ON THE RELATION BETWEEN WEAK AND STRONG INVARIANCE OF DIFFERENTIAL EQUATIONS

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Abstract. Some concepts of Lie algebra cohomology are used to systematize the search for differential equations invariant under a given Lie group G. In particular it is shown that if a "strongly invariant" equation exists, then all "weakly invariant" equations differ from it only by an arbitrary multiplicative factor. If no "strongly invariant" equation exists, then cohomology theory can be used to simplify the search for "weakly invariant" equations.

1. Introduction.

Lie group theory provides powerful tools for the study of differential equations (see e.g. [1-5] and references therein). In particular, for nonlinear partial differential equations, the use of Lie point symmetries makes it possible to obtain exact analytical solutions that are often otherwise inaccessible. The increased importance of nonlinear phenomena in physics and other sciences, together with developments in computer science (algebraic computing) and in Lie group theory itself (e.g. the theory of infinite dimensional Lie groups) have motivated a resurgence of interest in the symmetries of differential equations.

Lie group theory can also be used to classify differential equations into equivalence classes, or to construct equations with predetermined symmetries. The simplest method for doing this consists of three steps:

1. Postulate the general form of the invariant equation that we are looking for, i.e. the number of dependent and independent variables and the order n of the equation:

$$\Delta(x, u, u_{x_i}, u_{x_i x_j}, \dots , u_{x_{i_1}\cdots x_{i_n}}) = 0,\ \ x \in \mathbb{R}^p, u \in \mathbb{R}^q. \tag{1.1}$$

2. Realize the Lie algebra $\mathcal{G}$ of the considered Lie group G of local point transformations by vector fields of the form

$$X_a = \sum_{i=1}^{p} \xi_i^a(x,u)\,\frac{\partial}{\partial x_i} + \sum_{\alpha=1}^{q} \phi_\alpha^a(x,u)\,\frac{\partial}{\partial u_\alpha},\ \ a \in \mathcal{G}, \tag{1.2}$$

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and construct the n-th prolongation [1-5], $\widehat{X}_a = \mathrm{pr}^{(n)} X_a$, of these vector fields. Here the set $\{X_a\}$ are the fundamental vector fields corresponding to the elements a of the Lie algebra $\mathcal{G}$.

3. Request that the prolongation $\widehat{X}_a$ of these vector fields X_a should simultaneously annihilate the function $\Delta(x, u, u_{x_i}, \dots)$:

$$\widehat{X}_a \Delta = 0. \tag{1.3}$$

Thus, Δ viewed as a function of x, u and the derivatives of u up to order n, must satisfy the system of linear first order partial differential equations (1.3). Using the method of characteristics one finds a system of elementary solutions (if they exist) and Δ is then an arbitrary function of these elementary invariants.

This approach has recently been applied to find equations invariant under the symmetry group of the Kadomtsev-Petviashvili equation [6], the Poincaré, Similitude and Conformal groups of 1+1 dimensional Minkowski space [7], and under the Galilei, Galilei-Similitude and Schrödinger group of of $1+1$ dimensional space-time [8].

The approach described above has its limitations. Indeed, equation (1.3) corresponds to the fact that the function Δ is invariant under the corresponding local Lie point transformations. We shall call this "strong invariance" of equation (1.1). Equation (1.1) is constructed out of "universal invariants" of the action of the group G.

What is usually required is invariance of the solution set of the equation. The prolongations $\widehat{X}_a$ thus need not annihilate Δ universally, but only on the solution set of the equation (1.1), i.e. equation (1.3) is replaced by the weaker condition

$$\widehat{X}_a \Delta_{|\Delta=0} = 0. \tag{1.4}$$

To study the relation between eq. (1.3) and (1.4), which we shall call "weak invariance", we note that the function Δ is not uniquely defined. Thus, e.g.

$$\Delta = 0, \;\; \Delta^n = 0, \;\; (n > 0), \;\; e^f \Delta = 0, \tag{1.5}$$

will have the same solution sets. We shall require that the equation (1.1) define a manifold. The function Δ must be such that its differential does not vanish on the solution set

$$(d\Delta)_{|\Delta=0} \neq 0. \tag{1.6}$$

For instance $\Delta = u_{xx}$ satisfies (1.6), but $\Delta' = \Delta^2 = u_{xx}^2$ does not.

If condition (1.6) is satisfied, then the condition (1.4) of "weak invariance" is (at least locally) equivalent to

$$\widehat{X}_a \Delta = F_a \, \Delta, \tag{1.7}$$

where the functions $F_a(x, u, u_{x_i}, \dots)$ are some, *a priori* unspecified, functions of the same arguments as Δ that are not singular for $\Delta = 0$. We shall call equation (1.7) "weak invariance".

