THE BOND-TRUE SELF-AVOIDING WALK ON $\mathbb{Z}$.
II: LOCAL LIMIT THEOREM

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Abstract. The bond-true self-avoiding walk is a nearest neighbour random walk X_n on $\mathbb{Z}$, for which the probability of jumping along a bond of the lattice is proportional to $\exp(-g\cdot$ number of previous jumps along that bond). This paper is a continuation of Tóth (1993), where the local time process and first hitting times of X. were investigated. For formal definitions and notation see that paper. Here we prove a local limit theorem, as $\alpha\to\infty$, for the distribution of $\alpha^{-2/3}X_{\theta_{s/\alpha}}$, where $\theta_{s/\alpha}$ is a random time of geometric distribution with mean $\left(1-\mathrm{e}^{-s/\alpha}\right)^{-1}=\frac{\alpha}{s}+O(1)$.

1. Introduction and Result

We continue the investigation of BTSAW started in Tóth (1993), henceforth refered to as [I.]. For a general introduction, definitions and notation see that paper. We shall use consequently the notations of [I.], with one single exception: the local time process S defined in (2.3) below is essentially *twice* the process S of [I.]. In [I.] we proved a limit theorem for the local time process of BTSAW stopped at the first hitting of the site at distance n from the origin of the lattice. In the present paper we prove a local limit theorem for the distribution of the random walk itself.

Let $|W_y|\quad y\in(-\infty,\infty)$ be a (two-sided) reflected Brownian motion with an arbitrary starting point $|W_0|=h\in[0,\infty)$. For $x\in[0,\infty)$ define

$$\omega_x^-=\sup\{y<x\,:\,|W_y|=0\}\tag{1.1}$$

$$\omega_x^+=\inf\{y>x\,:\,|W_y|=0\}\tag{1.2}$$

and, for $x\geq 0$

$$T_x=\int_{\omega_0^-}^{\omega_x^+}|W_s|\,\mathrm{d}s.\tag{1.3}$$

For any initial condition $|W_0|=h$, T_x clearly has an absolutely continuous distribution. Let

$$\varrho(t;x,h)=\frac{\mathbf{P}\Big(T_x\in(t,t+\mathrm{d}t)\Big\|\,|W_0|=h\Big)}{\mathrm{d}t}\tag{1.4}$$

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be the density of the distribution of T_x. From scaling the Brownian motion we easily get

$$\alpha\varrho(\alpha t; \alpha^{2/3}x, \alpha^{1/3}h) = \varrho(t; x, h) \tag{1.5}$$

for any $\alpha > 0$. Define $\mathbb{R}_+ \times \mathbb{R} \ni (t, x) \mapsto \varphi(t, x) \in \mathbb{R}_+$ as follows

$$\varphi(t, x) = \int_0^\infty \varrho(\frac{t}{2}; |x|, h)\mathrm{d}h. \tag{1.6}$$

The finiteness of the integral on the right hand side will be seen soon. The scaling property (1.5) of ϱ implies

$$\alpha^{2/3}\varphi(\alpha t, \alpha^{2/3}x) = \varphi(t, x). \tag{1.7}$$

We shall denote by $\hat{\varrho}$ and $\hat{\varphi}$ the Laplace transforms of ϱ and φ:

$$\hat{\varrho}(s; x, h) = s\int_0^\infty \mathrm{e}^{-st}\varrho(t; x, h)\mathrm{d}t. \tag{1.8}$$

$$\hat{\varphi}(s, x) = s\int_0^\infty \mathrm{e}^{-st}\varphi(t, x)\mathrm{d}t = \int_0^\infty \hat{\varrho}(2s; |x|, h)\mathrm{d}h. \tag{1.9}$$

These functions scale as follows

$$\alpha\hat{\varrho}(\alpha^{-1}s; \alpha^{2/3}x, \alpha^{1/3}h) = \hat{\varrho}(s; x, h) \tag{1.10}$$

$$\alpha^{2/3}\hat{\varphi}(\alpha^{-1}s, \alpha^{2/3}x) = \hat{\varphi}(s, x). \tag{1.11}$$

