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Generalized Reduction Procedure: Symplectic and Poisson Formalism

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Abstract

We present a generalized reduction procedure which encompasses the one based on the momentum map and the projection method. By using the duality between manifolds and ring of functions defined on them, we have cast our procedure in an algebraic context. In this framework we give a simple example of reduction in the non-commutative setting.

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1 Introduction

It is well known that a constant of the motion can be used to "reduce the number of degrees of freedom" of a Hamiltonian dynamical system [Wi], [AM], [Ar], [MSSV]. An adequate number of constants of the motion may give rise to *action-angle* variables which are then used to analyze completely integrable systems.

In the Hamiltonian or Lagrangian formalism, constants of the motion are associated with symmetries of the equations of the motion and a reduction procedure relies on both concepts. First of all, constants of the motion provide invariant submanifolds, while symmetries, when acting on them, provide equivalence classes. The reduced dynamics becomes a dynamics on the quotient manifold of equivalence classes. Very often the reduced space can be imbedded as an invariant submanifold of the starting carrier space and one gets a dynamical evolution of initial data which have already a physical interpretation.

For instance, in the Hamiltonian formalism, a group G acts symplectically on the symplectic manifold (M,ω) and defines a momentum map $\mu : M \rightarrow \mathbf{g}^*$. This map gives the Hamiltonian generators of the action of G by setting $i_{X_j}\omega = d\mu(e_j)$, where $e_1, e_2, \ldots, e_r$, is a basis for the Lie algebra $\mathbf{g}$ of G, and $\mathbf{g}^*$ is the dual vector space. Given the regular value $k \in \mathbf{g}^*$ of μ, one considers the submanifold $\Sigma_k = \mu^{-1}(k) \subset M$ on which G_k, the stability group of k under the coadjoint representation, acts. The quotient manifold $\widetilde{\Sigma}_k = \Sigma_k / G_k$ is symplectic and is called the reduced phase space. If G is a symmetry group for the original dynamics, the latter preserves Σ_k and projects onto a Hamiltonian dynamics on $\widetilde{\Sigma}_k$. In this construction, the group G provides both the submanifold Σ_k and the subgroup G_k used to foliate it in order to get $\widetilde{\Sigma}_k$.

An interesting feature of the reduced system is that it may turn out to be non linear even if the starting one was linear and therefore integrable via exponentiation. Many completely integrable systems actually turn out to be reduced systems of free or simple ones [OP]. One may be tempted to conjecture that any completely integrable system should arise as reduction of a simple one. By simple here we mean a system which can be integrated via exponentiation (i.e. like going from a Lie algebra to a corresponding Lie group). It seems therefore convenient to elaborate a procedure which would be a kind of converse of the reduction procedure [KKS]. One hopes to unfold the non linearity of the dynamics on the quotient manifold by going to a higher dimensional carrier space so to get a system whose flow can be easily found. By applying the reduction procedure one then provides a flow for the starting non linear system. Of course there is no prescription to unfold nonlinearities, therefore a better understanding of the reduction procedure might help in suggesting a converse procedure.

Various descriptions of dynamical systems are available. One starts with a vector field Γ on a manifold M (carrier space) and by further qualification of M and Γ one finds a Lagrangian description on $M = TQ$ and Γ second order [AM], [MSSV], [MFLMR] or a symplectic description on (M, ω), with ω a symplectic structure and Γ a symplectic vector field. When (M, Γ) is thought of as a classical limit of a quantum system, M will be required to be a Poisson manifold and Γ to be a Hamiltonian vector field with respect to the given Poisson structure.

When dealing with the reduction of systems using additional structures one is obliged to worry also about them, i.e. the reduction procedure should be compatible with the additional structure on the carrier space. In some previous work we have considered some examples with the generalized reduction at work. We did not, however, consider the rôle of additional structures like the symplectic or the Poisson structure. Here we would like to take into account these additional structures. We shall also consider few examples where the Poisson structures are of the Lie-Poisson type and an example in the non-commutative (quantum) geometry.

As for notations we shall refer to [AM], [LM], [MSSV]. To help visualize how the paper is organized, we give a list of contents:

2 Generalized Reduction Procedure

Let us consider a dynamical system Γ on a carrier space M, namely an element $\Gamma \in \mathcal{X}(M)$. We shall indicate by Φ_t^Γ its flow.

A generalized reduction procedure is based on two steps [LMSV], [MM].

In the first one we consider:

i) A submanifold Σ of M which is invariant for Γ, namely to which Γ is tangent,

$$\Gamma(m) \in T_m\Sigma \ , \quad \forall \ \ m \in \Sigma \ . \tag{1}$$

In the second one we consider:

ii) An equivalence relation $\mathcal{R}$ on Σ which is compatible with Γ, namely such that

$$m \,\mathcal{R}\, m' \iff \Phi_t^\Gamma(m) \,\mathcal{R}\, \Phi_t^\Gamma(m') \ , \quad \forall \ \ m, m' \in \Sigma \ . \tag{2}$$

Given the previous ingredients, the *reduced* carrier space is the set of equivalences classes

$$\widetilde{\Sigma} = \Sigma / \mathcal{R} \ , \tag{3}$$

and the reduced dynamical system $\widetilde{\Gamma}$ will be the projection of Γ along the natural projection $\pi_\mathcal{R} : \Sigma \to \widetilde{\Sigma}_\mathcal{R}$.

Remark. Either one of the two previous steps could be trivial. Moreover, one could also proceede in opposite order, by giving first a compatible equivalence relation on M and then selecting an invariant submanifold in $M/\mathcal{R}$.

As we shall see, there are several ways to get an equivalence relation on invariant submanifolds. The most common ones are gotten by means of the action of a group or by involutive distributions.

Suppose we have a Lie group G which acts on Σ and which is a symmetry group for Γ restricted to Σ, (we do not require G to act on the full manifold M, and if it acts on M we do not require it to be a symmetry group for Γ on M but only for the restriction to Σ). On the quotient $\widetilde{\Sigma} = \Sigma/G$ we get a reduced dynamics.

We can 'dualize' the previous scheme by giving it on the algebra $\mathcal{F}(M)$ of observables on M. This dual view point is usefull e. g. when dealing with Poisson dynamics. In this algebraic context a dynamical system is seen as an element $\Gamma \in Der\mathcal{F}$ giving

$$\dot{f} = \Gamma \cdot f, \quad \forall \ \ f \in \mathcal{F} \ . \tag{4}$$

A reduction procedure will again require two steps.

ia) There exists an algebra $\mathcal{F}_\Sigma$ and a projection

$$\pi_\Sigma : \mathcal{F} \to \mathcal{F}_\Sigma \tag{5}$$

(this projection can be visualized in terms of the identification map $i_\Sigma : \Sigma \to M$ by setting $\pi_\Sigma(f) = i_\Sigma^*(f)$, so that $\mathcal{F}_\Sigma$ plays the rôle of the algebra of functions on Σ),

and a derivation $\Gamma_\Sigma \subset Der\mathcal{F}_\Sigma$ such that

$$\pi_\Sigma(\Gamma \cdot f) = \Gamma_\Sigma \cdot \pi_\Sigma(f) \quad \forall \ f \in \mathcal{F} \tag{6}$$

(this translates the invariance condition on Σ).

iia) There exists an invariant subalgebra of $\mathcal{F}_\Sigma$, namely a subalgebra $\widetilde{\mathcal{F}} \in \mathcal{F}_\Sigma$ such that

$$\Gamma_\Sigma \cdot \widetilde{\mathcal{F}} \subset \widetilde{\mathcal{F}} \tag{7}$$

(this translates the compatibility condition).

The restriction of Γ_Σ to $\widetilde{\mathcal{F}}$ can be denoted by $\widetilde{\Gamma}$ and provides us with the analog of the reduced dynamics we had an $\widetilde{\Sigma}$.

We can visualize our dual two steps reduction procedures with the help of the following diagrams

$$\begin{tikzcd}
 & (M,\Gamma) \arrow[dd] & & (\mathcal{F},\Gamma) \arrow[dr, "\pi_\Sigma"] & \\
(\Sigma,\Gamma_\Sigma) \arrow[ur, "i_\Sigma"] \arrow[dr, "\pi_\mathcal{R}"'] & & & & (\mathcal{F}_\Sigma,\Gamma_\Sigma) \\
 & (\widetilde{\Sigma},\widetilde{\Gamma}) & & (\widetilde{\mathcal{F}},\widetilde{\Gamma}) \arrow[uu] \arrow[ur, "i_\mathcal{R}"'] &
\end{tikzcd} \tag{8}$$

where $i_\mathcal{R}(\widetilde{\mathcal{F}}) \;=\; \pi^*_\mathcal{R}(\mathcal{F}(\widetilde{\Sigma}))$ and $\pi_\Sigma(\mathcal{F}) = i^*_\Sigma(\mathcal{F})$.

Remark. Here we should point out that if $i_\Sigma : \Sigma \to M$ is an embedding, then $\mathcal{F}_\Sigma \equiv \mathcal{F}(\Sigma)$, while if i_Σ is only an immersion, then $\mathcal{F}_\Sigma \subset \mathcal{F}(\Sigma)$.

Before we move on to take into account additional structures we illustrate the reduction procedure by using the ring of functions in the following examples.

2.1 Examples: rotationally invariant dynamics

In these examples we shall consider the less familiar dual point of view.

Suppose we have a dynamical system Γ on M which is invariant under the action of a Lie group G, namely $G_*\Gamma = \Gamma$. To construct a reduced dynamics we consider $\mathcal{F}^G = \{f \in \mathcal{F}(M) \ , \ G^* \cdot f = f\}$, i.e. the algebra of invariant functions under the action of G. Under some regularity assumptions on the action of G, the real spectrum $\mathcal{N}_G$ of $\mathcal{F}^G$ determines a differential manifold on which Γ defines a dynamical vector field Γ^*. Any ideal of constants of the motion for Γ^* in $\mathcal{F}^G$ gives a quotient algebra on which Γ^* defines a reduced dynamics $\widetilde{\Gamma}$. For details on these algebraic statements see section 6.

As an example let us consider a dynamical system Γ on $T\mathbf{R}^3$, invariant under the action of the rotation group $SO(3)$. If we suppose that Γ is second order, its general expression will be

$$\Gamma = \dot{\vec{r}} \cdot \frac{\partial}{\partial \vec{r}} + \vec{f}(r, \dot{r}, \vec{r} \cdot \dot{\vec{r}}) \cdot \frac{\partial}{\partial \dot{\vec{r}}} \ , \tag{9}$$

and for the moment we do not make any assumption on the explicit form of $\vec{f}$. The algebra $\mathcal{F}^G$ is made of functions which are invariant under $SO(3)$. The space of the orbits $\mathcal{N}_G$ can be parametrized by the three functions $\xi_1 = \vec{r}\cdot\vec{r}$, $\xi_2 = \dot{\vec{r}}\cdot\dot{\vec{r}}$ and $\xi_3 = \vec{r}\cdot\dot{\vec{r}}$ which are rotationally invariant. The reduced dynamics is then given by

$$
\begin{aligned}
\frac{d}{dt}\xi_1 &= 2\dot{\vec{r}}\cdot\vec{r} = 2\xi_3 \ , \\
\frac{d}{dt}\xi_2 &= 2\dot{\vec{r}}\cdot\ddot{\vec{r}} = 2\dot{\vec{r}}\cdot\vec{f} \ , \\
\frac{d}{dt}\xi_3 &= \dot{\vec{r}}\cdot\dot{\vec{r}} + \vec{r}\cdot\ddot{\vec{r}} = \xi_2 + \vec{r}\cdot\vec{f} \ ,
\end{aligned}
\tag{10}
$$

and the assertion that $\frac{d}{dt}\xi_j$ can be expressed in terms of the ξ's follows from the assumption of rotational invariance for the starting dynamics.

If one wants to produce reduced dynamics with additional structures (e.g. second order) one needs an extra step. One has to select a subset of variables (in this case just two) and express the dynamics in terms of them and of constants of the motion. To show this we shall specify our system as a very simple one, namely the free particles. In this case we have,

$$
\begin{aligned}
\frac{d}{dt}\xi_1 &= 2\xi_3 \ , \\
\frac{d}{dt}\xi_2 &= 0 \ , \\
\frac{d}{dt}\xi_3 &= \xi_2 \ .
\end{aligned}
\tag{11}
$$

Now we select an invariant two-dimensional submanifold in $\mathcal{N}_G$. This can be done in different ways.

