Extended application of high noise constructive criteria for interacting particle systems

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Abstract
We discuss computational aspects of verifying constructive criteria for the ergodicity of interacting particle systems. Both discrete time (probabilistic cellular automata) and continuous time spin flip dynamics are considered. We also investigate how the addition of an exchange or stirring dynamics influences the criteria and how the study of a corresponding percolation problem may sometimes improve the usual ergodicity criteria when considering dynamics on arbitrary graphs.

Key words: interacting particle systems, probabilistic cellular automata, ergodicity, constructive criteria, computational aspects, stirring dynamics, disagreement percolation.

1 Introduction

There are by now quite a number of examples of phase transitions in interacting particle systems and probabilistic cellular automata. These are large systems consisting of many individual components that, due to local interactions or transition rules, evolve stochastically from given initial data. After a long time, the system may or may not have forgotten where it started from, depending on certain parameters governing the updating.

The ergodic regime is that region in parameter space where, for all initial data, the system converges to the same and unique invariant measure. Just like in equilibrium systems where it concerns the uniqueness of Gibbs states, there are a number of explicit conditions on the transition rule that, if satisfied, guarantee that the model is ergodic. We concentrate in this paper on constructive criteria that where developed in [MS1], [MS2] and [MS3]. In particular we investigate how practical they are and how we best deal with them. In that sense, the paper has to do with computational aspects. The most non–trivial part of it concerns the continuous time case. So far, there was no example where the constructive criteria went substantially beyond the usual (single site) criterion. We analyze this problem here and give such an example.

The outline of the paper is as follows. In the next section we discuss the discrete time case of spin flip dynamics. Stirring (a spin exchange mechanism) is added in section 3. Section 4 is devoted to the relation of the question of ergodicity with a corresponding oriented percolation problem. Finally, in section 5, we treat the continuous time case.

2 Constructive criteria for PCA

First, we briefly recall some concepts, definitions and notation concerning probabilistic cellular automata (PCA, from now on), ergodicity and coupling. Secondly, we will state constructive criteria for exponential ergodicity of PCA. We then show that certain properties of PCA allow these criteria to be simplified. Finally, we will use these criteria by hand and with a computer to get bounds for critical parameter values for three simple one–dimensional PCA.

2.1 Introduction to PCA

Notation. L is a countably infinite set. Unless otherwise specified, we always take L to be the d–dimensional hypercubic lattice $\boldsymbol{Z^d}$. Attach to every $x \in L$ an Ising spin $\sigma(x)$. A configuration σ on L is a complete characterization of all of these spins:

$$\sigma : L \longrightarrow \{-,+\} \tag{1}$$

The set of all configurations is $\Omega = \{-,+\}^L$ with its usual product structure. The set of real valued continuous functions on Ω is denoted by $C(\Omega)$, it is equipped with the uniform norm

$$\|f\| = \sup_{\sigma\in\Omega} |f(\sigma)| \tag{2}$$

$E(\Omega)$ is the set of the probability measures on Ω. σ^y is the configuration that differs from σ only at y:

$$\sigma^y(x) = \begin{cases} \sigma(x) & \text{if } x \neq y \\ -\sigma(x) & \text{if } x = y \end{cases} \tag{3}$$

The maximal variation of a function $f \in C(\Omega)$ due to a spin flip at x is given by

$$\delta_x f = \sup_{\sigma\in\Omega} |f(\sigma^x) - f(\sigma)| \tag{4}$$

The total oscillation of f is then:

$$|||f||| = \sum_{x\in L} \delta_x f \tag{5}$$

The subspace of $C(\Omega)$ on which this semi–norm is finite will be called $D(\Omega)$. Since $D(\Omega)$ contains the local functions, it is dense in $C(\Omega)$.

What is a PCA? A PCA is a discrete time Markov process $\{\sigma_n\}_{n=0,1,\ldots}$ on Ω. It is characterized by transition probabilities

$$\{p_x(\pm|\tau)\}_{x\in L} \tag{6}$$

local in $\tau \in \Omega$.

$p_x(+|\tau)$ is the probability that the spin $\sigma(x)$ at place x will be $+$ if the configuration was τ at the previous time. Usually $p_x(+|\tau)$ is invariant under equal translations of x and τ. At every step all spins are simultaneously and independently updated, so that for every finite $\Lambda \subset L$

$$\text{Prob}[\forall x \in \Lambda : \sigma_n(x) = \upsilon(x)|\sigma_{n-1} = \tau] = \prod_{x\in\Lambda} p_x(\upsilon(x)|\tau) \tag{7}$$

Hence a PCA defines a map from Ω into $E(\Omega)$. The probability measure over the configurations σ, given the configuration τ at the previous time is denoted by $P(d\sigma|\tau)$. This kernel induces a mapping from $C(\Omega)$ into itself by

$$(Pf)(\cdot) = \int f(\sigma)P(d\sigma|\cdot) \tag{8}$$

and yet another map from $E(\Omega)$ into itself:

$$(\mu P)(f) = \mu(Pf) \tag{9}$$

Usually it is clear from the context what "P" means.

Ergodicity. It is interesting to know about the long time behavior of the PCA. Loosely speaking, a PCA is ergodic if the measure on the configuration space after infinitely many applications of the PCA–rule is independent of the initial measure. An ergodic PCA–dynamics "forgets" its past. More precisely

Definition *A PCA is ergodic iff there is a $\nu \in E(\Omega)$, so that for all $\mu \in E(\Omega)$*

$$\mu P^N \longrightarrow \nu \textit{ weakly as } N \longrightarrow \infty \tag{10}$$

2.2 Theory of constructive criteria

A good way to compare probability measures is to "couple" them. In the constructive criteria below, the influence of the choice of initial configuration on the long time behavior of the PCA is investigated by considering a "coupling" between the PCA and itself, starting with a pair of configurations disagreeing at one site only.

Coupling. Suppose we have two probability measures q and $\bar{q}$ on one and the same finite set $W = \{1, 2, \ldots, n\}$. Then a coupling $\boldsymbol{q}$ of them is a probability measure on $W \times W$ that has q and $\bar{q}$ as marginals:

$$\begin{aligned} \textstyle\sum_{j=1}^{n} \boldsymbol{q}(i,j) &= q(i) \\ \textstyle\sum_{i=1}^{n} \boldsymbol{q}(i,j) &= \bar{q}(j) \end{aligned} \tag{11}$$

We call a coupling optimal if the sum of the off–diagonal elements is minimal. If $n \geq 4$, there may be more than one optimal coupling $\boldsymbol{q}$. A prescription that is always successful in getting one is:

$$\boldsymbol{q}(i,j) = \begin{cases} \min\{q(i), \bar{q}(i)\} & \text{if } i = j \\ v^{-1}(q(i) - \boldsymbol{q}(i,i))(\bar{q}(j) - \boldsymbol{q}(j,j)) & \text{if } i \neq j \end{cases} \tag{12}$$

v is the variational distance between q and $\bar{q}$:

$$v = \mathrm{var}[q, \bar{q}] = \frac{1}{2} \sum_{i=1}^{n} |q(i) - \bar{q}(i)| = 1 - \sum_{i=1}^{n} \boldsymbol{q}(i,i) = \sum_{i \neq j} \boldsymbol{q}(i,j) \tag{13}$$

We see that v equals the sum of the off–diagonal elements of the optimal coupling.