Theorem 1. *Given $t \in (0, \infty)$ (resp. $s \in (0, \infty)$) fixed, $x \mapsto \varphi(t, x)$ (resp. $x \mapsto \hat{\varphi}(s, x)$) is a probability density. I.e. for any $t \in (0, \infty)$ (resp. $s \in (0, \infty)$):*

$$\int_{-\infty}^{\infty} \varphi(t, x)\, dx = 1 \tag{1.12}$$

$$\int_{-\infty}^{\infty} \hat{\varphi}(s, x)\, dx = 1. \tag{1.13}$$

Remark: (1.12) and (1.13) are, of course, equivalent statements. $\hat{\varphi}(s, \cdot)$ is the distribution $\varphi(t, \cdot)$ observed at a 'random time' of exponential distribution with mean value s^{-1}. The statement of this Theorem is quite surprising, since we could not find any direct intuitive way of its proof. The proof, postponed to the Appendix, relies partly on some probabilistic results concerning Brownian excursion theory and Bessel bridges, and on the other hand, on rather messy integrations involving Bessel- and hypergeometric functions.

We are ready now to formulate our main result. We denote by $P(n, k)$, $n \in \mathbb{N}$, $k \in \mathbb{Z}$ the distribution of our BTSAW at time n:

$$P(n, k) = \mathbf{P}\Big(X_n = k\Big) \tag{1.14}$$

and by $R(s,k)$, $s\in\mathbb{R}_+$, $k\in\mathbb{Z}$ the distribution of the BTSAW observed at a random time of geometric distribution with mean $(1-\mathrm{e}^{-s})^{-1}$:

$$R(s,k)=\big(1-\mathrm{e}^{-s}\big)\sum_{n=0}^{\infty}\mathrm{e}^{-sn}P(n,k). \tag{1.15}$$

We define the rescaled 'densities' of the above distributions

$$\varphi_\alpha(t,x)=\alpha^{2/3}P([\alpha t],[\alpha^{2/3}x]) \tag{1.16}$$
$$\hat\varphi_\alpha(s,x)=\alpha^{2/3}R(\alpha^{-1}s,[\alpha^{2/3}x]), \tag{1.17}$$

$t,s\in\mathbb{R}_+$, $x\in\mathbb{R}$. It is straightforward that $\hat\varphi_\alpha$ is exactly the Laplace transform of φ_α:

$$\hat\varphi_\alpha(s,x)=s\int_0^\infty \mathrm{e}^{-st}\varphi_\alpha(t,x)\mathrm{d}t. \tag{1.18}$$

Our main result is

Theorem 2. *For any $s\in\mathbb{R}_+$ and almost all $x\in\mathbb{R}$*

$$\hat\varphi_\alpha(s,x)\to\sigma^{2/3}\hat\varphi(s,\sigma^{2/3}x) \tag{1.19}$$

as $\alpha\to\infty$. The positive constant σ is explicitely given in (1.15) of [I.].

Remark: This is of course a *local limit theorem* for the BTSAW observed at an independent random time $\theta_{s/\alpha}$ of geometric distribution with mean $\big(1-\mathrm{e}^{-s/\alpha}\big)^{-1}$. In particular the (integral) limit law

$$\mathbf{P}\Big(\alpha^{-2/3}X_{\theta_{\frac{s}{\alpha}}}<x\Big)\to\int_{-\infty}^{x}\hat\varphi(s,\sigma^{2/3}y)\sigma^{2/3}\mathrm{d}y \tag{1.20}$$

follows. This is a little bit short of stating the limit theorem for deterministic time

$$\mathbf{P}\Big(\alpha^{-2/3}X_{[\alpha t]}<x\Big)\to\int_{-\infty}^{x}\varphi(t,\sigma^{2/3}y)\sigma^{2/3}\mathrm{d}y. \tag{1.21}$$

But, of course we can conclude that if $X_{[\alpha t]}/\alpha^{2/3}$ has got a limiting distribution then (1.21) also holds.