1. Fix $\xi_2 = k = const.$ On this invariant submanifold the reduced dynamics is

$$
\begin{aligned}
\frac{d}{dt}\xi_1 &= 2\xi_3 \ , \\
\frac{d}{dt}\xi_3 &= k \ .
\end{aligned}
\tag{12}
$$

If we set $x = \xi_1$ and $v = 2\xi_3$, the resulting system is a Lagrangian one with Lagrangian function given by

$$
\mathcal{L} = \frac{1}{2}v^2 + 2kx \ . \tag{13}
$$

2. Fix $\xi_1\xi_2 - \xi_3^2 = \ell^2 = const$ (this is the square of the angular momentum). We get $\xi_2 = \frac{1}{\xi_1}(\xi_3^2 + \ell^2)$ and as reduced dynamics

$$
\begin{aligned}
\frac{d}{dt}\xi_1 &= 2\xi_3 \ , \\
\frac{d}{dt}\xi_3 &= \frac{\xi_3^2 + \ell^2}{\xi_1} \ .
\end{aligned}
\tag{14}
$$

If we set again $x = \xi_1$ and $v = 2\xi_3$, the resulting system is a Lagrangian one with Lagrangian function given by

$$\mathcal{L} = \frac{1}{2}\frac{v^2}{x} - \frac{2\ell^2}{x} \ . \tag{15}$$

Let us consider now what happens if we try to get a dynamics on a different set of (invariant) variables while mantaining the same invariant submanifold. Suppose we take as variables $\eta = r = \sqrt{\xi_1}$ and $\dot{\eta}$. Starting from the free motion, after some algebra we find

$$\frac{d}{dt}\dot{\eta} = \frac{L^2}{\eta^3} \ , \tag{16}$$

where L^2 is the square of the angular momentum. If we now fix $L^2 = \ell^2$ we get a reduced system of Calogero-Moser type

$$\frac{d}{dt}\dot{\eta} = \frac{\ell^2}{\eta^3} \ , \tag{17}$$

which is quite different from the (14). The corresponding Lagrangian function is given by

$$\mathcal{L} = \frac{1}{2}(\dot{\eta}^2 - \frac{\ell^2}{\eta^2}) \ . \tag{18}$$

We can do similar considerations in the Hamiltonian framework. As an example let us consider now a dynamical system Γ on $T^*\mathbf{R}^3$, invariant under the action of the rotation group $SO(3)$. If we suppose that Γ is Hamiltonian with Hamiltonian function H invariant under the canonical action of $SO(3)$, its general expression, with respect to the standard symplectic structure, will be

$$\Gamma = \frac{\partial H}{\partial \vec{p}}(r, p, \vec{r}\cdot\vec{p}) \cdot \frac{\partial}{\partial \vec{r}} - \frac{\partial H}{\partial \vec{r}}(r, p, \vec{r}\cdot\vec{p}) \cdot \frac{\partial}{\partial \vec{p}} \ . \tag{19}$$

The space of orbits $\mathcal{N}_G$ can now be parametrized by the three functions $\xi_1 = \vec{r}\cdot\vec{r}$, $\xi_2 = \vec{p}\cdot\vec{p}$ and $\xi_3 = \vec{r}\cdot\vec{p}$. The reduced dynamics is then given by

$$\begin{aligned}
\frac{d}{dt}\xi_1 &= 2\dot{\vec{r}}\cdot\vec{r} = 2\vec{r}\cdot\frac{\partial H}{\partial \vec{p}} \ , \\
\frac{d}{dt}\xi_2 &= 2\vec{p}\cdot\dot{\vec{p}} = -2\vec{p}\cdot\frac{\partial H}{\partial \vec{r}} \ , \\
\frac{d}{dt}\xi_3 &= \dot{\vec{r}}\cdot\vec{p} + \vec{r}\cdot\dot{\vec{p}} = \vec{p}\cdot\frac{\partial H}{\partial \vec{p}} - \vec{r}\cdot\frac{\partial H}{\partial \vec{r}} \ ,
\end{aligned} \tag{20}$$

and the assertion that $\frac{d}{dt}\xi_j$ can be expressed in terms of the ξ's follows from the assumption of rotational invariance for the starting Hamiltonian.

We take again the free particle. Then the reduced system is formally the same as in (11),

$$\begin{aligned} \frac{d}{dt}\xi_1 &= 2\xi_3 \ , \\ \frac{d}{dt}\xi_2 &= 0 \ , \\ \frac{d}{dt}\xi_3 &= \xi_2 \ . \end{aligned} \tag{21}$$

In order to have a reduced dynamics which is symplectic, we have to select an invariant two-dimensional submanifold in $\mathcal{N}_G$.

1. Fix $\xi_2 = k = const.$ On this invariant submanifold the reduced dynamics is

$$\begin{aligned} \frac{d}{dt}\xi_1 &= 2\xi_3 \ , \\ \frac{d}{dt}\xi_3 &= k \ . \end{aligned} \tag{22}$$

This system is Hamiltonian with respect to the symplectic structure

$$\omega = d\xi_1 \wedge d\xi_3 \ , \qquad H = \xi_3^2 - k\xi_1 \ . \tag{23}$$

2. Fix $\xi_1\xi_2 - \xi_3^2 = \ell^2 = const.$ As reduced dynamics we get

$$\begin{aligned} \frac{d}{dt}\xi_1 &= 2\xi_3 \ , \\ \frac{d}{dt}\xi_3 &= \frac{\xi_3^2 + \ell^2}{\xi_1} \ . \end{aligned} \tag{24}$$

This is Hamiltonian with respect to the symplectic structure

$$\omega = \frac{1}{\xi_3^2 + \ell^2}\, d\xi_1 \wedge d\xi_3 \ , \qquad H = \frac{1}{m} ln \frac{\xi_3^2 + \ell^2}{|\xi_1|} \ . \tag{25}$$

The examples we have considered have shown that even from a simple system like the free particle, by using our reduction procedure it is possible to obtain a variety of interacting systems all of them however, completely integrable ones.

2.2 Reduction and symplectic structures

In the previous example we have already seen how one may get reduced Hamiltonian systems starting with a Hamiltonian one.

Here we make some general considerations. We have already emphasized that eventually we would like to reduce simple systems to get completely integrable ones. We would like to consider for instance, the reduction of free systems or harmonic oscillators. It is known [MSSV] that these systems admits many alternative Lagrangian or Hamiltonian descriptions. Therefore, it is interesting to investigate if these alternative descriptions survive the reduction procedure, i.e. do they provide alternative descriptions for the reduced dynamics? We do know that in many cases the answer is positive because many completely integrable systems do possess alternative descriptions [DMSV]. Let us then examine the situation.

We start with a symplectic manifold (M, ω) and require that the dynamics Γ is ω-Hamiltonian with Hamiltonian function H,

$$i_\Gamma \omega = dH \ . \tag{26}$$

On any invariant submanifold $\Sigma \stackrel{i_\Sigma}{\hookrightarrow} M$ we get a 2-form by pulling back ω, $\omega_\Sigma = i_\Sigma^* \omega$ which, however, will be degenerate in general. We can consider now the equivalence relation associated with the distribution defined by $ker\omega_\Sigma$. It is clear that if $i_X \omega_\Sigma = 0$ for some vector field $X \in \mathcal{X}(\Sigma)$, we find that $0 = L_\Gamma i_X \omega_\Sigma = i_{[\Gamma, X]} \omega_\Sigma$. With some abuse of notation we do not distinguish Γ from $\Gamma|_\Sigma$. Since Γ preserves the distribution $ker\omega_\Sigma$, it will be compatible with the associated equivalence relation.

¿From the dual point of view, we can consider all Hamiltonian vector fields on M which are tangent to Σ, i.e. $i_{X_f} \omega = df$ and $X_f(m) \in T_m \Sigma$, $\forall m \in \Sigma$. The pull back to Σ of all these Hamiltonian functions will provide us with a subalgebra of $\mathcal{F}(\Sigma)$ that will be denoted $\mathcal{F}(\omega, \Sigma)$. The dynamical vector field Γ on Σ will map this subalgebra into itself; therefore, its action on such a subalgebra provides the reduced dynamics $\widetilde{\Gamma}$. This statement follows easily from the fact that $i_{X_f} \omega = df$ can be restricted "term by term" to Σ, because X_f is assumed to be tangent to Σ and $i_{X_f|_\Sigma} \omega = df_\Sigma$. ¿From $L_\Gamma i_{X_f} \omega = i_{[\Gamma, X_f]} \omega = d(\Gamma \cdot f)$, followed by restriction to Σ we prove our statement.

It is clear now that if we consider our reduced dynamics $\widetilde{\Gamma}$ on the algebra $\mathcal{F}(\omega, \Sigma)$, it is very easy to find out if another symplectic structure ω_1 will be compatible with the reduction procedure and provide an invariant two form for $\widetilde{\Gamma}$. Indeed, if we construct $\mathcal{F}(\omega_1, \Sigma)$, the compatibility condition reads $\mathcal{F}(\omega_1, \Sigma) \subset \mathcal{F}(\omega, \Sigma)$. Of course, if the two algebrae coincide, $\mathcal{F}(\omega_1, \Sigma) = \mathcal{F}(\omega, \Sigma)$, ω_1 will project onto a symplectic structure which provides an alternative symplectic structure for $\widetilde{\Gamma}$. In terms of the distributions $ker\omega$ and $ker\omega_1$, the compatibility condition reads $ker\omega \subset ker\omega_1$. Again $ker\omega = ker\omega_1$ implies that ω_1 will project onto a symplectic structure on the reduced manifold.

Of course one can start with ω_1 instead of ω and the reduced carrier space could be different. Then one looks for all other symplectic structures that are going to be compatible with the given reduction.

Summing up, we have found that if $\widetilde{\Gamma}$ is the reduced dynamics associated with Σ and ω, any other symplectic description for Γ will provide an alternative description for $\widetilde{\Gamma}$ iff $\mathcal{F}(\omega_1, \Sigma) = \mathcal{F}(\omega, \Sigma)$.

Let us consider an example to illustrate the situation.

2.2.1 Reduction of free motion and symplectic description

On $T^*\mathbf{R}^3$ with standard symplectic structure ω_0 we consider free motion provided by the Hamiltonian

$$H = \frac{p^2}{2m} = \frac{1}{2m}\left(p_r^2 + \frac{L^2}{r^2}\right) \ , \tag{27}$$

where

$$p_r = \frac{\vec{p}\cdot\vec{r}}{r} \quad , \quad \vec{L} = \vec{p}\wedge\vec{r} \ . \tag{28}$$

We consider the invariant submanifold $\Sigma_{\vec{c}}$ obtained by fixing the value of the angular momentum $\vec{L}$,

$$\Sigma_{\vec{c}} = \left\{(\vec{r},\vec{p}) \vdash \vec{L} = \vec{c}\right\} \ . \tag{29}$$

The pull-back of ω_0 to $\Sigma_{\vec{c}}$ will be denoted by $\omega_{\vec{c}}$. In order to compute $ker\omega_{\vec{c}}$ we consider the infinitesimal generators X_1, X_2, X_3 of the rotation group on $T^*\mathbf{R}^3$. They are given by

$$X_i = \varepsilon_{ijk}(x_j\frac{\partial}{\partial x_k} + p_j\frac{\partial}{\partial p_k}) \ , \quad i = 1,2,3 \ . \tag{30}$$

Now $ker\omega_{\vec{c}}$ is generated by the Hamiltonian vector field Y associated with $\frac{1}{2}L^2$ when restricted to $\Sigma_{\vec{c}}$. It is given by

$$Y_{\vec{c}} = c_1X_1 + c_2X_2 + c_3X_3 \ . \tag{31}$$

The reduced manifold $\widetilde{\Sigma}_{\vec{c}}$ is defined as the submanifold of $\mathbf{R}^3$ endowed with coordinates $\xi_1 = r^2$, $\xi_2 = p^2$, $\xi_3 = \vec{r}\cdot\vec{p}$, with the relation $\xi_1\xi_2 - \xi_3^2 = c^2$. It is diffeomorphic with $T^*\mathbf{R}^+$ if we set $x = r$, $p_x = (\vec{p}\cdot\vec{r})/r$. The reduced dynamics is Hamiltonian and we have

$$\begin{aligned}
\widetilde{\Gamma} &= \frac{p_x}{m}\frac{\partial}{\partial x} + \frac{c^2}{mx^3}\frac{\partial}{\partial p_x} \ , \\
\widetilde{H} &= \frac{1}{2m}\left(p_x^2 + \frac{c^2}{x^2}\right) \ , \\
\widetilde{\omega} &= dp_x \wedge dx \ .
\end{aligned} \tag{32}$$

There are many alternative symplectic descriptions for the free motion on $T^*\mathbf{R}^3$. For instance,

$$\omega_F = d\left(\frac{\partial F}{\partial p_i}dq^i\right) \ , \quad \text{with} \quad det||\frac{\partial^2 F}{\partial p_i\partial p_j}|| \neq 0 \ , \tag{33}$$

provides a family of them. We shall consider a particular one just to illustrate the procedure. We consider

$$\omega_1 = \omega_0 + s dp^2 \wedge d(\vec{p}\cdot\vec{r}) \ , \tag{34}$$

with s a dimensional constant. The corresponding Hamiltonian for the free motion is given by

$$H_1 = \frac{p^2}{2m} + \frac{1}{2}s\frac{p^4}{m} \ . \tag{35}$$

By restricting ω_1 to $\Sigma_{\vec{c}}$ we get an alternative symplectic structure for $\widetilde{\Gamma}$ given by

$$\begin{aligned} \widetilde{\omega}_1 &= dp_x \wedge dx + s d(\frac{c^2}{x^2} + p_x^2) \wedge d(x p_x) \\ &= \left(1 + 2s(p_x^2 + \frac{c^2}{x^2})\right) dp_x \wedge dx \ . \end{aligned} \tag{36}$$

The corresponding Hamiltonian function is

$$\widetilde{H}_1 = \left(1 + s(p_x^2 + \frac{c^2}{x^2})\right) \frac{1}{2m}\left(p_x^2 + \frac{c^2}{x^2}\right) \ . \tag{37}$$

Of course, any other admissible symplectic structure ω' which does not satisfy

$$dL^2 \wedge (i_{L^j X_j}\omega') = 0, \tag{38}$$

will not be projectable. To satisfy the requirement that ω' restricted to $\Sigma_{\vec{c}}$ is projectable onto $\widetilde{\Sigma}$, we need that the foliating distribution is in the kernel of $\omega'|_{\Sigma}$. It may happen that the dimension of the kernel is too small for this to happen. Therefore, to get alternative symplectic structures for the reduced dynamics $\widetilde{\Gamma}$ on $\widetilde{\Sigma}$ we shall also allow to start with degenerate two forms on the initial carrier space as long as they are invariant under the dynamical evolution.