The purpose of this article is to show that if equation (1.3) has a nontrivial solution, then all solutions of equation (1.4) can be obtained directly from this solution. More specifically, if $\widetilde{\Delta}$ is the general solution of equation (1.3), and is not identically constant, then the general solution Δ of equation (1.7) will have the form

$$\Delta = e^f \, \widetilde{\Delta}, \tag{1.8}$$

where f is an arbitrary smooth function of the same arguments as $\widetilde{\Delta}$, not singular on the solution set of the equation $\widetilde{\Delta} = 0$.

If, on the other hand, the Lie group action is such that no strongly invariant equation (1.1) exists, weakly invariant equations may still be constructed.

In order to prove the essential equivalence of the set of all weakly and strongly invariant equations (when both exist), we shall make use of Lie algebra cohomology theory. Moreover, Lie algebra cohomology theory can be used to simplify the task of finding weakly invariant equations in the case when no strongly invariant ones exist.

2. Lie Algebra Cohomology

Let $\mathcal{G}$ be a Lie algebra and $\mathcal{A}$ a $\mathcal{G}$-module respectively. In other words, $\mathcal{A}$ is a module that is the carrier space for a linear representation Ψ of $\mathcal{G}$, i.e. $\Psi\colon \mathcal{G} \to \operatorname{End} \mathcal{A}$ satisfies

$$\Psi(a)\Psi(b) - \Psi(b)\Psi(a) = \Psi([a,b]).$$

By a n-cochain we mean an n-linear alternating mapping from $\mathcal{G} \times \ldots \times \mathcal{G}$ (n times) into $\mathcal{A}$. We denote by $C^n(\mathcal{G}, \mathcal{A})$ the space of n-cochains. For every $n \in \mathbb{N}$ we define $\delta_n\colon C^n(\mathcal{G}, \mathcal{A}) \to C^{n+1}(\mathcal{G}, \mathcal{A})$ by [9,10]

$$\begin{aligned}(\delta_n\alpha)(a_1,\ldots,a_{n+1}) :=& \sum_{i=1}^{n+1}(-1)^{i+1}\Psi(a_i)\alpha(a_1,\ldots,\widehat{a}_i,\ldots,a_{n+1})+\\ &+\sum_{i<j}(-1)^{i+j}\alpha([a_i,a_j],a_1,\ldots,\widehat{a}_i,\ldots,\widehat{a}_j,\ldots,a_{n+1}),\end{aligned} \tag{2.1}$$

where $\widehat{a}_i$ denotes, as usual, that the element a_i is omitted.

The linear maps δ_n can be shown to satisfy $\delta_{n+1} \circ \delta_n = 0$. The linear operator δ on $C(\mathcal{G}, \mathcal{A}) := \bigoplus_{n=0}^{\infty} C^n(\mathcal{G}, \mathcal{A})$ whose restriction to each $C^n(\mathcal{G}, \mathcal{A})$ is δ_n, satisfies $\delta^2 = 0$. We will then denote

$$\begin{aligned}B^n(\mathcal{G}, \mathcal{A}) &:= \{\, \alpha \in C^n(\mathcal{G}, \mathcal{A}) : \exists \beta \in C^{n-1}(\mathcal{G}, \mathcal{A}) \text{ such that } \alpha = \delta\beta \,\} = \operatorname{Im}\delta_{n-1},\\ Z^n(\mathcal{G}, \mathcal{A}) &:= \{\, \alpha \in C^n(\mathcal{G}, \mathcal{A}) : \delta\alpha = 0 \,\} = \operatorname{Ker}\delta_n.\end{aligned}$$

The elements of $Z^n(\mathcal{G}, \mathcal{A})$ are called n-cocycles, and those of $B^n(\mathcal{G}, \mathcal{A})$ are called n-coboundaries. Since $\delta^2 = 0$, we have $B^n(\mathcal{G}, \mathcal{A}) \subset Z^n(\mathcal{G}, \mathcal{A})$. The n-th cohomology group $H^n(\mathcal{G}, \mathcal{A})$ is defined as

$$H^n(\mathcal{G}, \mathcal{A}) := \frac{Z^n(\mathcal{G}, \mathcal{A})}{B^n(\mathcal{G}, \mathcal{A})},$$

and we will define $B^0(\mathcal{G}, \mathcal{A}) = 0$, by convention.

