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Quantum mechanics and quantum Hall effect on Riemann surfaces

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Abstract

The quantum mechanics of a system of charged particles interacting with a magnetic field on Riemann surfaces is studied. We explicitly construct the wave functions of ground states in the case of a metric proportional to the Chern form of the θ-bundle, and the wave functions of the Landau levels in the case of the the Poincaré metric. The degeneracy of the the Landau levels is obtained by using the Riemann-Roch theorem. Then we construct the Laughlin wave function on Riemann surfaces and discuss the mathematical structure hidden in the Laughlin wave function. Moreover the degeneracy of the Laughlin states is also discussed.

1 Introduction

We study here the quantum mechanics of a system of charged particles living on a two dimensional surface and interacting with a magnetic field orthogonal to the surface. When the surface is an infinite plane and the magnetic field is constant, we have the well known problem of the Landau levels. This problem has received a renewed interest in the context of the quantum Hall effect (QHE)(for a review, see Ref. [1]). In fact, the QHE appears to be related to a rich physical and also mathematical structure, which is worthwhile to investigate in various possible configurations.

A particular intriguing and interesting case occurs when the two dimensional surface is a Riemann surface of high genus. Although not directly accessible to experiments, the problem of the physics on the Riemann surface happens to have deep relations with modern investigations on some interesting problems, like the occurrence of chaos in the surface with a negative curvature [2], and recent developments in the theory of Riemann surfaces, for example, the moduli of the surface and the vector bundles defined on the moduli [3].

In this paper, we will explore the problem directly from the point of view of quantum mechanics, by defining the Hamiltonian and then constructing eigenfunctions of the Hamiltonian. In Ref. [4], the problem in the case of the surface being an open hyperbolic plane with a constant negative curvature was considered and Ref. [5] discussed the scattering on a hyperbolic torus. Instead, here we will mainly consider compact Riemann surfaces. Ref. [6] obtained some interesting results about eigenvalues and their multiplicity of a particle interacting with the magnetic field, in the case of Riemann surfaces of high genus with a constant curvature by using some results from the mathematical literature, for example by using Selberg trace formula (for a review on Selberg trace formula, see Ref. [7]). See also Ref. [8] for a related discussions.

We pursue this investigation by explicit construction, which allows us to derive, and in some sense, generalize all known results in a straightforward way and provides us also the wave functions of Landau levels. The wave functions turn out to be the holomorphic line bundle defined on the surface, or for the high Landau levels, they are obtained by applying some covariant derivatives on the holomorphic line bundle. Actually, the holomorphic line bundle can be defined without reference to a particular metric. Thus although we consider mainly two cases for the metric, we can make some possible interesting generalizations to metrics of other kinds, mainly for the case of the ground states.

It is known that the fractional quantum Hall effect (FQHE) is related to the properties of the ground states, through the Laughlin wave function. We will show the construction of the Laughlin wave function on Riemann surfaces and indicate some interesting relations with recent results appeared in the mathematical literature[3].

We organize the paper as follows. In section 2, we use the Riemann-Roch theorem to derive the degeneracy of the ground states of a particle interacting with a constant magnetic field As we will see, the natural definition of a constant magnetic field is to take it proportional to the area form. We consider first the case of a metric which is proportional to the Chern form of the θ-function line bundle [9], defined explicitly in section 2. Since it is expressed in terms of the canonical holomorphic one forms, we will call it "canonical θ-metric", abbreviated as CθM[1]. In section 3, we construct the Landau levels on the surface with the Poincaré metric. The eigenvalues of the Landau levels and their multiplicity are obtained. In section 4, we continue the discussion of section 3 to construct the wave functions of the the Landau levels. In section 5, we present the Laughlin wave function on high genus surfaces with particular metrics. The mathematical structure hidden in the Laughlin wave function is pointed out. In particular, we also discuss the degeneracy of the Laughlin states.

2 The lowest Landau level on Riemann surfaces with the CθM metric

We consider a particle on a Riemann surface interacting with a "monopole" field, that is the integral of the field strength out of the surface is different from zero. We use the metric $ds^2 = g_{z\bar{z}}dzd\bar{z}$ in complex coordinates and the volume form is $dv = [ig_{z\bar{z}}/2]dz \wedge d\bar{z} = g_{z\bar{z}}dx \wedge dy$. We apply a constant magnetic field on the surface. The natural definition of the constant magnetic

[1]we understand there is not a standard name for this metric in the literature. It is proportional to the so called Bergman kernel, which is also proportional to the curvature of the Arakelov metric.

field to the high genus Riemann surface[5, 10] is

$$F = Bdv = (\partial_z A_{\bar{z}} - \partial_{\bar{z}} A_z) dz \wedge d\bar{z},$$

with constant B. Thus $\partial_z A_{\bar{z}} - \partial_{\bar{z}} A_z = i g_{z\bar{z}} B/2$. The flux Φ is given by $2\pi\Phi = \int F = BV$, where V is the area of the surface and we assume here $B > 0$ ($\Phi > 0$). The Hamiltonian of an electron on the surface under the magnetic field is given by the Laplace-Beltrami operator,

$$\begin{aligned} H &= [1/2m\sqrt{g}](P_\mu - A_\mu) g^{\mu\nu} \sqrt{g}(P_\nu - A_\nu) \\ &= [g^{z\bar{z}}/m][(P_z - A_z)(P_{\bar{z}} - A_{\bar{z}}) \\ &\quad +(P_{\bar{z}} - A_{\bar{z}})(P_z - A_z)] \\ &= [2g^{z\bar{z}}/m](P_z - A_z)(P_{\bar{z}} - A_{\bar{z}}) + [B/2m] \end{aligned} \tag{1}$$

where $g^{z\bar{z}} = [1/g_{z\bar{z}}]$ and $P_z = -i\partial_z$, $P_{\bar{z}} = -i\partial_{\bar{z}}$ ($\partial_z = (\partial_x - i\partial_y)/2$). The inner product of two wave functions is defined as $< \psi_1|\psi_2 >= \int dv \bar{\psi}_1 \times \psi_2$.

$H' = [2g^{z\bar{z}}/m](P_z - A_z)(P_{\bar{z}} - A_{\bar{z}})$ is a positive definite hermitian operator because $< \psi|H'|\psi > \geq 0$ for any ψ. Thus if $H'\psi = 0$, then ψ satisfies $(P_{\bar{z}} - A_{\bar{z}})\psi = 0$. The solutions of this equation are the ground states of the Hamiltonian H or H', i.e. the lowest Landau level (LLL). The existence of the solutions of this equation is guaranteed by the Riemann-Roch theorem[11, 12]. The solutions belong to the sections of the holomorphic line bundle under the gauge field. The Riemann-Roch theorem tells us that $h^0(L) - h^1(L) = deg(L) - g + 1$, where $h^0(L)$ is the dimension of the sections of the holomorphic line bundle or the degeneracy of the ground states of the Hamiltonian H, $h^1(L)$ is the dimension of the holomorphic differential $(L^{-1} \times K)$ where K is the canonical bundle and $deg(L)$ is the degree of the line bundle which is equal to the first Chern number of the gauge field, or the magnetic flux out of the surface, Φ. When $deg(L) > 2g - 2$, $h^1(L)$ is equal to zero [11] and $h^0(L) = \Phi - g + 1$. As a consistent check, $h^0(L)$ indeed gives the right degeneracy of the ground states in the case of a particle on the sphere and torus interacting with a magnetic-monopole field.

