Nature of some random Jacobi matrices

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Abstract

In this paper we consider a random Jacobi matrix having purely absolutely continuous spectrum with a band structure and having possibly finitely many accumulation points of the spectrum. We show that such operators come from almost periodic potentials.

1 Introduction

In recent years, in the study of one-dimensional random Shrodinger operators, much interest has been shown in studying the influence of the nature of the spectrum on the randomness of the potential. Research in this direction has been carried out in both discrete and continuous settings. See for example the Kotani theory on the determinacy of potentials having some absolutely continuous spectrum; Kotani [6] for the continuous case and Simon [15] for the discrete case. Further it was shown by Kotani and Krishna [8] and Craig [3] that some random Shrodinger operators with purely absolutely continuous spectrum of a certain type are almost periodic of necessity. Proofs for such results consisted of setting up and solving the Dubrovin equation for some spectral parameters and Jacobi inversion on a Riemann surface. This was done for the continuous case by Levitan [11], Mc Kean and Moerbeke [12] and Mc Kean and Trubowitz [13].

In the discrete case Moerbeke [18] and Toda [17] have discussed the nature of the matrices constructed via inverse spectral theory. These turnout to be periodic. Carmona and Kotani [2] worked out the inverse spectral theory of random potentials, they construct an invariant probability measure associated with the relevant spectral functions of the random Jacobi matrix, through approximation by periodic ones. Kotani [7] has shown that if a random potential takes finitely many values and has some absolutely continuous spectrum then it is periodic. In [1] it was shown that for random Jacobi matrices , if the spectrum is purely absolutely continuous and consists of finite number of bands then the support of the random potential consists of only almost periodic sequences. In this paper we extend this result to the case where there are infinite number of bands with finite number of accumulation points. In [1] the proof was via Jacobi inversion on a Riemann surface using the Riemann theta function. But in the infinite bands case we have difficulty in showing the convergence of the series in the appropriate definition of the Riemann theta function unless we assume exponential decay of the gap lengths. So we use a direct method similar to the one used by Levitan in [11].

After this work was done, we came to know of the work of Oliver Knill [16] on the isospectral deformations of random Jacobi matrices. This paper in particular identifies a relevent Abel-Jacobi map and answers a question raised in [16], in a special case. We also received the paper of Gesztesy, Holden,Simon and Zhao [5] who discuss the relevent trace formulae and the inverse spectral theory in a more general context than that concerns us here.

Subsequently this paper contains five sections. In the first we define the random Jacobi matrix, state the assumptions on the spectrum and state the main theorem. In the second we describe the inverse spectral theory needed in the proof of the theorem. In the third we define a class of holomorphic functions on a subset of C and prove an interpolation theorem. In the fourth we construct a Riemann surface associated with the spectral functions and define a class of holomorphic differentials. Then we define the Abel-Jacobi map on a Banach space X and prove its properties. Finally we show the almost periodicity of the potential.

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2 Preliminaries

Consider $(\Omega, \mathcal{B}, \mathcal{P})$, where $\Omega =$ the space of real valued bounded sequences, $\mathcal{B}$ = the Borel σ -algebra on Ω generated by the topology of pointwise convergence on it and $\mathcal{P}$ = a probability measure on $(\Omega, \mathcal{B})$ which is invariant and ergodic with respect to the right translation T on Ω given by $(Tu)(n) = u(n-1)$. Corresponding to each $\omega \in \Omega$ we have a self-adjoint operator acting on $\ell^2(\mathbb{Z})$ given by

$$H^\omega = \Delta + \omega$$

where

$$(\Delta u)(n) = u(n-1) + u(n+1)$$

and

$$(\omega u)(n) = \omega(n)u(n); \quad u \in \ell^2(\mathbb{Z})$$

It is a result of P astur [14] that, under the assumptions on $\mathcal{P}$, the spectrum and spectral type of H^ω are constant for almost all ω, (see [6]); ie, there are subsets $\Sigma, \Sigma_{a.c}, \Sigma_{p.p}, \Sigma_{s.c}$ of R such that

$$\Sigma \;=\; \Sigma_{a.c} \cup \Sigma p.p \cup \Sigma s.c \;\;, \text{ (disjoint union)}$$

and there exists a subset Ω' of Ω with full measure such that for all $\omega \in \Omega'$

$$\sigma(H^\omega) \;=\; \Sigma,\; \sigma_{a.c}(H^\omega) \;=\; \Sigma_{a.c},\; \sigma_{p.p}(H^\omega) \;=\; \Sigma_{p.p} \quad \text{and} \sigma_{s.c} \;=\; \Sigma_{s.c}$$

In this paper we we assume that $\Sigma = \Sigma_{ac}$ and

$$\begin{aligned}\Sigma \;\; = \;\; & \{ \cup_{i=0}^{\infty}[\lambda_{2i+1}^{m}, \lambda_{2i}^{m}] \} \;\; \cup \;\; \{ \cup_{i=0}^{\infty}[\mu_{2i}^{0}, \mu_{2i+1}^{0}] \} \\ & \cup_{k=1}^{m-1}\{\cup_{i=0}^{\infty}[\lambda_{2i+1}^{k}, \lambda_{2i}^{k}] \;\; \cup_{i=0}^{\infty} [\mu_{2i}^{k}, \mu_{2i+1}^{k}]\} \\ & \cup_{k=1}^{m}\{\lambda_{\infty}^{k}\}\end{aligned}$$

where

$$\lambda_i^k > \lambda_{i+1}^k; \;\; \mu_i^k < \mu_{i+1}^k; \;\; k = 1, .., m-1; \;\; i = 0, ..$$

$$\lambda_i^m > \lambda_{i+1}^m; \;\; i = 1, ..$$

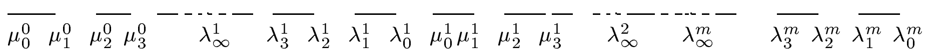
$$\mu_i^0 < \mu_{i+1}^0; \;\; i = 1, ..$$

$$\lambda_i^k < \mu_j^k; \;\; i, j = 1, ..., \;\; k = 1, .., m-1.$$

$$\mu_i^{k-1} \to \lambda_\infty^k \leftarrow \lambda_i^k \text{ as } i \to \infty; k = 1, .., m.$$

$$\lambda_\infty^k < \lambda_\infty^{k+1}; \;\; k = 1, .., m-1.$$

The spectrum looks as follows:



The gaps between any two consecutive spectral bands is called a spectral gap. We reindex the spectral gaps and write the i th gap as (τ_{2i-1}, τ_{2i}) and denote ℓ_i its length. It is clear that $\sum_{i=1}^{\infty} \ell_i < \infty$.

Assumptions 1 *There are points α_i and β_i on the spectral bands that are respectively to the left and right of the i'th spectral gap such that α_i lies to* *the right of the mid-point of the left spectral band and β_i lies to the left of the mid- point of the right spectral band ;*

$$s_i \equiv \tau_{2i-1} - \alpha_i = \beta_i - \tau_{2i}$$

such that

$\sum_i \frac{q_i}{s_i} < \infty$, *where* $q_i = \frac{\ell_i}{s_i}$.

Note that the above s_i will be used in the construction of a Banach space later on.

Before we proceed further, let us say that the condition on the summability of $\frac{q_i}{s_i}$ is used only in the proof of Lemma (3.5), in controlling some derivatives of Weyl functions around the accumulation points in the spectrum and nowhere else.

We now state our main

Theorem 2.1 *Consider $(\Omega, \mathcal{B}, \mathcal{P})$ be as above with $\mathcal{P}$ an ergodic probability measure. Suppose the spectrum of H^ω is purely absolutely continuous and satisfies assumptions 1. Then any $\omega \in \mathrm{supp}\mathcal{P}$ is an almost periodic sequence.*

Here we give a general outline of the proof of this theorem. Fix an $\omega \in \mathrm{supp}\mathcal{P}$ and consider the Green's kernel

$$g_\lambda^\omega(n,n) = \langle (H^\omega - \lambda)^{-1} \delta_n, \delta_n \rangle$$

It has exactly one zero $\xi_i(n)$ in the i'th spectral gap. Then the trace formula is that the potential is given by

$$\omega(n) = 1/2(\mu_0^0 + \lambda_0^m) + 1/2\sum(\tau_{2i} + \tau_{2i+1} - \xi_i^\omega(n))$$

So it is enough to show that the quantity on the R.H.S. of the above equation is an almost periodic sequence in n. Now each $(\xi_i(n))$, apart from being a zero of the Green's kernel, is a simple pole of exactly one of two Weyl functions which we will define in the next section and given this information it can be identified as a point on an infinite genus hyperelliptic Riemann surface $\mathcal{R}$ associated with the spectrum. We define the Abel-Jacobi map on X and show that the image of $(\xi_i^\omega(n))$ undergoes a linear motion. We then invert the Abel-Jacobi map and use the trace formula to show that $(\omega(n))$ is an almost periodic sequence.

3 Inverse spectral theory

Consider the operator H^ω defined in section 2, where $\omega \in \text{supp}\mathcal{P}$ the following facts are known from Simon [15]. Any $\lambda \in \mathbb{C}^+$ belongs to the resolvent of H^ω and we denote the greens kernel $\langle (H^\omega - \lambda)^{-1}\delta_n, \delta_n\rangle$ by $g_\lambda(n,n)$. Here δ_n is the basis vector which is 1 at the lattice point n and zero elsewhere. By Weyl theory, for the eigenvalue equation $(H^\omega - \lambda)\, u = 0$ there exists exactly one linearly independent solution each, which are integrable to the left and to the right respectively. Denote them by $u^\omega_{\pm,\lambda}$. Then we have

$$g_\lambda^\omega(n,n) = \frac{u^\omega_{+,\lambda}(n)u^\omega_{-,\lambda}(n)}{W} \qquad (1)$$

where W is the Wronskian of the solutions $u^\omega_{\pm,\lambda}$ given by $(u^\omega_{+,\lambda}(1)u^\omega_{-,\lambda}(0) - u^\omega_{+,\lambda}(0)u^\omega_{-,\lambda}(1))$. We also define the Weyl functions by

$$m^\omega_{\pm,n}(\lambda) = -\frac{u^\omega_{\pm,\lambda}(n \pm 1)}{u^\omega_{\pm,\lambda}(n)} \tag{2}$$

Then $m^\omega_{\pm,n}(\lambda)$ satisfies the equation

$$m^\omega_{\pm,n}(\lambda) = \omega(n) - \lambda - \left(m_{\pm,(n\mp1)}\right)^{-1} \tag{3}$$

In terms of the operators $H^\omega_{\pm,n}$ of H^ω restricted to the subspaces $\{u \in \ell^2(\mathbb{Z}^\pm)\ ;\ u(n) = 0\}$ we have

$$m^\omega_{\pm,n}(\lambda) = \langle (H^\omega_{\pm,n} - \lambda)^{-1}\delta_{n\pm1}, \delta_{n\pm1}\rangle \tag{4}$$

Definition 3.1 *F(z) defined on $\mathbb{C}^+$ is called Herglotz if it maps $\mathbb{C}^+$ into itself and is analytic.*

Theorem 3.1 *Let F(λ) be Herglotz. Then it has the representation*

$$F(\lambda) = a\lambda + b + 1/\pi \int_R \left(\frac{1}{\xi - \lambda} - \frac{\xi}{1+\xi^2}\right)\, d\mu(\xi)$$

where $a, b \in R,\ a \geq 0$ and $d\mu(\xi)$ is a non- negative measure such that

$$\begin{aligned} \text{supp } d\mu_{a.c} &= \{\ \xi\ :\ \lim_{\epsilon\to0}\ \text{Im } F(\xi + i\epsilon) \text{ exists and is finite }\} \\ \text{and supp } d\mu_{sing} &= \{\ \xi\ :\ \lim_{\epsilon\to0}\ \text{Im F}(\xi + i\epsilon) = \infty\ \} \\ \text{Moreover } d\mu_{a.c}(\xi) &= \text{Im } F(\xi + i\ 0).d\xi \end{aligned}$$

For a proof see [6] or [15].