The coupled PCA. The coupling between two PCA's P and $\bar{P}$ is a new PCA $\boldsymbol{P}$ on the product configuration space $\Omega \times \Omega$. $\boldsymbol{P}$ is characterized (in the same way as in (6) for the original PCA) by:

$$\{\boldsymbol{p}_x(\cdot, \cdot | \tau, \bar{\tau})\}_{x \in L} \tag{14}$$

where $\boldsymbol{p}_x(\cdot, \cdot | \tau, \bar{\tau})$ is the optimal coupling between $p_x(\cdot|\tau)$ and $\bar{p}_x(\cdot|\bar{\tau})$. So now we have a PCA with 4 instead of 2 spin values per site. The "marginals" of the coupled PCA $\boldsymbol{P}$ are the original processes: let $g(\sigma, \bar{\sigma}) = f(\sigma)$ for $f \in C(\Omega)$, then $\boldsymbol{P}g = Pf$. $\boldsymbol{P}$ is known as the *basic* coupling. In what follows we will always take the basic coupling of two PCA with identical transition rules but started from different initial configurations.

Constructive criteria. We need two more definitions. The first quantity expresses the maximal probability that a single disagreement at time 0 at site

x between the two initial configurations results in a disagreement between the spins at (N, y), under the stochastic evolution of the basic coupling:

$$\kappa_{yx}^N = \sup_\tau \mathrm{Prob}[\sigma_N(y) \neq \bar{\sigma}_N(y) | (\sigma_0, \bar{\sigma}_0) = (\tau, \tau^x)] \tag{15}$$

Secondly, a measure of the cumulative effect of disagreements at all sites on a spin, N time steps later, is given by

$$\gamma_N = \sup_y \sum_x \kappa_{yx}^N \tag{16}$$

Intuitively, it is clear that if γ_N is "small", then the influence of disagreements between the initial configurations is small, or, in other words, the evolution under the original PCA–rule is relatively independent of the initial configuration, which is close to saying that the PCA is ergodic. The following Theorem of [MS2] makes these statements precise.

Theorem 1 *If there is an $N = 1, 2, \ldots$ such that $\gamma_N < 1$ then the PCA is exponentially ergodic and for all functions $f \in D(\Omega)$*

$$\left\| P^{aN+b} f - \int f(\sigma) \nu(d\sigma) \right\| \leq \frac{\gamma_N^a}{1 - \gamma_N} |||P^b f||| \tag{17}$$

where ν is the unique invariant measure ($\nu = \nu P$), a is integer and $b = 0, 1, \ldots, N-1$.

Remarks.

1. Note that $Pf \in D(\Omega)$ whenever $f \in D(\Omega)$ so that the right hand side in (17) is finite.

2. These criteria are called constructive because they involve checking an inequality on explicitly defined quantities that depend on a finite number of spins only (see (15) and (16)).

3. As N increases, the criteria become better in a sense which is explained in [MS2]. In particular, for attractive PCA (see later), some criterium must be satisfied ($\gamma_N < 1$ for large enough N) whenever the PCA converges fast enough to its unique invariant measure.

2.3 Simplifications

We will investigate three simple one–dimensional PCA: Stavskaya's model, the asymmetric contact process and a hard–core model. Before applying the constructive criteria to these PCA we will look at some of their properties that will simplify the analysis.

2.3.1 Three simple one–dimensional PCA

Stavskaya's model. Stavskaya's model [T1] is one–dimensional, it has transition rules

$$p_x(+|\sigma(x),\sigma(x+1)) = \begin{cases} 1 \text{ if } \sigma(x)=\sigma(x+1)=+ \\ \lambda \text{ otherwise} \end{cases} \tag{18}$$

The all + configuration is invariant under application of the PCA–rule. If λ is close to 1, it is easy to see that the measure over the configurations tends to the δ–measure on the all + configuration after infinitely many applications of the PCA–rule, independently of the initial configuration. On the other hand, you can use Toom's argument (treated in full generality in [T1] and specifically for Stavskaya's model in [DKT]) to show that if λ is small enough, a non–zero fraction of $-$ will survive forever, if you start with the all $-$ configuration. It can be proved that there is a critical λ^* separating ergodic and non–ergodic behavior.

The asymmetric contact process. A generalization of Stavskaya's model is the asymmetric contact process. It is characterized by:

$$p_x(+|\sigma(x),\sigma(x+1)) = \begin{cases} 1 & \text{if } \sigma(x)=\sigma(x+1)=+ \\ 1-\delta M & \text{if } \sigma(x)=+ \text{ and } \sigma(x+1)=- \\ \delta & \text{otherwise} \end{cases} \tag{19}$$

If you interpret + and $-$ as healthy and sick individuals respectively and δ and δM as the recover and infection–by–the–right–neighbor rate, the process can be seen as a discrete time model for the spreading of an infectious disease through a population. We will always take $1-\delta M \geq \delta$. The spin flip probability is proportional to δ. The all + configuration is again invariant, and it is interesting to see how large you can choose M such that there is a δ for which this configuration is still the unique attractor of the system. The largest such M will be denoted by M^*. Other interest of the process lies in the fact that it is a discrete time approximation of a continuous time process. In Section 5 we will see how the analysis of this discrete time approximation can help us to investigate the behavior of the corresponding continuous time process.

The hard–core PCA. This PCA is again one–dimensional and it is defined by:

$$p_x(+|\sigma(x), \sigma(x+1)) = \begin{cases} \rho & \text{if } \sigma(x) = \sigma(x+1) = - \\ 0 & \text{otherwise} \end{cases} \tag{20}$$

The name "hard–core" comes from the interpretation of putting a particle on a $+$ site, and having a vacancy on a $-$ site. On the space–time square lattice, with $\{(t, x) = (-1, 1), (0, 1), (0, 0), (1, 0)\}$ as an elementary square, it is not possible to have two particles as nearest neighbors. One does however not recover the "usual" hard–core or hard square model as corresponding two–dimensional equilibrium statistical mechanical model. The difference is that here in some finite space–time box, not all allowed configurations with the same number of particles have the same probability of occuring. For the two–dimensional hard–core model (a zero temperature limit of the Ising anti–ferromagnet in an external field), [DKS] applied a computer assisted analysis to investigate the uniqueness of Gibbs measures with the aid of the so called Dobrushin–Shlosman constructive criteria. Recently van den Berg [B] and van den Berg and Steif [BS] improved on this analysis with a percolation argument. We define the quantity ρ^* as the largest density parameter ρ below which the PCA is certainly ergodic.