As an example let G be a transformation Lie group of a differentiable manifold M. We shall denote X_a the *fundamental vector field* associated to the element a of the Lie algebra $\mathcal{G}$ of G given by

$$(X_a f)(x) = \frac{d}{dt} f(\exp(-ta)x)\Big|_{t=0}. \tag{2.2}$$

The minus sign has been introduced for the map $X\colon \mathcal{G} \to \mathfrak{X}(M)$ defined by $a \mapsto X_a$ to be a Lie algebra homomorphism, i.e. $X_{[a,b]} = [X_a X_b]$. Let us consider $\mathcal{A} := \bigwedge^p(M)$, and define $\Psi(a)\beta := \mathcal{L}_{X_a}\beta$ for $\beta \in \bigwedge^p(M)$. This is a linear representation of $\mathcal{G}$ in $\mathcal{A}$ and the δ-operator takes the following form: if $\alpha \in C^n(\mathcal{G}, \bigwedge^p(M))$, then

$$\begin{aligned}(\delta_n\alpha)(a_1,\dots,a_{n+1}) = &\sum_{i=1}^{n+1}(-1)^{i+1}\mathcal{L}_{X_{a_i}}\alpha(a_1,\dots,\widehat{a}_i,\dots,a_{n+1})+\\ &+\sum_{i<j}(-1)^{i+j}\alpha([a_i,a_j],a_1,\dots,\widehat{a}_i,\dots,\widehat{a}_j,\dots,a_{n+1}).\end{aligned} \tag{2.3}$$

In particular,

$$\alpha \in C^1(\mathcal{G}, \bigwedge{}^p(M)) \implies (\delta\alpha)(a,b) = \mathcal{L}_{X_a}\alpha(b) - \mathcal{L}_{X_b}\alpha(a) - \alpha([a,b]). \tag{2.4}$$

The elements of $B^1(\mathcal{G}, \bigwedge^p(M))$ are those α for which $\exists\beta \in \bigwedge^p(M)$ with

$$\alpha(a) = \mathcal{L}_{X_a}\beta, \tag{2.5}$$

while the elements of $Z^1(\mathcal{G}, \bigwedge^p(M))$ are linear maps $\alpha\colon \mathcal{G} \to \bigwedge^p(M)$ satisfying

$$\mathcal{L}_{X_a}\alpha(b) - \mathcal{L}_{X_b}\alpha(a) = \alpha([a,b]). \tag{2.6}$$

We shall see that the case $p = 0$ is very relevant in the study of differential equations invariant under a Lie group.

3. Invariant differential equations

Jet bundle theory [11] furnishes the appropriate framework for the geometric setting of differential equations. If $\pi : E \to M$ is a vector bundle and $J^n(E,M)$ denotes the bundle of n-jets of sections for π, a n-th order (partial) differential equation is a hypersurface Σ in $J^n(E,M)$ (see e.g. [1]). By a solution of such an equation in $p \in M$ we mean a (local) section $\phi : M \to E$ such that its n-th prolongation $j^n\phi$ lies in Σ. An infinitesimal point symmetry of the equation Σ is a vector field $X \in \mathfrak{X}(E)$ such that its n-th prolongation $\widehat{X} \in \mathfrak{X}(J^n(E,M))$, when restricted to Σ, is tangent to Σ. The differential equation Σ is locally expressed in coordinates by the vanishing of a function $\Delta \in C^\infty(J^n(E,M))$ (see equation (1.1)). As mentioned above Δ is not uniquely determined. So, for instance the set $\Sigma' = {\Delta'}^{-1}(0)$, with $\Delta' = \Delta^2$ coincides with Σ. The geometric prescripton for Δ to define a subvariety structure on $\Delta^{-1}(0)$ is $d\Delta_{|\Delta^{-1}(0)} \neq 0$, a condition that Δ' would not satisfy. We can then assume that Σ is a regular submanifold and that the ideal of functions vanishing on Σ, in a neighborhood of the point we are considering, is principal. We will choose Δ as a generator of the ideal.

The invariance under a Lie group G of transformations of E means that the (prolonged) vector fields $\widehat{X}_a \in \mathfrak{X}(J^n(E,M))$ are tangent to the hypersurface Σ, what is expressed by the condition

$$\left(\widehat{X}_a\Delta\right)_{|\Sigma} = 0, \quad \forall a \in \mathcal{G}. \tag{3.1a}$$

Since X_a depends linearly on a it suffices to choose a basis $\{a_i\}$ of the Lie algebra and consider the system

$$\left(\widehat{X}_{a_i}\Delta\right)_{|\Sigma} = 0, \quad i = 1, \dots, l = \dim G, \tag{3.1b}$$

(see equation (1.4). Locally equation (3.1) is equivalent to the existence of a set of l regular functions $F_a \in C^\infty(J^n(E,M))$ such that equation (1.7) holds).