Let us illustrate the situation. For our free motion we consider the invariant submanifold defined by

$$\Sigma_\ell = \left\{(\vec{r},\vec{p}) \vdash p^2 r^2 - r^2 p_r^2 = \ell^2\right\} \ , \tag{39}$$

with equivalence relation provided by the rotation group. It is clear that the kernel of a symplectic structure on a codimension one submanifold must be one dimensional. It follows that there is no symplectic structure that we can restrict to Σ_ℓ and that project to $\widetilde{\Sigma}_\ell$. We have to start with a degenerate one. For instance we could take

$$\omega = f_1(p^2, \vec{p}\cdot\vec{r}) dp^2 \wedge d(\vec{p}\cdot\vec{r}) + f_2(p^2, r^2) dp^2 \wedge dr^2 + f_3(r^2, \vec{p}\cdot\vec{r}) dr^2 \wedge d(\vec{p}\cdot\vec{r}) \ . \tag{40}$$

In this case, to meet the invariance requirement under the dynamical evolution, we would require that $L_\Gamma \omega = 0$ on the submanifold Σ_ℓ. The choice $\omega = -\frac{1}{2r^2} dr^2 \wedge d(\vec{p}\cdot\vec{r})$ has a restriction to Σ_ℓ that is projectable onto

$$\widetilde{\omega} = dp_r \wedge dr \ . \tag{41}$$

with

$$\widetilde{H} = \frac{1}{2m}\left(p_r^2 + \frac{\ell^2}{r^2}\right) \ . \tag{42}$$

and

$$\widetilde{\Gamma} = \frac{p_r}{m}\frac{\partial}{\partial r} + \frac{\ell^2}{mr^3}\frac{\partial}{\partial p_r} \ . \tag{43}$$

A similar situation occurs if we fix the energy, so getting the invariant submanifold $\Sigma_E = \{(\vec{p},\vec{r}) \vdash p^2 = 2mE\}$. For the equivalence relation we still use the rotation group. In this case the reduced dynamics is given by

$$\widetilde{\Gamma} = \frac{p_r}{m}\frac{\partial}{\partial r} + \left(\frac{2E}{r} - \frac{p_r^2}{mr}\right)\frac{\partial}{\partial p_r} \ , \tag{44}$$

which is Hamiltonian on $T^*\mathbf{R}^+$ with structures

$$\widetilde{\omega} = r^2 dp_r \wedge dr \ , \qquad \widetilde{H} = Er^2 - \frac{r^2 p_r^2}{2m} \ . \tag{45}$$

Besides free motions on vector spaces, there are others dynamical systems which can be easily integrated. They are geodesical motions on Lie groups or on homogeneous spaces. Their flows are provided by the action of one-parameter subgroups. Therefore, in the following sections we are going to consider geodesical motions on these spaces.

3 Reducing geodesical motion on Lie groups

3.1 Few elements of Lagrangian formalism

We briefly recall few elements of the geometry of the tangent bundle [MFLMR]. Let TQ be the tangent bundle to an n-dimensional configuration space Q, with local coordinates $\{q^i, u^i \ , \ i \in \{1,\dots n\}\}$. On TQ there are two natural tensor fields which essentially characterize its structure. They are the *vertical endomorphism* S and the *dilation vector field* Δ which, in local coordinates are respectively given by $S = dq^i \otimes \frac{\partial}{\partial u^i}$, $\Delta = u^i \frac{\partial}{\partial u^i}$. A *second-order derivation*, or SODE for short, is any vector field Γ on TQ such that $S(\Gamma) = \Delta$. Locally a SODE is of the form $\Gamma = u^i \frac{\partial}{\partial q^i} + \Gamma^i(q,u)\frac{\partial}{\partial u^i}$.

There are two natural lifting procedures for vector fields from Q to TQ, namely the tangent and the vertical lifting. If $X = X^i(q)\frac{\partial}{\partial q^i} \in \mathcal{X}(Q)$, its *tangent lift* X^T and its *vertical lift* X^V are the elements in $\mathcal{X}(TQ)$ given by

$$X^T = X^j(q)\frac{\partial}{\partial q^j} + u^i \frac{\partial X^j}{\partial u^i}\frac{\partial}{\partial u^j} \ , \quad X^V = X^j(q)\frac{\partial}{\partial u^j} \ . \tag{1}$$

Let us take an element $\mathcal{L} \in \mathcal{F}(TQ)$. The 1-form $\theta_{\mathcal{L}}$ on TQ defined by $\theta_{\mathcal{L}} = d\mathcal{L} \circ S$ is the *Cartan 1-form* of the *Lagrangian* $\mathcal{L}$. The Lagrangian $\mathcal{L}$ is said to be regular, if the *Cartan 2-form* $\omega_{\mathcal{L}}$ on TQ defined by $\omega_{\mathcal{L}} = -d\theta_{\mathcal{L}}$, is non degenerate.

Given a Lagrangian $\mathcal{L}$, the *Euler-Lagrangian equations* for $\mathcal{L}$ are the following equations

$$L_{\Gamma}\theta_{\mathcal{L}} - d\mathcal{L} = 0 \; , \tag{2}$$

where the unknown quantity is the vector field Γ. In local coordinates (2) reads

$$\frac{d}{dt}q^i = u^i \; , \qquad \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial u^i} - \frac{\partial \mathcal{L}}{\partial q^i} = 0 \; . \tag{3}$$

If $\mathcal{L}$ is a regular Lagrangian, the solution of (2) turns out to be a SODE. In this case eqs. (2) can be written in an equivalent symplectic form by using the *energy* $E_{\mathcal{L}}$ of $\mathcal{L}$,

$$E_{\mathcal{L}} =: L_{\Delta}\mathcal{L} - \mathcal{L} = i_{\Gamma}\theta_{\mathcal{L}} - \mathcal{L} \; . \tag{4}$$

Then, by using Cartan identity, eqs.(2) are written as

$$i_{\Gamma}\omega_{\mathcal{L}} = dE_{\mathcal{L}} \; . \tag{5}$$

Sometimes it turns out to be useful to 'project' equations (2) along a non holonomic basis $X_1, \ldots, X_n$ of vector fields on Q. To do that one takes the contraction of (2) with the corresponding tangent lifts $X_i^T \in \mathcal{X}(TQ)$. When $X \in \mathcal{X}(Q)$, the contraction $i_{X^T}(L_{\Gamma}\theta_{\mathcal{L}} - d\mathcal{L}) = 0$ can be written as

$$L_{\Gamma} i_{X^T}\theta_{\mathcal{L}} - L_{X^T}\mathcal{L} = 0 \; . \tag{6}$$

If we choose a basis of vector fields, we obtain a global expression for the familiar Euler-Lagrangian equations.

3.2 Geodesical motion on Lie groups

Let G be a Lie group thought of as a subgroup of the group of matrices $GL(n, \mathbf{R})$, and $\mathbf{g}$ its Lie algebra with basis $\tau_1, \cdots, \tau_n$, $[\tau_i, \tau_j] = c_{ij}^k \tau_k$.

On G there is a canonical $\mathbf{g}$-valued left invariant 1-form

$$\alpha = g^{-1}dg \; , \tag{7}$$

which allows to define a basis $\{\theta^k\}$ of left invariant one forms in $\mathcal{X}^*G$ via

$$g^{-1}dg =: \tau_k \theta^k \; . \tag{8}$$

These forms satisfy the Maurer-Cartan equation

$$d\theta^k + \frac{1}{2}c_{ij}^k \theta^i \wedge \theta^j = 0 \; . \tag{9}$$

The dual basis of left invariant vector fields X_k, defined by

$$i_{X_k} g^{-1} dg = \tau_k \ , \tag{10}$$

satisfies $i_{X_k}\theta^j = \delta^j_k$, with commutations relations

$$[X_i, X_j] = c^k_{ij} X_k \tag{11}$$

along with

$$L_{X_i}\theta^k = -c^k_{ij}\theta^j \ . \tag{12}$$

Analogously, one can construct right invariant forms $\widetilde{\theta}^k$ and vector fields Y_k, starting from the right invariant $\mathbf{g}$-valued 1-form $\beta = (dg)g^{-1}$,

$$(dg)g^{-1} =: \tau_k \widetilde{\theta}^k \ , \tag{13}$$
$$i_{Y_k}(dg)g^{-1} = \tau_k \ . \tag{14}$$

Left and right invariant quantities are related by the adjoint rapresentation whose matrix elements D^j_i are defined by

$$g^{-1}\tau_i g = D^j_i(g)\ \tau_j \ . \tag{15}$$

In particular,

$$\theta^i = D^i_j(g)\widetilde{\theta}^j \ . \tag{16}$$

The left invariant vector fields are the generators of the right action of G on itself while the right invariant ones are the generators of the left action. Therefore, they mutually commute

$$[X_i, Y_j] = 0 \ . \tag{17}$$

We can construct a basis of vector fields and 1-forms for the tangent group TG starting with basis for G. This is done as follows[†]

$$\begin{aligned} \{X_k\} &\mapsto \{(X_k)^v, (X_k)^T\} \ , \\ \{\theta^k\} &\mapsto \{\tau^*_G\theta^k, d(i_\Gamma \tau^*_G\theta^k)\} \ , \end{aligned} \tag{18}$$

with Γ any SODE on TG. Notice the the basis in (18) are not dual to each other.

Let as suppose now that we have a metric m on G which, in the basis of left and right invariant 1-forms is written as

$$\begin{aligned} \mathsf{m} &= \mathsf{m}_{jk}\theta^j \otimes \theta^k \ , & (19) \\ &= \widetilde{\mathsf{m}}_{jk}\widetilde{\theta}^j \otimes \widetilde{\theta}^k \ , \qquad \widetilde{\mathsf{m}}_{jk} = D^s_j D^t_k\ \mathsf{m}_{st} \ . & (20) \end{aligned}$$

The associated geodesical motion on G is described by the Lagrangian

$$\mathcal{L}_{\mathsf{m}} = \frac{1}{2}\tau^*_G \mathsf{m}(\Gamma, \Gamma) = \frac{1}{2}\mathsf{m}_{jk}\dot{\theta}^j\dot{\theta}^k \ , \tag{21}$$

[†]With some abuse of notation we shall identify $\tau^*_G\theta^k$ with θ^k and write $i_\Gamma\tau^*_G\theta^k = \dot{\theta}^k$.

whose Cartan 1-form turns out to be

$$\theta_{\mathcal{L}} = \mathsf{m}(\Gamma, \ \cdot\ ) \ . \tag{22}$$

Right and left momenta associated with the respective actions are given by

$$P_k^{(R)} =: i_{X_k^T}\theta_{\mathcal{L}} = \mathsf{m}_{kj}\dot{\theta}^j \ , \tag{23}$$

$$P_k^{(L)} =: i_{Y_k^T}\theta_{\mathcal{L}} = \tilde{\mathsf{m}}_{kj}\dot{\tilde{\theta}}^j \ . \tag{24}$$

The associated matrix valued momenta are given by

$$\mathcal{P}^R =: \mathsf{m}^{ij}\tau_i P_j^{(L)} = g^{-1}\dot{g} \ , \tag{25}$$

$$\mathcal{P}^L =: \tilde{\mathsf{m}}^{ij}\tau_i P_j^{(R)} = \dot{g}g^{-1} \ . \tag{26}$$

Notice the $\mathcal{P}^R$ is invariant under *left* action while $\mathcal{P}^L$ is invariant under the *right* one.

Since TG is parallelizable in terms of left invariant (or right invariant) vector fields, it turns out to be more convenient to write the Euler-Lagrange equations in the corresponding non holonomic basis on G as in (6). With the Lagrangian (21), we have $L_{Z^T}\mathcal{L} = \frac{1}{2}(L_{Z^T}\mathsf{m})(\Gamma,\Gamma)$ so that the Euler-Lagrange equations become

$$\frac{d}{dt}\mathsf{m}(\Gamma, Z) = \frac{1}{2}(L_{Z^T}\mathsf{m})(\Gamma,\Gamma) \ . \tag{27}$$

As a consequence,

$$\frac{d}{dt}\mathsf{m}(\Gamma, Z) = 0 \quad \Longleftrightarrow \quad L_Z\mathsf{m} = 0 \ . \tag{28}$$

The quantity $\mathsf{m}(\Gamma, Z)$ can be thought of as the 'angle' between Γ and Z; it is a constant of the motion if and only if Z is a Killing vector field for m.

In particular, the time evolution of the left and right momenta are given by

$$\frac{d}{dt}P_k^{(L)} = \frac{1}{2}(L_{Y_k^T}\mathsf{m})(\Gamma,\Gamma) \ , \tag{29}$$

$$\frac{d}{dt}P_k^{(R)} = \frac{1}{2}(L_{X_k^T}\mathsf{m})(\Gamma,\Gamma) \ , \tag{30}$$

Therefore, left momenta are constants of the motion iff m is left invariant and this is equivalent to the components m_{ij} being numerical constants. As for the evolution of the right momenta, we find that

$$L_{X_k^T}\mathsf{m} = -\mathsf{m}_{ij}c^i_{kl}(\theta^j \otimes \theta^l + \theta^l \otimes \theta^j) \ , \tag{31}$$

and, after some algebra,

$$\frac{d}{dt}P_k^{(R)} = -c^i_{kr}\mathsf{m}^{rj}P_i^{(R)}P_j^{(R)} \ . \tag{32}$$

Analogously, right momenta are constants of the motion iff $\tilde{\mathsf{m}}$ is right invariant and this is equivalent to the components $\tilde{\mathsf{m}}_{ij}$ being numerical constants. As for the evolution of left momenta one would find an expression similar to (32).

We see that geodesical motion on Lie group associated with left invariant (or right invariant) metrics takes the form of first order equations in the appropriate momenta. In the language of momentum map, both left and right momenta give maps

$$\mathcal{P}^L \ , \quad \mathcal{P}^R \quad : T^*G \longrightarrow \mathbf{g}^* \ , \tag{33}$$

and the dynamics is projectable onto $\mathbf{g}^*$ where it is described by a Hamiltonian vector field with respect to the Konstant-Kirillov-Souriou Poisson structures (the + and the − structures) on $\mathbf{g}^*$.

¿From the expressions (25) and (26), in each of the previous cases (left or right invariance) the geodesical lines are just translations of one-parameter subgroups of the group G,

$$g(t) = g_0 e^{tA} \ , \quad A \in \mathbf{g} \ , \tag{34}$$

with A the corresponding conserved velocity.