2.3.2 Attractivity and anti–attractivity

Attractivity. Although we can apply the constructive criteria to the above PCA directly, a better look at them pays off. An important property of Stavskaya's model and the asymmetric contact process is their attractivity. A PCA is called attractive if

$$\sigma \geq \bar{\sigma} \Longrightarrow p(+|\sigma) \geq p(+|\bar{\sigma}) \tag{21}$$

(Of course $\sigma \geq \bar{\sigma}$ means that $\forall x \in L : \sigma(x) \geq \bar{\sigma}(x)$.) This order will persist in time under application of the basic coupling as stated in the following well–known

Lemma 1 *In the basic coupling of an attractive PCA with itself, order is conserved in time:*

$$\sigma_0 \geq \bar{\sigma}_0 \Longrightarrow \mathrm{Prob}[\sigma_N \geq \bar{\sigma}_N | \sigma_0, \bar{\sigma}_0] = 1, \forall N = 1, 2, \ldots \tag{22}$$

Attractivity greatly simplifies the expression for κ_{yx}^N:

$$\kappa_{yx}^N = \sup_{\sigma_0} |\mathrm{Prob}[\sigma_N(y) = + | \sigma_0] - \mathrm{Prob}[\bar{\sigma}_N(y) = + | \sigma_0^x]| \tag{23}$$

The proof is a simple consequence of (22). Therefore, we do not have to look at the basic coupling in order to determine γ_N for an attractive PCA. By this observation, the number of space–time configurations to consider, reduces to its square root.

Anti–attractivity. The hard–core PCA has a property which is opposite to attractivity:

$$\sigma \geq \bar{\sigma} \Longrightarrow p(+|\sigma) \leq p(+|\bar{\sigma}) \tag{24}$$

We propose to call this property "anti–attractivity". The order in the basic coupling oscillates:

$$\sigma_0 \geq \bar{\sigma}_0 \Longrightarrow \left\{ \begin{array}{r} \mathrm{Prob}[\sigma_{2N-1} \leq \bar{\sigma}_{2N-1} | \sigma_0, \bar{\sigma}_0] = 1 \\ \mathrm{Prob}[\sigma_{2N} \geq \bar{\sigma}_2 | \sigma_0, \bar{\sigma}_0] = 1 \end{array} \right. \quad (N = 1, 2, \ldots) \tag{25}$$

For the same reasons as in the attractive case, the expression for κ_{yx}^N reduces to (23) again.

2.3.3 The worst initial configuration

The calculation for κ_{yx}^N decreases if you know for which initial configuration σ_0 the supremum in its expression is reached. For the contact process and Stavskaya's model we know this "worst" initial configuration.

Lemma 2 *The initial configuration $\sigma_0 \equiv +$ yields the supremum in the expression for κ_{yx}^N for the asymmetric contact process and Stavskaya's model.*

Proof. We give the proof for the contact process. Define for $\sigma_0 \geq \bar\sigma_0$:

$$Q_y^N(\sigma_0, \bar\sigma_0) = \mathrm{Prob}[\sigma_N(y) = +|\sigma_0] - \mathrm{Prob}[\bar\sigma_N(y) = +|\bar\sigma_0] \qquad (26)$$

We proof by induction that Q_y^N increases under the following types of "spin flips":

$$(\sigma_0(x), \bar\sigma_0(x)) = (-,-) \longmapsto (+,-) \text{ or } (+,+) \qquad (27)$$

(We use quotation marks to distinguish these special spin flips.) This is clear for $N = 1$. Express Q^N in Q^{N-1}:

$$Q_y^N(\sigma_0, \bar\sigma_0) = \iint Q_y^{N-1}(\sigma_1, \bar\sigma_1)\boldsymbol{P}(d\sigma_1, d\bar\sigma_1|\sigma_0, \bar\sigma_0) \qquad (28)$$

By attractivity, the "spin flips" on $(\sigma_0, \bar\sigma_0)$ cause $\boldsymbol{P}(d\sigma_1, d\bar\sigma_1|\sigma_0, \bar\sigma_0)$ to shift some weight from configurations $(\sigma_1, \bar\sigma_1)$ to "spin flipped" (in the sense of (27)) versions of them. So $\boldsymbol{P}$ will be more concentrated on configurations where, by the induction hypothesis, Q^{N-1} is large; thus Q^N increases by "spin flips". One can prove this by checking what happens to $\boldsymbol{p}_x(\sigma_1(x), \bar\sigma_1(x)|\sigma_0, \bar\sigma_0)$. For instance, if $(\sigma_0(0), \sigma_0(1),\ \bar\sigma_0(0), \bar\sigma_0(1))$ flips from $(-+,--)$ to $(++,+-)$ then

$$\begin{array}{rrcr} \boldsymbol{p}_0(--|\sigma_0, \bar\sigma_0) = & 1-\delta & \longmapsto & 0 \\ \boldsymbol{p}_0(+-|\sigma_0, \bar\sigma_0) = & 0 & \longmapsto & \delta M \\ \boldsymbol{p}_0(++|\sigma_0, \bar\sigma_0) = & \delta & \longmapsto & 1-\delta M \end{array} \qquad (29)$$

Weight is transported from $(-,-)$ to $(+,-)$ and $(+,+)$. Thus a "spin flip" on $(\sigma_0, \bar\sigma_0)$ causes "spin flips" on $(\sigma_1, \bar\sigma_1)$. The other configurations $(\sigma_0(0), \sigma_0(1), \bar\sigma_0(0), \bar\sigma_0(1))$ can be checked in the same way. Under the condition $\bar\sigma_0 = \sigma_0^x$, $Q_y^N(\sigma_0, \bar\sigma_0)$ obviously reaches its maximum at $\bar\sigma_0 \equiv +$, it coincides with κ_{yx}^N then. ●

2.4 Results obtained with the constructive criteria

In this section we will use the constructive criteria to get bounds for critical parameter values for Stavskaya's model, the one–dimensional asymmetric contact process and the hard–core PCA. Basically, there are three ways to apply the criteria to specific PCA. The first is to do the calculations by hand, this method works only for simple PCA and small N. Much higher order criteria can be obtained by programming the calculations on a computer. The third method estimates the κ_{yx}^N by observing simulations of the basic coupling on a computer.

2.4.1 Application of the constructive criteria by hand

With the simplifications of the previous subsection in mind, it is relatively easy to calculate γ_N for $N \leq 3$ by hand, for the three PCA mentioned before. We give the analytic expressions for γ_N together with bounds for critical parameter values obtained with these criteria.