If the Riemann surface Σ has g ($g > 0$) handles, there exist abelian differentials, g holomorphic and g anti-holomorphic closed 1-forms, ω_i and $\bar{\omega}_i$. They are normalized by

$$\int_{A_i} \omega_j = \delta_{ij} \ , \int_{B_i} \omega_j = \Omega_{ij} \tag{2}$$

where A_i, B_i are a canonical homology basis or closed loops around handles on Σ, and the imaginary part of Ω is a positive matrix. We consider the CθM metric given by

$$g_{z\bar{z}} = \bar{\omega}(Im\Omega)^{-1}\omega, \tag{3}$$

which is always greater than zero. This metric is proportional to the Bergman reproducing Kernel and it is also proportional to the curvature of the Arakelov metric[3]. Its most interesting feature (for our study here) is that it is proportional to the Chern form of the θ-function line bundle[9], implying the covariant derivatives match the transformation properties of the θ-function.

We note that [9]

$$V = \int_{\Sigma} dxdy\bar{\omega}(Im\Omega)^{-1}\omega = g. \tag{4}$$

Let us take Φ equal to γg. Because of Dirac quantization, γg must be an integer and here we assume that γg is a positive integer. We will explicitly construct the ground states in the case of γ being a positive integer here and the case of fractional γ will be discussed in section 5.2. Now we have $B = 2\pi\gamma$ and $F_{z\bar{z}} = \partial_z A_{\bar{z}} - \partial_{\bar{z}} A_z = i\pi\gamma\bar{\omega}(Im\Omega)^{-1}\omega$. We can take $A_{\bar{z}} = i\pi\gamma\bar{\omega}(Im\Omega)^{-1}u/2$, where $u^i = \int_{z_0}^{z}\omega^i$ and $A_{\bar{z}} = \bar{A}_z$ are the gauge potentials in a certain gauge.

The ground states satisfy the equation,

$$\bar{D}\Psi = (\bar{\partial} + \frac{\pi}{2}\gamma\bar{\omega}(Im\Omega)^{-1}u)\Psi = 0. \tag{5}$$

Because

$$D(u + \Omega n + m) = K(n,m)^{-1}D(u)K(n,m) \tag{6}$$

with

$$K(n,m) = \exp\gamma[S(n,m) - \bar{S}(n,m)]$$

and

$$S(n,m) = (\pi/2)\bar{u}(Im\Omega)^{-1}(\Omega n + m),$$

thus we can choose the boundary condition as [13],

$$K(n,m)\Psi(u + \Omega n + m) = \exp(i\alpha(n,m))\Psi(u). \tag{7}$$

Generally we can take $\alpha(n,m)$ as

$$\alpha(n,m) = i\gamma\pi nm + 2\pi i\gamma a_0 m - 2\pi i b_0 n. \tag{8}$$

We define the function

$$\Psi = \exp[-(\pi/2)\gamma\bar{u}(Im\Omega)^{-1}u + (\pi/2)\gamma u(Im\Omega)^{-1}u]F. \tag{9}$$

Then the function F satisfies the equation,

$$F(u + \Omega n + m) = \exp(-i\pi\gamma n\Omega n - 2\pi\gamma nu - i\gamma\pi nm + i\alpha(n, m))F(u). \tag{10}$$

The solutions of Eq. (10) are

$$F_i(u) = \theta \begin{bmatrix} a \\ b \end{bmatrix} (\gamma u|\gamma\Omega) \tag{11}$$

with $b = b_0$ and $a_i = (a_0)_i + l_i/\gamma$, $l_i = 0, \cdots, \gamma - 1$ and $i = 1, \cdots, g$. It seems that there are γ^g solutions. However, from the Riemann-Roch theorem, we know that the degeneracy is $\gamma g - g + 1$ when $\gamma > 1$ (remind that γ is a positive integer here, then we have $\gamma g > 2g - 2$ and $h^1 = 0$). We observe that when $g = 1$, $\gamma^g = \gamma g - g + 1$ but $\gamma^g > \gamma g - g + 1$ for $\gamma > 1$ and $g > 1$. In fact, generally the solutions given by Eq. (11) are not linear-independent. Take, for example, F_1 given by

$$F_1 = \theta \begin{bmatrix} a_1 \\ b \end{bmatrix} (\gamma u|\gamma\Omega)\,, \tag{12}$$

then consider F_i/F_1 which are the meromorphic functions on the Riemann surface Σ. Because F_1 has γg zeros, for example, at $z_i, i = 1, \cdots, \gamma g$, the meromorphic functions will have possible poles at points z_i. The dimension of such meromorphic functions is given by the Riemann-Roch theorem, the number of the possible poles, γg, minus $g - 1$ in the case of $\gamma > 1$, which is equal to $\gamma g - g + 1$ and is the degeneracy of the ground states. If $\gamma = 1$, according to Eq. (11) there is only one solution (it is possible that this solution is identical to zero by the Riemann vanishing theorem and thus there will be no solutions). We remark that if $(a_0)_i = 1/2\,, (b_0)_i = \gamma/2$, the wave functions are transformed covariantly by the modular transformations (for the case of $g = 1$, see Ref. [13]).