Lemma 3.1 *The functions $g^\omega_\lambda(n,n)$ and $m^\omega_{\pm,n}(\lambda)$ are Herglotz.*

Proof: Follows easily from the spectral representation for Green's kernels.

Using the continuity properties of the green functions, in the potential, one can easily see the following theorem on the lines of [3],[6] or [15].

Theorem 3.2 *Suppose the spectrum of H^ω is purely absolutely continuous and is the union of closed intervals (finitely or infinitely many). Then*

$$\begin{aligned} \mathrm{Re} g^\omega_{\lambda+i0}(n,n) &= 0 \ \ \forall\ \omega \in \mathrm{supp}\mathcal{P} \text{ and } \forall\ n \in Z, \lambda \in \mathrm{Int}\ \sigma(H^\omega) \\ \mathrm{Im}\ g^\omega_{\lambda+i0}(n,n) &= 0\ \forall \omega \in \mathrm{supp}\mathcal{P} \text{ and } \forall\ n \in Z, \lambda \in R - \sigma(H^\omega) \end{aligned}$$

Here $g^\omega_{\lambda+i0}$ etc. denote the limits $\lim_{\epsilon\to 0} g^\omega_{\lambda+i\epsilon}$ etc. and $\mathrm{Int}\ \sigma(H^\omega)$ denotes the interior of the spectral bands.

Henceforth we fix an $\omega \in$ supp P and drop the subscript ω from the spectral functions. We also note that the green function is positive in the interval $(-\infty, \mu^0_0)$. We can use this fact in fixing the sign of the square root in the following lemma. We recall that (τ_{2i-1}, τ_{2i}) is the i'th spectral gap.

Lemma 3.2 *$g_\lambda(n,n)$ has exactly one zero $\xi_i(n)$ in the i'th spectral gap and has the expression*

$$\frac{1}{\sqrt{(\lambda-\mu^0_0)\ (\lambda-\lambda^m_0)}} \prod_{i=1}^{\infty} \frac{(\lambda-\xi_i(n))}{\sqrt{(\lambda-\tau_{2i-1})\ (\lambda-\tau_{2i})}} \quad ; \lambda \in \mathbb{C}^+ \tag{5}$$

Proof: First of all we show that the infinite product given above converges. For that we note that

$$\begin{aligned}
\left|\frac{\lambda-\xi_i(n)}{\lambda-\tau_{2i-1}}\right| &\leq 1+\left|\frac{\tau_{2i-1}-\xi_i(n)}{\lambda-\tau_{2i-1}}\right| \\
\text{and } \left|\frac{\tau_{2i-1}-\xi_i(n)}{\lambda-\tau_{2i-1}}\right| &\leq 1+q_i \text{ for all i except finitely many} \\
\text{and similarly } \left|\frac{\lambda-\xi_i(n)}{\lambda-\tau_{2i}}\right| &\leq 1+q_i \text{ for all i except finitely many.}
\end{aligned}$$

So the infinite product converges since the product $\prod_{i=1}^{\infty}(1+q_i)$ converges. Now in each spectral gap $g_\lambda(n,n)$ is an increasing function since $\frac{d}{d\lambda}g_\lambda(n,n) = \langle n,\ (H-\lambda)^{-2}\ n\rangle > 0$ and hence we can have at most one zero in each spectral gap. First let us assume that it has ,for each i, one zero $\xi_i(n)$in the i'th spectral gap; i=1,2,.... Now $g_\lambda(n,n)$ and log $g_\lambda(n,n)$ are Herglotz functions. By Theorem 3.1, $\lim_{\epsilon\to 0}\ g_{\lambda+i\epsilon}(n,n)$ exists and is finite for almost all $\lambda \in R$. Now by the continuity of the function log, $\lim_{\epsilon\to 0}$ log $g_{\lambda+i\epsilon}(n,n)$ exists wherever $\lim_{\epsilon\to 0}\ g_{\lambda+i\epsilon}(n,n)$ exists and is finite or infinite according as the latter limit is finite or infinite. Now by Theorem 3.2 the latter limit is finite at all points on the real line except perhaps at the spectral end-points and at the accumulation points. However these points won't contribute to the measure appearing in the Herglotz representation for log $g_\lambda(n,n)$ since it does not have exponential behaviour at infinity. So by Theorem 3.1

$$\log g_\lambda(n,n) = a\lambda + b + \frac{1}{\pi}\int_R \left(\frac{1}{(\xi-\lambda)} - \frac{\xi}{(1+\xi^2)}\right) \tag{6}$$

$$\mathrm{Im}\ \log\ (g_{\xi+i0}(n,n))\ d\xi \tag{7}$$

Now by Theorem 3.2, Im $g_{\xi+i0}(n,n) = 0$ for all $\xi \in \mathbb{R} - \Sigma$ and Re g_{xi+i0}(n,n) $= 0$ for all $\xi \in \Sigma$. Hence noting that $g_\lambda(n,n)$ is an increasing function of λ in each spectral gap we write down the density Im log $g_{\xi+i0}(n,n)$ as

$$\operatorname{Im}\log g_{\xi+i0}(n,n) = \begin{cases} 0 & \text{for} \quad \xi < \mu_0^0 \\ \pi & \text{for} \quad \tau_{2i-1} \leq \xi \leq \xi_i(n) \\ 0 & \text{for} \quad \xi_i(n) < \xi \leq \tau_{2i} \\ \pi & \text{for} \quad \xi \geq \lambda_0^m \\ \frac{\pi}{2} & \text{for} \quad \xi \in \Sigma \end{cases}$$

Substituting in (7) we get

$$\begin{aligned} \log g_\lambda(n,n) &= a\lambda + \tilde{b} + \frac{1}{2}\sum_{i=0}^{\infty}\int_{\tau_{2i}}^{\tau_{2i+1}} \frac{1}{(\xi-\lambda)}d\xi + \sum_{i=1}^{\infty}\int_{\tau_{2i-1}}^{\xi_i(n)} \frac{1}{(\xi-\lambda)}d\xi \\ &\quad + \int_{\lambda_0^m}^{\infty}\left(\frac{1}{\xi-\lambda} - \frac{\xi}{1+\xi^2}\right)d\xi \\ &= a\lambda + \tilde{b} + \sum_{i=0}^{\infty}\log\left(\sqrt{\frac{\tau_{2i+1}-\lambda}{\tau_{2i}-\lambda}}\right) + \sum_{i=1}^{\infty}\log\left(\frac{\xi_i(n)-\lambda}{\tau_{2i-1}-\lambda}\right) \\ &\quad + \left[\log\frac{(\xi-\lambda)}{\sqrt{1+\xi^2}}\right]_{\lambda_0^m}^{\infty} \\ &= a\lambda + d + \log\frac{1}{\sqrt{(\lambda-\mu_0^0)\,(\lambda-\lambda_0^m)}}\prod_{i=1}^{\infty}\frac{(\lambda-\xi_i(n))}{\sqrt{(\lambda-\tau_{2i-1})\,(\lambda-\tau_{2i})}} \end{aligned}$$

Now as $\lambda \to \infty$, $g_\lambda(n,n) \sim -\frac{1}{\lambda}$. Hence we get (5). In the proof we have assumed that in each spectral gap (τ_{2i-1}, τ_{2i}) there is a zero $\xi_i(n)$. Suppose it is not so. Then in such gaps $g_\lambda(n,n)$ is either non-negative or non-positive. In either case we can write down the density Im $\log_{\xi+i0}(n,n)$, perform the integration and will get the same expression as (5) with $\xi_i(n) = \tau_{2i-1}$ or τ_{2i}

according as $g_\lambda(n,n)$ is non-negative or non-positive in the i'th spectral gap. Hence the lemma.

As a corollary we have the following trace formula.

Corollory 3.1 *For any* $\omega \in \mathrm{supp}\mathcal{P}$ *we have*

$$\omega(n) = \frac{1}{2}(\mu_0^0 + \lambda_0^m) + \frac{1}{2}\sum_{i=1}^{\infty}((\tau_{2i} + \tau_{2i-1}) - 2\xi_i(n)) \tag{8}$$

Proof: $\omega(n)$ is the coeficient of $\frac{1}{\lambda^2}$ in the asymptotic expansion of $\langle (H^\omega - \lambda)^{-1}\delta_n, \delta_n\rangle$ and the R.H.S of (8) is the coefficient of $\frac{1}{\lambda^2}$ in the asymptotic expansion of the expression for $g_\lambda(n,n)$

Lemma 3.3 *For* $\lambda \in \mathbb{C} \setminus \Sigma$*,*

$$-g_\lambda(n,n)^{-1} = m_{+,n}(\lambda) + m_{-,n}(\lambda) + \lambda - \omega(n)$$

Proof: Follows from the definition of the above spectral functions and the Weyl formula (2) for the green functions.

Lemma 3.4 *Let the spectrum of* H^ω *be purely absolutely continuous and consists of bands; finitely or infinitely many. Then* $\forall \omega \in \mathrm{supp}\mathcal{P}$ *and for all* λ *belonging to the interior of any spectral band we have*

$$\mathrm{Im}\ m^\omega_{+,n}(\lambda) \ = \ \mathrm{Im}\ m^\omega_{-,n}(\lambda)$$

Proof: See [6] or [15].