Stavskaya's model.

$$\gamma_1 = 2(1-\lambda) \implies \lambda^* \leq 0.5 \tag{30}$$

$$\gamma_2 = (1-\lambda)^2(3+\lambda) \implies \lambda^* < 0.4626 \tag{31}$$

$$\gamma_3 = 4(1-\lambda)^3(1+\lambda) \implies \lambda^* < 0.4425 \tag{32}$$

Asymmetric contact process.

$$\begin{aligned}
\gamma_1 &= 1+\delta(M-1) && (33)\\
&\implies M^* \geq 1 \\
\gamma_2 &= 1+2\delta(M-1)+\delta^2(1-2M)+\delta^3 M^2 && (34)\\
&\implies M^* > 1.15 \quad (\delta = 0.46) \\
\gamma_3 &= 1+3\delta(M-1)+3\delta^2(1-2M)+ \\
&\quad\ \delta^3(-1+3M+4M^2)-5\delta^4 M^2+\delta^5(M^2+M^3) \\
&\implies M^* > 1.26 \quad (\delta = 0.43) && (35)
\end{aligned}$$

(In [MS2] and [MS3] there was an error in the expression for γ_3.)

Hard–core PCA.

$$\gamma_1 = 2\rho \implies \rho^* \geq 0.5 \tag{36}$$

$$\gamma_2 = \rho^2(4-\rho) \implies \rho^* > 0.5374 \tag{37}$$

$$\gamma_3 = 2\rho^3(2-\rho)^2 \implies \rho^* > 0.649 \tag{38}$$

Calculations by hand become very tedious and unreliable for higher order criteria. At this point *Mathematica* helps us to get analytic expressions for γ_N for larger N. We are, however, mainly interested in the paramater values for which $\gamma_N \approx 1$, so we can better do calculations with fixed parameters with a conventional programming language like *Pascal*.

2.4.2 Calculations of γ_N by computer

We have written simple *Pascal* programs for the computation of γ_N. Central in our calculations is of course the Markov property which leads to recursive relations which for the above PCA are

$$\mathrm{Prob}[\sigma_N(0)=+|\sigma_0(0),\sigma_0(1),\ldots,\sigma_0(N)] = \sum_{\sigma_1(0)=-}^{+}\sum_{\sigma_1(1)=-}^{+}\cdots\sum_{\sigma_1(N-1)=-}^{+}$$
$$\mathrm{Prob}[\sigma_N(0)=+|\sigma_1(0),\sigma_1(1),\ldots,\sigma_1(N-1)]\prod_{x=0}^{N-1}p(\sigma_1(x)|\sigma_0) \tag{39}$$

Some relevant results of these computer–calculations are as follows:

Stavskaya' model.

$$\gamma_{28}(\lambda=0.358)=0.986 \Longrightarrow \lambda^* < 0.358 \tag{40}$$

Asymmetric contact process.

$$\gamma_{24}(M=1.84)=0.9993 \Longrightarrow M^* > 1.84 \quad (\delta=0.23) \tag{41}$$

Hard–core PCA.

$$\gamma_{17}(\rho=0.83)=0.9947 \Longrightarrow \rho^* > 0.83 \tag{42}$$

2.4.3 Checking the constructive criteria by means of simulations.

The last method to investigate a PCA's behavior with regard to ergodicity is to look at simulations of the coupled PCA. κ_{yx}^N is then obtained by recording the frequency of the event $\sigma_N(y) \neq \bar{\sigma}_N(y)$ for all relevant different initial configurations (σ_0,σ_0^x), and taking the maximum over these frequencies.

We will explain this method for the asymmetric contact process in some more detail. Because of the properties mentioned before, the expression for γ_N reduces to

$$\gamma_N = \sum_{x=0}^{-N}\mathrm{Prob}[\sigma_N(x)=-|\sigma_0=+^0] \tag{43}$$

where $+^0$ is the configuration which is $+$ everywhere except at $x = 0$. In a simulation of the PCA, the configuration at the next time n is obtained in the following way: draw for each site x a (pseudo–)random number s, with $0 \leq s < 1$, and apply then the rule

$$\begin{array}{ll} \sigma_n(x) = + & \text{if } s \leq p_x(+|\sigma_{n-1}) \\ \sigma_n(x) = - & \text{if } s > p_x(+|\sigma_{n-1}) \end{array} \tag{44}$$

Let the PCA "run" many times inside the "triangle"

$$\{(t, x)|t \in \{1, 2, \ldots, N\}, x \in \{-t, -t+1, \ldots, 0\}\} \tag{45}$$

(everything outside the triangle is $+$), and count each time the number of $-$ in the configuration σ_N. γ_N is simply the average. This procedure can easily be programmed on a computer. We have done that for Stavskaya's model and the contact process. Significant results are given below.

Stavskaya's model.

$$\gamma_{100}(\lambda = 0.331) = 0.98 \pm 0.01 \overset{\text{very probably}}{\Longrightarrow} \lambda^* < 0.331 \tag{46}$$

Asymmetric contact process.

$$\gamma_{100}(M = 2.22) = 0.976 \pm 0.006 \overset{\text{very probably}}{\Longrightarrow} M^* > 2.22 \quad (\delta = 0.15) \tag{47}$$

If you do not know the worst initial configuration, simulation techniques become much harder to apply. One complication is the large number of initial configurations to look at, this number increases exponentially in N. Another complication arises when a lot of initial configurations are (almost) the worst. Because of statistical fluctuations, taking simply the worst–looking initial configuration will yield an overestimation of κ_{yx}^N and γ_N. One would have to do an immense number of simulations to get rid of this effect. It is preferable to apply "smart" methods: first simulate all initial configurations equally often, later more and more concentrate the simulations on the seemingly bad ones.

In this technique of checking the MSCC with simulations, one does finite time observations and one obtains statements, with well–defined confidence, about the long time behavior of the PCA.

3 Combinations of spin flip and stirring processes

If we imagine spin values as indicating different types of molecules, spin flip processes model chemical reactions and stirring processes can be seen as models for the motion of the molecules. In some contexts combinations of spin flip and stirring processes are therefore called "reaction–diffusion" processes. We investigate here which implications the addition of stirring has for the constructive criteria. At the end of this section we will compare our results with similar work by Ferrari [F2].

First we will introduce the concept "mixer", this is the stochastic transformation rule that defines the stirring process. Usually we denote a mixer by an "M". M is related to the stirring process as the PCA–rule P is to the spin flip process.

Definition. To start with, we need a partition $\{A_i\}_{i\in I}$ of our lattice. The sets A_i in this partition are finite. Suppose that for each $i \in I$ we have a probability distribution α_i on the set $\mathcal{S}_i$ of permutations on A_i. We then define the mixer M acting on $f \in C(\Omega)$ as

$$Mf(\sigma) = \int f(\eta) M(d\eta|\sigma) \tag{48}$$

where, for every $\sigma \in \Omega, M(d\eta|\sigma)$ is the product measure $\prod_{i\in I} M_i(\eta|\sigma)$ with marginals $M_i(\cdot|\sigma)$ on $\{-,+\}^{A_i}$ giving probability $\alpha_i(q), q \in \mathcal{S}_i$, to the configuration

$$\eta(x) = \sigma(q(x)), x \in A_i \tag{49}$$

Example. M interchanges with probability $\frac{1}{2}$ spins in sets of type $A_i = \{2i, 2i+1\}, i \in \boldsymbol{Z}$.