3 The Landau levels on Riemann surfaces with the Poincaré metric

When $g > 1$, the simply connected covering space of the surface Σ is the upper half plane H (for example, Refs. [12, 9]), and Σ is equal to H/Γ, where Γ is the discreet subgroup of the isometry group of H, isomorphic to the first homotopy of Σ. Γ is generated by the Fuchsian transformations around a canonical homology basis, $\Gamma_{A_i}, \Gamma_{B_i}$ with

$$\prod_{i=1}^{g} \Gamma_{A_i}\Gamma_{B_i}\Gamma_{A_i}^{-1}\Gamma_{B_i}^{-1} = 1. \tag{13}$$

The metric is given by the Poincaré metric, $ds^2 = y^{-2}(dx^2 + dy^2)$, and we note that $\int dv = \int y^{-2}dxdy = 2\pi(2g-2)$ (without punctures). In the case of the Poincaré metric $g_{z\bar{z}} = y^{-2}$, the curvature is constant:

$$g^{z\bar{z}}R_{z\bar{z}} = -2g^{z\bar{z}}\partial\bar{\partial}ln g_{z\bar{z}} = -1.$$

We take $F = Bdv$, and thus $A_z = -iB\partial(\ln g_{z\bar{z}})/2$ and the flux Φ is equal to $2B(g-1)$. Then we define $D = \partial - (B/2)\partial \ln g_{z\bar{z}}$ and $\bar{D} = \bar{\partial} + (B/2)\bar{\partial}\ln g_{z\bar{z}}$. The Hamiltonian is (we take $m = 2$ in Eq. (2) for the simplicity),

$$H = -g^{z\bar{z}}D\bar{D} + (B/4). \tag{14}$$

The eigenfunctions satisfy

$$H\Psi = E\Psi. \tag{15}$$

If $\tilde{z}$ is another local coordinate on Σ and the domain of $\tilde{z}$ intersects non-trivially the domain of z, $g_{z\bar{z}}dzd\bar{z}$ is invariant under coordinate changes, or

$$g_{z\bar{z}}dzd\bar{z} = g_{\tilde{z}\bar{\tilde{z}}}d\tilde{z}d\bar{\tilde{z}} \tag{16}$$

on the intersection of the domains of z and $\tilde{z}$. D and $\bar{D}$ are transformed as

$$\tilde{D} = (dz/d\tilde{z})U^{-1}DU \ , \bar{\tilde{D}} = (d\bar{z}/d\bar{\tilde{z}})U^{-1}\bar{D}U \tag{17}$$

where $U(z,\tilde{z}) = (dz/d\tilde{z})^{-B/2}(d\bar{z}/d\bar{\tilde{z}})^{B/2}$. The Hamiltonian is transformed as

$$\tilde{H} = U^{-1}HU, \tag{18}$$

thus the wave function is transformed as

$$\tilde{\Psi} = U^{-1}\Psi \tag{19}$$

or $\Psi(dz)^{B/2}(d\bar{z})^{-B/2}$ is invariant under the transformation. So we conclude that Ψ is a differential form of type $T_{B/2}^{\bar{B}/2}$. Furthermore, the wave function is transformed under the Fuchsian transformations as

$$\Psi(\gamma z) = u(\gamma, z)\Psi(z),\ \ u(\gamma, z) = \nu(B, \gamma)(cz + d)^{2B}/|cz + d|^{2B} \tag{20}$$

where γ is a Fuchsian group element $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\gamma z = (az + b)/(cz + d)$. $\nu(B, -1) = e^{-i2\pi B}$ and $u(\gamma_1\gamma_2, z) = u(\gamma_1, \gamma_2 z)u(\gamma_2, z)$ are the consistency conditions ensuring univaluedness of the wave function on the universal covering space [7]. The boundary condition is twisted if $\nu(B, \gamma) \neq 1$. The ground states are given by the solutions of the following equation,

$$\bar{D}\Psi_0 = 0. \tag{21}$$

The solutions of this equation are $\Psi_0 = g_{z\bar{z}}^{(-B/2)}\tilde{\Psi}_0$ with $\bar{\partial}\tilde{\Psi}_0 = 0$. $\tilde{\Psi}_0$ belongs to a differential form of type T_B. According to Eq. (20), $\tilde{\Psi}_0$ is transformed under the Fuchsian transformations as

$$\tilde{\Psi}_0(\gamma z) = u'(\gamma, z)\tilde{\Psi}_0(z),\ \ u'(\gamma, z) = \nu(B, \gamma)(cz + d)^{2B} \tag{22}$$

with $\nu(B, \gamma)$ defined in Eq. (20). When $B = 1$, T_B is the canonical holomorphic line bundle and $\tilde{\Psi}_0$ is given by the sections of the canonical holomorphic line bundle. By the Riemann-Roch theorem, we have $dimT_B - dimT_{1-B} = (2B - 1)(g - 1)$, where $dimT_B$ is the dimension of the sections of the holomorphic bundle T_B. The dimension (or the degeneracy of the ground states) of T_1 is g for the non-twisted boundary condition and is $g - 1$ for the twisted boundary condition ($\nu \neq 1$), because $dimT_0 = 1$ for the non-twisted boundary condition and $dimT_0 = 0$ for the twisted boundary condition. When B is an positive integer which is greater than one, the dimension of T_B is $(2B - 1)(g - 1)$ by the Riemann-Roch theorem ($dimT_{1-B} = 0$, as $1 - B$ is negative). When $B = 1/2$, $T_{1/2}$ is 1/2-differentials (the spin bundle). The dimension of $T_{1/2}$ is generically one for the odd-spin structures and zero for the even-spin structures. The energy of the ground states is $B/4$. An expression for the wave functions will be described in the next section. There we will also indicate generalizations to the case of fractional B, provided that $(2B - 1)(g - 1)$ is integer, and also possible generalizations to surfaces with punctures.

Here and in the following, we call $g = g_{z\bar{z}}$ for short. We introduce the covariant derivative, ∇_z, and its Hermitian conjugate $(\nabla_z)^\dagger = -\nabla^z$,

$$\begin{aligned} \nabla_z \quad &: T_k^l \to T_{k+1}^l \;, \nabla_z = g^k \partial g^{-k}, \\ (\nabla_z)^\dagger \quad &: T_k^l \to T_{k-1}^l \;, (\nabla_z)^\dagger = -g^{-l-1} \bar{\partial} g^l. \end{aligned} \tag{23}$$

Note that D is the covariant operator ∇_z acting on $T_{B/2}^{\bar{B}/2}$ ($\bar{D} = g\nabla^z$ where ∇^z acts on $T_{B/2}^{\bar{B}/2}$).

Let us next discuss the higher Landau levels. By writing

$$H - B/4 = -\nabla_z \nabla^z,$$

we notice that if Ψ_1 is an eigenfunction of H with eigenvalue $E_1 > B/4$, then $\Psi_1 = -\nabla_z \nabla^z \Psi_1 / \epsilon_1$ (where $\epsilon_1 = E_1 - B/4 \neq 0$). Therefore $\Psi_1 = \nabla_z \Phi$ for some Φ. Of course, since Ψ_1 is of the form $T_{B/2}^{\bar{B}/2}$, then Φ will be of the form $T_{B/2-1}^{\bar{B}/2}$. Thus we have, more explicitly,

$$\Psi_1 = (\partial - (B/2 - 1)\partial lng)\Phi.$$

Due to the property of the Poincaré metric $\partial\bar{\partial} lng = g/2$, one can easily show that

$$-\nabla_z \nabla^z \Psi_1 = \frac{B-1}{2} \Psi_1 + \nabla_z(-\nabla_z \nabla^z \Phi).$$

When $B \geq 1$, one can show that $< \Psi_1 | \nabla_z(-\nabla_z \nabla^z \Phi) > \geq 0$. It is thus clear that the states of the lowest excited level are obtained, if there exist Φ such that $\nabla_z \nabla^z \Phi = 0$, i.e. $\bar{D}\Phi = 0$. This means that $\Phi = \Phi_0 = g^{-B/2} \tilde{\Phi}_0$ with $\bar{\partial}\tilde{\Phi}_0 = 0$. Since $\tilde{\Phi}_0$ is of the form T_{B-1}, there exist solutions of the equation $\bar{\partial}\tilde{\Phi}_0 = 0$ for $B \geq 1$. The energy of the lowest excited states is thus

$$E_1 = \frac{3}{4}B - \frac{1}{2}.$$

The degeneracy of this Landau level is the dimension of the sections of the holomorphic bundle T_{B-1} (which is equal to $(2B-3)(g-1)$ if $B > 2$). When $B < 1$, there is only the *zero'th* "Landau level" (the lowest Landau level). Beyond the "Landau levels", little is know about the spectrum. We will make a comment about this point at the end of the present section.