The crucial point is that the functions F_a are not arbitrary. The vector fields $\widehat{X}_a$ give a realization of the Lie algebra $\mathcal{G}$ and therefore we have

$$[\widehat{X}_{a_i}, \widehat{X}_{a_j}] = f_{ijk}\, \widehat{X}_{a_k}, \tag{3.2}$$

where f_{ijk} are the structure constants of $\mathcal{G}$ relative to the basis $\{a_i\}$, namely $[a_i, a_j] = f_{ijk}\, a_k$.

Using equations (3.2) and taking the difference between X_b applied to both sides of (1.7) and the corresponding expression with a and b interchanged, we find that the functions F_a must satisfy

$$\widehat{X}_a F_b - \widehat{X}_b F_a = F_{[a,b]} = f_{abc}\, F_c. \tag{3.3}$$

Equation (3.3) shows that the map $F : \mathcal{G} \to C^\infty(J^n(E,M))$ defined by $a \mapsto F_a$ is a 1-cocycle, i.e., $F \in Z^1(\mathcal{G}, C^\infty(J^n(E,M)))$. Indeed, since X_a depends linearly on a, the same is true for F_a, and we have

$$(\delta F)(a,b) = 0 \Longleftrightarrow F_{[a,b]} = X_a F_b - X_b F_a. \tag{3.4}$$

Moreover, $\Delta = 0$ and $\widetilde{\Delta} = e^{-f}\Delta = 0$ define the same differential equation, for $f \in C^\infty(J^n(E,M))$, $e^f{}_{|\Sigma} \neq 0$. The invariance condition (1.7) for equation $\widetilde{\Delta} = 0$ is

$$\widehat{X}_a \widetilde{\Delta} = (-(\widehat{X}_a f) + F_a)\widetilde{\Delta}. \tag{3.5}$$

Thus, we have

$$\widetilde{F}_a = F_a - X_a f, \tag{3.6}$$

which shows that $\widetilde{F} = F + \delta f$, i.e., $\widetilde{F}$ is cohomologous to F. We see that every invariant differential equation defines an element of $H^1(\mathcal{G}, C^\infty(J^n(E,M)))$. If we have $H^1(\mathcal{G}, C^\infty(J^k(E,M))) = 0$ (all 1-cocycles are 1-coboundaries), and if the equations $X_a\widetilde{\Delta} = 0$ have a nontrivial solution, then all solutions Δ of equation (1.7) have the form (1.6).

In order to decide when we have $H^1(\mathcal{G}, C^\infty(J^k(E,M))) = 0$ and when it is possible to replace weak invariance by strong invariance, we shall need some results from the geometric theory of linear and quasi-linear partial differential equations (see e.g. [12]).

We recall that any solution f of a given system of first order linear differential equations

$$X_i f = 0, \; i = 1, \dots, l, \qquad X_i \in \mathfrak{X}(N), \tag{3.7a}$$

must also satisfy the compatibility conditions $[X_i, X_j]f = 0$. A necessary condition for the existence of a solution f is the existence of a function annihilated by all vector fields of the Lie algebra generated by the given vector fields X_i. A sufficient condition is that the dimension d of such an involutive distribution (assumed to be constant) is lower than the dimension of N,

$$d < \dim N, \tag{3.8}$$

(d is the rank of the matrix of coefficients of equation (3.7a)). Indeed as the distribution generated by $\{X_i\}$ is involutive, the system (3.7a) can be replaced by an equivalent one

$$Y_i f = 0, \qquad i = 1, \dots, d \tag{3.7b}$$

in such a way that $[Y_i, Y_j] = 0$. Moreover, then (see e.g. [13]), there will be a new coordinate system in a neighborhood of the point we are studying, $(\xi_1, \dots, \xi_n)$, with $Y_i = \frac{\partial}{\partial \xi_i}$, for $i = 1, \dots, d$. The solution of (3.7b) is an arbitrary function of $\xi_{d+1}, \dots, \xi_n$. These last coordinate functions are $\dim N - d$ functionally independent elementary solutions to be determined by using the method of characteristics.

We shall also be dealing with a quasi-linear (or inhomogeneous) system

$$X_i f = F_i, \; i = 1, \dots, l, \qquad X_i \in \mathfrak{X}(N). \tag{3.9}$$

This inhomogeneous system can be reduced to a homogeneous one by introducing an auxiliary variable z and putting

$$\widetilde{X}_i = X_i + F_i \frac{\partial}{\partial z}, \qquad \widetilde{X}_i \phi = 0, \; i = 1, \dots, l. \tag{3.10}$$

If $\phi \in C^\infty(N \times \mathbb{R})$ is a solution of (3.10) satisfying $\frac{\partial \phi}{\partial z} \neq 0$ in a neighborhood of a point, then, a solution of equation (3.9) is defined implicitly by $\phi(x, f(x)) = 0$, where C is an arbitrary constant, and all solutions are found in this way by an appropriate choice of the constant C.