Remark 1. In a stirring process obtained by consecutive application of a mixer M, there is no exchange of spins between different sets A_i of the partition. To get a more thorough stirring one can take a convex combination or a product of M and its "translates", instead of the single mixer M. The translate M' of the mixer in the above example is: M' interchanges with probability $\frac{1}{2}$ spins in sets of type $A'_i = \{2i-1, 2i\}, i \in \boldsymbol{Z}$. The results of the analysis below will not alter if we take such a generalized mixer instead of a single mixer.

Remark 2. Note that within each set of the partition, a mixer conserves the number of plus spins. So, a stirring process cannot have the same ergodic behavior as a spin–flip process (see the previous section).

The stochastic transformation rule of a spin flip stirring process is a convex combination of products of mixers and PCA. We are interested in the long time behavior of these processes. Although we can directly apply the constructive criteria to these mixer–PCA to detect ergodicity, the addition of stirring makes calculations considerably larger. Fortunately however, we can get criteria for mixer–PCA from the constructive criteria for the PCA alone, with help of the following

Lemma 3

$$|||M||| \leq 1 \tag{50}$$

Proof. Suppose that $x \in A_i$. Take $\boldsymbol{M}$ to be the coupled mixer that acts exactly the same on both configurations, i.e., $\boldsymbol{M} = \prod_{i \in I} \boldsymbol{M}_i$ and $\boldsymbol{M}_i(\cdot,\cdot|\sigma,\bar{\sigma})$ gives probability $\alpha_i(q), q \in S_i$ to the (new) configuration $(\tau(y), \bar{\tau}(y)) = (\sigma(q(y)), \bar{\sigma}(q(y)))$ (analogous to (49)).

$$\begin{aligned}
|Mf(\sigma^x) - Mf(\sigma)| &= \left|\int (f(\bar{\tau}) - f(\tau))\ \boldsymbol{M}(d\bar{\tau} d\tau|\sigma^x \sigma)\right| && (51)\\
&\leq \sum_{y \in L} \delta_y f \int \mathbf{1}(\tau_y \neq \bar{\tau}_y)\ \boldsymbol{M}(d\bar{\tau} d\tau|\sigma^x \sigma) && (52)\\
&= \sum_{y \in A_i} \delta_y f \sum_{q:q(y)=x} \alpha_i(q) && (53)
\end{aligned}$$

Take suprema and sum over x:

$$|||Mf||| \leq \sum_{i \in I} \sum_{y \in A_i} \delta_y f \sum_{x \in A_i} \sum_{q:q(y)=x} \alpha_i(q) = |||f||| \tag{54}$$

●

Remark 1. Although we do not need it, equality in the Lemma holds: $|||M||| = 1$.

Remark 2. The proof of the Lemma depends crucially on the fact that M is independent of the configuration. If it is not, $|||M||| \leq 1$ does not need to be true.

Implications of the constructive criteria and Lemma 3. We illustrate what happens with a few examples. We find upper bounds for the triple norms of some combinations of PCA and mixers. The first example is simply the product of a PCA and a mixer:

$$S_1 = PM \tag{55}$$

Because

$$|||S_1||| \leq |||P||| \cdot |||M||| = |||P||| \tag{56}$$

S_1 is ergodic if $\gamma_1 < 1$ is satisfied for the PCA. Here higher order constructive criteria for the PCA cannot be used to get stronger results. However, for the example

$$S_2 = MP^5 \tag{57}$$

it is enough that $\gamma_5 < 1$. The next mixer–PCA:

$$S_3 = M_1 P^3 M_2^2 P^2 \tag{58}$$

is ergodic if

$$\gamma_3 \cdot \gamma_2 < 1 \tag{59}$$

Finally we look at a stochastic transformation rule that chooses randomly at each time to be a PCA or to be a mixer:

$$S_4 = \lambda M + (1 - \lambda)P, \text{ with } \lambda \in [0, 1) \tag{60}$$

S_4 will be ergodic if the criterion $\gamma_1 < 1$ for the PCA is satisfied.

Comparison with Ferrari's results. The constructive criteria for pure PCA are better in detecting ergodic behavior than Ferrari's generalized duality argument of [F2]. This argument says that a PCA is exponentially ergodic if:

$$r(1-2k) < 1 \tag{61}$$

with

$$r = \sup_{x,\sigma} |R_x| \tag{62}$$

with R_x the neighborhood of x where $p_x(+|\sigma)$ depends on σ, and

$$k = \min_{x,\sigma}\{p_x(+|\sigma), 1 - p_x(+|\sigma)\} \tag{63}$$

It is easy to check that already the $\gamma_1 < 1$ criterion is more powerful than Ferrari's criterion. Higher order constructive criteria are even better. Both criteria remain valid if we combine our PCA with mixers. As we have seen, for some combinations it is even possible to use higher N MSCC. Both methods fail if the mixer depends on the configuration. So, we improve on Ferrari's results.

About continuous time. Ferrari also considered combinations of spin flip and stirring dynamics in continuous time [F2]. He shows there that if the spin flip process satisfies some criterion stronger than the so–called "$M < \epsilon$"–criterion (see section 5) the spin flip stirring process is still ergodic. In this section we have seen that the $\gamma_1 < 1$ criterion of the spin flip process alone remains valid if we add stirring. Since this criterion implies the "$M < \epsilon$"–criterion for continuous time we also improve on Ferrari's results for continuous time.

4 Ergodicity of PCA and oriented percolation

Ergodicity means forgetting about initial conditions. Intuitively, we can picture this as absence of percolation for paths starting at a space–time point and running backwards in time thereby connecting points that have undergone the influence of the initial state. We make this more precise following an construction analogous to the one made for Markov fields in [BM]. The main advantage of this approach here is that it allows to connect quite generally critical behavior in an oriented percolation problem to phase transitions for PCA.