The next section is devoted to the proof of Theorem 2, Theorem 1 is proved in the Appendix.

2. Proof of Theorem 2

We shall prove (1.19) for $x\geq 0$. For $x\leq 0$ the same proof holds with slightly changed notation. In [I.] we proved a limit theorem for the local time process of the BTSAW stopped at the first hitting of a distant site of the lattice. Now we first reformulate that theorem for the BTSAW stopped at the m-th hitting (coming from right or left) of that site. Unfortunately the notation becomes more complicated.

Let $k \in \mathbb{Z} \cap (0, \infty)$ be a lattice site and $m \in \mathbb{N}$. We denote by $T^{(\mathrm{l})}_{k,m}$ and $T^{(\mathrm{r})}_{k,m}$ the time of the $m+1$-th arrival to the lattice site k from left respectively right:

$$T^{(\mathrm{l,r})}_{k,-1} = 0 \tag{2.1}$$

$$T^{(\mathrm{l,r})}_{k,m+1} = \min\{i > T^{(\mathrm{l,r})}_{k,m} : X_{i-1} = k \mp 1,\ X_i = k\}. \tag{2.2}$$

(Note that T_k defined in (1.6) of [I.] is $T^{(\mathrm{l})}_{k,0}$ in this notation.) The bond local time process of the BTSAW stopped at $T^{(*)}_{k,m}$, $* = \mathrm{l}, \mathrm{r}$, is

$$S^{(*)}_{k,m}(l) = v(k - l|\underline{X}_0^{T^{(*)}_{k,m}}), \quad l \in \mathbb{Z}. \tag{2.3}$$

We remind that $v(k|\underline{x}_0^i)$, defined formally in (1.1)-(1.3) of [I.], is the the total number of jumps along the bond $(k-1,k)$ performed by the nearest neighbour walk $\underline{x}_0^i$. Let

$$\omega^{(*)+}_{k,m} = \min\{l \geq k : S^{(*)}_{k,m}(l) = 0\} \tag{2.4}$$

$$\omega^{(*)-}_{k,m} = \max\{l \leq 0 : S^{(*)}_{k,m}(l) = 0\}. \tag{2.5}$$

In plain words: $k - \omega^{(*)+}_{k,m}$ and $k - \omega^{(*)-}_{k,m} - 1$ are the leftmost, respectively rightmost sites visited by the walk $\underline{X}_0^{T^{(*)}_{k,m}}$. The following Theorem is a natural generalization of Theorem 1 in [I.]:

Theorem 3. *Let $x \in [0, \infty)$, $h \geq 0$ and $* = \mathrm{l}, \mathrm{r}$.*

$$\left(\frac{\omega^{(*)-}_{[\alpha x],[\sqrt{\alpha}h]}}{\alpha}, \frac{\omega^{(*)+}_{[\alpha x],[\sqrt{\alpha}h]}}{\alpha}, \frac{S^{(*)}_{[\alpha x],[\sqrt{\alpha}h]}([\alpha y])}{2\sigma\sqrt{\alpha}} : \frac{\omega^{(*)-}_{[\alpha x],[\sqrt{\alpha}h]}}{\alpha} \leq y \leq \frac{\omega^{(*)+}_{[\alpha x],[\sqrt{\alpha}h]}}{\alpha} \right)$$
$$\Rightarrow \left(\omega_0^-, \omega_x^+, |W_y| : \omega_0^- \leq y \leq \omega_x^+ \Big\| \; |W_y| = h \right) \tag{2.6}$$ ∎

in $\mathbb{R}_- \times \mathbb{R}_+ \times D(-\infty, \infty)$ as $\alpha \to \infty$. The positive constant σ is defined in (1.15) of [I.].

Remark: Theorem 1 of [I.] was formulated for $h = 0$ and $* = \mathrm{l}$, and consequently the formulation and notation was simpler. But one can easily see that except the more complicated notation no new element arises in this general setup. We refer the reader to the proof in [I.].