We can generalize the above discussion to high Landau levels. The wave function of the $k'th$ Landau level is given by

$$\begin{aligned}\Psi_k &= (\nabla_z)^k \Phi_0 \\ &= (\partial - (B/2-1)\partial \ln g)(\partial - (B/2-2)\partial \ln g) \\ &\quad \cdots (\partial - (B/2-k)\partial \ln g)\Phi_0\end{aligned} \tag{24}$$

with $\tilde{\Phi}_0 = g^{B/2}\Phi_0$ and $\bar{\partial}\tilde{\Phi}_0 = 0$. $\tilde{\Phi}_0$ is a differential form of the type T_{B-k}. Notice that this construction generalizes the standard construction for the harmonic oscillator. By using the relation, which holds for the Poincaré metric,

$$[\nabla^z \ \nabla_z]T^m_n = -(m+n)/2, \tag{25}$$

one can explicitly check that Ψ_k is the eigenfunction of the Hamiltonian, with the eigenvalue as

$$E_k = [B(2k+1) - k(k+1)]/4. \tag{26}$$

The degeneracy of the $k'th$ Landau level is given by the dimension of the sections of the holomorphic bundle of the type T_{B-k}, which is equal to $(2B-2k-1)(g-1)$ when $B-k>1$. Because the dimension of T_n is zero when n is negative, k must not be greater than B. Hence there is only a finite number of "Landau levels".

When B is an integer, k can take value from 0 to B. When $k=B$, the corresponding $\tilde{\Psi}_0$ is the (holomorphic) differential form of the type T_0. T_0 is a constant function on the surface. We can also include twisted boundary conditions, which would physically correspond to the presence of some magnetic flux through the handles. If the boundary condition of the wave function is the twisted one, there does not exist a non-zero constant function which satisfies the twisted boundary condition. Thus the dimension of T_0 is zero in this case and there is not the $B'th$ Landau level. When $k=B-1$, the degeneracy of this Landau level is the dimension of the canonical bundle T_1, which is equal to g for the non-twisted boundary condition and is equal to $g-1$ for the twisted boundary condition. B can be also an half-integer. Then k can take value from 0 to $B-(1/2)$. When $k=B-(1/2)$, the degeneracy of this Landau level is the dimension of the spin bundle $T_{1/2}$. The dimension of the holomorphic sections of the spin bundle generically is zero for the even-spin structures and one for the odd ones (or for twisted ones).

In the next section, we will show a construction of the wave functions and we will see that it is possible to generalize the present scheme also to the case of B fractional, provided a condition is satisfied, and to include also "punctures" on the surface. We will also discuss the resulting spectrum of the Landau levels in the general case.

Of course, the "Landau levels" that we have found by the above method do not exhaust the spectrum. In fact, when k has reached the maximum value for which $s = B - k$ is positive or zero, we can still express $\Psi = \nabla_z^k g^{-B/2} \tilde{\Phi}$ where $\tilde{\Phi}$ is a T_s differential and get an additional infinity of levels and corresponding wave functions by the solutions of the eigenvalue equation for $-\nabla_z \nabla^z \tilde{\Phi}_n = E_n \tilde{\Phi}_n$. The corresponding eigenvalues for Ψ will be $E = \frac{1}{4}(B(2k+1) - k(k+1)) + E_n$. In particular, for an integer B, this will relate the general solution of our problem to the eigenvalues and eigenfunctions of the Laplacian on the scalar (i.e. the zero forms) on the Riemann surface with the Poincaré metric, a problem which is not completely solved and for which there exists a vast literature (for a recent review see Ref. [14]).

4 The wave functions of the Landau levels

In order to complete the construction of the wave functions of the "Landau levels" of the last section, we would like to describe $\tilde{\Phi}_0$, that is the holomorphic sections of the bundle corresponding to the differential of the type T_s. We will present a formula for the determinant $\det h_i(z_j)$, where h_i are the independent holomorphic sections and i, j run over the degeneracy of the Landau level. To get a particular wave function, it is of course enough to consider this determinant as a function of a particular z, fixing arbitrarily the remaining ones. We have anticipated from the Riemann-Roch theorem that the degeneracy is $N = (2s-1)(g-1)$, for s integer or half integer greater than 1, the cases $s = 1$, $s = 1/2$, $s = 0$ corresponding respectively to the g abelian differential, to the holomorphic spin structure(s) and to the constant respectively, as recalled above. The following formula does not make reference to any particular metric, as the notion of holomorphic differentials is introduced in a metric independent fashion.

The formula can be read from Ref. [15], and it has been obtained in a contest of String theory, following the work of Knizhnik[16]. It is:

$$\det h_i(z_j) = \theta\begin{bmatrix} a \\ b \end{bmatrix}\left(\sum u_i - (2s-1)\sum_{i=1}^{g-1} r_i\right)\prod_{i=1}^{g-1}(\nu_0(z_i))^{2s-1}$$
$$\times \prod_{i<j}^{N} E(z_i, z_j) / \prod_{i=1}^{N}\prod_{l=1}^{g-1}(E(z_i, r_l))^{2s-1}, \tag{27}$$

where $\nu_0(z)$ is a holomorphic half-differential with $g-1$ zeros, corresponding to an arbitrary (but fixed) odd characteristic and $\nu_0(r_i) = 0,\ i = 1, \cdots, g-1$. The symbol $E(z_i, z_j)$ denotes the "prime form", which is a $-1/2$ differential and it has only one zero for $z_i = z_j$ (for a review of definitions and transformation properties of the theta functions, prime forms etc., see for instance Ref. [17]). Recall that $h_i(z)$ is the solution of the following equation,

$$\bar{D} g^{-s/2} h(z) = g^{-s/2} \bar{\partial} h(z) = 0 \tag{28}$$

with $\bar{D} = \bar{\partial} + (s/2)\bar{\partial} \ln g$. What we discuss now is valid for any metric, not only for the Poincaré metric. The above equation implies that

$$\bar{D}_{z_k} \prod_{i=1}^{N} g^{-s/2}(z_i, \bar{z}_i) \det h_i(z_j) = 0. \tag{29}$$

Thus with respect to the coordinate z_i, the function $\det h_i(z_j)$ shall be a form of type T_s, which we can show directly from Eq. (27). Because the prime form is a $(-1/2)$ form and ν_0 is a $1/2$ form,

$$\frac{1}{2}(2s-1) - \frac{1}{2}\left((2s-1)(g-1) - 1\right) + (2s-1)(g-1) = s, \tag{30}$$

which is the type of the form with respect to every coordinate z_i.