Lemma 3.5 *The Weyl-functions have the expressions*

$$m_{\pm,n}(\lambda) = \frac{1}{2}\{ \quad \sqrt{(\lambda-\mu_0^0)(\lambda-\lambda_0^m)}\prod_{i=1}^{\infty}\frac{\sqrt{(\lambda-\tau_{2i-1})(\lambda-\tau_{2i})}}{(\lambda-\xi_i(n))}$$
$$-(\lambda-\omega(n)) \pm \sum_{i=1}^{\infty} c_{i,n}\frac{\mathrm{Res}(\xi_i(n))}{(\lambda-\xi_i(n))}\ \}$$

Proof: ¿From lemmas 3.3 and 3.2 we get an expression for $m_{+,n}(\lambda) + m_{-,n}(\lambda); \lambda \in \mathbb{C}^+$. So it is enough to obtain $m_{+,n}(\lambda) - m_{-,n}(\lambda)$ explicitly. From these two relations the lemma follows. To do this we proceed as follows.

We note that since $g_\lambda(n,n)$ has a simple zero at $\xi_i(n)$ in each gap, its inverse has simple poles at these points. Let $\mathrm{Res}(\xi_i(n))$'s denote the corresponding residues at these points. Define variables $c_{i,n} : \mathbb{Z}^+ \to \{-1,+1\}$ by $c_{i,n} = \pm 1$ according as $\xi_i(n)$ is a pole of $m_{\pm,n}(\lambda)$. Then by lemma (3.3) the residue of $(m_{+,n}(\lambda) - m_{-,n}(\lambda))$ at $\xi_i(n)$ is given by $c_{i,n}$ $\mathrm{Res}\xi_i(n)$. It will be proved in lemma (3.6) that $\sum_i \mathrm{Res}\xi_i(n)$ is absolutely convergent. Consider the function $f(\lambda) = m_{+,n}(\lambda) - m_{-,n}(\lambda) - \sum_{i=1}^{\infty} c_{i,n}\frac{\mathrm{Res}(\xi_i(n))}{(\lambda-\xi_i(n))}$. Apriori this is a holomorphic function defined on $\mathbb{C}\setminus\Sigma$. Now by lemma (3.4), Im $f(\lambda)$=0 forall λ lying in the interior of the spectral bands. Hence $f(\lambda)$ has analytic continuation through the interior of the spectral bands as well. We claim that it is analytic at the end-points of the bands also. For that it is enough to show that it is bounded in some neighbourhood of each of these points. If any of these points is a zero of $g_\lambda(n,n)$ then it is a simple pole of $m_{+,n}(\lambda) - m_{-,n}(\lambda)$ and hence from the definition of $f(\lambda)$ we see that it is bounded around that point. Now consider the end-points which are not zeroes of $g_\lambda(n,n)$. We have the following relationship

$$(m_{+,n}(\lambda) - m_{-,n}(\lambda) - (\lambda - \omega(n)))^2 = g_\lambda(n,n)^{-2} - 4\,\frac{g_\lambda(n+1,n+1)}{g_\lambda(n,n)} \quad (9)$$

This follows easily from the definition of $m_{\pm,n}(\lambda)$ and from the identity $(a-b)^2 = (a+b)^2 - 4ab$. From the expressions for the quantities appearing in the the R.H.S. of (9) we see that they are bounded in some neighbourhood of each of these end-points. Hence the L.H.S. of (9) and therefore $f(\lambda)$ are bounded around these points. Thus $f(\lambda)$ can be extended as a analytic function on $\mathbb{C} \setminus \cup_{k=1}^{m} \lambda_\infty^k$. We claim that it is analytic at λ_∞^k's also. For that we show that the non-holomorphic terms in the Laurent series expansion of $f(\lambda)$ vanish. For each k; k=1,...,m, define a sequence of rectangles R_i^k around λ_∞^k as follows: R_i^k has its sides parallel to the axes; the sides perpendicular to the x-axis cuts through the mid-points of suitable spectral bands to the left and right of λ_∞^k in such a way that the width of R_i^k shrinks to zero as $i \to \infty$, and the height above and below the x-axis will be the maximum of the distance of λ_∞^k from the sides parallel to the x-axis. Let $\lambda \in R_i^k$, and consider

$$\left|\frac{\lambda - \tau_i}{\lambda - \xi_i}\right| = 1 + \left|\frac{\xi_i - \tau_i}{\lambda - \xi_i}\right| \leq 1 + q_i$$

Doing the same analysis for the other terms of $g_\lambda(n,n)^{-2}$ and using the fact that the infinite product $\prod_i (1+q_i)$ converges we see that the first term on the R.H.S. of (9) is bounded uniformly on the R_i^k. Using an exactly similar argument we see that the second term also is bounded on the R_i^k. Now in the proof of the next lemma it will be shown that $\mathrm{Res}(\xi_i(n)) \sim \ell_i$. So $\sum_{i=1}^{\infty} c_{i,n} \frac{\mathrm{Res}(\xi_i(n))}{(\lambda - \xi_i(n))} \leq \text{const.} \sum_i q_i$. Thus $f(\lambda)$ is bounded on the R_i^k and hence

$\int_{R_i^k} f(\lambda) \to 0$ as $i \to \infty$. Thus the residue of $f(\lambda)$ at λ_∞^k is zero. Now multiplying $f(\lambda)$ by various powers of $(\lambda - \lambda_\infty^k)$ and doing the same analysis as above we see that the negative terms in the Laurent series expansion of $f(\lambda)$ around λ_∞^k vanish for each k=1,..,m. Hence $f(\lambda)$ is an entire function. Also, it vanishes at infinity since $m_{\pm,n}(\lambda)$ vanish there. Hence by Liouville's theorem $f(\lambda) \equiv 0$. From this and the expressions for the green's kernels, the lemma follows.

We now prove lemma (3.6) which was required in the proof of the above lemma.

Lemma 3.6 $\sum_i \mathrm{Res}(\xi_i(n))$ *is absolutely convergent.*

Proof:

$$\begin{aligned}
\mathrm{Res}\ \xi_i(n) &= \left(-\sqrt{(\xi_i(n)-\mu_0^0)(\xi_i(n)-\lambda_0^m)}\right) \\
&\quad \times \left(\prod_{j\neq i} \frac{\sqrt{(\xi_i(n)-\tau_{2j-1})(\xi_i(n)-\tau_{2j})}}{(\xi_i(n)-\xi_j(n))}\right) \\
&\quad \times \left(\sqrt{(\xi_i(n)-\tau_{2i-1})(\xi_i(n)-\tau_{2i})}\right)
\end{aligned}$$

The quantity in the first bracket is bounded by a constant independent of i while that in the third bracket is bounded by ℓ_i. Now for $j \neq i$

$$\left|\frac{\xi_i(n)-\tau_{2j-1}}{\xi_i(n)-\xi_j(n)}\right| \leq 1 + \frac{|\xi_i(n)-\tau_{2j-1}|}{|\xi_i(n)-\xi_j(n)|} \leq 1 + q_j$$

Similarly

$$\left|\frac{\xi_i(n)-\tau_{2j}}{\xi_i(n)-\xi_j(n)}\right| \leq 1 + q_i$$

So the quantity in the second bracket is bounded by $\prod_j (1 + q_j)$. Hence $|\text{Res}\xi_i(n)| \leq c\ell_i$, where c is a constant independent of i. The lemma follows since $\sum_i \ell_i < \infty$. We have as a corollory of lemma 3.5 the following ,

Theorem 3.3 *Consider P an ergodic potential as above and Σ its absolutely continuous spectrum, then the quantities $\{\xi(0), c_{i,0}\}$ determine the potential ω uniquely.*

Proof From the above proof it is clear that the data Σ, $\{\xi_i(0), c_{i,0}\}$ determine the m functions uniquely and also by the trace formula, $\omega(0)$ uniquely. Hence the theorem.

As a consequence of the proof of lemma 3.5, we have the following interesting corollory.

Corollory 3.2 *The m functions $m_{\pm,n}(\lambda)$ and $(m_{\pm,n}(\lambda))^{-1}$ are uniformly bounded on the rectangles R_i^k defined in the lemma 3.5. Moreover the derivatives of $m_{\pm,n}(\lambda)$ are uniformly bounded on $R_i^k \cap (\mathbb{C} \setminus \Sigma)$ for each i and k.*

Proof: Clearly the bounds obtained in the proof of lemma 3.5 proves the corollory for $m_{\pm,n}(\lambda)$. On the other hand for the inverses of the m functions, we use the relations of equation 3 on the portion of R_i^k 's in the resolvent of $H^{\pm}$ and extend the bounds by continuity of $m_{\pm,n}$ on the points in the spectrum. We have the expression for $m_{+,n}(\lambda)$ from Lemma 3.5 as

$$m_{+,n}(\lambda) = -g_\lambda(n,n)^{-1} + \sum \frac{Res\xi_i(n)}{\lambda - \xi_i(n)}$$

Therefore it is enough to show the boundedness of the derivatives of each of the terms on the right hand side of the above equation. As for the boundedness of the second term it follows from the assumptions and the estimates as in Lemma 3.5 and regarding the boundedness of the derivative of the inverse of the green function, we note that it is a meromorphic funciton in $\mathbb{C} \setminus \Sigma$ and the derivative is given by

$$\begin{aligned} \frac{d}{d\lambda} g_\lambda(n,n)^{-1} &= \frac{1}{2} g_\lambda(n,n)^{-1} \times \Bigg(\frac{1}{(\lambda-\lambda_0^0)^2} + \frac{1}{(\lambda-\lambda_0^m)^2} \\ &\quad + \sum_i \frac{\tau_{2i-1}-\xi_i(n)}{(\lambda-\tau_{2i-1})(\lambda-\xi_i(n))} + \frac{\tau_{2i}-\xi_i(n)}{(\lambda-\tau_{2i})(\lambda-\xi_i(n))} \Bigg) \end{aligned}$$

Now note that the above equation shows that using a computation similar to the one used in the Lemma 3.5 , that apriori the derivative is bounded on the rectangles lying in any open set of $\mathbb{C} \setminus \Sigma$, and with a uniform bound coming from the proof of Lemma 3.5. We note that hence, even the lim sup of the derivative along any sequence on the rectangle approaching the real axis, both from $\mathbb{C}^+$ and from $\mathbb{C}^-$ is uniformly bounded. Hence the result.

Remark 3.1 *For future use we note the following*

$$\begin{aligned} -m_{-,n}(\lambda) - (\lambda - \omega(n)) &= \frac{1}{2}\Big\{ \sqrt{(\lambda-\mu_0^0)(\lambda-\lambda_0^m)} \prod_{i=1}^{\infty} \frac{\sqrt{(\lambda-\tau_{2i-1})(\lambda-\tau_{2i})}}{(\lambda-\xi_i(n))} \\ &\quad -(\lambda-\omega(n)) + \sum_{i=1}^{\infty} c_{i,n} \frac{\mathrm{Res}(\xi_i(n))}{(\lambda-\xi_i(n))} \Big\} \end{aligned}$$

That is, the expressions for $m_{+,n}(\lambda)$ and $-m_{-,n}(\lambda) - (\lambda - \omega(n))$ differ only in the sign of the radical appearing in them.