We shall use the following straightforward

Corollary 1. *For $x \in [0, \infty)$, $h \geq 0$ and $* = \mathrm{l}, \mathrm{r}$.*

$$\frac{T^{(*)}_{[\alpha x],[\sqrt{\alpha}h]}}{2\sigma\alpha^{3/2}} \Rightarrow \left(T_x \Big\| \; |W_0| = h \right) \tag{2.8}$$

as $\alpha \to \infty$.

Remark: This is of course the natural generalization of Corollary 2 in [I.].

To prove Theorem 2 we note first that

$$P(n,k)=\mathbf{P}\Big(X_n=k\Big)=\sum_{m=0}^{\infty}\Big[\mathbf{P}\Big(T^{(\mathrm{l})}_{k,m}=n\Big)+\mathbf{P}\Big(T^{(\mathrm{r})}_{k,m}=n\Big)\Big]. \tag{2.9}$$

On the other hand, from the definition of $\hat\varphi$

$$\hat\varphi_\alpha(s,x)=\frac{1-\mathrm{e}^{-s/\alpha}}{s/\alpha}s\alpha^{-1/3}\sum_{n=0}^{\infty}\mathrm{e}^{-ns/\alpha}P(n,[\alpha^{2/3}x]). \tag{2.10}$$

Combining (2.9) and (2.10) we are lead to

$$\hat\varphi_\alpha(s,x)=\frac{1-\mathrm{e}^{-s/\alpha}}{s/\alpha}\times \tag{2.11}$$
$$s\alpha^{-1/3}\sum_{m=0}^{\infty}\Big[\mathbf{E}\Big(\exp\Big(-sT^{(\mathrm{l})}_{[\alpha^{2/3}x],m}/\alpha\Big)\Big)+\mathbf{E}\Big(\exp\Big(-sT^{(\mathrm{r})}_{[\alpha^{2/3}x],m}/\alpha\Big)\Big)\Big].$$

Defining

$$\hat\varrho^{(\mathrm{l,r})}_\alpha(s;x,h)=s\mathbf{E}\Big(\exp\Big(-sT^{(\mathrm{l,r})}_{[\alpha^{2/3}x],[\alpha^{1/3}h]}/(2\sigma\alpha)\Big)\Big) \tag{2.12}$$

(2.11) reads

$$\hat\varphi_\alpha(s,x)=\frac{1-\mathrm{e}^{-s/\alpha}}{s/\alpha}\frac{1}{2\sigma}\int_0^\infty\Big(\hat\varrho^{(\mathrm{l})}_\alpha(2\sigma s;x,h)+\hat\varrho^{(\mathrm{r})}_\alpha(2\sigma s;x,h)\Big)\,\mathrm{d}h \tag{2.13}$$

¿From Corollary 1. follows that for any $s>0$, $x\in[0,\infty)$ and $h>0$

$$\hat\varrho^{(\mathrm{l,r})}_\alpha(s;x,h)\to\hat\varrho(s;x,h) \tag{2.14}$$

as $\alpha\to\infty$. (2.13) and (2.14) imply

$$\liminf_{\alpha\to\infty}\hat\varphi_\alpha(s,x)\geq\sigma^{-1}\int_0^\infty\hat\varrho(2\sigma s;|x|,h)\mathrm{d}h=\sigma^{2/3}\hat\varphi(s,\sigma^{2/3}x). \tag{2.15}$$

On the other hand

$$\int_{-\infty}^{\infty}\hat\varphi_\alpha(s,x)\mathrm{d}x=1=\int_{-\infty}^{\infty}\hat\varphi(s,x)\mathrm{d}x. \tag{2.16}$$

¿From (2.15) and (2.16) follows the statement of Theorem 2. $\square$

Appendix: Proof of Theorem 1

Throughout this section we shall use the notations of Revuz and Yor (1991), chapters XI and XII, in particular we denote by $\rho_\nu(t)$, $t \in [0,1]$ the standard Bessel bridge of index ν (i.e. of 'dimension' $\delta = 2(\nu+1)$) over the time interval $[0,1]$.