It is seen that it is indeed holomorphic, the zeroes in the denominator are canceled by corresponding zeroes in the numerator.

The variable in the theta function is $(\sum u_i - (2s-1)\sum_{i=1}^{g-1} r_i)$. We write the theta function in this way because the phase obtained by moving the coordinates around handles will be independent on the zeros r_i in this format (this is only for the convenience). We can also absorb it into the characteristics. In contrast to the case when the magnetic field is proportional to the CθM metric, in the present case the gauge potential $A_z = -is\partial(\ln g)/2$ is single-valued around the handles. Thus the boundary condition shall be

$$h_i(u + \Omega n + m) = \exp(\alpha n + \beta m) h_i(u). \tag{31}$$

The above equation implies that

$$\det h_i(u_j + \Omega n + m) = \exp(\alpha n + \beta m) \det h_i(u_j). \tag{32}$$

The right side of Eq. (27) indeed transforms around the handles in the way described by Eq. (32). The values of α, β are determined by the type of the odd characteristic ν_0 and the values of the characteristics a, b of the theta function. Therefore by using Eq. (32), the characteristics a and b in Eq. (27) can be fixed uniquely.

By Eq. (22), it is easy to show that $\det h_i(z_j)$ shall transform in the case of the Poincaré metric under the Fuchsian transformations as

$$\det h_i(\gamma z_j) = u'(\gamma, z) \det h_i(z_j), \tag{33}$$

where $u'(\gamma, z)$ is given by Eq. (22). We can check that Eq. (27) transforms in the way of Eq. (33) in the case of the Poincaré metric. The transformations of the prime form and half -differential under the Fuchsian transformations can be found in Ref. [3]. $\nu(B, \gamma)$ corresponds to the boundary condition parameters and the characteristics a and b in Eq. (27) are fixed by $\nu(B, \gamma)$.

The formula Eq. (27) makes sense also for s fractional, provided that $N = (2s-1)(g-1)$ is a positive integer, giving the multiplicity of the level (apart from the particular case $s = 1$ recalled above), for generic characteristics a, b corresponding to twisted boundary conditions. The value of s for this case is $s > 1/2$. The dual line bundle will be a form of the type T_v with $v = 1-s < 1/2$. If $s > 1$, since there are no holomorphic T_v forms with $v < 0$, N is the multiplicity for any characteristic. In the case where $1 > s > 1/2$, then $1/2 > v > 0$, and for the generic moduli of the Riemann surface, there will be one holomorphic T_v form for some characteristics. Thus, for these characteristics, the multiplicity of the level corresponding to T_s will be $N+1$. The formula for the wave function would be in this case a generalization of Eq. (27)[3]. For those characteristics there exists also a Landau level, generically with multiplicity 1, for those $s < 1/2$ for which $(1-2s)(g-1)$ is a positive integer.

Since $s = B - k$ (recalled from the last section), this means that we find Landau levels provided that $(2B-1)(g-1)$ is a positive integer, or, with some characteristics, that $(1-2B)(g-1)$ is a nonnegative integer (remember that we assume $B > 0$). Therefore the general condition for the existing of

Landau levels is that $2B(g-1)$ is integer, that is the Dirac quantization condition. We remind that the energy of the level is

$$E_k = \frac{1}{4}(B(2k+1) - k(k+1))$$

corresponding to $s = B - k$. Thus we see that the Landau level of maximal energy is obtained for the value of k which is nearest to $B - 1/2$. If B is integer then the maximal energy is $B^2/4$, if B is half integer the maximal energy is $\frac{1}{4}(B^2 + 1/4)$. If B is another allowed fraction, the maximal energy is intermediate between the previous two.

This discussion about Eq. (27) can be further generalized to the case of "punctures", which formally corresponds to the possibility of allowing poles for Φ_0 at some points of the surface, with the understanding that those points are infinitely far (with the Poincaré metric) from any other points. This means that the puncture can be taken at infinity or on the real axis in the upper half-plane, the surface making a narrow cusp there such that the area is still finite. Thus this discussion makes now use of a particular metric on the Riemann surface. Quantum mechanically we require the wave function to be normalizable and (taking the puncture at infinity) this implies for a differential T_s requiring $(y^2)^{s-2}|T_s|^2$ to be integrable in y for $y \to \infty$. This means that the poles of T_s can be of order s at most, since a pole of order r at $z = \infty$ would imply $\lim_{y\to\infty} T_s \sim y^{r-2s}$ as it is seen by the appropriate change of chart. If we allow for punctures at say $w_1, ..., w_n$, we have the freedom of generalizing Eq. (27) by multiplying the r.h.s by $\prod_i \prod_l (E(z_i, w_l))^{-s}$ and subtracting $(s \sum w_l)$ from the argument of the theta function (this insures that (32) continues to hold). This construction gives again a T_s differential provided now $N = (2s-1)(g-1) + ns$, which is the new multiplicity of the level (we should be aware that $N = (2s-1)(g-1) + ns$ may not be true in the case of $s \leq 1$. See the previous discussion).

Finally, a further generalization could consist in allowing for some twists on the punctures, corresponding to considering branch points rather than poles at w_l.

5 The Laughlin wave function on Riemann surfaces

5.1 The Laughlin wave functions in the constant field on the surface with the Poincaré metric

In the present section, we will discuss the Laughlin wave functions in the constant field on the surface with the Poincaré metric. In the next subsection, the Laughlin states in the constant magnetic field will be worked out in the case of the magnetic field which is proportional to the CθM metric. The mathematical structure behind the Laughlin wave functions will be discussed in the end of this subsection. We shall remark that the following discussions can be generalized to the case of the magnetic field being proportional to the curvature, if we take the Hamiltonian in a special ordering,

$$H = -g^{z\bar{z}} D\bar{D} \tag{34}$$

with $A_z = -iB\partial(\ln g)/2$ and g is an arbitrary metric. The magnetic field is a constant one if the curvature is constant, for example, in the case of the Poincaré metric (see the last section). The generalization is straight forward and we will not discuss it here.