Also from (4) we note that $m_{+,n}(\lambda) \sim -\frac{1}{\lambda}$ *as* $\lambda \to \infty$ *and* $-m_{-,n}(\lambda) - (\lambda - \omega(n)) \sim -\lambda$ *as* $\lambda \to \infty$.

4 An interpolation theorem

Consider the domain $\Pi = \mathbb{C} \setminus \Sigma$, Σ the spectrum. We define a class of functions on Π having zeroes "not more than the number of spectral gaps minus one". For $i = 1, 2, .., \infty$, let (ξ_j^i) be such that ξ_j^i belongs to the i'th spectral gap. We consider functions of the form

$$\begin{aligned} \omega_{i,\xi} &= \frac{1}{\sqrt{(\lambda-\mu_0^0)(\lambda-\lambda_0^m)}} \left(\prod_{j \neq i} \frac{(\lambda-\xi_j^i)}{\sqrt{(\lambda-\tau_{2j})(\lambda-\tau_{2j-1})}} \right) \\ &\quad \times \frac{1}{\sqrt{(\lambda-\tau_{2i})(\lambda-\tau_{2i-1})}} \end{aligned}$$

Let κ_i be complex numbers such that $|\kappa_i| < c\ \ell_i; i = 1, 2, ...$ where c is a constant. Then we have

Lemma 4.1 *Let* $\omega_{i,\xi}$ and κ_i *be as above. Then* $\sum_i \kappa_i\ \omega_{i,\xi}$ *is a holomorphic function on* Π.

Proof: Since each $\omega_{i,\xi}$ is holomorphic in Π it is enough to show that $\sum_i \kappa_i\ \omega_{i,\xi}$ converges uniformly on compact sets. Let λ belong to a fixed compact set in Π. The following estimates are clear, except possibly for finitely many inidces i and j.

$$\begin{aligned}
\left|\frac{(\lambda-\xi_j^i)}{\sqrt{(\lambda-\tau_{2j})(\lambda-\tau_{2j-1})}}\right| &\leq \sqrt{1+\frac{\xi_j^i-\tau_{2j}}{\lambda-\tau_{2j}}}\sqrt{1+\frac{\xi_j^i-\tau_{2j-1}}{\lambda-\tau_{2j-1}}} \\
&\leq \sqrt{1+q_j}\sqrt{1+q_j} \\
&= 1+q_j
\end{aligned}$$

and

$$\frac{1}{\sqrt{(\lambda-\tau_{2i})(\lambda-\tau_{2i-1})}} \leq \frac{1}{b_i}$$

Here b_i is the minimum of the lengths of the spectral bands to the left and right of the i'th spectral gap. Hence,

$$\left|\frac{\kappa_i}{\sqrt{(\lambda-\tau_{2i})(\lambda-\tau_{2i-1})}}\right| \leq c\,\frac{\ell_i}{b_i} = c\ q_i \text{ except possibly for finitely many i's.}$$

Hence we see that the series $\sum_i \kappa_i\ \omega_{i,\xi}$ is majorised by the series $d\sum_i q_i$, where d is a suitable constant, and this series is convergent. Hence the lemma.

Definition 4.1 *Let Ψ be the collection of sequences (ξ_j^i)'s indexed by the superscripts i's such that for each i, ξ_j^i belongs to the closure of the j'th spectral gap. We define a class $\mathcal{M}$ of holomorphic functions on Π by*

$$\mathcal{M} = \left\{\sum_i \alpha_i\ \omega_{i,\xi}\ ; (\xi) \in \Psi \text{ and } \kappa_i \text{ are such that } |\kappa_i| \leq c\ \ell_i,\ \text{for all i.}\right\}$$

Now we will show that the elements of $\mathcal{M}$ have the property that they can be interpolated by their value at one point each in each spectral gap.
Let $(\mu_i) \in \Psi$. Define

$$\Omega_\mu(\lambda) = \frac{1}{\sqrt{(\lambda-\mu_0^0)(\lambda-\lambda_0^m)}}\prod_i \frac{(\lambda-\mu_i)}{\sqrt{(\lambda-\tau_{2i})(\lambda-\tau_{2i-1})}}$$

By an argument similar to one used in the proof of Lemma 3.2 we can see that the infinite product converges uniformly and absolutely on compact subsets of Π and hence defines a holomorphic function there since the finite products are holomorphic. Consider an element of $\mathcal{M}$ given by

$$f(\lambda) = \sum_i \kappa_i \; \omega_{i,\xi}$$

Now fix a $\lambda \in R_i$, where R_i is the union of the contours R_i^k encircling the λ_∞^k's as given in Lemma 3.5. Then

$$\left|\frac{\lambda - \xi_j^i}{\lambda - \mu_j}\right| \leq 1 + \left|\frac{\xi_j^i - \mu_j}{\lambda - \mu_j}\right| \leq c \; q_i,$$

for all j and i

$$\left|\frac{\kappa_i}{\lambda - \mu_i}\right| \leq c \frac{\ell_i}{|\lambda - \mu_i|} \leq c \; q_i$$

for all i. Hence we see that

$$\mathrm{Sup}_{\lambda \in \cup_i R_i} \left|\frac{f(\lambda)}{\Omega_\mu(\lambda)}\right| \leq d \sum_i q_i \leq \tilde{d}, \quad \text{say.}$$

That is, $\left|\frac{f(\lambda)}{\Omega_\mu(\lambda)}\right|$ is bounded uniformly on $\sum_i R_i$, and this bound is independent of μ. Also we see that as $\lambda \to \infty$, $\frac{f(\lambda)}{\Omega_\mu(\lambda)} \sim \frac{1}{\lambda^2}$.
These two properties enable us to prove the

Theorem 4.1 *(Interpolation theorem)*
Let $f(\lambda) \in \mathcal{M}$ and let $(\mu) \in \Psi$. Then

$$f(\lambda) = \left(\sum_i \frac{1}{\Omega'_\mu(\mu_i)} \frac{f(\mu_i)}{(\lambda - \mu_i)}\right) \Omega_\mu(\lambda), \quad \text{for all} \lambda \in \Pi$$

Here Ω'_μ denotes $\frac{d}{d\lambda}\Omega_\mu(\lambda)$.

Proof: Consider

$$\frac{f(\lambda)}{\Omega_\mu(\lambda)} = \sum_i \kappa_i \prod_{j\neq i} \left(\frac{\lambda - \xi^i_j}{\lambda - \mu_j}\right) \frac{1}{(\lambda - \mu_i)} \tag{10}$$

Consider λ belonging to a compact subset K of C excluding the μ_i's and λ^k_∞'s. Then

$$\begin{aligned} \left|\frac{\lambda - \xi^i_j}{\lambda - \mu_i}\right| &\leq 1 + \left|\frac{\mu_i - \xi^i_j}{\lambda - \mu_i}\right| \\ &\leq 1 + q_i \text{ for all i and j except finitely many,} \\ \text{and } \sup_{i,\lambda\in K} \frac{1}{|\lambda - \mu_i|} &< \infty \end{aligned}$$

Hence, since $\sum_i \kappa_i$ converges, the R.H.S of (10) converges absolutely and uniformly for all $\lambda \in K$. Thus it defines an analytic function on the complex plane minus the μ_i's and the λ^k_∞'s. Therefore $\frac{f(\lambda)}{\Omega_\mu(\lambda)}$ which is apriori a meromrphic function on Π can be extended as a meromorphic function on $\mathbb{C}$ with simple poles at μ_i and with λ^k_∞'s as other singularities. Now fix $\lambda \in \Pi$. Then $g(z) = \frac{f(z)}{\Omega_\mu(z)} \frac{1}{z-\lambda}$ is a meromorphic function on $\mathbb{C} \setminus \{\cup_{k=1}^m \lambda^k_\infty\}$ with simple poles at the μ_i's. The residue at any μ_i is $\frac{f(\mu_i)}{\Omega'_\mu(\mu_i)(\lambda-\mu_i)}$ while the residue at λ is $\frac{f(\lambda)}{\Omega(\lambda)}$ and the residue at infinity is zero. Also since $\frac{f(z)}{\Omega_\mu(z)}$ is bounded uniformly on R_i, $\int_{R_i} \frac{f(z)}{\Omega_\mu(z)} \frac{1}{(z-\lambda)} \to 0$, as $i \to \infty$.
Hence by residue theorem

$$\frac{f(\lambda)}{\Omega_\mu(\lambda)} + \sum_i \frac{f(\mu_i)}{\Omega'^{/}_\mu(\mu_i)\ (\mu_i - \lambda)} = 0$$

Therefore

$$f(\lambda) = \Omega_\mu(\lambda) \left(\sum_i \frac{f(\mu_i)}{\Omega'_\mu(\mu_i)\ (\lambda - \mu_i)}\right)$$

Corollary 4.1 *Consider $f(\lambda) \in \mathcal{M}$. Suppose f has a zero, say η_i in each I_i, the closure of the i th spectral gap. Then* $f(\lambda) \equiv 0$ on Π

Proof: Consider $\Omega_{(\eta)}$. Then by theorem 4.1

$$f(\lambda) = \Omega_{(\eta)}(\lambda) \left(\sum_i \frac{f(\eta_i)}{\Omega'_\eta(\eta_i)\ (\lambda - \eta_i)} \right) = 0$$

There is a very interesting corollory of the interpolation theorem that we will need in the Jacobi inversion.

Corollory 4.1 *Consider a set of points θ_j one in each $[0, \pi]$ Then there exists a sequence of numbers C_i^j , with $C_i^j \leq c\ \ell_i$, j,i =1,2,.., $\sum_{i=1}^{\infty} C_i^j f_{ik}(\theta_k) = 0$ for each $k \neq j$ and $\sum_{i=1}^{\infty} C_i^j f_{ij}(\theta_j) \neq 0$. where f_{ij} are the functions appearing in equation (23).*

Proof: Consider the change of variables,

$$x_i = \tau_{2i-1} + \ell_i sin^2(\theta_i)$$

from S_i to $I_i = [\tau_{2i-1}, \tau_{2i}]$,where S_i is one of $[0, \frac{\pi}{2}], [\frac{\pi}{2}, \pi]$, and consider the analytic functions

$$\begin{aligned} g_j(\lambda) &= \frac{1}{\sqrt{(\lambda - \mu_0^0)(\lambda - \lambda_0^m)}} \left(\prod_{i \neq j} \frac{(\lambda - x_i)}{\sqrt{(\lambda - \tau_{2i})(\lambda - \tau_{2i-1})}} \right) \\ &\quad \times \frac{1}{\sqrt{(\lambda - \tau_{2j})(\lambda - \tau_{2j-1})}} \end{aligned}$$

on Π. By the interpolation theorem, these can be expressed as a linear combination of f_i's appearing in the equation (11). Let the constants

$$\int_{\tau_{2i-1}}^{\tau_{2i}} g_j d\lambda = C_i^j$$

Then clearly , by a simple estimate on the functions g_i it follows that , $C_i^j \leq c\ \ell_i$ and therefore by the interpolation theorem that $g_j(\lambda) = \sum C_i^j f_i(\lambda)$ for each λ in any I_k. Write λ in I_k as $\lambda = \tau_{2k-1} + \ell_k sin^2(\theta)$. The the above relation takes the form

$$\sqrt{(\lambda - \tau_{2k-1})(\lambda - \tau_{2k})}g_j(\lambda) = \sum_i C_i^j f_{ik}(\theta)$$

The result now follows if we take $\lambda = x_k = \tau_{2k-1} + \ell_k sin^2(\theta_k)$, for $k \neq j$ since g_j has a zero there. As for the case k = j it is obvious since g_j does not have a zero there.