The converse is also true, namely any translation of a 1-parameter subgroup of G represents a geodesical motion with respect to a suitable metric on G. Let us suppose we have a translation of a 1-parameter subgroup of G, $g(t) = Be^{tA}$. By writing $\dot{g} = Ag$ we see that the left velocity $\dot{g}g^{-1} = A$ is a constant of the motion. Take now any scalar product on the Lie algebra $\mathbf{g}$, $n_{ij} = <\tau_i, \tau_j>$ and extend it to the unique left invariant metric on G, given by $\mathsf{m} = n_{ij}\theta^i \otimes \theta^j$. The associated geodesical motion admits the starting 1-parameter subgroup as a possible motion. Had we written $\dot{g} = gA$ we should have extended the scalar product to the unique right invariant metric. It should be noticed, however, that in general it is not possible to have a bi-invariant metric on G since it is not possible to have an adjoint invariant scalar product on $\mathbf{g}$ in general. This is however true if G is compact or semisimple.

Let us write the metric in its diagonal form. Then we have

$$\mathsf{m} = \theta^1 \otimes \theta^1 + \cdots + \theta^n \otimes \theta^n \ , \tag{35}$$

$$\mathcal{L}_{\mathsf{m}} = \frac{1}{2}(\dot{\theta}^1\dot{\theta}^1 + \cdots + \dot{\theta}^n\dot{\theta}^n) \ , \tag{36}$$

$$\theta_{\mathcal{L}} = \dot{\theta}^1\theta^1 + \cdots + \dot{\theta}^n\theta^n \ . \tag{37}$$

If the left invariant vector fields $X_{k+1}, \cdots, X_n$ are Killing vectors, also their commutators are Killing vectors; therefore we can assume that they close on a Lie subalgebra. We denote by K the corresponding subgroup in G. The associated momenta $P_j^{(R)}$, $j \in \{k+1, \cdots, n\}$ will be constants of the motion for the geodesical motion. If we fix their values, say $P_j^{(R)} = c_j$, we get a submanifold $\Sigma_{\mathbf{c}}$ of TG. We can consider now the pullback of $\omega_{\mathcal{L}}$ to $\Sigma_{\mathbf{c}}$ to get $\omega_{\mathbf{c}}$. Its kernel $ker\omega_{\mathbf{c}}$ determines an involutive distribution $\mathcal{D}$ on $\Sigma_{\mathbf{c}}$ and we get a reduced space $\tilde{\Sigma}_{\mathbf{c}} = \Sigma_{\mathbf{c}}/\mathcal{D}$ diffeomorphic to $T(G/K)$.

As for the reduced dynamics we can make few general considerations. The energy

$$E_{\mathbf{c}} = \frac{1}{2}\sum_{i=1}^{k}(\dot{\theta}^i)^2 + \frac{1}{2}\sum_{k+1}^{n}(c_j)^2 \ , \tag{38}$$

will be projectable on $T(G/K)$. We can associate a metric tensor to it by setting $\mathsf{m}_{\mathbf{c}} = \sum_{i=1}^{k} \theta^i \otimes \theta^i$. We consider also the metric Lagrangian $\mathcal{L}_{\mathbf{c}} = \frac{1}{2}\sum_{i=1}^{k}(\dot{\theta}^i)^2$ and the associated 2-form $\omega_{\mathcal{L}_{\mathbf{c}}}$. We see that if we consider the pullback of $\omega_{\mathcal{L}_{\mathbf{c}}}$ to $\Sigma_{\mathbf{c}}$ and compare it with $\omega_{\mathbf{c}}$ we find that they differ by a term containing 2-forms like $c_j d\theta^j$. Since our dynamics on $\Sigma_{\mathbf{c}}$ satisfies both equations $i_{\Gamma_{\mathbf{c}}}\omega_{\mathcal{L}_{\mathbf{c}}} = dE_{\mathbf{c}}$ and $i_{\Gamma_{\mathbf{c}}}\omega_{\mathbf{c}} = dE_{\mathbf{c}}$, this implies that the additional forces, with respect to the geodesical ones on G/K, represent gyroscopic forces or forces of the magnetic type. In the latter case, the values of the c_j's may be interpreted as contributing a 'magnetic charge'.

We are going to illustrate these general considerations on a familiar example in the following section.

3.3 Geodesical motion on $SU(2)$

As an example we consider geodesical motion on $SU(2)$ and project it onto motions on the 2-dimensional sphere.

We think of $SU(2)$ in terms of unitary 2×2 matrices with determinant equal to one. As a basis for its Lie algrebra we take $\tau_k = \frac{i}{2}\sigma_k, \ , k \in \{1,2,3\}$ the three Pauli matrices. Then $[\tau_j, \tau_k] = \varepsilon_{jk}^{l}\tau_l$.

Given any triple $(n_1, n_2, n_3) \in \mathbf{R}^3$, we consider the left-invariant metric on $SU(2)$ given by

$$\mathsf{m} = n_1\theta^1 \otimes \theta^1 + n_2\theta^2 \otimes \theta^2 + n_3\theta^3 \otimes \theta^3 \ . \tag{39}$$

The associated Lagrangian

$$\mathcal{L} = \frac{1}{2}(n_1\dot{\theta}^1\dot{\theta}^1 + n_2\dot{\theta}^2\dot{\theta}^2 + n_3\dot{\theta}^3\dot{\theta}^3) \tag{40}$$

has the following Euler-Lagrange equations:

$$\begin{aligned} \frac{d}{dt}P_k^{(L)} &= 0 \ , \quad k \in \{1,2,3\} \ , \\ \frac{d}{dt}P_1^{(R)} &= \frac{n_2 - n_3}{n_2 n_3}P_2^{(R)}P_3^{(R)} \ , \\ \frac{d}{dt}P_2^{(R)} &= \frac{n_3 - n_1}{n_3 n_1}P_3^{(R)}P_1^{(R)} \ , \\ \frac{d}{dt}P_3^{(R)} &= \frac{n_1 - n_2}{n_1 n_2}P_1^{(R)}P_2^{(R)} \ , \end{aligned} \tag{41}$$

namely the Euler equations for a top.

On $SU(2)$ there is a natural bi-invariant metric corresponding to the Cartan-Killing form on its Lie algebra which, when dealing with matrices, can be expressed in terms of the trace. The associated Lagrangian is given by

$$\mathcal{L} = tr\dot{g}(\dot{g^{-1}}) = -trg^{-1}\dot{g}g^{-1}\dot{g} = -tr\dot{g}g^{-1}\dot{g}g^{-1} \ , \tag{42}$$

which in terms of left or right invariant forms is

$$\mathcal{L} = \frac{1}{2}(\dot{\theta}^1\dot{\theta}^1 + \dot{\theta}^2\dot{\theta}^2 + \dot{\theta}^3\dot{\theta}^3) = \frac{1}{2}(\dot{\tilde{\theta}}^1\dot{\tilde{\theta}}^1 + \dot{\tilde{\theta}}^2\dot{\tilde{\theta}}^2 + \dot{\tilde{\theta}}^3\dot{\tilde{\theta}}^3) \ . \tag{43}$$

The associated geodesical equations of the motion are

$$\frac{d}{dt}g^{-1}\dot{g} = 0 \ , \qquad \frac{d}{dt}\dot{g}g^{-1} = 0 \ , \tag{44}$$

or, equivalently,

$$\frac{d}{dt}\dot{\theta}^k = 0 \ , \qquad \frac{d}{dt}\dot{\tilde{\theta}}^k = 0 \ , \quad k \in \{1,2,3\} \ . \tag{45}$$

Let us consider now the projection

$$\pi : SU(2) \to S^2 \ , \quad g \mapsto g\sigma_3 g^{-1} =: \vec{x}\cdot\vec{\sigma} \ . \tag{46}$$

One can verify that $\vec{x}\cdot\vec{x} = 1$ so that x^1, x^2, x^3 are coordinates on S^2. By identifying them with their pullback to $SU(2)$, the Lagrangian (43) and the associated Cartan 1-form $\theta_{\mathcal{L}}$ can be written as

$$\mathcal{L} = \frac{1}{2}\left((\dot{x}^1)^2 + (\dot{x}^2)^2 + (\dot{x}^3)^2\right) + \frac{1}{2}(\dot{\theta}^3)^2 \ , \tag{47}$$

$$\theta_{\mathcal{L}} = \dot{x}^1 dx^1 + \dot{x}^2 dx^2 + \dot{x}^3 dx^3 + \dot{\theta}^3\theta^3 \ . \tag{48}$$

Let us now inquire about the projectability of the geodesical motion (44)-(45) to TS^2. We consider

$$\dot{x}^j = \frac{d}{dt}x^j \ , \tag{49}$$

$$\ddot{x}^j = \frac{d}{dt}\dot{x}^j \ , \tag{50}$$

where $\frac{d}{dt}$ is the time-derivative along the geodesical motion on $SU(2)$. This motion would be projectable if $\ddot{x}^j$ could be expressed in terms of the variables $\dot{x}^j$ and x^j alone. In terms of left invariant forms, after some algebra one arrives at

$$\dot{\vec{x}}\cdot\vec{\sigma} = g(\dot{\theta}^2\sigma_1 - \dot{\theta}^1\sigma_2)g^{-1} \ , \tag{51}$$

$$\ddot{\vec{x}}\cdot\vec{\sigma} = -((\dot{\theta}^1)^2 + (\dot{\theta}^2)^2)g\sigma_3 g^{-1} + \dot{\theta}^3 g(\dot{\theta}^1\sigma_1 + \dot{\theta}^2\sigma_2)g^{-1} \ . \tag{52}$$

We find that, in general, the motion (52) is not projectable onto a motion on S^2. The term that cannot be expressed in terms of quantities on TS^2 is $\dot{\theta}^3$. However, since $\dot{\theta}^3$ is a constant of the motion, we can fix a value for it which determines an invariant submanifold Σ of $TSU(2)$ which covers TS^2. The motion restricted to Σ is projectable to TS^2. For each submanifold Σ we find a different dynamical system on TS^2 which is parametrized by the values of $\dot{\theta}^3$. We find two classes of motions which are quite different depending on whether $\dot{\theta}^3$ vanishes or not:

1. $\dot{\theta}^3 = 0$.

The resulting motion is the geodesical motion on S^2. From (51) we get that

$$(\dot{\theta}^1)^2 + (\dot{\theta}^2)^2 = (\dot{x}^1)^2 + (\dot{x}^2)^2 + (\dot{x}^3)^2 \ . \tag{53}$$

As a consequence,

$$\frac{d}{dt}\dot{\vec{x}} = -(\dot{\vec{x}}^2)\vec{x} \ . \tag{54}$$

2. $\dot{\theta}^3 = k \neq 0$.

The resulting motion is the motion of a charged particle on S^2 moving in the field of a magnetic monopole situates at the center of the sphere. To better see this fact let us start again from the Lagrangian (47). We find that

$$\theta_{\mathcal{L}}|_\Sigma = \dot{x}^1 dx^1 + \dot{x}^2 dx^2 + \dot{x}^3 dx^3 + k\theta^3|_\Sigma \ , \tag{55}$$

$$\omega_{\mathcal{L}}|_\Sigma = dx^1 \wedge d\dot{x}^1 + dx^2 \wedge d\dot{x}^2 + dx^3 \wedge d\dot{x}^3 - k d\theta^3|_\Sigma \ . \tag{56}$$

Now, θ^3 is just the monopole connection on the $U(1)$ bundle $SU(2) \to S^2$ with $d\theta^3$ the corresponding curvature. It turns out that the latter is the volume form on S^2,

$$d\theta^3 = \frac{1}{2}\varepsilon_{jlm} x^j dx^l \wedge dx^m \ . \tag{57}$$

The 2-form (56) becomes

$$\omega_{\mathcal{L}}|_\Sigma = dx^1 \wedge d\dot{x}^1 + dx^2 \wedge d\dot{x}^2 + dx^3 \wedge d\dot{x}^3 - \frac{k}{2}\varepsilon_{jlm}\, x^j dx^l \wedge dx^m \ . \tag{58}$$

As for the energy, we find

$$E_{\mathcal{L}}|_\Sigma = \frac{1}{2}\left((\dot{x}^1)^2 + (\dot{x}^2)^2 + (\dot{x}^3)^2\right) + k^2 \ . \tag{59}$$

The dynamical system determined by the couple $(\omega_{\mathcal{L}}|_\Sigma, E_{\mathcal{L}}|_\Sigma)$ is found to be

$$\widetilde{\Gamma} = \dot{x}^j \frac{\partial}{\partial x^j} - k\varepsilon_{jlm}\, x^j \dot{x}^l \frac{\partial}{\partial \dot{x}^m} \ , \tag{60}$$

with equations of the motion given by

$$\frac{d}{dt}\dot{x}^l = -k\varepsilon_{ljm}\, x^j \dot{x}^m \ . \tag{61}$$

4 The Calogero-Moser system

We may parametrize elements in $\mathbf{R}^3$ using 2×2 symmetric matrices

$$\mathbf{X} = \left\{ \begin{array}{cc} x_1 & \frac{1}{\sqrt{2}} x_2 \\ \frac{1}{\sqrt{2}} x_2 & x_3 \end{array} \right\} . \qquad (1)$$

Free motion on $T\mathbf{R}^3$ can be written as

$$\ddot{\mathbf{X}} = 0 \; , \qquad (2)$$

with standard Lagrangian function

$$\mathcal{L} = \frac{1}{2} tr \dot{\mathbf{X}}^2 = \frac{1}{2}(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2) \; . \qquad (3)$$

The matrix

$$\mathbf{M} = [\mathbf{X}, \dot{\mathbf{X}}] \qquad (4)$$

is then a constant of the motion. Since $\mathbf{M}$ is an antisymmetric matrix it can be written as

$$\mathbf{M} = \ell \, \sigma \; , \qquad \sigma = \left\{ \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right\} , \qquad (5)$$

with ℓ the absolute value of the total angular momentum.