4.1 Disagreement percolation for PCA

As we have seen before, the basic coupling for a PCA defines a new PCA on the product space. Starting at time zero from the pair $(\sigma, \bar{\sigma})$ of configurations $\sigma, \bar{\sigma} \in \Omega$, we obtain a Markov process $(\sigma_N, \bar{\sigma}_N), N = 0, 1, 2, \ldots$, with $(\sigma_0, \bar{\sigma}_0) = (\sigma, \bar{\sigma})$, whose transition probabilities satisfy

$$\mathrm{Prob}[\sigma_N(i) \neq \bar{\sigma}_N(i) | \sigma_{N-1} = \xi, \bar{\sigma}_{N-1} = \bar{\xi}] = \mathrm{var}[p_i(\cdot|\xi), p_i(\cdot|\bar{\xi})] \tag{64}$$

the right hand side being the variational distance between the (original) PCA transition probabilities $p_i(\cdot|\xi)$ and $p_i(\cdot|\bar{\xi})$ for given $\xi, \bar{\xi} \in \Omega$, (see (13)). For every realization $(\sigma_N(i), \bar{\sigma}_N(i)), i \in \boldsymbol{Z}^d, N = 0, 1, 2, \ldots$, let us now define new variables

$$s_N(i) = \begin{cases} 1 \text{ if } \sigma_N(i) \neq \bar{\sigma}_N(i) \\ 0 \text{ otherwise} \end{cases} \tag{65}$$

We say that the space–time point (N, i) is *open* (respectively closed) if $s_N(i) = 1$ (respectively $s_N(i) = 0$). ¿From (64) we can see that, with probability one with respect to the basic coupling, the point (N, i) is open only if there is a point $(N-1, j)$ with j in the neighborhood of i, which is also open. But then, there must be an *oriented path of disagreement* from the point (N, i) to some point $(0, j)$ on the zero–time layer, that is, a sequence of open points $v_0, v_1, \ldots, v_N$ with $v_0 = (N, i)$ and $v_n = (N-n, j_n)$ such that j_n is in the neighborhood of $j_{n-1}, n = 1, \ldots, N$. In fact, we have that in the basic coupling

$$\mathrm{Prob}[\sigma_N(i) \neq \bar{\sigma}_N(i) | \sigma, \bar{\sigma}] = \mathrm{Prob}[(N, i) \rightsquigarrow (0, \cdot) | \sigma, \bar{\sigma}] \tag{66}$$

where $(N, i) \rightsquigarrow (0, \cdot)$ stands for the event that there is an oriented path of disagreement from (N, i) to some $(0, j), j \in L$. Since we have that

$$|P^N f(\sigma) - P^N f(\bar{\sigma})| \leq \sum_i \delta_i f \ \mathrm{Prob}[\sigma_N(i) \neq \bar{\sigma}_N(i) | \sigma, \bar{\sigma}] \tag{67}$$

for functions $f \in D(\Omega)$, we see from (66) that the absence of oriented percolation for the field defined in (65) will imply uniqueness of the invariant state for our PCA. To make this more precise and to turn this into a simple uniqueness criterion, we define the quantity

$$q_i = \max_{\xi,\bar{\xi}} \ \mathrm{var}[p_i(\cdot|\xi), p_i(\cdot|\bar{\xi})] \tag{68}$$

which of course always exceeds (64). We can then consider the product measure $\nu_{\{q_i\}}$: zero–one variables $\tilde{s}_N(i)$ are assigned to the vertices of the same space–time lattice as before:

$$\tilde{s}_N(i) = \begin{cases} 1 \text{ with probability } q_i \\ 0 \text{ with probability } (1-q_i) \end{cases} \tag{69}$$

independently for all $i \in L$, $N = 1, 2, \ldots$. For completeness, we put $\tilde{s}_0(i) = 1$. Now note that because of (68) the $\{s_N(i)\}$ process and the $\{\tilde{s}_N(i)\}$ process can be coupled in such a way that $s_N(i) \leq \tilde{s}_N(i)$ with probability one for every space–time point. This, in turn, implies that (66) is dominated by the probability

$$\nu_{\{q_i\}}[(N,i) \rightsquigarrow (0,\cdot)] \tag{70}$$

of *independent* oriented percolation (here, an open path is one on which all of the $\tilde{s}_N(i)$ are 1). Hence, from (67),

$$\sup_{\sigma,\bar{\sigma}} |P^N f(\sigma) - P^N f(\bar{\sigma})| \leq \sum_i \delta_i f \ \nu_{\{q_i\}}[(N,i) \rightsquigarrow (0,\cdot)] \leq c_N |||f||| \tag{71}$$

where

$$c_N = \sup_j \nu_{\{q_i\}}[(N,j) \rightsquigarrow (0,\cdot)] \tag{72}$$

If the $\{q_i\}$ are small enough so that

$$c_N \longrightarrow 0 \text{ as } N \uparrow \infty \tag{73}$$

then the PCA is ergodic and converges uniformly to the unique invariant measure. We have thus proven

Theorem 2 *If in (72) $c_N \longrightarrow 0$ as $N \uparrow \infty$, then the PCA is ergodic and for all $f \in D(\Omega)$,*

$$\|P^N f(\sigma) - \nu(f)\| \leq c_N |||f||| \tag{74}$$

where ν is the unique invariant measure.

4.2 Applications

We have in mind two sorts of applications. The first consists in applying the uniqueness criterion (72) to various examples to obtain bounds on the critical parameters in terms of percolation thresholds. The second application is to consider PCA on arbitrary graphs and random PCA.

4.2.1 Examples.

Naturally, all the information we have on the threshold q_c for independent oriented percolation can be translated into information about the ergodic regime of our PCA. This works best in those cases where in fact (68) equals

$$q_i = \max_{\xi} \operatorname{var}[p_i(\cdot|\xi), p_i(\cdot|\xi^j)] \tag{75}$$

for all j in the neighborhood of i. If we assume that each neighborhood contains exactly R sites, then the Dobrushin single–site criterion ($N = 1$ in Theorem 1) gives

$$R \sup_i q_i < 1 \tag{76}$$

while we know from a Peierls–type estimate that this inequality certainly implies that $c_N \longrightarrow 0$. In fact the estimate in (76) is expected to be rather bad for low dimensional graphs. Stavskaya's model and the hard–core PCA are two examples where the situation as in (75) occurs and for which $R = 2$. In both models the space–time graphs are the same and they are equivalent to the usual square lattice.

In the figure below, a part of this space–time graph is depicted. Open points are marked with open circles. An oriented path of disagreement from $(7, 0)$ to $(0, 3)$ is shown.

Defining q_c to be the threshold for independent site oriented percolation on this space–time lattice, we certainly have that $q_c \geq p_c(2)$, the threshold for independent site percolation on the square lattice. It is rigorously (and computer–assisted) known that $p_c(2) \geq 0.54\ldots$ (see Menshikov and Pelikh [MP]) so that (76) is indeed *strictly* improved by Theorem 2.