5 The Riemann surface

In this section we construct a Riemann surface $\mathcal{R}$ whose branch points are the λ_i's and the μ_i's. We take two copies of the Riemann sphere and delete the points λ_∞^k's from each. The rest of the procedure is similar to the construction of a hyperelliptic Riemann surface with finite number of branch points. On each sphere we cut the real line along the spectral bands and glue the resulting spheres together by joining the upper lips of the cuts of the first one to the corresponding lower lips of the cuts of the second one and vice versa. Thus we get a non-compact hyperelliptic surface of infinite genus. Also, we denote the points at infinity of the upper and lower sheets by ∞_1 and ∞_2 respectively.

We denote by λ the projection map from $\mathcal{R}$ into the Riemann sphere. On $\mathcal{R}$ we give a coordinate system as follows. At ∞_1 we give local coordinates by mapping a neighbourhood of it to a neighbourhood of the origin in the Riemann sphere by the map $\infty_1 \to 0$ and $p \to \frac{1}{\lambda(p)}$ for other points p in the neighbourhood. Similarly we give coordinates around ∞_2. At point p_n corresponding to a spectral end point τ_n we give the coordinate system given by $z^2 = \lambda(p) - \tau_n$. For the rest of the points we give coordinate systems via the projection map λ. Thus the Riemann surface is a two sheeted branched cover of $\hat{\mathbb{C}} \setminus \cup\{\lambda_\infty^k\}$ with the points $\{\mu_0^0, \lambda_0^m, \tau_{2i}$ *and* $\tau_{2i+1}\}$ i = 1,2,.... as the branch points.

We now give some definitions.

Definition 5.1 *The closed loop which starts from* τ_{2i-1} *, goes to* τ_{2i} *through the embedded spectral gap on the lower sheet and comes back to* τ_{2i-1} *through the similar spectral gap on the upper sheet is called the i'th* β*- cycle and is denoted by* β_i*.*

Let P_0 *be any point on the embedded real line on the lower sheet to the left of* μ_0^0 *such that the distance between* P_0 *and* μ_0^0 *is greater than the the distance between* μ_0^0 *and* λ_0^m*. The closed loop, lying on the lower sheet, that starts from some point on* β_i *going to* P_0 *and coming back to the same point through the other side of the lower sheet is called the i'th alpha cycle and is denoted by* α_i*.*

Now we define certain open subsets $\mathcal{R}_N$ of $\mathcal{R}$ which are (non-compact) Riemann surfaces in their own right. Consider the sequence R_N^k k=1,..,m of rectangles considered in the lemma 3.5. Consider the closed region $\bar{R}_N^k$ enclosed by the rectangles R_N^k in $\mathbb{C}$ for each k = 1,..,m. We shall take henceforth the rectangles with smoothed out corners so that the boundary of each of these regions is a smooth curve. Let $S_N = \cup_k \bar{R}_N^k$. Let $\mathcal{R}_N = \mathcal{R} \setminus S_N$. Then $\mathcal{R}_N$ is an open region of the Riemann surface $\mathcal{R}$ and is a Riemann surface by its own right for each N. Also $\mathcal{R}_N \subset \mathcal{R}_M$ if N $<$ M.

Now each of the $\mathcal{R}_N$'s is an open Riemann surface of fininte type. That is it is of finite genus with finitely many holes. This fact will be needed later. First we shall consider a class of differentials on $\mathcal{R}$ which will be normalized on using the α and β cycles considered above.

Lemma 5.1 *Each of the functions of class $\mathcal{M}$ considered in the Definition (4.1) gives rise to a holomorphic differential on $\mathcal{R}$.*

Proof: Consider the differentials $f(\lambda)d\lambda$ on $\mathcal{R}$ with f $\in \mathcal{M}$. Apriori this is a differential defined on $\lambda^{-1}(\Pi)$ and a computation in each of the local coordinate charts on $\mathcal{R}$ shows that it extends to a holomorphic differential on the whole of $\mathcal{R}$.

Now we fix a basis of differentials on $\mathcal{R}$.

Lemma 5.2 *For each i=1,2,.. we can find sequences (η_j^i); $j = 1, 2, .., j \neq i$,* with *η_j^i in the j'th spectral gap such that on $\mathbb{C} \setminus \Sigma$ if we define the differential, for some numbers σ_i,*

$$\begin{aligned} d\omega_i &= \frac{\sigma_i d\lambda}{\sqrt{(\lambda - \mu_0^0)(\lambda - \lambda_0^m)}} \frac{1}{\sqrt{(\lambda - \tau_{2i-1})(\lambda - \tau_{2i})}} \prod_{j \neq i} \frac{(\lambda - \eta_j^i)}{\sqrt{(\lambda - \tau_{2j-1})(\lambda - \tau_{2j})}} \\ &= f_i(\lambda)d\lambda \end{aligned} \tag{11}$$

then $\int_{\tau_{2j-1}}^{\tau_{2j}} d\omega_i = 0$ for *$j \neq i$; $i, j = 1, 2, ...$ and $\int_{\tau_{2i-1}}^{\tau_{2i}} d\omega_i$ is nonzero.*

For the proof of the above lemma we need the following

Lemma 5.3 *Let $I_k = [a_k, b_k]$; $k = 1, 2, .., n$ be intervals in the real line. Define $I = \prod_{k=1}^{n} I_k$. Let $F : I \rightarrow R^n$ be a continuous function such that $F_k(x_1, .., a_k, .., x_n) < 0$ and $F_k(x_1, .., b_k, .., x_n) > 0$ $\forall k = 1, .., n$, where $F_k(.)$ denotes the k'th coordinate of F. Then F has a zero in the interior of I.*

Proof: Without loss of generality we can assume that $a_k < 0$ and $b_k > 0$ $\forall k$. The straight line homotopy $x \rightarrow tF(x) + (1-t)x; 0 \leq t \leq 1$ gives a homotopy between F and the identity map [4]. The conditions $F(x_1, .., a_k, .., x_n) < 0$ and $F(x_1, .., b_k, .., x_n) > 0$ ensure that throughout the homotopic deformation no point in the the boundary of I goes to zero. Hence the topological degree of F is the same as that of the identity map which is 1. So F has at least one zero in the interior of I.

Proof of Lemma 5.2: Fix an i. Let I_k denote the closure of the k'th spectral gap and corresponding to each (n-1) tuple $(x_1, .., x_k, .., x_n)$; $x_k \in I_k$; $k \neq i$ define a function of a real variable λ by

$$f^n(x_1, .., x_n)(\lambda) = \frac{1}{\sqrt{(\lambda - \mu_0^0)(\lambda - \lambda_0^m)}} \frac{1}{\sqrt{(\lambda - \tau_{2i-1})(\lambda - \tau_{2i})}} \left(\prod_{j \neq i_{j=1}}^{n} \frac{(\lambda - x_j)}{\sqrt{(\lambda - \tau_{2j-1})(\lambda - \tau_{2j})}} \right)$$

Define $F^n : \prod_{k=1 k \neq i}^{n} I_k \rightarrow R^{n-1}$ by

$$F^n(x_1, ..., x_n) = \left(\int_{I_1} f^n(x_1, .., x_n)(\lambda) d\lambda, ..., \int_{I_n} f^n(x_1, .., x_n)(\lambda) d\lambda \right)$$

We claim that F^n has a zero. It is clear that if we multiply the integrals occuring in the definition of F^n by scalars (in particular -1), the zeros of

F^n are unaffected. So, after such multiplications if necessary, we see that F^n satisfies the conditions of the lemma 5.3. Hence it has a zero $(x_1^n, .., x_n^n)$. This way corresponding to each n we get an n-tuple. Since x_1^n's move in a compact set as n varies, we can find a subsequence n_1 such that the corresponding $x_1^{n_1}$'s converge to a point η_1^i. From the above subsequence we can find another subsequence n_2 such that the corresponding $x_2^{n_2}$'s converge to a point η_2^i. This way inductively we can define a sequence of points η_k^i's. Using these η_k^i's we define $d\omega_i$ as in (11). We claim that

$$\int_{I_k} d\omega_i = 0, \ \forall k \neq i$$

Fix a k. Let $\epsilon > 0$ be given. Using estimates similar to those in lemma 3.2 we see that there exists an n_0 such that, $\forall \lambda \in I_k$ all the tail products of the R.H.S. of (11) after n_o terms differ from 1 at most by ϵ. So

$$|\int_{I_k} d\omega_i - \int_{I_k} d\omega_i^n| < \epsilon.\ \ell_k;\ \forall n > n_0$$

where $d\omega_i^n$ denote the n'th partial product of $d\omega_i$. Again using estimates similar to those in lemma 3.2 we see that uniformly in I_k any two partial products of $d\omega_i^n$ or f^n exceeding the n_0'th one differ only by ϵ. So choosing n large enough so that the n_0'th partial product of $d\omega_i^n$ and an $f^m(.,,.)(\lambda)$ differ uniformly in I_k by ϵ we see that

$$|\int_{I_k} d\omega_i| < 4\epsilon\ .\ \ell_k$$

Since ϵ is arbitrary the lemma follows for j$\neq$i. For the case j = i , it is clear that the integrand has a definite sign as the beta cycle is traversed, since the change of sign in the integrand as one changes sheets is compensated by the reversal of the contour of integration.

Lemma 5.4 *The differentials $d\omega_i$ satisfy the normalizations*

1. $\int_{\beta_j} d\omega_i \; = \; \pi \; \delta_{ij}$

2. $\int_{\alpha_j} d\omega_i \; = \; \pi_{ij}$ *with* π_{ij} *purely imaginary.*

Proof: Let $\int_{\alpha_j} d\omega_i \; = \; \pi_{ij}$. Since $\int_{\beta_i} d\omega_i$ is real, the normalizing constant is real. Therefore chosing σ_i appropriately we will get the stated normalization on the beta cycles in view of Lemma (5.2). Now we can deform α_i into the closed loop that starts from τ_{2i}, goes to λ_0^m and comes back to τ_{2i} entirely through the embedded real line in such a way that while coming back the embedded spectral gaps it traverses are the same as those of the onward journey whereas the embedded spectral bands it traverses are the ones other than the ones already traversed. Hence we see that π_{ij} is two times the sum of the integrals of $d\omega_i$ over the spectral bands lying to the right of τ_{2i}. So from (11) we see that π_{ij} is purely imaginary.