Due to the scaling property (1.7) of $\hat\varphi$ it is enough to prove (1.13) for one particular value of s, say $s=1$. Changing the order of integration we easily get:

$$\int_0^\infty \Big(\int_0^\infty \hat\varrho(1;x,h)\mathrm{d}h \Big)\mathrm{d}x = \int_0^\infty \mathbf{E}\Big(\int_0^\infty \mathrm{e}^{-T_x}\mathrm{d}x \Big\| |W_0| = h \Big)\mathrm{d}h \tag{A.1}$$

Using Itô's and Bismut's characterizations of the 1-d Brownian excursions (i.e. Theorems XII.4.2. and XII.4.7. in Revuz and Yor (1991)) we can express the integral on the right hand side of (A.1) as follows:

$$\int_0^\infty \hat\varphi(s,x)\mathrm{d}x =$$
$$\frac{1}{2\sqrt{2\pi}}\left[\frac{1}{2}\int_0^\infty \mathbf{E}\Big(\exp(-\tau b^{3/2})\Big) b^{1/2}\mathrm{d}b + \frac{\Big(\int_0^\infty \mathbf{E}\Big(\exp(-\tau b^{3/2})\Big) b^{-1/2}\mathrm{d}b\Big)^2}{\int_0^\infty \mathbf{E}\Big(1-\exp(-\tau b^{3/2})\Big) b^{-3/2}\mathrm{d}b} \right] \tag{A.2}$$

where

$$\tau = \int_0^1 \rho_{\frac12}(s)\mathrm{d}s \tag{A.3}$$

is the area under the three dimensional Bessel bridge. Performing the integrals in (A.2) we get

$$\int_{-\infty}^\infty \hat\varphi(s,x)\mathrm{d}x = \frac{1}{\sqrt{2\pi}}\left[\frac{1}{3}\mathbf{E}\Big(\tau^{-1}\Big) + \frac{2}{9}\frac{\Gamma^2(1/3)}{\Gamma(2/3)}\frac{\mathbf{E}^2\Big(\tau^{-1/3}\Big)}{\mathbf{E}\Big(\tau^{1/3}\Big)} \right]. \tag{A.4}$$

Let the random variables X and Y be defined as follows:

$$X = \int_0^1 \Big(\rho_{\frac13}(s)\Big)^{-2/3}\mathrm{d}s \tag{A.5}$$
$$Y = \Big(\frac32\Big)^{2/3}\Big(\int_0^1 \rho_{\frac12}(s)\mathrm{d}s\Big)^{-2/3} = \Big(\frac32\Big)^{2/3}\tau^{-2/3}. \tag{A.6}$$

According to Theorem XI.3.5 in Revuz and Yor (1991), for any measurable function $f:\mathbb{R}_+\to\mathbb{R}_+$

$$\mathbf{E}\Big(f(Y)\Big) = \frac{2^{1/6}\Gamma(1/2)}{\Gamma(1/3)}\mathbf{E}\Big(X^{1/2}f(X)\Big). \tag{A.7}$$

Choosing $f(y) = y^{3/2}$, $y^{1/2}$ and $y^{-1/2}$ we get in turn

$$\mathbf{E}\Big(\tau^{-1}\Big) = \left(\frac{3}{2}\right)^{-1} \frac{2^{1/6}\Gamma(1/2)}{\Gamma(1/3)} \mathbf{E}\Big(X^2\Big) \tag{A.8}$$

$$\mathbf{E}\Big(\tau^{-1/3}\Big) = \left(\frac{3}{2}\right)^{-1/3} \frac{2^{1/6}\Gamma(1/2)}{\Gamma(1/3)} \mathbf{E}\Big(X\Big) \tag{A.9}$$

$$\mathbf{E}\Big(\tau^{1/3}\Big) = \left(\frac{3}{2}\right)^{1/3} \frac{2^{1/6}\Gamma(1/2)}{\Gamma(1/3)}. \tag{A.10}$$