Following section 3, we take $F_{z\bar{z}} = iB\bar{\partial}\partial \ln g$, $A_z = -iB\partial(\ln g)/2$ and $g > 1$. The ground states satisfy the equation, $\bar{D}\Psi_0 = 0$ and the solution of the equation is $\Psi_0 = g^{-B/2}\tilde{\Psi}_0$ with $\partial\tilde{\Psi}_0 = 0$. Ψ_0 is T_B differentials. $h_i(z)$ are the solutions of the equation $\partial\tilde{\Psi}_0 = 0$. In the FQHE, the magnetic field applied is very strong. Thus B is a very large number and the number of the sections of the holomorphic T_B differentials is equal to $(2B-1)(g-1)$. If the ground states are completely filled, which corresponds to the case of the quantum Hall state with filling $\nu = 1$, the wave function of the quantum Hall state is given by $\Psi_{JL} = \prod_{i=1} g^{-B/2}(z_i\bar{z}_i) \det h_i(z_j)$ where $i = 1, \cdots, (2B-1)(g-1)$ (Ψ_{JL} stands for the Jastrow-Laughlin type wave function).

A formula for $\det h_i(z_j)$ has been shown and discussed in the previous section, see Eq. (27). For any quantum Hall state, we write $\Psi_{JL} = \prod_{i=1} g^{(-B/2)}(z_i\bar{z}_i)\Psi'_J$ and Ψ'_J is a holomorphic function of the coordinates of any electrons, as Ψ_{JL} satisfies the equation $\bar{D}_{z_i}\Psi_{JL} = 0$ (we have this equation because every electron stays in the LLL). The boundary condition, Eq. (20) or Eq. (22) for the single particle implies a boundary condition on the many-body wave function (here is the wave function of the Hall state),

$$\Psi'_{JL}(\gamma z_j) = u'(\gamma, z)\Psi'_{JL}(z_j), \tag{35}$$

with $u'(\gamma, z)$ given by by Eq. (22). Furthermore, by using Eq. (35), the characteristics a and b in Eq. (27) can be fixed uniquely. Hence the degeneracy of the Hall state at filling $\nu = 1$ is one (from the physical points of view, there is only one way to completely fill the lowest Landau levels).

If the filling is equal to $1/m$, we make an Ansatz for the wave function, $\Psi_{JL} = \prod_{i=1} g^{(-B/2)}(z_i\bar{z}_i)\Psi'_{JL}$,

$$\begin{aligned} \Psi'_{JL} &= \theta\left[\begin{array}{c} a \\ b \end{array}\right]\left(m\sum u_i - Q\sum_{i=1}^{g-1} r_i|m\Omega\right)\prod_{i=1}^{g-1}(\nu_0(z_i))^{Q_1} \\ &\times \prod_{i<j}^{N}(E(z_i, z_j))^m \,/ \prod_{i=1}^{N}\prod_{l=1}^{g-1}(E(z_i, r_l))^{Q_2}, \end{aligned} \tag{36}$$

where N is the number of the electrons. Q_1 must be equal to Q_2, otherwise, there will be singularities at r_i. The way we write the theta function in Eq. (36) is based on the intuition from the wave function on the torus. Moreover, under the Fuchsian transformation, Eq. (36) shall be transformed as Eq. (35). This implies that $Q_1 = Q_2 = 2B - m$ and $m(N-1+g) = 2B(g-1) = \Phi$. And we take $Q = Q_1$ for the same reason in the case of $\nu = 1$. The characteristics a and b are determined by the boundary condition, which gives

$$b = b_0,\ a = a_0 + l/m,\ l_i = 1, \cdots, m,\ i = 1, \cdots, g, \tag{37}$$

where a_0 and b_0 depend on the boundary value parameters α and β. The above equation does not imply that the degeneracy of the Laughlin wave functions is m^g. According to the second section, the linear-independent number of the functions

$$\theta\left[\begin{array}{c} a \\ b \end{array}\right]\left(m\sum u_i - Q\sum_{i=1}^{g-1} r_i|m\Omega\right) \tag{38}$$

is $mg - g + 1$ and thus the degeneracy is actually $mg - g + 1$

There is a deep mathematical structure behind the Laughlin wave functions. In the case of $\nu = 1$, we know the wave function is given by the determinant of the sections of the holomorphic line bundle. In the case of $\nu = m$ with m greater than 1, the Laughlin wave functions is given by the determinant of the sections of the holomorphic rank-m vector bundle. The discussion about the determinant of the sections of the holomorphic vector bundle can be found in Ref. [3].

We consider a line bundle with connection $A'_z = -iB'\partial \ln g/2$ (thus $\Phi' = 2B'(g-1)$ is the corresponding magnetic flux and is a positive integer). We take $B' = B/m$ for the reasons to be seen later. Under the Fuchsian transformations, the transformations of the line bundle are given by Eq. (20) with the replacement of B by B'. We also introduce a flat vector bundle of rank-m with a flat connection. The flat vector bundle $\Psi(\gamma z)_k$ transforms under the Fuchsian transformations as

$$\Psi(\gamma z)_k = \sum_{i=1}^{m} \chi(\gamma)_{kj} \Psi(z)_j,$$

where $\chi(\gamma)$ is a constant matrix. We define a vector bundle E as the tensor product of the line bundle and the flat vector bundle. The vector bundle then is transformed under the Fuchsian transformations as,

$$\Psi(\gamma z)_k = \sum_{l=1}^{m} u(\gamma, z)_{kl} \Psi(z)_l, \;\; u(\gamma, z) = \nu(B', \gamma)(cz+d)^{2B'}/|cz+d|^{2B'} \quad (39)$$

where γ is a Fuchsian group element and $\nu(B', \gamma)$ is now a $m \times m$ matrix. Following the discussion about the line bundle, we shall have $\nu(B', -1) = e^{-i2\pi B'}$ and $u(\gamma_1\gamma_2, z) = u(\gamma_1, \gamma_2 z) u(\gamma_2, z)$ are the consistency conditions ensuring univaluedness of the wave function on the universal covering space. The holomorphic sections of the vector bundle is the solution of the following equation,

$$(P_{\bar{z}_i} - A'_{\bar{z}})\vec{\Psi}(z) = 0. \quad (40)$$

where $\vec{\Psi}(z)$ is a m-dimension vector with the component $\Psi(z)_k$. The degree of the vector bundle is $deg(E) = m \times \Phi' = \Phi$. We assume here Φ' or Φ is much larger than one (because we apply a very strong magnetic field in the FQHE), h^1 is zero by the Kodaira vanishing theorem, which states that there do not exist the sections of the one-form holomorphic vector bundle or the holomorphic vector bundle $E^{-1} \times K$ where E is the vector bundle and K is the canonical bundle. It is possible to see that the vector bundle $E^{-1} \times K$

is negative[18]. Then by the Kodaira vanishing theorem, the dimension of the sections of the holomorphic vector bundle $E^{-1} \times K$ is zero[18]. From the Riemann-Roch theorem for the vector bundles, the dimension of the sections of the holomorphic vector bundle is $h^0(E) = h^1(E) + deg(E) + m(1-g) = \Phi + m(1-g)$. Suppose the basis of the vector bundle is given by $\vec{\Psi}_i(z)$ with $i = 1, \cdots, h^0(E)$, we can construct a determinant, $\det \vec{\Psi}_i(z_j)$ with $j = 1, \cdots, N$ and $N = h^0(E)/m$.