These general results, when applied to our particular case, in which we are trying to cancel the functions F_a by choosing a solution of the system

$$\widehat{X}_a f - F_a = 0, \tag{3.11}$$

tell us that we should replace each vector field $\widehat{X}_a$ by $\widetilde{X}_a = \widehat{X}_a + F_a \frac{\partial}{\partial z}$. Notice now that

$$[\widetilde{X}_a, \widetilde{X}_b] = [\widehat{X}_a, \widehat{X}_b] + \widehat{X}_a F_b - \widehat{X}_b F_a,$$

and using that F defines a cocycle we see that

$$[\widetilde{X}_a, \widetilde{X}_b] = \widetilde{X}_{[a,b]},$$

i.e., the vector fields $\widetilde{X}_a$ satisfy the same commutation relations as $\widehat{X}_a$ and therefore no new compatibility conditions will arise for the system (3.10). Therefore, a nontrivial solution f of (3.11) will exist when the dimension $\widetilde{d}$ of the involutive distribution generated by $\{\widetilde{X}_a\}$ is lower than $n + 1$. Notice that since $\widetilde{d} \leq d + 1$, condition (3.8) is again a sufficient condition for the existence of a nontrivial solution of (3.11). Again an arbitrary function of the elementary solutions of equation (3.11) is involved, making it possible to impose smoothness conditions.

In other words, if the system $\widehat{X}_a \widetilde{\Delta} = 0$ has a solution, then the system (3.11) also has a solution, and therefore the problem of finding weakly invariant differential equations is reduced to that of determining the strongly invariant differential equations, because the general solution of (1.4) is given by (1.6).

The preceding result is summarized as follows:

Theorem 1. *Let the equation $\Delta = 0$ be weakly invariant under the Lie transformation group G, i.e. satisfy equation (1.7). Let there exist an equation $\widetilde{\Delta} = 0$ that is strongly invariant under the same group action, i.e. $\widetilde{\Delta} = 0$ satisfies equation (1.3) for the same vector fields $\widehat{X}_a$. Then there exists a function $f \in C^\infty(J^n(E, M))$ such that $\widetilde{\Delta} = e^{-f}\Delta$. All weakly invariant equations will have the form $\Delta = e^g\widetilde{\Delta}$, where $\widetilde{\Delta}$ is the general solution of equation (1.3) and g is arbitrary.*

Proof. By assumption the function $\Delta \in C^\infty(J^n(E, M))$ satisfies equation (1.7) The functions $F_a \in C^\infty(J^n(E, M))$ must satisfy the 1-cocycle condition (3.3). Now consider the strongly invariant equation $\widetilde{\Delta} = 0$ and put $\widetilde{\Delta} = e^{-f}\Delta$. We have $\widehat{X}_a\widetilde{\Delta} = e^{-f}(F_a - \widehat{X}_a f)\Delta$. In order to have $\widehat{X}_a\widetilde{\Delta} = 0$ the function f must hence satisfy the system

$$\widehat{X}_a f + F_a = 0, \forall a \in \mathcal{G}. \tag{3.12))}$$

This system always has a solution f. To see this we note:

i) Condition (3.3) guarantees that compatibility between the different equations of the system (3.11) does not lead to new equations. The distribution generated by $\{\widetilde{X}_a\}$ is involutive.

ii) The fact that a strongly invariant equation $\widetilde{\Delta} = 0$ exists, guarantees that the homogeneous system $\widehat{X}_a f = 0$ has a nontrivial solution. This implies that $d < \dim M$ where d is the dimension of the corresponding involutive distribution. As analyzed above, condition (3.3) guarantees that the inhomogeneous system (3.11) also has a nontrivial solution.