We shall now show how to reduce the dynamics (2) to the Calogero-Moser dynamics on $T\mathbf{R}^2$. We recall that this dynamics is associated with the following Lagrangian on $T\mathbf{R}^2$ (see for instance [Mo])

$$\mathcal{L}_{CM} = \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2) - \frac{\ell^2}{(q_2 - q_1)^2} \; . \qquad (6)$$

Since $\mathbf{X}$ is a symmetric matrix, it can be diagonalized by elements in the rotation group $SO(2)$ by a similarity transformation,

$$\mathbf{X} = \mathbf{G} \; \mathbf{Q} \; \mathbf{G}^{-1} \; , \qquad (7)$$

with

$$\mathbf{Q} = \left\{ \begin{array}{cc} q_1 & 0 \\ 0 & q_2 \end{array} \right\} , \qquad \mathbf{G} = \left\{ \begin{array}{cc} cos\varphi & sin\varphi \\ -sin\varphi & cos\varphi \end{array} \right\} . \qquad (8)$$

As a consequence,

$$\dot{\mathbf{X}} = [\dot{\mathbf{G}} \; \mathbf{G}^{-1}, \mathbf{X}] + \mathbf{G} \; \dot{\mathbf{Q}} \; \mathbf{G}^{-1} = \mathbf{G}\{[\mathbf{G}^{-1} \; \dot{\mathbf{G}}, \mathbf{Q}] + \dot{\mathbf{Q}} \; \}\mathbf{G}^{-1} \qquad (9)$$

and

$$\mathbf{M} = [\mathbf{X}, [\dot{\mathbf{G}} \; \mathbf{G}^{-1}, \mathbf{X}]] = \mathbf{G} \; [\mathbf{Q}, [\mathbf{G}^{-1} \; \dot{\mathbf{G}}, \mathbf{Q}]]\mathbf{G}^{-1} \; . \qquad (10)$$

Using the fact that

$$\mathbf{G}^{-1} \; \dot{\mathbf{G}} \; = \; \dot{\mathbf{G}} \; \mathbf{G}^{-1} \; = \; \dot{\varphi}\sigma \; , \qquad (11)$$

after some algebra, we can derive the ℓ in (5) as

$$\ell = -\frac{1}{2} tr\mathbf{M}\sigma = \dot{\varphi}(q_2 - q_1)^2 \ . \tag{12}$$

By deriving (9) once more with respect to time, after some algebra we get the following equation

$$\ddot{\mathbf{Q}} - \dot{\varphi}^2[\sigma, [\sigma, \mathbf{Q}]] = 0 \ . \tag{13}$$

¿From which, if we derive $\dot{\varphi}$ from (12), we get the equations

$$\begin{aligned} \ddot{q}_1 &= -\frac{2\ell^2}{(q_2 - q_1)^3} \ , \\ \ddot{q}_2 &= \frac{2\ell^2}{(q_2 - q_1)^3} \ . \end{aligned} \tag{14}$$

By using the fact that ℓ is a constant of the motion, for each value $\ell = const$ we find an invariant submanifold Σ_ℓ in the tangent bundle of the space of symmetric matrices, which cover $T\mathbf{R}^2$. Projecting from each one of these submanifolds we find a family of dynamical systems for the variables q_1, q_2 which we can read as a second order dynamical systems on $T\mathbf{R}^2$. We have two classes:

1. $\ell = 0$. Free motion on $\mathbf{R}^2$.
2. $\ell \neq 0$. Calogero-Moser systems on $\mathbf{R}^2$.

Remark. Equations (14) on $T\mathbf{R}^2$ admit a Lagrangian description with Lagrangian function given by (6). However, the latter cannot be gotten by reduction of the starting free Lagrangian (3). Indeed, using the coordinates q_1, q_2 and φ, the Lagrangian (3) can be written as

$$\begin{aligned} \mathcal{L} &= \frac{1}{2} tr\dot{\mathbf{G}}^2 + \frac{1}{2} tr[\mathbf{G}^{-1}\dot{\mathbf{G}}, \mathbf{Q}]^2 \\ &= \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2) + \frac{\ell^2}{(q_2 - q_1)^2} \ . \end{aligned} \tag{15}$$

If we fix now the value of ℓ to be a constant one, the resulting function is a Lagrangian on $T\mathbf{R}^2$ which does not coincide with (6) since the potential term has the opposite sign.

4.1 Symplectic reduction and deformation for the Calogero-Moser system

Let us start with the canonical symplectic structure ω on $T^*\mathbf{R}^3$ given in canonical coordinates (x, y) by $\omega = dy_i \wedge dx_i$. Identifying $\mathbf{R}^3$ with the space of symmetric matrices as in (1), consider the symplectic action of the group $SO(2)$ on this cotangent bundle implemented by the action on $\mathbf{R}^3$ defined by $\mathbf{X} \mapsto \mathbf{G}\mathbf{X}\mathbf{G}^{-1}$, for $\mathbf{G}$ being as in (7). The infinitesimal generator Y of this action is a Hamiltonian vector field with the Hamiltonian

$$p_0 = \sqrt{2}(x_2(y_1 - y_3) + (x_3 - x_1)y_2) \ . \tag{16}$$

Introducing new coordinates (φ, q_1, q_2) in $\mathbf{R}^3$ (cf. (8)) by

$$\begin{aligned} x_1 &= q_1 cos^2\varphi + q_2 sin^2\varphi \ , \\ x_3 &= q_1 sin^2\varphi + q_2 cos^2\varphi \ , \\ x_2 &= \frac{sin 2\varphi}{\sqrt{2}}(q_2 - q_1) \ , \end{aligned} \tag{17}$$

we get conjugate coordinates in the cotangent bundle of the form

$$\begin{aligned} p_\varphi = p_0 &= (q_2 - q_1)(y_1 - y_3) sin 2\varphi + \sqrt{2}(q_2 - q_1) y_2 cos 2\varphi \ , \\ p_{q_1} = p_1 &= y_1 cos^2\varphi + y_3 sin^2\varphi - y_2 \frac{sin 2\varphi}{\sqrt{2}} \ , \\ p_{q_2} = p_2 &= y_1 sin^2\varphi + y_3 cos^2\varphi + y_2 \frac{sin 2\varphi}{\sqrt{2}} \ , \end{aligned} \tag{18}$$

i.e. $\omega = dp_i \wedge dq_i$, were we put $q_0 = \varphi$. In the new coordinates $Y = \partial_\varphi$. The standard reduction with respect to the action of $SO(2)$ is then the following. For a regular value k of the momentum mapping p_0 consider the submanifold $\Sigma_k = p_0^{-1}(k)$. Then the subbundle of kernels of $\omega_{|\Sigma_k}$ in $T\Sigma_k$ is generated by Y and we can pass to the quotient manifold $(\widetilde{\Sigma}_k, \widetilde{\omega})$. Since (q_1, q_2, p_1, p_2) are clearly generators of the algebra of Y–invariant functions on each $\widetilde{\Sigma}_k$, we can think of $\widetilde{\Sigma}_k$ as $T^*\mathbf{R}^2$ with canonical coordinates (q, p). We can also reduce to $\widetilde{\Sigma}_k$ any dynamics represented by a Y–invariant Hamiltonian on $T^*\mathbf{R}^3$. For example, the Hamiltonian $H_0 = \frac{1}{2}\sum_i y_i^2$ of the free motion Γ is Y–invariant and we can write it in the new coordinates in the form

$$H_0 = \frac{1}{2}(p_1^2 + p_2^2) + \frac{p_0^2}{4(q_2 - q_1)^2} \ , \tag{19}$$

so the reduced Hamiltonian describes the Calogero–Moser system (14) with $\ell = \frac{p_0}{2}$. We can also reduce all Y–invariant potentials, i.e. potentials of the form

$$V = V(x_1 + x_3, (x_1 - x_3)^2 + 2x_2^2) \ . \tag{20}$$

The reduced potential is then $\widetilde{V}(q_1, q_2) = V(q_1 + q_2, (q_2 - q_1)^2)$. Observe now that we can reduce to $\widetilde{\Sigma}_k$ not only ω but any symplectic form ω' such that $\omega' = \alpha \wedge dp_0 + \omega''$, where α is any 1–form and $i_Y\omega'' = 0$. Consider now functions

$$\begin{aligned} P &= p_1 + p_2 = y_1 + y_2 \ , \\ F &= (p_1 - p_2)^2 + \frac{p_0^2}{(q_2 - q_1)^2} = (y_3 - y_1)^2 + 2y_2^2 \ , \end{aligned} \tag{21}$$

and the 2–form

$$\eta = dp_2 \wedge dq_1 + dp_1 \wedge dq_2 \ . \tag{22}$$

They are clearly projectable and one can check that

$$\begin{aligned} i_\Gamma \eta &= dG \ , \\ G &= p_1 p_2 - \frac{p_0^2}{4(q_2-q_1)^2} \ . \end{aligned} \tag{23}$$

Then, if we put now

$$\omega_{\lambda,s,t} = \lambda\omega + s\eta + t dP \wedge dF \ , \tag{24}$$

we get a closed projectable and non–degenerate (at least for $\lambda \neq 0$ and small s) 2–form, i.e. a new projectable symplectic structure. Our free motion Γ can be now equivalently described by the Hamiltonian $H_{\lambda,s} = \lambda H_0 + sG$ with respect to the symplectic form $\omega_{\lambda,s,t}$ (F, P are constants of the motion). All this project to $\widetilde{\Sigma}_k$, so the Calogero–Moser dynamics $\widetilde{\Gamma}$ can be equivalently described by a three–parameter family of symplectic structures $\omega_{\lambda,s,t}$ on $T^*\mathbf{R}^2$ and two–parameter family of corresponding Hamiltonians $H_{\lambda,s}$. Observe that the form $\omega_{0,s,t}$ is degenerate but its projection $\widetilde{\omega}_{0,s,t}$ is symplectic for $s \neq 0$, so that $s\widetilde{G}$ is also an alternative Hamiltonian for $\widetilde{\Gamma}$ and $\omega_{0,s,t}$.

The Poisson structure $\widetilde{\Lambda}_{\lambda,s,t}$ on $T^*\mathbf{R}^2$ corresponding to $\widetilde{\omega}_{\lambda,s,t}$ is given by

$$\begin{aligned} \widetilde{\Lambda}_{\lambda,s,t} &= \frac{1}{\lambda^2 - s^2}\left((\lambda - tf)\partial_{p_1} \wedge \partial_{q_1} - (s - tf)\partial_{p_2} \wedge \partial_{q_1}\right. \\ &\qquad \left. -(s + tf)\partial_{p_2} \wedge \partial_{q_1} + (\lambda + tf)\partial_{p_2} \wedge \partial_{q_2}\right) \ , \end{aligned} \tag{25}$$

where $f = \frac{2p_0^2}{(q_2-q_1)^3}$.

By using the generalization of the Calogero-Moser system to n-particles described in [OP], we can extend our considerations to n-particles. In particular we can exhibit alternative Lagrangian, Hamiltonian and Poisson structures like in the case we have considered.

In the previous examples we have studied a reduced motion along homogeneous spaces of the form M/K. In the following examples we shall consider a complementary situation in the splitting $K \to M \to M/K$. We shall consider the projection of the motion on M along the leaves of the foliation. In this way M/K will parametrize a family of dynamical systems.

5 Reduction of geodesical motions

We shall first consider the simple case of geodesical motion on the Euclidean spheres and then we shall treat the more general case of geodesical motion on general hyperboloids.

5.1 Geodesical motions on spheres

The geodesical motion on S^2 can also be obtained by reducing a dynamical system different by the one considered in section 3.3. The setting can be easily generalized to the n-dimentional sphere so we shall describe the general case.

On $T^*\mathbf{R}^{n+1}$ with standard symplectic structure $\omega = dp_i \wedge dq^i$, we consider the following Hamiltonian

$$H = \frac{1}{2} Gram(\vec{p}, \vec{q}) = \frac{1}{2}\left(p^2 q^2 - (\vec{p}\cdot\vec{q})^2\right) \ . \tag{1}$$

The corresponding equations of motion can be integrated by exponentiation. Indeed, on any submanifold that we obtain by fixing the values of p^2, q^2 and $\vec{p}\cdot\vec{q}$ (they are constants of the motion) the dynamics becomes linear (depending on the specific values of the constants of the motion). Our system is then a simple system. We fix now the submanifold $\Sigma = \{(\vec{p}, \vec{q}) \vdash q^2 = 1\}$ and as an equivalence relation on Σ we take $(\vec{p}_1, \vec{q}_1)\mathcal{R}(\vec{p}_2, \vec{q}_2)$ if $\vec{q}_1 = \vec{q}_2$ and $\vec{p}_1 = \vec{p}_2 + s\vec{q}$, for some $s \in \mathbf{R}$. It is possible to embedd the quotient manifold into $T^*\mathbf{R}^{n+1}$ by selecting in every equivalence class an element $\vec{\pi}$ such that $\vec{\pi}\cdot\vec{q} = 0$. This means that in the equivalence class of $(\vec{p}, \vec{q})$ we consider $\vec{\pi} = \vec{p} - (\vec{p}\cdot\vec{q})\vec{q}$ with $q^2 = 1$. In this way the quotient manifold becomes T^*S^n. The starting dynamical vector field

$$\Gamma = \left(q^2\vec{p} - (\vec{q}\cdot\vec{p})\vec{q}\right)\cdot\frac{\partial}{\partial\vec{q}} + \left((\vec{q}\cdot\vec{p})\vec{p} - p^2\vec{q}\right)\cdot\frac{\partial}{\partial\vec{p}} \ , \tag{2}$$

on the submanifold T^*S^n becomes

$$\tilde{\Gamma} = \vec{\pi}\cdot\frac{\partial}{\partial\vec{q}} - \pi^2\vec{q}\cdot\frac{\partial}{\partial\vec{\pi}} \ . \tag{3}$$

The corresponding equations of motion in the second order form

$$\ddot{\vec{q}} = -(\dot{q})^2\vec{q} \ , \tag{4}$$

represent the geodesical flow on the unitary sphere S^n.