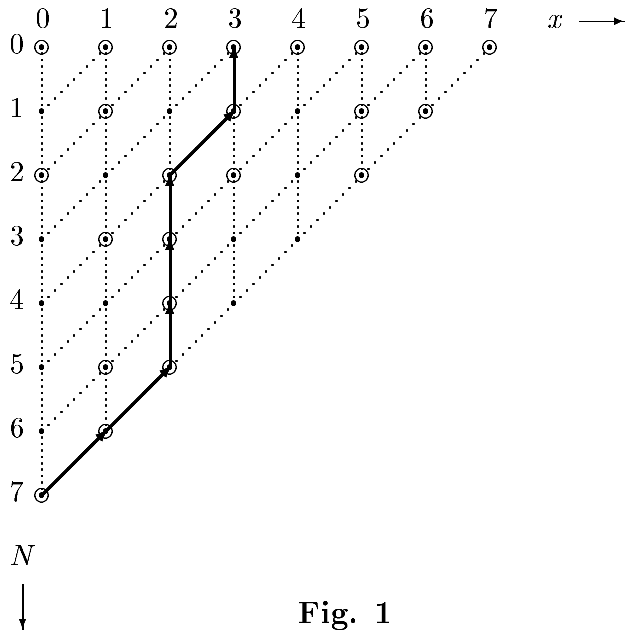

Fig. 1

4.2.2 PCA on arbitrary graphs

Let L be an arbitrary (infinite) graph which is locally finite. That is, every site $i \in L$ has a finite neighborhood consisting of R_i neighbors j for which the right hand side of (75) is non–zero. One can ask whether there is an $\epsilon > 0$ such that, if

$$q_i \equiv \max_{\xi, \bar{\xi}} \ \mathrm{var}[p_i(\cdot|\xi), p_i(\cdot|\bar{\xi})] \leq \epsilon \tag{77}$$

for all $i \in L$, then the PCA has a unique invariant measure. In the case where the R_i are uniformly bounded, this is certainly so, as follows already from Dobrushin's single–site condition ($N = 1$ in (Theorem 1)). But without this restriction, this answer is certainly false in general and must depend on the particularities of the graph L. In fact, answering this question goes beyond the constructive criteria that we discussed before, as these criteria (see Theorem 1) always suppose uniform bounds. The situation is reminiscent of the problem of uniqueness of Gibbs measures on arbitrary graphs. There the problem was solved by an analysis of Bassalygo and Dobrushin in [BD] who found a natural condition on the graph for which the corresponding question to (77) is positive. In that respect, the Theorem above sheds some light on this question for PCA. In order to apply it, consider the oriented percolation problem on the space–time graph generated by L. Put ν_q the Bernoulli measure on this graph which makes every space–time point open (respectively closed) with probability q (respectively $1-q$). Then, putting all $q_i = q$ in (72),

$$q_c = \inf\{q \in [0,1] : c_N \longrightarrow 0 \text{ as } N \uparrow \infty\} \tag{78}$$

is the usual threshold probability. The answer to the question in (77) is then positive whenever $q_c > 0$. It is an intrinsic property of the space–time graph.

4.2.3 Random PCA.

A logical continuation of the previous line of thinking would be to consider the problem of ergodicity for random PCA (see the analog for Gibbs measures in [BD]). These are PCA for which the transition rules $p_i(\pm|\xi)$ depend on random variables (quenched disorder). This would however lead us too far away from the main subject of this paper; we prefer to postpone this analysis to later work.

5 An application of the constructive criteria for interacting particle systems

Steif's idea [S] to approximate continuous time interacting particle sytems (IPS) by PCA is used by [MS2] and [MS3] to obtain ergodicity criteria for IPS. We treat a specific example to show how to improve drastically on the well–known "$M < \epsilon$" criterion.

5.1 Constructive criteria in continuous time

Brief introduction to IPS. An IPS is a continuous–time Markov process $\{\sigma_t\}_{t\geq 0}$ on Ω. It is characterized by a collection of translation invariant, local, bounded and nonnegative functions

$$\{c_x(\sigma)\}_{x\in L} \tag{79}$$

$c_x(\sigma)$ has to be interpreted as the flip rate of the spin at site x if the IPS is in state σ:

$$\text{Prob}[\sigma_t(x) \neq \sigma_0(x) | \sigma_0 = \sigma] = t c_x(\sigma) + o(t) \tag{80}$$

The range r of the IPS is the side length of a minimal hypercube

$$C_{z,r} = \{y \in \boldsymbol{Z^d} | z_i \leq y_i \leq z_i + r, i \in \{1, \ldots, d\}\} \tag{81}$$

in which spin flips do not alter $c_0(\sigma)$. The construction of the Markov semi–group $S(t)$ on $C(\Omega)$ (the analogue of P^N) is accomplished via the generator L of the IPS. L acts on local functions as

$$Lf(\sigma) = \sum_{x \in L} c_x(\sigma)[f(\sigma^x) - f(\sigma)] \tag{82}$$

Details of this construction can be found in [L]. Ergodicity is defined analogously as for PCA in (10).

The $M < \epsilon$ criterion. We define

$$M = \sum_{x \neq 0} \sup_{\sigma \in \Omega} |c_0(\sigma^x) - c_0(\sigma)| \tag{83}$$

$$\epsilon = \inf_{\sigma \in \Omega} |c_0(\sigma^x) + c_0(\sigma)| \tag{84}$$

$$B = \sup_{\sigma \in \Omega} c_0(\sigma) \tag{85}$$

M measures how strongly the flip rates depend on the configuration, ϵ represents the part of the behavior of a single spin that does not depend on the configuration at all and B is just a time scale factor. It is well–known that $M < \epsilon$ implies ergodicity [L]. Later we will give a short proof of this criterion.

Just like we did in section 2 we introduce quantities expressing the propagation of disagreements under the evolution of the basic coupling of the IPS. Define for continuous time t,

$$\gamma_t = \sum_{x \in L} \sup_{\sigma \in \Omega} Q_0^t(\sigma, \sigma^x) \tag{86}$$

where

$$Q_0^t(\sigma, \sigma^x) = \text{Prob}[\sigma_t(0) \neq \bar{\sigma}_t(0) | (\sigma_0, \bar{\sigma}_0) = (\sigma, \sigma^x)] \tag{87}$$

is calculated in the basic coupling $(\sigma_t, \bar{\sigma}_t)$ for IPS with the same flip rates but started from different initial conditions (see [L] and [M]). A complication is that the above quantity $Q_0^t(\sigma, \sigma^x)$ cannot be calculated directly, because in any time interval of nonzero length, any spin can flip unboundedly often. However, since the probability that a lot of spin flips occur in a bounded region in a short time is small, it is possible to approximate IPS with processes that do not allow too many spin flips within small time intervals. It turns out that PCA are suitable approximations.

Approximating IPS by PCA. Choose $\delta > 0$ small. Given an IPS with rates $c_x(\sigma)$, we define its δ–approximating PCA (δ–PCA) by taking transition probabilities

$$p_x^{(\delta)}(+|\sigma) = \begin{cases} \frac{1}{2}(1 - e^{-2\delta c_x(\sigma)}) & \text{if } \sigma(x) = -, \\ 1 - \frac{1}{2}(1 - e^{-2\delta c_x(\sigma)}) & \text{if } \sigma(x) = + \end{cases} \tag{88}$$

So, to first order in δ, $p_x^{(\delta)}(+|\sigma)$ goes as

$$\begin{cases} \delta c_x(\sigma) & \text{if } \sigma_x = - \\ 1 - \delta c_x(\sigma) & \text{if } \sigma_x = + \end{cases} \tag{89}$$

The reason for not taking (89) as definition of the δ-PCA is that (88) approximates the IPS slightly better (see [MS2]). Intuitively, it is clear that the δ–PCA approximates the IPS. Technical details can be found in [M], [MS2] and [S].