Comments Theorem 1 greatly facilitates the search for invariant equations for any transformation Lie group G and its Lie algebra $\mathcal{G}$. Two situations can arise:

1. The vector fields $\{X_1, \ldots, X_l\}$ are such that a strongly invariant equation $\widetilde{\Delta} = 0$ satisfying equation (1.3) exists. The procedure then is: Find the most general function $\widetilde{\Delta} \in C^\infty(J^n(E, M))$ satisfying equation (1.3), then write $\Delta = e^{-f}\widetilde{\Delta}$ where $f \in C^\infty(J^n(E, M))$ is arbitrary. The most general weakly invariant equation is $\Delta = 0$ and it satisfies $\widehat{X}_a\Delta = -(\widehat{X}_a f)\Delta$.
2. No strongly invariant equation exists. In this case Theorem 1 is not directly applicable and weakly invariant equations may exist. To find them, we first take a subalgebra $\mathcal{G}_0 \subset \mathcal{G}$ for which a strongly invariant equation does exist ($\mathcal{G}_0$ should be a maximal subalgebra for which this property holds). We find the most general function $\widetilde{\Delta}_0$ annuled by the subalgebra, say $\{X_1, \ldots, X_{l_0}\}$, $l_0 < l$, and then use the fact that the equation we are looking for must have the form $\Delta = e^f\widetilde{\Delta}_0$ to annul the functions $F_1, \ldots, F_{l_0}$. This will greatly simplify the 1-cocycle conditions (3.3) and the search for the weakly invariant equation $\Delta = 0$.

It is also worth remarking that $X_a\Delta = 0$ if and only if the set of equations $\Delta = k$, with any $k \in \mathbb{R}$, is weakly invariant. This is so because if $X_a\Delta_{|\Delta=0} = 0$, there will be a function F_a such that $X_a\Delta = F_a\,\Delta$ and then, if $\Delta' = \Delta - k$, as $X_a\Delta' = F_a(\Delta' + k)$, the weak invariance of the equation $\Delta' = 0$ is equivalent to $F_a = 0$. As a consequence of this remark and the result of the preceding comments, if $\Delta = 0$ is invariant under G, there will exist a function f such that all equations $e^f\Delta = k$ are invariant under G.

Let us now illustrate the general results by some examples.

4. Examples

4.1. Invariant Ordinary Differential Equations.

We shall consider only first and second order ODEs

$$\Delta(x,u,u_x)=0,\qquad \frac{\partial\Delta}{\partial u_x}\neq 0, \tag{4.1}$$

$$\Delta(x,u,u_x,u_{xx})=0,\qquad \frac{\partial\Delta}{\partial u_{xx}}\neq 0, \tag{4.2}$$

We shall consider several different Lie algebras $\mathcal{G}$.

1.- $\dim\mathcal{G}=2$, $\mathcal{G}$ abelian, equation(4.1).

Up to equivalence under diffeomorphisms of $\mathbb{R}^2$, we have only one realization, namely

$$X_1=\frac{\partial}{\partial x},\qquad X_2=\frac{\partial}{\partial u} \tag{4.3}$$

We have $\widehat{X}_1=X_1$, $\widehat{X}_2=X_2$, for any order prolongation. Let us find all weakly invariant first order ODEs. We have:

$$\widehat{X}_1\Delta=F_1\Delta,\qquad \widehat{X}_2\Delta=F_2\Delta, \tag{4.4}$$

$$\frac{\partial F_1}{\partial u}-\frac{\partial F_2}{\partial x}=0, \tag{4.5}$$

where Δ, F_1 and F_2 are functions of x, u and u_x. From the cocycle condition of equation (4.5) we find

$$F_1=\phi_x(x,u,u_x),\quad F_2=\phi_u(x,u,u_x)+\psi_u(u,u_x), \tag{4.6}$$

where ϕ and ψ are arbitrary. Substituting equation (4.6) into equation (4.4) and solving for Δ, we find

$$\Delta=\widetilde{\Delta}(u_x)e^{\phi+\psi}, \tag{4.7}$$

where $\widetilde{\Delta}(u_x)$ is arbitrary.

Theorem 1 gives the same result directly, namely $\widetilde{\Delta}(u_x)=0$ is strongly invariant and we have

$$\Delta=e^{f(x,u,u_x)}\widetilde{\Delta}(u_x)=0,\ X_1\Delta=f_x\Delta,\ X_2\Delta=f_u\Delta, \tag{4.8}$$

i.e. $f=-\phi-\psi$.

2.- $\dim\mathcal{G}=4$, $\mathcal{G}$ nonabelian, equation (4.1).

We have

$$[X_1,X_2]=X_1,\ X_1=\frac{\partial}{\partial x},\ X_2=x\frac{\partial}{\partial x}+u\frac{\partial}{\partial u}. \tag{4.9}$$

The first prolongations satisfy $\widehat{X}_1=X_1$, $\widehat{X}_2=X_2$, so first order weakly invariant equations satisfy equation (4.4) with the cocycle condition

$$X_1F_2-X_2F_1=F_1. \tag{4.10}$$

The general solution of (4.10) is

$$F_1 = X_1\phi,\ F_2 = X_2\phi + u\psi_u(u,u_x). \tag{4.11}$$

Solving equation (4.4) for X_i as in (4.9), F_i as in (4.11) we obtain

$$\Delta = e^{f(x,u,u_x)}\widetilde{\Delta}(u_x), \tag{4.12}$$

with f and $\widetilde{\Delta}$ arbitrary, in agreement with Theorem 1.

