of length $3L$ at time t. Vasilyev [20] has given an example of a transition kernel, involving the same kind of rescaling as in renormalization transformations, for which two different Bernoulli measures are stationary (i.e. two Gibbs measures for different interaction). There is thus an essential difference between these renormalization transformations and the other stochastic transformations considered in the paper. For related questions in renormalization group techniques, we refer to [1]. However, these two kinds of transformations become very similar in the so called mean field limit. In many cases, a first approach to study (complicated) PCA dynamics consists in replacing the space–time lattice (say, $^{d+1}$) by a Cayley graph or tree, for which the neighborhoods of any pair of sites in one time layer are non–overlapping. Applying the PCA rules in this situation yields a closed relation for the magnetization $\langle\sigma_x\rangle$ in a stationary state. An example of this for the Toom model can be found in [6].

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