Now the following lemma shows that while taking estimates we can omit this normalizing constant and in future, for $d\omega_i$ we use the same expression as on the R.H.S. of (11).

Lemma 5.5 *There exists constants $c_1, c_2 > 0$ such that*

$$c_1 < |(\int_{\beta_i} d\omega_i)| < c_2$$

Proof: First note that $\int_{\beta_i} d\omega_i \; = \; 2 \int_{\tau_{2i-1}}^{\tau_{2i}} (d\omega_i)$. Since $d\omega_i$ has the same sign in the interval of integration it is enough to show that there exists $c_1, c_2 > 0$ such that

$$c_1 \; < \int_{\tau_{2i-1}}^{\tau_{2i}} |d\omega_i| \; < c_2$$

We split the integrand into $f_1(\lambda)$. $f_2(\lambda)$ where

$$f_1(\lambda) = \frac{1}{\sqrt{(\lambda-\mu_0^0)(\lambda-\lambda_0^m)}} \prod_{j\neq i} \frac{(\lambda-\eta_j^i)}{\sqrt{(\lambda-\tau_{2j-1})(\lambda-\tau_{2j})}} \tag{12}$$

$$\text{and } f_2(\lambda) = \frac{1}{\sqrt{(\lambda-\tau_{2i})(\lambda-\tau_{2i-1})}} \tag{13}$$

Now

$$\left|\frac{\lambda-\eta_j^i}{\lambda-\tau_{2j-1}}\right|^{\frac{1}{2}} = \left|1+\frac{\tau_{2j}-\eta_j^i}{\lambda-\tau_{2j}}\right|^{\frac{1}{2}}$$

Now for all j except finitely many we have $|q_j| < 1$. Hence for such j

$$(1-q_j)^{\frac{1}{2}} < \left|\frac{\lambda-\eta_j^i}{\lambda-\tau_{2j-1}}\right|^{\frac{1}{2}} < (1+q_i)^{\frac{1}{2}}, \text{ since} \left|\frac{\tau_{2j}-\eta_j^i}{\lambda-\tau_{2j}}\right| < q_j$$

Similarly

$$(1-q_j)^{\frac{1}{2}} < \left|\frac{\lambda-\eta_j^i}{\lambda-\tau_{2j}}\right|^{\frac{1}{2}} < (1+q_j)^{\frac{1}{2}}$$

Hence throughout the interval of integration

$$c_3 \prod_{j\neq i}(1-q_j) < |f_1(\lambda)| < \prod_{j\neq i}(1+q_j)\ c_4, \text{ where } c_3, c_4 \text{ are constants.}$$

Therefore

$$c_3 \prod_{j}(1-q_j) < |f_1(\lambda)| < \prod_{j}(1+q_j)\ c_4$$

We evaluate the integral $\int_{\tau_{2i-1}}^{\tau_{2i}} f_2(\lambda)\ d\lambda$ by the change of variable

$\lambda = \tau_{2i-1} + \ell_k sin^2\phi$ and get the value to be $\frac{\pi}{2}$. From the above estimates the lemma follows.

The following lemma gives estimates for the integral of $d\omega_i$ over other beta cycles.

Lemma 5.6 *There exists a positive constant c such that*

$$\int_{\beta_j} |d\omega_i| < c\ q_j \text{ all } j \neq i$$

Proof: Since $\int_{\beta_j} d\omega_i$ is $2\int_{\tau_{2j-1}}^{\tau_{2j}}\ d\omega_i$ it is enough to consider the integral on the spectral gap (τ_{2j-1}, τ_{2j}). Then using estimates simlar to the ones used in Lemma 5.5 the lemma follows.

We now consider on the Riemann surface $\mathcal{R}$ a meromorphic function ψ_i which agrees with either $m_{+,i}$ or $-m_{-,i} - \lambda + \omega(i)$ on a given sheet of the surface. Consider the product

$$\phi_n = \prod_{i=0}^{n-1} \psi_i$$

and consider the meromorphic differentials $\frac{d\phi_n}{\phi_n}$. This differential has no zeros and its poles are precisely at $\{\xi_i(0), \xi_i(n)\}$, i= 1,2,.. and at ∞_1 and ∞_2. In the following we obtain the Abel-Jacobi map, evauated at these points on $\mathcal{R}$. To do this we consider the meromorphic differentials $\omega_i \cdot \frac{d\phi_n}{\phi_n}$ i = 1,2,.. where ω_i is the holomorphic function corresponding to which $d\omega_i$ is the holomorphic differential. We then have the following proposition to compute the Abel-Jacobi image.

Proposition 5.1 *For each $i \in \mathbb{Z}^+$, there exists an integer $m_i(n)$ such that*

$$\sum_{k=1}^{\infty} \int_{\xi_k(0)}^{\xi_k(n)} d\omega_i = -n\ .\ c_i + m_i(n) \tag{14}$$

where $c_i = \int_{\infty_2}^{\infty_1} d\omega_i$ and $m_i(n)$'s are bounded uniformly in i. Here the integrations in the beta cycles are taken in the anti-clockwise direction.

Proof: We first note that since the Riemann surface we have is of infinite genus, we need to do an approximation to obtain the relation (14). To this end we consider the open region $\mathcal{R}_N$ of $\mathcal{R}$ obtained by considering the closed rectangles R_N^k , bounded by the rectangular curves considered in the lemma (3.5) (if necessary with smoothed out to make the boundry smooth) and removing the preimages, under the λ map , of $\cup\bar{R}_N^k$. That is

$$\mathcal{R}_N = \mathcal{R} \setminus \left(\cup\lambda^{-1}\left[\bar{R}_N^k\right]\right)$$

$\mathcal{R}_N$ is a Riemann surface by its own right and it is of finite type open Riemann surface. It can be compactified by adding 2m disks (or some topological deformations of them). Therefore there is a normal domain D_N which is a polygon with the number of sides equal to 4 times the number of beta cycles in $\mathcal{R}_N$ and having 2m holes corresponding to the number of punctures in $\mathcal{R}_N$ (which is 2m by construction). Suppose that the number of beta cycles in $\mathcal{R}_N$ is k(N). Then the 4k(N) sides of the polygon are indexed as a_i or $b_i$according to whether they correspond to traversing an alpha or a beta cycle in the anticlockwise or as a_i^{-1} or b_i^{-1} corresponding to traversing them in the clockwise direction. Further the sides are such that $a_i, b_i, a_i^{-1} and b_i^{-1}$ occur as sides of the polygon in that order for each i.

We consider a simply connected region obtained as follows. We cut open the above planar region along curves starting from a vertex of the polygon and going to the lift of one of the R_N^k. Let us call this line as δ_k for each k. We can then obtain a simply connected region, as D_N. This region has sides in addition to $a_i, b_i, a_i^{-1} and b_i^{-1}$, the curves $\delta_k, \gamma_k, \delta_k^{-1} and \gamma_k^{-1}$ k = 1,..,m. where γ_k's are the lifts of R_N^k 's. This construction is done so that the above sides occur consecutively in the same order as written above.

Since the simply connected domain so obtained is a cover for $\mathcal{R}_N$, meromorphic funcntions, differentials etc., can be lifted to the covering space and will retain their properties. Now we choose a closed curve on D_N that traverses the sides of the polygon and compute the integral of $\omega_i \frac{d\phi_n}{\phi_n}$. Since the lift of $d\omega_i$ is considered on a simply connected domain in $\mathbb{C}$, we can integrate them along any path to obtain a holomorphic function, so we take an arbitrary point λ_0 in D_N and define

$$\omega_i(\lambda) = \int_{\lambda_0}^{\lambda} d\omega_i$$

Now we integrate the differential $\omega_i \frac{d\phi_n}{\phi_n}$, which incidentally has finitely many poles in D_N, along a closed curve traversing the boundary of D_N such that none of the poles of the differential fall on it, then we have, by the residue theorem,

$$\frac{1}{2\pi i} \int_C \omega_i \frac{d\phi_n}{\phi_n} \quad = \quad \sum_{j=1}^{k(N)} [\omega_i(\xi_j(n)) - \omega_i(\xi_j(0))] + n[\omega_i(\infty_1) - \omega_i(\infty_2)] \tag{15}$$

Now we compute the quantities on the left directly. It is clear the integral over the curve C can be written as

$$\int_C \omega_i \frac{d\phi_n}{\phi_n} = \sum_{i=1}^{k(N)} \left[\int_{a_i} + \int_{b_i} + \int_{a_i^{-1}} + \int_{b_i^{-1}} \right] + \sum_{i=1}^{m} \left[\int_{\delta_i} + \int_{\gamma_i} + \int_{\delta_i^{-1}} + \int_{\gamma_i^{-1}} \omega_i \frac{d\phi_n}{\phi_n} \right] \tag{16}$$

by a slight deformation if necessary. To compute this integral explicitly we use the following properties of the values of ω_i on the corresponding points on the a,b 's and a^{-1}, b^{-1} s. We denote the points on a_j^{-1} corresponding to the point p in a_j as p^* and similarly for the b's and δ's.

$$
\begin{aligned}
\frac{(\phi_n)^{'}}{\phi_n}(p*) &= \frac{{\phi_n}^{'}}{\phi_n}(p), \\
\omega_i(p*) &= \omega_i(p) + \pi\delta_{ij},\ if p \in a_j \\
\frac{(\phi_n)^{'}}{\phi_n}(p*) &= \frac{{\phi_n}^{'}}{\phi_n}(p), \\
\omega_i(p*) &= \omega_i(p) + \pi_{ij},\ if p \in b_j \\
\frac{(\phi_n)^{'}}{\phi_n}(p*) &= \frac{{\phi_n}^{'}}{\phi_n}(p), \\
\omega_i(p*) &= \omega_i(p) + \int_{\gamma_k} d\omega_i,\ if p \in \delta_k
\end{aligned} \tag{17}
$$

Using these relations , and the fact that integrating $\frac{(\phi_n)^{'}}{\phi_n}$ over an alpha or a beta cycle gives the change in the argument of ϕ_n, which is $2\pi i$ times an integer, we compute the integrals in equation (16) to obtain the value of the integral as

$$
\int_C \omega_i \frac{d\phi_n}{\phi_n} = \pi\delta_{ij}(2\pi i m_i(n)) + \sum_{j=1}^{k(N)} \pi_{ij}(2\pi i l_j(n)) \tag{18}
$$