3.- $\dim\mathcal{G} = 3$, $\mathcal{G}$ solvable, equation (4.1).

We have

$$[X_1,X_2] = 0,\ [X_1,X_3] = \alpha X_1 + \beta X_2,\ [X_2,X_3] = \gamma X_1 + \delta X_2. \tag{4.13}$$

The first prolongation of the vector fields is

$$\widehat{X}_1 = \frac{\partial}{\partial x}, \widehat{X}_2 = \frac{\partial}{\partial u}, \widehat{X}_3 = (\alpha x+\gamma u)\frac{\partial}{\partial x}+(\beta x+\delta u)\frac{\partial}{\partial u}+[\beta+(\delta-\alpha)u_x-\gamma u_x^2]\frac{\partial}{\partial u_x}. \tag{4.14}$$

The weakly invariant equations satisfy

$$X_i\Delta = F_i\Delta,\ i = 1,2,3, \tag{4.15}$$

$$\begin{aligned} &F_{2,x} - F_{1,u} = 0, \qquad F_{3,x} - (xF_{1,x} + uF_{1,u}) = \alpha F_1 + \beta F_2,\\ &F_{3,u} - (xF_{2,x} + uF_{2,u}) = \gamma F_1 + \delta F_2. \end{aligned} \tag{4.16}$$

Two cases must be considered separately.

i) $\alpha = \delta = 1,\ \beta = \gamma = 0$ (4.17)

A strongly invariant equation exists in this case, namely $\widetilde{\Delta}(u_x) = 0$, where $\widetilde{\Delta}$ is arbitrary. The general solution of equation (4.16) in this case is

$$\begin{aligned} &F_1 = \phi_x(x,u,u_x),\ F_2 = \phi_u(x,u,u_x) + \psi_u(u,u_x),\\ &F_3 = x\phi_x(x,u,u_x) + u\phi_u(x,u,u_x) + u\psi_u(u,u_x) + \sigma(u_x), \end{aligned} \tag{4.18}$$

Solving equations (4.15) directly in this case yields

$$\Delta = e^{\phi+\psi}\widetilde{\Delta}(u_x) = e^{f(x,u,u_x)}\widetilde{\Delta}(u_x), \tag{4.19}$$

and requires $\sigma(u_x) = 0$. Equation (4.17) again confirms Theorem 1.

ii) $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \neq \alpha\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ (4.20).

In this case no strongly invariant equation exists. Let us find all weakly invariant equations. The algebra $\{X_1,X_2\}$ allows a strongly invariant equation $\widetilde{\Delta}_0(u_x) = 0$ and all weakly invariant equations have the form $\Delta = e^f\widetilde{\Delta}_0$ (see equation (4.12)). This allows us to set $F_1 = F_2 = 0$ in equations (4.15) and (4.16). We hence have $F_{3,x} = F_{3,u} = 0$ and

$$X_1\Delta = 0,\ X_2\Delta = 0,\ X_3\Delta = F_3(u_x)\Delta, \tag{4.21}$$

i.e.,

$$[\beta+(\delta-\alpha)u_x-\gamma u_x^2]\frac{\partial\Delta}{\partial u_x}(u_x)=F_3(u_x)\Delta \tag{4.22}$$

and hence

$$\Delta = A\exp\left[\int \frac{du_x\, F_3(u_x)}{\beta+(\delta-\alpha)u_x-\gamma u_x^2}\right]. \tag{4.23}$$

In particular, a "reasonable" weakly invariant equation is

$$\widetilde{\Delta}=u_x-C, \tag{4.24}$$

with

$$\begin{cases} \delta-\alpha=-\frac{\beta}{C}+C\gamma, \text{for } C\neq 0 \\ \beta=0, \qquad\qquad \text{for } C=0 \end{cases}. \tag{4.25}$$

Further weakly invariant equations are

$$\widetilde{\Delta}=e^{f(x,u,u_x)}(u_x-C), \tag{4.26}$$

with f arbitrary (nonsingular for $u_x=C$).

4.- $\mathcal{G}=\mathfrak{sl}(3,\mathbb{R})$, equation(4.2).