Inserting these expressions into (A.3) we get

$$\int_{-\infty}^{\infty} \hat{\varphi}(s,x)\mathrm{d}x = \frac{2^{2/3}}{9\Gamma(1/3)} \left[\mathbf{E}\Big(X^2\Big) + \frac{2}{3}\frac{\Gamma^2(1/3)}{\Gamma(2/3)}\mathbf{E}^2\Big(X\Big)\right]. \tag{A.11}$$

We sketch the lengthy calculations of $\mathbf{E}\Big(X\Big)$ and $\mathbf{E}\Big(X^2\Big)$. We shall use the notations of Erdély et al. (1953): we denote by J_ν the Bessel function of order ν, by I_ν the modified Bessel function of order ν and by $\Phi(a,c;\cdot)$ the confluent hypergeometric function with indices a and c. Let $\pi_t(x)$, $t \in (0,1)$, $x \geq 0$ and $\pi_{s,t}(x,y)$, $s,t \in (0,1)$, $s+t<1$, $x,y \geq 0$ be the densities of the distribution of $\rho_{\frac{1}{3}}(t)$ respectively of the joint distribution of $\Big(\rho_{\frac{1}{3}}(s), \rho_{\frac{1}{3}}(1-t)\Big)$ (see Itô and McKean (1965) or Revuz and Yor (1991)):

$$\pi_t(x) = \frac{3}{2^{1/3}\Gamma(1/3)} \frac{x^{5/3}}{[t(1-t)]^{4/3}} \mathrm{e}^{-\frac{x^2}{2t(1-t)}} \tag{A.12}$$

$$\pi_{t,s}(x,y) = \frac{3}{2^{1/3}\Gamma(1/3)} \left(\frac{x}{s}\right)^{4/3} \left(\frac{y}{t}\right)^{4/3} \mathrm{e}^{-\frac{x^2}{2s}} \mathrm{e}^{-\frac{y^2}{2t}} \times \frac{1}{1-t-s}\mathrm{e}^{-\frac{x^2+y^2}{2(1-t-s)}} I_{\frac{1}{3}}\left(\frac{xy}{1-t-s}\right). \tag{A.13}$$

The first two moments and correlations of $\rho_{\frac{1}{3}}^{-2/3}$ will be denoted as follows:

$$m_1(t) = \mathbf{E}\Big(\Big(\rho_{\frac{1}{3}}(t)\Big)^{-2/3}\Big) = \int_0^\infty x^{-2/3}\pi_t(x)\mathrm{d}x \tag{A.14}$$

$$m_2(t) = \mathbf{E}\Big(\Big(\rho_{\frac{1}{3}}(t)\Big)^{-4/3}\Big) = \int_0^\infty x^{-4/3}\pi_t(x)\mathrm{d}x \tag{A.15}$$

$$c(s,t) = \mathbf{E}\Big(\Big(\rho_{\frac{1}{3}}(s)\Big)^{-2/3}\Big(\rho_{\frac{1}{3}}(1-t)\Big)^{-2/3}\Big) = \int_0^\infty\int_0^\infty (xy)^{-2/3}\pi_{s,t}(x,y)\mathrm{d}x\mathrm{d}y. \tag{A.16}$$

Clearly

$$\mathbf{E}\Big(X\Big) = \int_0^1 m_1(t)\mathrm{d}t \tag{A.17}$$

$$\mathbf{E}\Big(X^2\Big) = 2\int_0^1\int_0^{1-t} c(s,t)\mathrm{d}s\mathrm{d}t. \tag{A.18}$$

m_1 and m_2 are easily computed:

$$m_1(t) = \frac{3}{2^{1/3}\Gamma(1/3)}[t(1-t)]^{-1/3} \tag{A.19}$$

$$m_2(t) = \frac{3\Gamma(2/3)}{2^{2/3}\Gamma(1/3)}[t(1-t)]^{-2/3}. \tag{A.20}$$

¿From (A.16) and (A.18)

$$\mathbf{E}\Big(X\Big) = \frac{9\Gamma^2(2/3)}{2^{1/3}\Gamma^2(1/3)} \tag{A.21}$$