This example is to be kept in mind when addressing the unfolding procedure: the same dynamical system can be unfolded in different ways.

5.2 Geodesical motions on hyperboloids

In our example on the geodesical motion on S^n we have started with the Hamiltonian function (1) on the cotangent bundle $T^*\mathbf{R}^{n+1}$. We have used the Euclidean metric to lower and raise indices so as to consider $\vec{q}$ and $\vec{p}$ as vectors having the same (co-) variance transformation properties.

Since we would like to go beyond the Euclidean metric, we cannot assume $\vec{q}$ and $\vec{p}$ to be vectors with identical transformation properties if they are going to provide a symplectic chart for the symplectic structure we are going to use. In order to avoid this problem we shall work on the tangent bundle $T\mathbf{R}^{n+1}$ instead of the cotangent bundle. However, we do not have any Lagrangian corresponding to the Hamiltonian (1) which we could use to construct a corresponding Lagrangian formalism on $T\mathbf{R}^{n+1}$. What seems to be a natural choice, namely

$$\mathcal{L} = \frac{1}{2}\left(\dot{q}^2 q^2 - (\vec{q}\cdot\dot{\vec{q}})^2\right) \ , \tag{5}$$

turns out to be singular and it does not give rise to a symplectic structure on $T\mathbf{R}^{n+1}$. We are forced therefore, to work on $T\mathbf{R}^{n+1}$ with a symplectic structure defined autonomously, even though in a non natural way. We are going to use the so called symplectic formalism for the tangent bundle [MSSV]. This means that there is no connection between the generating function of the dynamics and the Legendre map which is used to pull-back to the tangent bundle the canonical symplectic structure on the cotangent bundle.

Let us start then with $T\mathbf{R}^{n+1} \equiv \mathbf{R}^{n+1} \times \mathbf{R}^{n+1}$ with coordinates $(\mathbf{q}, \dot{\mathbf{q}}) = (q^i, \dot{q}^i)$ and consider any non degenerate quadratic form m on $\mathbf{R}^{n+1}$,

$$\mathsf{m}(u,v) = u^i v^j \mathsf{m}(e_i, e_j) = u^i v^j \mathsf{m}_{ij} \ , \quad \mathsf{m}_{ij} \in \mathbf{R} \ , \quad det|\mathsf{m}_{ij}| \neq 0 \ . \tag{6}$$

We can construct a symplectic form ω_{m} and a function E_{m} (generating function for the dynamics) by

$$\omega_{\mathsf{m}} = \mathsf{m}(d\mathbf{q} \stackrel{\wedge}{,} d\dot{\mathbf{q}}) = \mathsf{m}_{ij} dq^i \wedge d\dot{q}^j \ , \tag{7}$$

$$E_{\mathsf{m}} = \frac{1}{2}\left(\mathsf{m}(\mathbf{q},\mathbf{q})\mathsf{m}(\dot{\mathbf{q}},\dot{\mathbf{q}}) - \mathsf{m}(\mathbf{q},\dot{\mathbf{q}})^2\right) \ . \tag{8}$$

The dynamical vector field Γ determined by $i_\Gamma \omega_{\mathsf{m}} = dE_{\mathsf{m}}$ turns out to be

$$\Gamma_{\mathsf{m}} = \left(\mathsf{m}(\mathbf{q},\mathbf{q})\dot{q}^j - \mathsf{m}(\mathbf{q},\dot{\mathbf{q}})q^j\right)\frac{\partial}{\partial q^j} + \left(\mathsf{m}(\mathbf{q},\dot{\mathbf{q}})\dot{q}^j - \mathsf{m}(\dot{\mathbf{q}},\dot{\mathbf{q}})q^j\right)\frac{\partial}{\partial \dot{q}^j} \ . \tag{9}$$

The three quantities $\mathsf{m}(\mathbf{q},\mathbf{q})$, $\mathsf{m}(\mathbf{q},\dot{\mathbf{q}})$ and $\mathsf{m}(\dot{\mathbf{q}},\dot{\mathbf{q}})$ are constants of the motion for Γ. On any submanifold Σ of $T\mathbf{R}^{n+1}$ determined by $\mathsf{m}(\mathbf{q},\mathbf{q}) = c_1$, $\mathsf{m}(\mathbf{q},\dot{\mathbf{q}}) = c_2$, $\mathsf{m}(\dot{\mathbf{q}},\dot{\mathbf{q}}) = c_3$, $c_1, c_2, c_3 \in \mathbf{R}$, the dynamics becomes linear and can be integrated by exponentiation, i.e. it is a simple system.

We identify a particular submanifold by setting $c_1 = 1, c_2 = 0$ and we get a restricted vector field

$$\widetilde{\Gamma}_{\mathsf{m}} = \dot{q}^j \frac{\partial}{\partial q^j} - \mathsf{m}(\dot{\mathbf{q}},\dot{\mathbf{q}})\ q^j \frac{\partial}{\partial \dot{q}^j} \ , \tag{10}$$

giving

$$\ddot{q}^j = -\mathsf{m}(\dot{\mathbf{q}},\dot{\mathbf{q}})\ q^j \ , \tag{11}$$

namely, the geodesical motion on the 'pseudo-sphere' (hyperboloid) in $\mathbf{R}^{n+1}$ determined by $\mathsf{m}(\mathbf{q},\mathbf{q}) = 1$.

Remark. As $\mathsf{m}(\dot{\mathbf{q}},\dot{\mathbf{q}})$ is a constant of the motion, on each level set determined by $\mathsf{m}(\dot{\mathbf{q}},\dot{\mathbf{q}}) = c_3$, we can still reduce the dynamics (10) to

$$\tilde{\tilde{\Gamma}}_{\mathsf{m}} = \dot{q}^j \frac{\partial}{\partial q^j} - c_3 q^j \frac{\partial}{\partial \dot{q}^j} \ , \tag{12}$$

which represents n isotropic harmonic oscillators.

If we select $\mathsf{m}(\mathbf{q},\mathbf{q}) > 0$ and the orbit goes through the point $\sqrt{\mathsf{m}(\mathbf{q},\mathbf{q})}(1,0,\cdots,0)$, the reduced space can be identified with

$$T^* \left(SO(p,(n+1)-p)/SO(p-1,(n+1)-p) \right) \ , \tag{13}$$

where $(p,(n+1)-p)$ is the signature of m. The reduced motion takes place on a homogeneous space which carries an action of $SO(p,(n+1)-p)$.

Remark. We would like to mention that one could start with $\mathsf{m}(X,Y) = X^\dagger M Y$ where $X, Y \in \mathbf{C}^{n+1}$ and M is an Hermitiam matrix, $M^\dagger = M$. The superscript † denotes Hermitian conjugation. In this case we would consider the group $SU(p,q)$ acting on $\mathbf{C}^{n+1}$ and proceed to define homogeneous spaces like in the real case. Systems based on these spaces have been considered in [ORW].

By looking at our construction of ω_{m}, we notice that the Lagrangian function $\mathcal{L}_{\mathsf{m}} = \frac{1}{2}\mathsf{m}_{ij}\dot{q}^i\dot{q}^j$ gives rise exactly to our symplectic structure as $\omega_{\mathsf{m}} = -d(\frac{\partial \mathcal{L}_{\mathsf{m}}}{\partial \dot{q}^j} dq^j)$. It is therefore natural to consider the free motion associated with $\mathcal{L}_{\mathsf{m}}$ and to compare it with the reduced geodesical motion on the hyperboloids. We notice that one has also $E_{\mathcal{L}} = \dot{q}^j \frac{\partial \mathcal{L}}{\partial \dot{q}^j} - \mathcal{L}_{\mathsf{m}} = \frac{1}{2}\mathsf{m}_{ij}\dot{q}^i\dot{q}^j$. We can apply the same decomposition we used for the free particle motion in $\mathbf{R}^3$ in section 2.2.1 and write

$$E_{\mathcal{L}} = \frac{1}{2}\dot{\mathbf{q}}^2 + \frac{1}{2}\frac{(\mathbf{q}\wedge\dot{\mathbf{q}})^2}{\mathbf{q}^2} \ . \tag{14}$$

Since our quadratic form is not necessarily positive definite and we are not in three dimensions,we have to say what we mean by $\dot{\mathbf{q}}^2$ and what is the meaning of $(\mathbf{q}\wedge\dot{\mathbf{q}})^2$. To make things similar to the three dimensional case we have to digress a little to introduce the appropriate calculus.

Digression: Calculus with vector valued forms.

We start with our non degenerate quadratic form on $\mathbf{R}^{n+1}$,

$$\mathsf{m}(u,v) = \mathsf{m}(e_i,e_j)e^i \otimes e^j \ , \quad \mathsf{m}_{ij} = \mathsf{m}(e_i,e_j) \ . \tag{15}$$

By using vector valued forms, we can write

$$\mathbf{q}\wedge d\mathbf{q} = q^i dq^j e_i \wedge e_j = \frac{1}{2}(q^i dq^j - q^j dq^i) e_i \wedge e_j \ , \tag{16}$$

$$\mathsf{m}(d\mathbf{q} \overset{\otimes}{,} d\mathbf{q}) = dq^i \otimes dq^j \mathsf{m}(e_i,e_j) = dq^i \otimes dq^j \mathsf{m}_{ij} \ , \tag{17}$$

$$\mathsf{m}(\mathbf{q}\wedge d\mathbf{q} \overset{\otimes}{,} \mathbf{q}\wedge d\mathbf{q}) = \frac{1}{4}(q^i dq^j - q^j dq^i) \otimes (q^m dq^n - q^n dq^m) \times$$
$$\times \mathsf{m}(e_i \wedge e_j, e_m \wedge e_n) \ , \tag{18}$$

where $\mathsf{m}(v\wedge u, w\wedge z)$ is the natural extension of the quadratic form to $\Lambda^2\mathbf{R}^{n+1}$ defined by

$$\mathsf{m}(e_i\wedge e_j, e_m\wedge e_n) = \mathsf{m}(e_i, e_m)\mathsf{m}(e_j, e_n) - \mathsf{m}(e_i, e_n)\mathsf{m}(e_j, e_m)\ . \tag{19}$$

After some algebra we get the following decomposition

$$\mathsf{m}(d\mathbf{q} \overset{\otimes}{,} d\mathbf{q}) = dq\otimes dq + \frac{1}{q^2}\mathsf{m}(\mathbf{q}\wedge d\mathbf{q} \overset{\otimes}{,} \mathbf{q}\wedge d\mathbf{q})\ , \tag{20}$$

where

$$\begin{aligned} q^2 &= \mathsf{m}(\mathbf{q},\mathbf{q}) = \mathsf{m}_{ij}q^iq^j\ , \\ q &= \sqrt{|q^2|}\ . \end{aligned} \tag{21}$$

Consider now the free motion on $\mathbf{R}^{n+1}$ described by the Lagrangian

$$\mathcal{L}_{\mathsf{m}} = \frac{1}{2}\mathsf{m}(d\mathbf{q} \overset{\otimes}{,} d\mathbf{q})(\Gamma,\Gamma) = \frac{1}{2}\mathsf{m}_{ij}\dot{q}^i\dot{q}^j\ . \tag{22}$$

It gives

$$\omega_{\mathcal{L}_{\mathsf{m}}} = \mathsf{m}_{ij}dq^i\wedge d\dot{q}^j\ , \tag{23}$$

$$\begin{aligned} E_{\mathcal{L}_{\mathsf{m}}} &= \frac{1}{2}\mathsf{m}_{ij}\dot{q}^i\dot{q}^j \\ &= \frac{1}{2}\dot{q}^2 + \frac{1}{2}\frac{\mathsf{m}(\mathbf{q}\wedge\dot{\mathbf{q}},\mathbf{q}\wedge\dot{\mathbf{q}})}{\mathsf{m}(\mathbf{q},\mathbf{q})} \\ &= E^{rad}_{\mathsf{m}} + E^{ang}_{\mathsf{m}}\ . \end{aligned} \tag{24}$$

Consider now the vector field Γ_{ang} defined by

$$i_{\Gamma_{ang}}\omega_{\mathcal{L}_{\mathsf{m}}} = dE^{ang}_{\mathsf{m}}\ . \tag{25}$$

We find

$$\begin{aligned} \Gamma_{ang} &= \frac{1}{\mathsf{m}(\mathbf{q},\mathbf{q})}\left[\Big(\mathsf{m}(\mathbf{q},\mathbf{q})\dot{q}^j - \mathsf{m}(\mathbf{q},\dot{\mathbf{q}})q^j\Big)\frac{\partial}{\partial q^j} + \Big(\mathsf{m}(\mathbf{q},\dot{\mathbf{q}})\dot{q}^j - \mathsf{m}(\dot{\mathbf{q}},\dot{\mathbf{q}})q^j\Big)\frac{\partial}{\partial\dot{q}^j}\right] \\ &+ \frac{1}{\mathsf{m}(\mathbf{q},\mathbf{q})}\mathsf{m}(\mathbf{q}\wedge\dot{\mathbf{q}},\mathbf{q}\wedge\dot{\mathbf{q}})q^j\frac{\partial}{\partial\dot{q}^j} \\ &= \frac{1}{\mathsf{m}(\mathbf{q},\mathbf{q})}\Gamma_{\mathsf{m}} + \frac{1}{\mathsf{m}(\mathbf{q},\mathbf{q})}\mathsf{m}(\mathbf{q}\wedge\dot{\mathbf{q}},\mathbf{q}\wedge\dot{\mathbf{q}})q^j\frac{\partial}{\partial\dot{q}^j}\ , \end{aligned} \tag{26}$$

where Γ_{m} is the same as in (9). Therefore, the dynamical system associated with the angular part of the energy does not coincide with our previous dynamics Γ_{m} and moreover is quite cumbersome. Let us consider however the projection onto the hyperboloid $\mathbf{H}^n = \{\mathsf{m}(\mathbf{q},\mathbf{q}) = 1\}$,

$$\begin{aligned} &\pi : \mathbf{R}^{n+1} \longrightarrow \mathbf{H}^n\ , \quad \mathbf{q}\mapsto\tilde{\mathbf{q}} = \frac{\mathbf{q}}{\sqrt{|\mathsf{m}(\mathbf{q},\mathbf{q})|}}, \\ &T\pi : T\mathbf{R}^{n+1} \longrightarrow T\mathbf{H}^n\ , \quad (\mathbf{q},\dot{\mathbf{q}})\mapsto(\tilde{\mathbf{q}},\dot{\tilde{\mathbf{q}}})\ . \end{aligned} \tag{27}$$

Then $T\pi(q^j \frac{\partial}{\partial \dot{q}^j}) = 0$, so that Γ_{ang} and Γ_{m} project onto the same vector field on $T\mathbf{H}^n$, i.e. both give the same geodesical motion on $\mathbf{H}^n$.