Define $\gamma_N^{(\delta)}$ as in (16) but corresponding to the δ–PCA as in (88). Moreover, put

$$E(N, \delta) = \min_{n \in \{1,2,\ldots\}} E_n(N, \delta) \tag{90}$$

$$E_n(N, \delta) = 2(2nr + 1)^d z(N, \delta) + e^{-\epsilon N \delta} \sum_{k \geq n} \frac{(MN\delta)^k}{k!} \tag{91}$$

$$z(N, \delta) = (e^{MN\delta} - 1)\left(\frac{B}{M} - \frac{1 - e^{-2B\delta}}{2(e^{M\delta} - 1)}\right) \tag{92}$$

Theorem 3 *If we can find (N, δ) such that*

$$\gamma_N^{(\delta)} \leq 1 - E(N, \delta) \tag{93}$$

then the IPS is uniformly exponentially ergodic, that is, there are constants $c > 0, \lambda \leq \infty$, so that for all $f \in D(\Omega)$,

$$\left\| S(t) - \int \nu(d\sigma) f(\sigma) \right\| \leq c e^{-\lambda t} |||f|||. \tag{94}$$

where ν is the unique invariant measure ($\nu L = 0$).

Remark 1. Actually, from the original arguments in [MS2], it is clear that we can slightly improve the Theorem by replacing $\gamma_N^{(\delta)}$ in (16) by

$$\gamma_{N,n}^{(\delta)} = \sum_{x \in C_{nz,nr}} \kappa_{0x}^{(\delta)\ N} \tag{95}$$

Remark 2. Using the definition of the δ–PCA (88) and $\gamma_1^{(\delta)}$ (16) we easily find

$$\gamma_1^{(\delta)} = 1 + \delta(M - \epsilon) \tag{96}$$

On the other hand $E_2(1, \delta)$ is of order 2 in δ, so if $M < \epsilon$, the MSCC for IPS will be fullfilled for $N = 1$ and δ small enough.

For better criteria we have to use higher order MSCC. In order to let these be effective, we may not take δ too small because the terms in $\gamma_N^{(\delta)}$ responsible for making the $\gamma_N^{(\delta)}$–criterion better than the $\gamma_1^{(\delta)}$–criterion would vanish then.

5.2 The continuous–time asymmetric contact process

Definition. The flip rates of the continuous–time asymmetric contact process are

$$c_x(\sigma) = \begin{cases} 0 & \text{if} \quad \sigma(x-1) = \sigma(x) = + \\ M & \text{if} \quad \sigma(x-1) = -, \sigma(x) = + \\ \epsilon & \text{if} \quad \sigma(x) = - \end{cases} \tag{97}$$

It is easy to see that M and ϵ have the same meaning as in the $M < \epsilon$ criterion, see (85).

Approximation by the discrete–time asymmetric contact process. The δ–PCA has transition probabilities

$$p_x^{(\delta)}(\sigma) = \begin{cases} 0 & \text{if} \quad \sigma(x-1) = \sigma(x) = + \\ 1-\alpha & \text{if} \quad \sigma(x-1) = -, \sigma(x) = + \\ \beta & \text{if} \quad \sigma(x) = - \end{cases} \tag{98}$$

with

$$\alpha = \frac{1}{2}(1 - e^{-2\delta M}), \ \beta = \frac{1}{2}(1 - e^{-2\delta\epsilon}) \tag{99}$$

We have studied the above PCA, the discrete–time asymmetric contact process, in subsections 2.3 and 2.4.

Calculating $\gamma_N^{(\delta)}$. With the help of these subsections we calculate $\gamma_N^{(\delta)}$ to second order in δ:

$$1 + N(M-\epsilon)\delta + N(\epsilon^2 - M^2)\delta^2 + \frac{1}{2}N(N-1)(\epsilon^2 - 2\epsilon M)\delta^2 + o(\delta^2) \tag{100}$$

$E_3(N,\delta)$ is to second order in δ equal to $12NM^2\delta^2$. It is discouraging to see that the smallest value of N for which the coefficient of the second order in δ of $\gamma_N^{(\delta)} + E_3(N,\delta)$ becomes negative, is 26; we will have to know $\gamma_N^{(\delta)}$ for N larger than we have computed it before.

We are now facing a similar task as in subsections 2.3 and 2.4 of finding a δ^* such that M can be as large as possible under the conditions $\epsilon = 1$ and $\gamma_N^{(\delta)} + E(N,\delta) < 1$. The extra term $E(N,\delta)$ decreases this δ^*. It turns out that the hypercube $\{-n,\ldots,0\}$ (see (81)) that roughly contains the infection will grow so slowly that it becomes feasible to calculate with a computer all $\kappa_{y0}^{(\delta)\ N}$ with y in this hypercube for N as large as $10,000$.

About the calculations. For the asymmetric contact process the expression for $\gamma_{N,n}^{(\delta)}$ simplifies to

$$\gamma_{N,n}^{(\delta)} = \sum_{y=0}^{-n} \mathrm{Prob}[\sigma_N(y) = - | \sigma_0 = +^0] \tag{101}$$

The probability distribution over the configurations on $\{-n,\ldots,0\}$ at $t=N$ can be expressed in the distributions at $t = N-1$:

$$P^{(\delta)\ N}(\sigma_N(-n),\ldots,\sigma_N(0)|\sigma_0=+^0)=$$

$$\sum_{\sigma_{N-1}(-n)=-}^{+}\cdots\sum_{\sigma_{N-1}(0)=-}^{+}\prod_{x=-n}^{0}p_x^{(\delta)}(\sigma_N(x)|\sigma_{N-1}(x),\sigma_{N-1}(x+1))$$

$$P^{(\delta)\ N-1}(\sigma_{N-1}(-n+1),\ldots,\sigma_{N-1}(0)|\sigma_0=+^0) \tag{102}$$

Results. We have programmed the above recursion relation on a computer. Some relevant results are

ϵ	δ	M	N	n	$1-\gamma_{N,n}^{(\delta)}-E_n(N,\delta)$
1	3.5×10^{-3}	1.066	10^2	3	4.6×10^{-4}
1	1.1×10^{-3}	1.28	10^3	5	4.8×10^{-3}
1	1.65×10^{-4}	1.5	10^4	8	3.8×10^{-3}

(103)

(The computation for $N=10^4$ took a few hours on a *Sun SPARC station 1+*.) We conclude that the continuous–time asymmetric contact process is ergodic for $M<1.5\epsilon$.

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