Let us realize $SL(3,\mathbb{R})$ as the group of projective transformations of a plane. The second prolongation of the corresponding vector fields is:

$$\begin{aligned}
\hat{X}_1&=\frac{\partial}{\partial x},\\
\hat{X}_2&=\frac{\partial}{\partial u},\\
\hat{X}_3&=x\frac{\partial}{\partial u}+\frac{\partial}{\partial u_x},\\
\hat{X}_4&=x\frac{\partial}{\partial x}+2u\frac{\partial}{\partial u}+u_x\frac{\partial}{\partial u_x},\\
\hat{X}_5&=u\frac{\partial}{\partial u}+u_x\frac{\partial}{\partial u_x}+u_{xx}\frac{\partial}{\partial u_{xx}},\\
\hat{X}_6&=u\frac{\partial}{\partial u}-u_x^2\frac{\partial}{\partial u_x}-3u_xu_{xx}\frac{\partial}{\partial u_{xx}},\\
\hat{X}_7&=xu\frac{\partial}{\partial x}+u^2\frac{\partial}{\partial u}+(uu_x-xu_x^2)\frac{\partial}{\partial u_x}-3xu_xu_{xx}\frac{\partial}{\partial u_{xx}},\\
\hat{X}_8&=x^2\frac{\partial}{\partial x}+xu\frac{\partial}{\partial u}+(u-xu_x)\frac{\partial}{\partial u_x}-3xu_{xx}\frac{\partial}{\partial u_{xx}},
\end{aligned} \tag{4.27}$$

There is no strongly invariant equation of the form of equation (4.2). The algebra $\{X_1,X_2,X_3,X_4\}$ does allow a strongly invariant equation, namely $\widetilde{\Delta}(u_{xx})=0$ with $\widetilde{\Delta}$ arbitrary. Now let us look for a weakly invariant equation $\Delta=e^f\widetilde{\Delta}=0$. Putting

$X_i\Delta = 0$ for $i = 1, 2, 3, 4$, we find $f = f(u_{xx})$. Thus, we can simply look for an equation of the form $\Delta(u_{xx}) = 0$. We have

$$\begin{gathered} X_a\Delta(u_{xx}) = F_a\Delta, \quad a = 5, 6, 7, 8 \\ F_5 = -\frac{F_6}{3u_x} = -\frac{F_7}{3xu_x} = -\frac{F_8}{3x} \end{gathered} \tag{4.28}$$

and

$$X_5\Delta = u_{xx}\frac{\partial\Delta}{\partial u_{xx}} \tag{4.29}$$

Hence $\Delta(u_{xx})$ is any function satisfying

$$\left(\frac{\partial\Delta}{\partial u_{xx}}\right)_{\big|u_{xx}=0} < \infty \tag{4.30}$$

Any such equation can be transformed (by a point transformation) into

$$u_{xx} = 0. \tag{4.31}$$

That any second order ODE, invariant under $SL(3, \mathbb{R})$ can be transformed into equation (4.31) is of course well known [14].

4.2. Invariant Partial Differential Equations.

Here we restrict ourselves to a single example, namely the conformal group Conf$(1, 1)$ in 1+1 dimensional Minkowski space. It was shown in Ref.[7] that 5 different inequivalent realizations of the Lie algebra conf $\approx \mathfrak{so}(2, 2)$ exist (in the space of triplets (x, t, u). Let us consider the first realization [7] K_1. The vector fields are

$$\begin{gathered} P_0 = \frac{\partial}{\partial t}, \quad P_1 = \frac{\partial}{\partial x}, \quad K = x\frac{\partial}{\partial t} + t\frac{\partial}{\partial x}, \quad D = x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t}, \\ C_0 = (x^2 + t^2)\frac{\partial}{\partial t} + 2xt\frac{\partial}{\partial x}, \quad C_1 = -(x^2 + t^2)\frac{\partial}{\partial x} - 2xt\frac{\partial}{\partial t}. \end{gathered} \tag{4.32}$$

The most general strongly invariant equation of the form

$$\Delta(x, t, u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}) = 0 \tag{4.33}$$

is [7]

$$\Delta\left(u, \frac{u_{tt} - u_{xx}}{u_t^2 - u_x^2}\right) = 0, \tag{4.34}$$

or equivalently, if $\Delta_u \neq 0$,

$$\frac{u_{tt} - u_{xx}}{u_t^2 - u_x^2} - F(u) = 0, \tag{4.35}$$

where F is any function. Theorem 1 shows that this result is stronger than was taken credit for in Ref [7]. Indeed, up to an arbitrary multiplicative factor, equation (4.35) is the most general equation of the form (4.33), invariant under the Conf$(1, 1)$ group action corresponding to the realization (4.32), both strongly and weakly.

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