If we consider these examples in the framework of the unfolding, we see here that our geodesical motion can be unfolded in two different ways, one is a simple system, the other one is the angular part of the free motion.

6 Reduction procedure in algebraic terms

To illustrate the reduction procedure from the dual view point, we shall start with the ring $\mathcal{F}$ of smooth functions on a manifold M.

Let us consider again the projection $\pi_\Sigma : \mathcal{F} \to \mathcal{F}_\Sigma$ in (5). Clearly $ker\pi_\Sigma$ is an ideal in $\mathcal{F}$ and consists exactly of functions vanishing on $\mathcal{F}_\Sigma$, and we have a corresponding sequence of associative and commutative algebrae $0 \to ker\pi_\Sigma \to \mathcal{F} \to \mathcal{F}_\Sigma \to 0$. The idea is then that a starting point for an algebraic approach could be a sequence of associative, commutative algebrae

$$0 \longrightarrow \mathcal{I} \stackrel{i}{\longrightarrow} \mathcal{F} \stackrel{\pi}{\longrightarrow} \mathcal{Q}_\mathcal{I} \longrightarrow 0 \ , \tag{1}$$

where $\mathcal{I}$ is an ideal (with respect to the associative structure) of $\mathcal{F}$ and $\mathcal{Q}_\mathcal{I} = \mathcal{F}/\mathcal{I}$. Both i and π are algebra homomorphisms, e.g. $\pi(f_1 \cdot f_2) = \pi(f_1) \cdot \pi(f_2)$, $\pi(f_1 + f_2) = \pi(f_1) + \pi(f_2)$.

Any derivation X on $\mathcal{F}$ will define a derivation on $\mathcal{Q}_\mathcal{I}$ if and only if $X \cdot \mathcal{I} \subset \mathcal{I}$, so that the algebra $\mathcal{Q}_\mathcal{I}$ of equivalence classes is taken into itself. We shall indicate by $\mathcal{X}(\mathcal{I})$ the Lie subalgebra of these derivations

$$\mathcal{X}(\mathcal{I}) = \{X \in Der\mathcal{F} \vdash X \cdot \mathcal{I} \subset \mathcal{I}\} \ . \tag{2}$$

A sequence is compatible with a dynamical vector field Γ if $\Gamma \in \mathcal{X}(\mathcal{I})$. Such a sequence replaces the choice of an invariant submanifold Σ. The analog of the equivalence relation is provided by any subalgebra of $\mathcal{Q}_\mathcal{I}$ which is invariant under the action of Γ. The restriction of Γ to any such subalgebra represents a reduced dynamics.

Having recalled our algebraic framework for reduction we are now ready to add additional structures, namely Poisson brackets. As any symplectic structure is associated with a (non-degenerate) Poisson bracket we shall not consider symplectic structures separately.

6.1 Poisson reduction

Let us suppose that the algebra $\mathcal{F}$ in (1) is a Poisson algebra. We recall that this means a Lie algebra structure $\{\ ,\ \}$ on $\mathcal{F}$ with the additional requirement that the map $f \longmapsto X_f$, defined by $X_f \cdot g := \{f, g\}$, is a Lie algebra homomorphism from $\mathcal{F}$ into $Der\mathcal{F}$. We shall first study conditions under which we can reduce also the Poisson structrure.

We call *reduction* of this Poisson structure any quotient Poisson structure of a Poisson subalgebra $\mathcal{F}'$ of $\mathcal{F}$ by a Poisson ideal (i.e. an ideal with respect to both the structures, associative and Lie).

We describe standard ways to obtain reductions. Let us consider a "submanifold", namely an associative ideal $\mathcal{I}$ in $\mathcal{F}$. If $\mathcal{I}$ is a Poisson ideal we can pass to the quotient. In most cases, however, this is not true and we must look for a Poisson subalgebra $\mathcal{F}'$ in $\mathcal{F}$ such that $\mathcal{F}' \cap \mathcal{I}$ is a Poisson ideal. The algebra $\mathcal{F}'$ can be chosen to be the normalizer $\mathcal{N}_\mathcal{I}$ of $\mathcal{I}$ under Poisson bracket,

$$\mathcal{N}_\mathcal{I} =: \{f \in \mathcal{F} \vdash \{f, \mathcal{I}\} \subset \mathcal{I}\} \ . \tag{3}$$

It is easy to see that $\mathcal{N}_\mathcal{I}$ is a Poisson algebra [Gr]. Elements in $\mathcal{N}_\mathcal{I}$ are generating functions of Hamiltonian derivations X_f which project to derivations of $\mathcal{Q}_\mathcal{I} = \mathcal{N}_\mathcal{I}/\mathcal{I}$. We select a Poisson subalgebra in $\mathcal{I}$ by intersecting the latter with $\mathcal{N}_\mathcal{I}$.

The following statement is true:

The intersection

$$\mathcal{I}' = \mathcal{I} \cap \mathcal{N}_\mathcal{I} \ , \tag{4}$$

is a Poisson ideal in $\mathcal{N}_\mathcal{I}$.

Proof. $\mathcal{I}'$ is already a Lie ideal. From the properties of the Poisson brackets: $\{\mathcal{F}\mathcal{I}', \mathcal{I}\} \subset \mathcal{F}\{\mathcal{N}_\mathcal{I}, \mathcal{I}\} + \mathcal{I}\{\mathcal{F}, \mathcal{I}\} \subset \mathcal{I}$, so that $\mathcal{F}\mathcal{I}' \subset \mathcal{N}_\mathcal{I}$.

As an obvious consequence we have:

The space $\mathcal{Q}'_\mathcal{I} =: \mathcal{N}_\mathcal{I}/\mathcal{I}'$ is a Poisson algebra such that the following is an exact sequence of Poisson algebrae

$$0 \longrightarrow \mathcal{I}' \xrightarrow{i} \mathcal{N}_\mathcal{I} \xrightarrow{\pi} \mathcal{Q}_{\mathcal{I}'} \longrightarrow 0 \ . \tag{5}$$

Given any two elements (equivalence classes) $[f], [g] \in \mathcal{Q}'_\mathcal{I}$, their Poisson brackets is given by

$$\{[f], [g]\}_{\mathcal{Q}'_I} = [\{f, g\}_{\mathcal{N}_\mathcal{I}}] \ . \tag{6}$$

Remark. In general $\mathcal{Q}_\mathcal{I}$ will be different from $\mathcal{Q}'_\mathcal{I}$.

Remark. Let $\mathcal{I}$ be the ideal $\mathcal{F} = C^\infty(M)$ consisting of smooth functions vanishing on a closed embedded submanifold Σ. The normalizer $\mathcal{N}(\mathcal{I})$ consists of all functions whose Hamiltonian vector fields are tangent to Σ. The quotient algebra $\mathcal{Q}_\mathcal{I}$ consists of smooth functions on Σ.

The second, more geometric method of reducing Poisson structures is to determine the subalgebra $\mathcal{F}'$ by means of a distribution.

Definition By *distribution* on $\mathcal{F}$ we shall mean any subset $\mathcal{D}$ of the algebra $Der\mathcal{F}$. We say that $\mathcal{D}$ is *integrable* if it is a subalgebra of $Der\mathcal{F}$.

Having a distribution $\mathcal{D} \subset Der\mathcal{F}$, define the associative algebra

$$\mathcal{F}_{\mathcal{D}} = \{f \in \mathcal{F} \vdash \mathcal{D} \cdot f = 0\} . \tag{7}$$

Note that $\mathcal{F}_{\mathcal{D}} = \mathcal{F}_{\hat{\mathcal{D}}}$, where $\hat{\mathcal{D}}$ is the integrable distribution generated by $\mathcal{D}$ so we can start with integrable distributions.

We say that $\mathcal{D}$ is compatible with the Poisson bracket $\{\cdot,\cdot\}$ if $\mathcal{F}_{\mathcal{D}}$ is a Poisson subalgebra.

We shall give now an example of reduction associated with a Poisson bracket that arises as the limit of a quantum group structure on $SU_q(2)$.

6.1.1 Example: Lie-Poisson structure for SU(2)

Consider a quadratic Poisson structure on $\mathbf{R}^4$ given by

$$\begin{array}{lll} \{y_1,y_2\} = 0 & \{y_1,y_3\} = y_1y_4 & \{y_1,y_4\} = -y_1y_3 \\ \{y_2,y_3\} = y_2y_4 & \{y_2,y_4\} = -y_2y_3 & \{y_3,y_4\} = y_1^2+y_2^2 . \end{array} \tag{8}$$

It is easy to show that $y_1^2+y_2^2+y_3^2+y_4^2$ is a Casimir function for the bracket. Therefore, the latter can be reduced to a bracket on the unit sphere S^3 . By identifying this sphere with the group $SU(2)$ we get what is known as a Lie-Poisson structure on $SU(2)$. We identify the latter with S^3 via the map

$$(y_1,y_2,y_3,y_4) \mapsto s = y_4\mathbf{1} + i(\vec{y}\cdot\vec{\sigma}) = \begin{pmatrix} y_4+iy_3 & -y_2+iy_1 \\ y_2+iy_1 & y_4+iy_3 \end{pmatrix} , \tag{9}$$

with σ_i , $i=1,2,3$, the three Pauli matrices.

Consider now the compatible distribution $\mathcal{D}$ generated by the vector field

$$X = -y_2\frac{\partial}{\partial y_1} + y_1\frac{\partial}{\partial y_2} + y_4\frac{\partial}{\partial y_3} - y_3\frac{\partial}{\partial y_4} . \tag{10}$$

The reduced algebra $\mathcal{F}_D$ can be regarded as the algebra of functions of variables

$$u = -y_1^2 - y_2^2 + y_3^2 + y_4^2 , \quad v = 2(y_1y_3+y_2y_4) , \quad z = 2(y_1y_4 - y_2y_3) , \tag{11}$$

with brackets

$$\begin{aligned} \{v,u\} &= 2(1-u)z \\ \{u,z\} &= 2(1-u)v \\ \{z,v\} &= 2(1-u)u \end{aligned} \tag{12}$$

One finds that $u^2 + v^2 + z^2 = 1$ so that the reduced space of $SU(2)$ is the unit sphere S^2 and the reduced bracket is singular at the north pole $(u = 1, v = z = 0)$. The stereographic projection from the north pole maps this structure onto the standard one on $\mathbf{R}^2$.

Usually, it is more convenient, for computational purposes, to mix algebraic and geometrical notions, i.e. we use an ideal (submanifold) and a distribution.

As an example we consider the Poisson reduction of Marsden and Ratiu [MR].

6.1.2 Example

Let us suppose we are given a Poisson manifold $(P, \{\cdot, \cdot\})$ and a submanifold Σ of P. Suppose there is a subbundle $E \subset TP|_\Sigma$, with the following properties

1. $E \cap T\Sigma$ is an integrable subbundle of $T\Sigma$, so defining a foliation Φ on Σ;

2. the foliation Φ is regular, so that the space of leaves Σ/Φ is a manifold and the projection $\pi : \Sigma \to \Sigma/\Phi$ is a submersion;

3. the bundle E leaves the Poisson structure invariant in the sense that for any two functions F, G whose differentials dF, dG vanish on E, the Poisson bracket $\{F, G\}$ has differential vanishing on E;

Define $\mathcal{F}'$ to be the algebra of functions on P whose differential vanishes on E. Then, condition 3. assures that $\mathcal{F}'$ is a Poisson subalgebra of $\mathcal{F} = C^\infty(P)$. If $\mathcal{I}$ is the ideal of functions vanishing on Σ, conditions 1. and 2. assure that $\mathcal{F}'/\mathcal{I} \cap \mathcal{F}'$ can be thought of as the algebra of smooth functions on Σ/Φ. To have a Poisson structure on $\mathcal{F}'/\mathcal{I} \cap \mathcal{F}'$ we must have that $\mathcal{I} \cap \mathcal{F}'$ is a Poisson ideal in $\mathcal{F}'$, which is clearly equivalent to the fact that Hamiltonian vector fields associated with $\mathcal{F}'$ when restricted to Σ belong to $T\Sigma \oplus E$. This is just the last assumption of Marsden and Ratiu.

The coming example is the typical situation we face when dealing with the constraint formalism in the Dirac-Bergmann approach. It should be noticed however that in our approach a dynamics is already given, it is not to be determined like in the constraint formalism.