We consider the real parts of the integral to obtain the relation

$$
\begin{aligned}
\sum_{j=1}^{k(N)} [\omega_i(\xi_j(n)) - \omega_i(\xi_j(0))] &= -n[\omega_i(\infty_1) - \omega_i(\infty_2)] + \pi\delta_{ij}m_i(n) \\
&+ \sum_{j=1}^{k(N)} \pi_{ij}l_j(n) - \sum_{j=1}^{2m} \ Re\ \frac{1}{2\pi i}\int_{\gamma_j+\gamma_j^{-1}} \omega_i \frac{d\phi_n}{\phi_n} \\
&- \sum_{j=1}^{2m} \ Re\ \frac{1}{2\pi i}\int_{\delta_j+\delta_j^{-1}} \rho_{ji}\frac{d\phi_n}{\phi_n}
\end{aligned} \tag{19}
$$

where we have taken $\rho_{ji} = \int_{\gamma_j}\ or \int_{\gamma_j^{-1}}$ of $d\omega_i$. In short we can write the above equation as

$$
\sum_{j=1}^{k(N)} [\omega_i(\xi_j(n)) - \omega_i(\xi_j(0))] = -n[\omega_i(\infty_1) - \omega_i(\infty_2)] + \pi\delta_{ij}m_i(n)
$$

$$+\sum_{j=1}^{k(N)} \pi_{ij} l_j(n) + E_N \tag{20}$$

where we have grouped all the error terms in E_N. To show that E_N goes to zero in absolute value as N goes to ∞ we note that the integral over each of the γ_j's in the equation (18) is estimated using the lemma (3.2) as in the proof of lemma (3.5) , to show that it goes to zero as N goest to ∞. For similar reasons, the integral over δ's also goes to zero, since ρ_{ji} can be shown to go to zero and the contours δ_i 's can be chosen so that $\frac{(\phi_n)'}{\phi_n}$ is uniformly bounded on it. This proves the lemma since the remaining sum converges absolutely as can be seen from the estimates of lemma (5.6). Thus the equation (20) itself shows that the quantities $m_i(n), l_i(n)$ are uniformly bounded in i for each n.

In the above proposition, the integral on the left hand side is done on deforming the β cycles to fall on the real part of $\mathcal{R}$, infact precicely on the preimages of the gaps. Therefore we use a continuous parametrization for the β cycles and rewrite the equation (14) as follows. For convenience, let us denote the upper and lower sheets of $\mathcal{R}$ as $\mathcal{R}^{\pm}$ respectively.

Denote a point x_i on β_i by

$$x_i = \tau_{2i-1} + \ell_i \; sin^2(\theta_i) \tag{21}$$

where $0 \leq \theta_i < \frac{\pi}{2}$, if $x_i \in \mathcal{R}^+$ and $\frac{\pi}{2} \leq \theta_i < \pi$, if $x_i \in \mathcal{R}^-$.

Then in this parametrization the equation (14) becomes

$$\sum_{k=1}^{\infty} \int_{\theta_k(0)}^{\theta_k(n)} f_{ik}(\theta_k) d\theta_k = -n \; . \; c_i + m_i(n) \tag{22}$$

where, denoting $s(\theta) = sin^2(\theta)$,

$$\begin{aligned} f_{ik}(\theta) &= \frac{\sigma_i}{\sqrt{\ell_k\ (s(\theta) + \tau_{2k-1} - \mu_0^0)(\ell_k\ (s(\theta) + \tau_{2k-1} - \lambda_0^m)}} \\ &\times \frac{(\ell_k\ (s(\theta) + \tau_{2k-1} - \eta_k^i)}{\sqrt{(\ell_k\ (s(\theta) + \tau_{2k-1} - \tau_{2i-1})(\ell_k\ (s(\theta) + \tau_{2k-1} - \tau_{2i})}} \\ &\times \prod_{j \neq i,k} \frac{(\ell_k\ (s(\theta) + \tau_{2k-1} - \eta_j^i)}{\sqrt{(\ell_k\ (s(\theta) + \tau_{2k-1} - \tau_{2j-1})(\ell_k\ (s(\theta) + \tau_{2k-1} - \tau_{2j})}} \end{aligned} \tag{23}$$

We note here that the functions f_{ik} are periodic of period π. The estimates of lemmas (5.5,5.6) translate for f_{ik}'s into

$$c_1 \leq sup_{\theta \in \mathbb{R}} |f_{ii}(\theta)| \leq c_2 \ and \ sup_{\theta \in \mathbb{R}} |f_{ik}(\theta)| \leq c\ q_k \tag{24}$$

for some positive constants c_1, c_2 independent of i, k etc.,

6 Almost periodicity

In this section we show the almost periodicity of the potentials $\omega(n)$. To do this we start by defining a Banach space X as follows.

$$X = \{x_i \in \mathbb{R} : \sum_i s_i |x_i| < \infty, i = 1, 2, ...\}$$

and we also define the sets

$$X_{\mathbb{Z}} = \{\pi x : x \in X \ and \ x_i \in \mathbb{Z}, i = 1, 2, ..\}$$

$$\mathrm{X}/X_{\mathbb{Z}} = \{x \in X : 0 \leq x_i \leq \pi, \forall i\}$$

where s_i is the sequence as in the Assumptions on the spectrum. To start with we have the following compactness result.

Proposition 6.1 *The set $S = \{x \in X : sup_i|x_i| \leq d < \infty\}$ is compact in X.*

Proof: Note that each point in the set S has coordinates which are uniformly bounded by d and the set is closed. We note that $\sum_{i>k} s_i$ goes to zero as k goes to ∞. Now given any sequence x^n, we can find coordinate wise convergent subsequences. Therefore by a diagonal sequence argument gives the result

Lemma 6.1 *The linear operator A_x on X given by*

$$(A_x)_{ij} = f_{ij}(x_j), i \neq j \ and \ = 0 \ for \ i = j \tag{25}$$

is compact on X , where f_{ij}'s are as in equation (23).

Proof: We will show that the closure of the image of the unit ball under A is compact. Clearly for any $y \in X$,

$$sup_i|\left(A_x y\right)_i| = \sup_i |\sum_j \left(A_x\right)_{ij} y_j| = \sup_i |\sum_j f_{ij}(x_j) y_j|$$

$$\leq sup_i sup_{j \neq i}|s_j^{-1} f_{ij}| \sum s_j |y_j| \leq \left(sup_i sup_j |s_j^{-1} f_{ij}|\right) \|y\| < \infty$$

in the above inequality we have used $s_i \geq q_i$ and the equation (24). From the above estimates the result follows.

In the following to distinguish vectors in X from points in $\mathbb{R}$ we use the vector notaion. Let $\vec{\theta} \in$ X and consider the map

$$\mathcal{A}(\vec{\theta})(i) = \sum_{k=1}^{\infty} \int_0^{\theta_k} f_{ik}(\theta) d\theta, \ i \in \mathbb{Z}^+ \tag{26}$$

where f_{ik} 's are defined periodically on the whole of $\mathbb{R}$. Then it is clear that in view of the normalizations, lemma 5.5, we have, that $\mathcal{A}$ maps X into itself and for later convenience we write $\mathcal{A}$ as $\mathcal{A}_1+\mathcal{A}_2$, where $\mathcal{A}_1$ and $\mathcal{A}_2$ are given by

$$\mathcal{A}_1(\vec{\theta})(i)=\int^{\theta_i} f_{ii}(\theta)d\theta \ and\ \mathcal{A}_2(\vec{\theta})=\mathcal{A}(\vec{\theta})-\mathcal{A}_1(\vec{\theta}) \tag{27}$$

These two maps satisfy the following property with respect to addition by elements from $X_{\mathbb{Z}}$.

$$\mathcal{A}_1(\vec{\theta}+\vec{m})=\mathcal{A}_1(\vec{\theta})+\vec{m},\ and\ \mathcal{A}_2(\vec{\theta}+\vec{m})=\mathcal{A}_2(\vec{\theta}),\ for\ each\ \vec{m}\in X_{\mathbb{Z}} \tag{28}$$

Then we have the following theorem

Theorem 6.1 *The map $\mathcal{A}$,$\mathcal{A}_1$ and $\mathcal{A}_2$ have the following properties as maps from X to X.*

1. *$\mathcal{A}_1$ is one-one ,onto and $\mathcal{A}_1$, ${\mathcal{A}_1}^{-1}$ are uniformly Lipschitz.*
2. *$\mathcal{A}_2$, $\mathcal{A}_2\cdot{\mathcal{A}_1}^{-1}$ are strongly differentiable and the derivatives at each $x \in X$ are compact operators.*
3. *$\mathcal{B}=I+\mathcal{A}_2\cdot{\mathcal{A}_1}^{-1}$ is strongly differentiable with the derivative $(I+B_x)$ invertible for each $x\in X$ and $sup_{x\in \mathrm{X}/X_{\mathbb{Z}}}\|(I+B_x)^{-1}\|<\infty$.*
4. *$\mathcal{A}$ and $\mathcal{B}$ are one to one and onto.*
5. *$\mathcal{A}^{-1}$ is uniformly Lipshitz on $\mathrm{X}/X_{\mathbb{Z}}$.*

Proof:

1. The map $\mathcal{A}_1$ is diagonal and in view of the equation (28) above , it is enough to look at $\mathcal{A}_1$ on $\mathrm{X}/X_{\mathbb{Z}}$. Here $\mathcal{A}_1(x)(i)=g_i(x_i)=\int^{x_i} f_{ii}(\theta)d\theta$ is

an increasing function on $[0,\pi]$ with $g_i(0)=0$ and $g_i(\pi)=\pi$ in view of the normalization 5.5. Therefore it is one one and onto coordinatewise. By equation (24), $f_{ii}(x_i)$ and their inverses are uniformly bounded for each i and $x_i \in [0,\pi]$, hence the mean value theorem applied to each g_i, we see that both $\mathcal{A}_1$ and ${\mathcal{A}_1}^{-1}$ are uniformly Lipshitz. coordinatewise.