One can show that

$$\bar{\partial} \det(\Lambda_{i,j}) = \sum_k \det(\Lambda'_{i,j}(k)) \tag{41}$$

where the matrix $\Lambda'_{i,j}(k)$ is given by $\Lambda'_{i,j}(k) = \Lambda_{i,j},\ i \neq k$ and $\Lambda'_{k,j}(k) = \bar{\partial}\Lambda_{k,j}$. By using Eq. (41) and Eq. (40), we can prove that

$$(P_{\bar{z}_i} - mA'_{\bar{z}}) \det \vec{\Psi}_i(z_j) = 0. \tag{42}$$

$\det \vec{\Psi}_i(z_j)$ is an anti-symmetric functions with respect to the interchange of any coordinates z_i and z_j. Thus the above equation shows that $\det \vec{\Psi}_i(z_j)$ is the wave function of the electrons interacting with the magnetic field $mA'_z = A_z$. The flux out of surface of this magnetic field A_z is equal to $\Phi = m\Phi'$. We shall show that $\det \vec{\Psi}_i(z_j)$ is a Laughlin type wave function. Furthermore, we have a relation $m(N-1+g) = \Phi$ and the above discussions offer a mathematical explanation of this relation for the Laughlin states. This relation had been used to calculate the spin of the quasiparticle in Ref. [19] and the value of the spin is found to be topological independent.

Now we shall prove that $\det \vec{\Psi}_i(z_j)$ is a Laughlin type wave function. By Eq. (39), one can also show that

$$\det \vec{\Psi}_i(\gamma z_j) = \det(\nu(B,\gamma))(cz_j + d)^{2B}/|cz_j + d|^{2B} \det \vec{\Psi}_i(z_j) \tag{43}$$

with $\det(\nu(B,-1)) = e^{-i2\pi B}$. Thus $\det \vec{\Psi}_i(z_j)$ transforms in the same way as the Laughlin wave function Ψ_{JL}. Moreover, when $z_i \to z_j$, we can easily show that $\det \vec{\Psi}_i(z_j) \to (z_i - z_j)^m$. Thus the function obtained by taking the ratio of $\det \vec{\Psi}_i(z_j)$ with

$$\prod_{i=1}^{g-1} (\nu_0(z_i))^{2B-m} \prod_{i<j}^{N} (E(z_i,z_j))^m \Big/ \prod_{i=1}^{N} g^{B/2}(z_i,\bar{z}_i) \prod_{i=1}^{N} \prod_{l=1}^{g-1} (E(z_i,r_l))^{2B-m}\,,$$

has no poles. By using Eq. (40), one can show that this function is an holomorphic function of coordinates z_i. Furthermore, by using Eq. (43), we find that this function transforms exactly in the way as the theta functions in Eq. (38). This holomorphic function must be equal to one of the theta functions in Eq. (38). Thus we complete our prove that $\det \vec{\Psi}_i(z_j)$ is a Laughlin type wave function. Similar arguments, for example in Ref. [20], had been used in proving some identities.

¿From the discussion of the previous section, B can be fractional in the above formulae (the above Laughlin wave functions make sense even B is fractional), but N and Φ must be integers.

We comment that the construction of the Laughlin wave function by the determinant of the sections of the holomorphic vector bundle can also apply to the case of $g = 0, 1$. It seems to us that the degeneracy of Laughlin wave functions is related to the different choice of the basis of the sections of the holomorphic vector bundle. Finally we shall point out that the hierarchical wave function on the Riemann surface can also be constructed by following the method developed in Ref. [21]

5.2 The Laughlin wave functions in the constant field on the surface with the CθM metric

Following the first section, we take $F_{z\bar{z}} = \partial_z A_{\bar{z}} - \partial_{\bar{z}} A_z = i\pi\gamma\bar{\omega}(Im\Omega)^{-1}\omega$. $A_{\bar{z}} = i\pi\gamma\bar{\omega}(Im\Omega)^{-1}U/2$. The ground states are given by

$$\begin{aligned} \Psi_i &= X(u)F_i(u), \\ X(u) &= \exp[-(\pi/2)\gamma\bar{u}(Im\Omega)^{-1}u + (\pi/2)\gamma u(Im\Omega)^{-1}u], \end{aligned} \tag{44}$$

where $i = 1, \cdots, \gamma g - g + 1$ and we assume here that $\gamma > 1$ and γ is an integer. We remark that the following formula for the wave function is true also in the case of fractional γ and $\gamma g > 2g - 2$ (remind that γg is always an integer). F_i are the linear independent solutions of Eq. (10). The wave function of the electrons when the first Landau levels (or ground states) are completely filled (the filling is equal to 1 in this case) is then

$$\Psi_{\nu=1} = \det\left(X(u_j)F_i(u_j)\right) = \prod_i X(u_i)\det\left(F_i(u_j)\right). \tag{45}$$

$\det\left(F_i(u_j)\right)$ can be calculated even we do not know how to select $F_i(u)$, the linear independent solutions of Eq. (10). According to Ref. [3], $\det\left(F_i(u_j)\right)$

is equal to

$$\theta \left[\begin{array}{c} a \\ b \end{array} \right] \left(\sum u_i\right) \prod_{i<j}^{N} E(z_i, z_j) f(z_1, \cdots, z_N), \tag{46}$$

where $N = \gamma g - g + 1$, $f(z_i, z_j) = f(z_j, z_i)$ and $f(z_i)$ has no zeros with respect to any coordinates. The function $f(z_i)$ can be determined by the boundary condition. From Eq. (10), we have

$$\begin{aligned} \det\left(F_i(u_j + \Omega n + m)\right) &= \exp(-i\pi\gamma n\Omega n - 2\pi\gamma n u_j \\ &\quad -i\gamma\pi n m + i\alpha(n,m)) \det\left(F_i(u_j)\right). \end{aligned} \tag{47}$$

Remarkably, we find that $\det F_i(u_j)$ is given by the same formula as that in Eq. (27), that is $\det F_i(u_j) = \det h_i(z_j)$. One has only to replace $2s$ by γ and take $N = \gamma g - g + 1$ (instead of $N = 2s(g-1) - g + 1$, as it was in Eq. (27)). One can then verify that Eq. (47) is indeed satisfied. Moreover, the characteristics a and b are uniquely fixed by the boundary condition. We are not surprised that $\det F_i(u_j)$ is given by Eq. (27) because both of them are the determinants of the holomorphic sections of bundles.