The computation $c(s,t)$ is much more involved. Using identity (7.7.38) of Erdélyi et al. (1953), vol. 2, we first rewrite $\pi_{s,t}$

$$\begin{aligned}\pi_{t,s}(x,y) =& \frac{3}{2^{1/3}\Gamma(1/3)}(st)^{-4/3}\times \\ & \int_0^\infty e^{(1-s-t)v^2/2}\Big(x^{4/3}e^{-\frac{x^2}{2s}}J_{\frac{1}{3}}(xv)\Big)\Big(y^{4/3}e^{-\frac{y^2}{2t}}J_{\frac{1}{3}}(yv)\Big)v\mathrm{d}v.\end{aligned} \tag{A.22}$$

Next we use the Hankel transform (8.6.14) from Erdélyi et al. (1954), vol. 2 and the identity (6.3.7) from Erdélyi et al. (1953), vol. 1 to get

$$c(s,t) = \frac{3^3}{2^{2/3}\Gamma^3(1/3)}(st)^{-1/3}\int_0^\infty u^{1/3}e^{-u}\Phi(\tfrac{1}{3},\tfrac{4}{3};su)\Phi(\tfrac{1}{3},\tfrac{4}{3};tu)\mathrm{d}u. \tag{A.23}$$

In the following step we use the basic integral representation of the confluent hypergeometric function, (6.5.1) from Erdélyi et al. (1953), vol. 1:

$$c(s,t) = \frac{1}{2^{2/3}\Gamma^2(1/3)}(st)^{-2/3}\int_0^s\int_0^t(xy)^{-2/3}(1-x-y)^{-4/3}\mathrm{d}y\mathrm{d}x. \tag{A.24}$$

In the last expression the integration with respect to the y variable can be performed:

$$c(s,t) = \frac{9}{2^{2/3}\Gamma^2(1/3)}t^{-1/3}\frac{s^{-2/3}}{3}F(s,t) \tag{A.25}$$

where the function F is defined on the domain $\{(s,t)\in(0,1)\times(0,1) : s+t<1\}$ as follows:

$$F(s,t) = \int_0^s x^{-2/3}(1-x)^{-1}(1-t-x)^{-1/3}\mathrm{d}x. \tag{A.26}$$

Integrating by parts with respect to the s variable we are lead to

$$\begin{aligned}\int_0^1\int_0^{1-t}c(s,t)\mathrm{d}\mathrm{d}t =& 3\int_0^1(1-t)c(1-t,t)\mathrm{d}t \\ & -\frac{9}{2^{2/3}\Gamma^2(1/3)}\int_0^1\int_0^{1-t}t^{-1/3}s^{-1/3}(1-s)^{-1}(1-t-s)^{-1/3}\mathrm{d}s\mathrm{d}t.\end{aligned} \tag{A.27}$$ ∎

Inserting $c(1-t,t) = m_2(t)$ from (A.19) and performing some straightforward transformations in the last integral on the right hand side of (A.26) we finally get

$$\begin{aligned}\mathbf{E}\Big(X^2\Big) =& \frac{9\cdot 2^{1/3}\Gamma(2/3)}{\Gamma(1/3)}\int_0^1 t^{-2/3}(1-t)^{1/3}\mathrm{d}t \\ &- \frac{9\cdot 2^{1/3}}{\Gamma^2(1/3)}\int_0^1 s^{-1/3}(1-s)^{-2/3}\mathrm{d}s\int_0^1 t^{-1/3}(1-t)^{-1/3}\mathrm{d}t = \\ &\frac{9\Gamma(1/3)}{2^{2/3}} - \frac{27\cdot 2^{1/3}\Gamma^3(2/3)}{\Gamma^2(1/3)}.\end{aligned} \tag{A.28}$$

Inserting $\mathbf{E}\Big(X\Big)$ from (A.20) and $\mathbf{E}\Big(X^2\Big)$ from (A.27) into (A.10) we get (1.13) indeed. □

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