2. We shall prove that $\mathcal{A}_2$ is strongly differentiable , the proof for $\mathcal{A}_2 \cdot {\mathcal{A}_1}^{-1}$ is similar. Consider the matrix $A_{2,x}$ of partial derivatives of $\mathcal{A}_2$ at x. Explicitly we have

$$(A_{2,x})_{ij} = f_{ij}(x_j) \ for i \neq j \ and \ = 0 for i = j$$

Then we have for any h $\in$ X

$$\|\mathcal{A}_2(x+h) - \mathcal{A}_2(x) - A_{2,x}h\| \tag{29}$$

$$\begin{aligned}
&= \sum_{i=1}^{\infty} s_i \mid \mathcal{A}_2(x+h)(i) - \mathcal{A}_2(x)(i) - (A_{2,x}h)(i) \mid \\
&= \sum_{i=1}^{\infty} s_i \mid \sum_{k\neq i} [\int_{x_k}^{x_k+h_k} f_{ik}(\theta_k) d\theta_k - f_{ik}(x_k)h_k] \mid \\
&= \sum_{i=1}^{\infty} s_i \mid \sum_{k\neq i} [f_{ik}(\tilde{\theta}_k) - f_{ik}(x_k)]h_k \mid \\
&\leq \sum_{i=1}^{M(\epsilon)} s_i \mid \sum_{k\neq i}^{\ell(\epsilon)} [f_{ik}(\tilde{\theta}_k) - f_{ik}(x_k)]h_k \mid \\
&+ \sum_{i=M(\epsilon)+1}^{\infty} s_i \mid \sum_{k\neq i} [f_{ik}(\tilde{\theta}_k) - f_{ik}(x_k)]h_k \mid \\
&+ \sum_{i=1}^{M(\epsilon)} s_i \mid \sum_{k(\neq i)=\ell(\epsilon)+1} [f_{ik}(\tilde{\theta}_k) - f_{ik}(x_k)]h_k \mid \\
&\leq \epsilon \|h\|
\end{aligned}$$

where $\tilde{\theta}_k$ is some point in $[x_k, x_k + h_k]$ for each k. The first sum in the last of the inequalities is small since h_k goes to zero for each k as $\|h\|$ goes to zero and that each of the f_{ik}'s is uniformly continuous on compacts. The second and the third of the inequalities are small from equations (24) and the summability of the quantities s_i. The compactness of the operator $A_{2,x}$ for each x follows from lemma (6.1).

3. The strong differentiablity of $\mathcal{B}$ follows from (2) above and its derivative is the operator $(I + A_2 \cdot {A_1}^{-1})$ for each x in X. Let us call the compact operator $A_2 \cdot {A_1}^{-1}$ as B_x for each x for convenience. Then we show the uniform boundedness of the inverse $(I + B_x)^{-1}$ as follows. First note that since B_x is a compact operator, by the Fredholm alternative, it is enough to show that $I + B_x$ has no zero eigen value to show the invertibility and the boundedness of the inverse at x. Therefore consider

$$(I + A_2 \cdot (A_1)^{-1})y = 0, \ for\ y \in X$$

Since A_1 is invertible and maps X to itself, it is enough to show that

$$(A_1 + A_2)y = 0, \ for\ y \in X$$

Writing this explicitly in terms of the entries of the matrices we have that

$$\sum_{j=1}^{\infty} f_{ij}(x_j)y_j = 0\ for\ each\ i$$

Now from Lemma (4.1) we can choose numbers C_i^k for each k such that

$$\sum_j \left(\sum_i C_i^k f_{ij}(x_j) \right) y_j = \left(\sum_i C_i^k f_{ik}(x_k) \right) y_k = 0, \ for\ each\ k \qquad (30)$$

Since the sum in the parantheses is non zero again by Lemma (4.1), the result follows. Now showing the continuity of $(I+B_x)^{-1}$ shows the uniform boundedness of its norm on compacts, hence in particular on X/$X_{\mathbb{Z}}$. The norm continuity of $(I+B_x)^{-1}$ follows from the relation

$$(I+B_y)^{-1}-(I+B_x)^{-1}=(I+B_x)^{-1}\left[\left[1+(I+B_x)^{-1}(B_y-B_x)\right]-I\right]$$

As y goes to x in X, the right hand side of the above equation is a bounded operator and goes to zero in norm , hence the result.

4. We shall show that $\mathcal{A}$ is one-one and that $\mathcal{B}$ is onto. Then from (1) it follows that $\mathcal{B}$ is one one and since $\mathcal{A}=\mathcal{B}\cdot\mathcal{A}_1$ that $\mathcal{A}$ is also onto. Now suppose there are x and y in X such that

$$\mathcal{A}(x)=\mathcal{A}(y)$$

This would imply that

$$\sum_j\int_{x_j}^{y_j} f_{ij}(\theta_j)d\theta_j=0\ foreach\ i$$

That is for $a_j=\tau_{2j-1}+\ell_j sin^2(x_j)$ and similarly defining b_j with x_j replaced by y_j , we have that

$$\sum_j\int_{a_j}^{b_j} g_i(\lambda)d\lambda=0\ for\ each\ i$$

where g_i are the functions appearing in equation (11). By the interpolation theorem, then this would imply that for each function in the class $\mathcal{M}$ the above would be true. This gives a contradiction as we can choose a specific function from that class with prescribed zeros such that the integral is nonzero for distinct x and y. To show that $\mathcal{B}$ is onto , we note that (3) already implies, by the inverse function theorem, that the range of $\mathcal{B}$ is open. So it is enough to that the range is closed to conclude $\mathcal{B}$ is onto. Now $\mathcal{B}$ satisfies $\mathcal{B}(\theta + \vec{m}) = \mathcal{B}(\theta) + \vec{m}$ for $\vec{m} \in X_{\mathbb{Z}}$. So by the uniqueness of the decomposition of a number into it's fractional and integer parts, it is enough to show that $Ran(\mathcal{B}) \cap \mathrm{X}/X_{\mathbb{Z}}$ is closed. To this end consider a sequence y^n in the above set. Let it converge to some point y in $\mathrm{X}/X_{\mathbb{Z}}$. Since, y^n is in the range of $\mathcal{B}$ there is a sequence x^n such that $\mathcal{B}(x^n) = y^n$. It is easy to verify that such an x^n satisfies $sup_n \; sup_i \; |x_i^n|$ is uniformly bounded. Hence as in the proof of lemma (6.1) , we can find a convergent subsequence x^{n_k} converging to some point x . Hence the result.

5. From (4) we see that $\mathcal{A}^{-1}$ exists and also satisfies $\mathcal{A}^{-1}(\theta + \vec{m}) = \mathcal{A}^{-1}(\theta) - \vec{m}$. We shall show that $\mathcal{B}^{-1}$ is uniformly Lipschitz on $\mathrm{X}/X_{\mathbb{Z}}$ and since $\mathcal{A}_1$ is one one from $\mathrm{X}/X_{\mathbb{Z}}$ onto itself, the result for $\mathcal{A}^{-1}$ follows. First note that $\mathrm{X}/X_{\mathbb{Z}}$ is a closed convex set. Consider any x, y in $\mathrm{X}/X_{\mathbb{Z}}$. Then by the fundamental theorem for integral calculus we have that

$$\begin{aligned}
\mathcal{B}^{-1}(x) - \mathcal{B}^{-1}(y) &= \int_0^1 \frac{d}{dt}\mathcal{B}^{-1}\left(tx + (1-t)y\right) dt \\
&= \int_0^1 \left(1 + B_{(tx+(1-t)y)}\right)^{-1} (x-y)dt
\end{aligned}$$

the integrals being defined in the Bochner sense. Then the stated Lipschitz condition follows from the uniform boundedness of $\|(I + B_z)^{-1}\|$ on $\mathrm{X}/X_{\mathbb{Z}}$.

Now we are ready to prove the almost periodicity of the potentials.

Theorem 6.2 *The quantities $\omega(n)$ given by the trace formula are almost periodic in n.*

Proof: Recall the trace formula

$$\omega(n) = \sum_i \frac{1}{2}(\tau_{2i} + \tau_{2i-1} - 2\xi_i(n)) = \sum_i \frac{1}{2}\ell_i \ cos^2(\theta_i(n))$$

where we have written $\xi_i(n) = \tau_{2i-1} + \ell_i \ sin^2(\theta_i(n))$. So that we have for each n and k in $\mathbb{Z}$ that

$$\omega(n) - \omega(n+k) = \sum_i \frac{1}{2}\ell_i \ \Big(cos^2(\theta_i(n)) - cos^2(\theta_i(n+k)\Big)$$

Now the stated almost periodicity is immediate from the above equation if we show that given $\epsilon > 0$ there is an N independent of n and a sequence $m_i(n, N)$ of integers , i in $\mathbb{Z}^+$ such that

$$sup_{n\in\mathbb{Z}} \left[\sum_{i=1}^{\infty} \ell_i|\theta_i(n) - \theta_i(n+N) + \pi m_i(n,N)|\right] < \epsilon \tag{31}$$

We shall prove this as follows. Consider the ϵ and let $k(\epsilon)$ be such that

$$\sum_{i=k+1}^{\infty} \ell_i < \frac{\epsilon}{2} \tag{32}$$

Then , for each M we have,

$$\begin{aligned} |\omega(n) - \omega(n+M)| \ &\leq \ \left[\sum_{1}^{k(\epsilon)} + \sum_{k(\epsilon)+1}^{\infty}\right] \ell_i \Big(cos^2(\theta_i(n) - cos^2(\theta_i(n+M))\Big) \\ &\leq \ \sum_{i=1}^{k(\epsilon)} \ell_i \Big(cos^2(\theta_i(n) - cos^2(\theta_i(n+M))\Big) + \epsilon \end{aligned}$$

Now it is enough to show that there exist integers N and $m_i(n, N)$ for each i such that

$$\sum_{i=1}^{k(\epsilon)} \ell_i |\theta(n) + \pi m_i(n, N) - \theta(n + N)| < \epsilon \tag{33}$$

It is clear that this will follow if we show that, there exists an N and $m(\vec{n, N}) \in X_{\mathbb{Z}}$ such that,

$$\left(sup_{i=1,..,k(\epsilon)} \frac{\ell_i}{s_i} \right) \|\theta(n) + m(\vec{n, N}) - \theta(n, N)\| < \epsilon \tag{34}$$

Consider for any $\delta > 0$ an ℓ such that $\sum_{i=\ell}^{\infty} \pi s_i < \delta$. Now given ℓ and δ it is possible to choose integers M, $M_1, .., M_\ell$ such that

$$sup_{i=1,..,\ell} |Mc_i - \pi M_i| < \delta$$

as can be seen by a simple application of Poincare recurrence theorem. Now we look at

$$\mathcal{A}(\theta(n)) - \mathcal{A}(\theta(n + M)) = \vec{y} + \vec{m} \tag{35}$$

where

$$y_i = Mc_i - \pi M_i \ for \ 0 \leq i \leq \ell \ and \ y_i = Mc_i - \pi[\frac{Mc_i}{\pi}] \ if \ \ i > \ell$$

where $\vec{c}$ is the vector appearing in Proposition (5.1). Therefore we choose the δ in the above to be $(\sum s_i)^{-1}(inf_{i=1,..,k}(\frac{s_i}{\ell_i}))\epsilon$, take the corresponding M in the equation (35) to be N and use the Lipschitz estimate

$$\|\theta(n) + m(\vec{n, N}) - \theta(n + N)\| \leq C\|\mathcal{A}(\theta(n) + m(\vec{n, N})) - \mathcal{A}(\theta(n + N))\| \leq C\|y\|$$

to obtain the estimate required in equation (34) .

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