If filling $\nu = 1/m$, the Laughlin wave functions are given by, $\Psi_{JL} = \prod_i X(u_i)\Psi'_J$ with Ψ'_J taking the same form as Eq. (36). However, now we shall take $Q = Q_1 = Q_2 = \gamma - m$, $\gamma = \Phi/g$ compared with $Q = 2B - m$ and $B = \Phi/2(g-1)$ in the case of the Poincaré metric. Always, we have the relation $m(N - 1 + g) = \Phi$. The wave functions shall satisfy the boundary condition

$$\begin{aligned} \Psi'(u_i + \Omega n + m) &= \exp(-i\pi\gamma n\Omega n - 2\pi\gamma n u_i \\ &\quad -i\gamma\pi n m + i\alpha(n,m))\Psi'(u_i). \end{aligned} \tag{48}$$

and the characteristics a and b in Eq. (36) can be fixed by the boundary condition, e.g., Eq. (48). The characteristics are given $b = b_0,\ a = a_0 + l/m.\ l_i = 1, \cdots, m$ and $i = 1, \cdots, g$, where a_0 and b_0 depend on the boundary value parameters $\alpha(n,m)$. It is easy to see that the degeneracy of the Laughlin wave functions is also equal to $mg - g + 1$.

By making continuation of γ, γ can be fractional, although N and Φ are always integers. Now we can show how to obtain the ground state wave functions of a particle interacting with a constant magnetic field with the CθM metric in the case of fractional γ. When $\gamma g > 2g - 2$ and γ being fractional, the degeneracy of the ground states is still given by $N=\gamma g-g+1$ and the wave function of the electrons at filling $\nu=1$ is still given by Eq. (27) multiplied by $\prod_i X(u_i)$. The wave function of a single particle can be obtained by fixing the coordinates of other particles in the wave function of the electrons at filling $\nu=1$ in this case.

5.3 The degeneracy of the Laughlin state, a general discussion

We have show that the degeneracy of some Laughlin states in the last two subsections is $mg-g+1$ and we will try to show here that, generally, the degeneracy of Laughlin states is $mg-g+1$ under some reasonable assumptions. The Laughlin type wave function is the many particle wave function which looks like $\Psi_{JL}=F(z_1,\cdots,z_N)\prod_{i<j}[f(z_i,z_j)]^m$, where $f(z_i,z_j)=-f(z_j,z_i)$ and it is a function on holomorphic coordinates z_i and z_j. Furthermore when z_i approaches z_j, $f(z_i,z_j)$ is proportional to z_i-z_j and only when $z_i=z_j$, $f(z_i,z_j)=0$. The only function with those properties on Riemann surfaces is the prime form function $E(z_i,z_j)$. Because every particle stays in the lowest Landau level, so $(P_{\bar{z}_i}-A_{\bar{z}_i})\Psi_{JL}=0$, Hence the Laughlin wave function will be

$$\Psi_{JL}=\prod_{i=1}^{N}\exp[\int A_{\bar{z}_i}d\bar{z}_i]\prod_{i<j}^{N}[E(z_i,z_j)]^m\times F'(z_1,\cdots,z_N) \tag{49}$$

where $F'(z_i)$ is a function of holomorphic coordinates z_i. F' shall be determined by the boundary condition on the surface which is compatible with the Hamiltonian. Suppose that the magnetic field is smooth enough, we expect that Ψ_{JL} has no poles and $\exp[\int A_{\bar{z}_i}d\bar{z}_i]$ has no zeros and poles. We have shown that $m(N-1+g)=\Phi$ for the Laughlin states in the last two subsection and offered a mathematical explanation of this relation. Thus we can assume that $m(N-1+g)=\Phi$ is true for any Laughlin states. Because the degree is equal to Φ, so Ψ_{JL} has Φ zeros with respect to every coordinate. The number of zeros (counting the order of the zeros) in the function $\prod_{i<j}[E(z_i,z_j)]^m$ is $m(N-1)$ with respect to z_i. So the remaining $\Phi-m(N-1)=mg$ zeros are contained in the function F'. Suppose we have solutions F'_i, thus $G_i=\frac{F'_i}{F'_1}$ is a meromorphic function with respect to z_i, where F'_1 is one of the solutions (possibly some zeros in F'_i cancel some zeros in F'_1). ¿From the previous discussion, we make an assumption that G_i is a

function of the center coordinate $\sum_i z_i$. Because the meromorphic function can be always given by one θ function divided by another θ function, so the meromorphic function with poles at the points which lie on some zeros of the function F_1' is given by

$$G_i = \frac{\theta\begin{bmatrix} a_i \\ b_i \end{bmatrix}(mu|m\Omega)}{\theta\begin{bmatrix} a_1 \\ b_1 \end{bmatrix}(mu|m\Omega)} \tag{50}$$

where now $u = \sum_i \int^{z_i} \omega$ and $\theta\begin{bmatrix} a_1 \\ b_1 \end{bmatrix}(mu|m\Omega)$ has same zeros as F_1'. By Riemann-Roch theorem, the number of such linear independent meromorphic functions is $mg - g + 1$ for $m > 1$ ($m = 1$ is the case of the integer QHE and the degeneracy of the Hall state is one). Thus the number of the linear independent functions F_i' or the degeneracy of the Laughlin states is $mg - g + 1$. However in Refs. [22, 23], it was pointed out that the degeneracy of the Laughlin states is s m^g on the surface with g handles. In Ref. [23], Wen and Niu analyzed the Chern-Simons effective theory of the FQHE to get the degeneracy of the Laughlin states. In the Chern-Simons theory, there are so called large components of gauge fields (for example, Ref. [24]) and one part of the wave function is $F_l'(\mathcal{A}) = \theta\begin{bmatrix} a_0 + \frac{l}{m} \\ b_0 \end{bmatrix}(mu + m\mathcal{A}|m\Omega)$, where $\mathcal{A}$ is the large component of the gauge field, $a_0 + \frac{l}{m}, b_0$ take values in $\mathbf{R}^g/\mathbf{Z}^g$ and a_0, b_0 is dependent on the boundary condition. The phase space of $\mathcal{A}$ is Jacobian variety $\mathbf{C}^g/\mathbf{Z}^g + \Omega\mathbf{Z}^g$. Because $\mathcal{A}$ is now a dynamical variable, all m^g functions $F_l'(\mathcal{A})$ are independent with each other. Thus the degeneracy of the Laughlin states is m^g. However if we suppose that $\mathcal{A}$ is a constant vector, the number of the linear independent functions among $F_l'(\mathcal{A})$ is $mg - g + 1$.

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