6.1.3 Example

Let us suppose we are on a Poisson manifold $(M, \{\cdot, \cdot\})$. We take $\mathcal{F} \equiv \mathcal{F}(M)$. Given k functions $f_j \in \mathcal{F}(M)$ $j \in \{1, \cdots k\}$ we take the ideal $\mathcal{I} = \mathcal{I}_a$ generated by

$$\{f_i - a_i \ \vdash \ a_i \in \mathbf{R}, \quad i = 1, \cdots, k\} \ . \tag{13}$$

Suppose we have any compatible distribution $\mathcal{D}$ generated by vector fields $\{X_i\ ,\ i = 1, \cdots, n\}$. If the X_i's are Hamiltonian vector fields, then the distribution is automatically compatible. Put now

$$\begin{aligned} \mathcal{F}' &= \mathcal{F}_{\mathcal{D}} \cap \mathcal{N}(\mathcal{I}_a)\ , \\ \mathcal{I}' &= \mathcal{I}_a \cap \mathcal{F}'\ . \end{aligned} \tag{14}$$

It is easy to see that $\mathcal{F}'$ is a Poisson algebra and that $\mathcal{I}'$ is a Poisson ideal in $\mathcal{F}'$, so that we can reduce the Poisson bracket to $\mathcal{F}'/\mathcal{I}'$.

Having an Ad^*-equivariant momentum map $f = (f_1, \cdots, f_k) : M \to \mathbf{g}^*$ associated with a Hamiltonian action of the group G, we can take $\mathcal{D}$ generated by the Hamiltonian vector fields X_{f_j}. Then $\mathcal{F}_{\mathcal{D}} \subset \mathcal{N}(\mathcal{I}_a)$ and we get the classical reduction with $\mathcal{F}'/\mathcal{I}' = \mathcal{F}_{\mathcal{D}}/\mathcal{I}_a$ interpreted as the algebra of functions on the space of orbits Σ_a/G_a.

6.1.4 Example

Consider the cotangent bundle T^*G of a group G with the canonical Poisson structure. Let $\{x_i\}$ be a basis of the Lie algebra $\mathbf{g}$ and $\{e^i\}$ the dual basis of $\mathbf{g}^*$. Notice that $\{x_i\}$ can be regarded as a coordinate system on $\mathbf{g}^*$. The bundle T^*G can be naturally considered as a Lie group. In 'body' coordinates $T^*G \equiv G \times \mathbf{g}^*$, and the multiplication has the form

$$(h_1, \mu_1) \cdot (h_2, \mu_2) = (h_1 h_2, \mu_1 + Ad^*_{h_2}(\mu_2))\ . \tag{15}$$

The Lie algebra of this group can be identified with the semidirect product $\mathbf{g} \oplus \mathbf{g}^*$, with $\mathbf{g}^*$ thought of as a commutative algebra.

Let $\{X_i, \theta^i\}$ be the corresponding left, and $\{\widetilde{X}_i, \widetilde{\theta}^i\}$ the corresponding right invariant vector fields on T^*G. In term of them the symplectic Poisson structure on T^*G can be written as

$$\Lambda = \frac{1}{2}(X_i \wedge \theta^i + \widetilde{X}_i \wedge \widetilde{\theta}^i)\ . \tag{16}$$

In body coordinates $\{h, x_i\}$ for T^*G,

$$\theta^i = \frac{\partial}{\partial x_i}\ , \quad X_i = X_i^G - c_{ij}^k x_k \frac{\partial}{\partial x_j}\ , \tag{17}$$

$$\widetilde{\theta}^i = Ad^*_h(\frac{\partial}{\partial x_i})\ , \quad \widetilde{X}_i = \widetilde{X}_i^G\ , \tag{18}$$

where X_i^G and $\widetilde{X}_i^G$ are the corresponding left and right invariant vector fields on G. Therefore,

$$\Lambda = \frac{1}{2}(X_i^G \wedge \frac{\partial}{\partial x_i} + \widetilde{X}_i^G \wedge \widetilde{\theta}^i + c_{ij}^k x_k \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j})\ . \tag{19}$$

The algebra of functions $\mathcal{F}(T^*Q)$ can be reduced to the $\widetilde{X}_i$-invariant functions $\mathcal{F}'$ using the projection $\pi : T^*G \equiv G \times \mathbf{g}^* \to \mathbf{g}^*$. These functions can be identified with smooth functions on $\mathbf{g}^*$ equipped with the Konstant-Kirillov-Souriou Poisson structure

$$\Lambda' = \frac{1}{2} c_{ij}^k x_k \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} \ . \tag{20}$$

By reducing to the X_i-invariant functions, we get the opposite Poisson structure on $\mathbf{g}^*$.

A different way to get the previous structures on $\mathbf{g}^*$ is to consider the left and right invariant momentum maps

$$\mathcal{P}^L \ , \ \mathcal{P}^R : T^*G \longrightarrow \mathbf{g}^* \ , \tag{21}$$

whose components have Poisson brackets

$$\begin{aligned} \{P_i^L, P_j^L\} &= -c_{ij}^k P_k^L \ , \\ \{P_i^R, P_j^R\} &= c_{ij}^k P_k^R \ , \\ \{P_i^L, P_j^R\} &= 0 \ . \end{aligned} \tag{22}$$

The algebra $\mathcal{F}''$ of polynomials in (P_i^L, P_j^R) is a Poisson algebra. The ideals $\mathcal{I}^L$ and $\mathcal{I}^R$ generated by left and right momenta are Poisson ideals, so we can pass to the quotients $\mathcal{F}''/\mathcal{I}^L$ and $\mathcal{F}''/\mathcal{I}^R$ getting as a result the KKS Poisson structures on $\mathbf{g}^*$.

7 An example of non-commutative reduction

Let us see finally, how our algebraic reduction procedure fits into the non-commutative setting, i. e. to quantum spaces and, after passing to the semi–classical limit, gives us a "usual" Poisson reduction. Roughly speaking, we obtain quantum spaces deforming corresponding "dual" algebraic objects, as for example the commutative algebrae of a given class functions on a space (cf. Gel'fand-Najmark functor) can be deformed into noncommutative ones. This class may be arbitrarily chosen; Woronowicz [Wo] prefers to work with C^*-algebrae what is more difficult, but fruitful, while Drinfel'd works with purely algebraic version only. To make the whole thing transparent, let us consider the very classical example of Woronowicz [Wo] for the group $SU(2)$ which topologically is a three dimensional sphere. The *-algebra $\mathcal{A}$ generated by matrix elements is dense in $C(SU(2))$ and can be characterized as the "maximal" unital commutative *-algebra $\mathcal{A}$ generated by elements α, ν and satisfying $\alpha^*\alpha + \nu^*\nu = I$. Dropping the assumption about commutativity, Woronowicz proposed to consider the algebra $\mathcal{A}_q$ as the unital non-commutative *-algebra generated by α, ν satisfying $\alpha^*\alpha + \nu^*\nu = I$ and additionally the commutation relations

$$\begin{aligned} &\alpha\alpha^* - \alpha^*\alpha = (2q - q^2)\nu^*\nu \ , && \nu^*\nu - \nu\nu^* = 0 \ , \\ &\nu\alpha - \alpha\nu = q\nu\alpha \ , && \nu^*\alpha - \alpha\nu^* = q\nu^*\alpha. \end{aligned} \tag{1}$$

It is clear that for $q = 0$ we get the previous commutative algebra $\mathcal{A}$, so $\mathcal{A}_q$ is a one-parameter deformation of $\mathcal{A} = \mathcal{A}_0$. The crucial point here is that the algebrae $\mathcal{A}_q$ do not "collapse" for $1 > q \geq 0$, i.e. the basis remains the same (for example $\{\nu^{*m}\nu^n\alpha^k,\ \nu^{*m}\nu^n\alpha^{*l} : k, m, n = 0, 1, 2, ...\ \text{and}\ \ l = 1, 2, ...\}$ is a basis in $\mathcal{A}_q$ for all $1 > q \geq 0$), and we may consider the algebrae $\mathcal{A}_q$ not as different objects, but as different multiplications $\circ_q$ on the same object, namely the space $\mathcal{A}$. In this way we get the so called *formal deformations* of $\mathcal{A}$, since the $\circ_q$-product of elements $u, v \in \mathcal{A}$ reads

$$u \circ_q v = uv + \sum_{n=1}^{\infty} q^n P_n(u, v)\ . \tag{2}$$

For example, using the identification of $\mathcal{A}_q$ with $\mathcal{A}$ via the basis as above, we get

$$\alpha \circ_q \nu = \nu \circ_q \alpha - (\nu \circ_q \alpha - \alpha \circ_q \nu) = \nu\alpha - q\nu\alpha \tag{3}$$

and

$$\alpha \circ_q \alpha^* = \alpha^* \circ_q \alpha + (\alpha \circ_q \alpha^* - \alpha^* \circ_q \alpha) = (I - \nu^*\nu) + 2q\nu^*\nu - q^2\nu^*\nu\ . \tag{4}$$

Since the commutator bracket $[u, v]_q = u\circ_q v - v\circ_q u$, as for any associative algebra, is a biderivation and satisfies the Jacobi identity, we get easily that

$$\{u, v\} := P_1(u, v) - P_1(v, u) \tag{5}$$

is a Poisson bracket on the original commutative algebra $\mathcal{A}$. Hence the first non-trivial term of a commutator in the deformed algebra gives us a Poisson structure on the original manifold. ¿From the commutation relations (1) defining the Woronowicz' quantum 3-sphere we get easily the corresponding Poisson bracket which on matrix elements of $SU(2)$ has the form

$$\begin{array}{ll} \{\alpha, \bar{\alpha}\} = 2\bar{\nu}\nu\ , & \{\nu, \bar{\nu}\} = 0\ , \\ \{\nu, \alpha\} = \nu\alpha\ , & \{\bar{\nu}, \alpha\} = \bar{\nu}\alpha\ . \end{array} \tag{6}$$

Passing to real functions $\alpha = y_4 + iy_3\ ,\ \nu = y_2 + iy_1$, we get a purely imaginary bracket with the imaginary part being exactly the Poisson bracket described in (8).

Consider now the subalgebra $\mathcal{A}'_q$ in $\mathcal{A}_q$ generated by the elements $u = I - 2\nu^*\nu = \alpha^*\alpha - \nu^*\nu$, $w = 2\nu^*\alpha$ and w^*. One can easly see that $uu^* + w^*w = I$ and that $\mathcal{A}'_q$ is a formal deformation of the algebra $\mathcal{A}'_0$ generated by two complex functions u and w satisfying $|u|^2 + |w|^2 = 1$, namely the algebra of polynomials on the two-dimensional sphere S^2. Therefore, the algebra $\mathcal{A}'_q$ can be regarded as a quantum 2-sphere. Since any algebra with a commutator bracket is a Poisson algebra, our quantum sphere carries a quantum Poisson structure. One easily finds that

$$\begin{aligned} &[w, u] = (q^2 - 2q)(1 - u)w\ , \\ &[w^*, u] = -(q^2 - 2q)(1 - u)w^*\ , \\ &[w, w^*] = -2(q^2 - 2q)(1 - u) + (4q - 6q^2 + 4q^3 - q^4)(1 - u)^2\ . \end{aligned} \tag{7}$$

Passing to the semi-classical limit and real functions, we get the functions u, $v = Re(w)$ and $z = -Im(w)$ with a purely imaginary bracket. The brackets of the basic variables are found to be

$$
\begin{aligned}
\{v,u\} &= 2(1-u)z \ , \\
\{u,z\} &= 2(1-u)v \ , \\
\{z,v\} &= 2(1-u)u
\end{aligned}
\tag{8}
$$

which is exactly the structure obtained in (12). Therefore, we get the same reduction of the Lie-Poisson structure using the reduction at the quantum (non-commutative) level and then passing to the semi-classical limit.

Let us consider now the dynamics on $\mathcal{A}_q$ described by the Hamiltonian

$$
H = \frac{1}{2}u = \frac{1}{2}(I - 2\nu^*\nu) = \frac{1}{2}(\alpha^*\alpha - \nu^*\nu) \ . \tag{9}
$$

We have

$$
\begin{aligned}
&[H,\nu] = 0 \ , \\
&[H,\nu^*] = 0 \ , \\
&[H,\alpha] = (q^2 - 2q)\nu^*\nu\alpha \ , \\
&[H,\alpha^*] = -(q^2 - 2q)\nu^*\nu\alpha^* \ ,
\end{aligned}
\tag{10}
$$

so that the dynamics e^{itad_H} is of the form

$$
\begin{aligned}
&\nu(t) = \nu \ , \\
&\nu^*(t) = \nu^* \ , \\
&\alpha(t) = e^{it(q^2-2q)\nu^*\nu}\alpha \ , \\
&\alpha^*(t) = e^{-it(q^2-2q)\nu^*\nu}\alpha^* \ .
\end{aligned}
\tag{11}
$$

Passing to the classical limit, we get the Hamiltonian

$$
H = \frac{1}{2}(y_4^2 + y_3^2 - y_2^2 - y_1^2) \ , \tag{12}
$$

which, with respect to the Poisson structure on S^3 obtained by reducing the structure (8), corresponds to the vector field

$$
\Gamma = 2(y_1^2 + y_2^2)(y_4\frac{\partial}{\partial y_3} - y_3\frac{\partial}{\partial y_4}) \ . \tag{13}
$$

The corresponding trajectories are given by

$$
\begin{aligned}
y_1(t) &= y_1(0) \ , \\
y_2(t) &= y_2(0) \ , \\
y_3(t) &= cos(2t(y_1^2 + y_2^2))y_3(0) + sin(2t(y_1^2 + y_2^2))y_4(0) \ , \\
y_4(t) &= -sin(2t(y_1^2 + y_2^2))y_3(0) + cos(2t(y_1^2 + y_2^2))y_4(0) \ .
\end{aligned}
\tag{14}
$$

These trajectories are really the limits of (11) when we take the limit $q^2/q \to 0$, $q/q \to 1$, since then $\nu^*\nu = y_1^2 + y_2^2$ and $\alpha = y_4 + iy_3$. It is easy to see that Γ commutes with the vector field X given by (10). Therefore we can reduce the system by the distribution generated by X.

On the other hand, the Hamiltonian (9) is an element of the reduced algebra $\mathcal{A}'_q$ and we get a reduced dynamics on $\mathcal{A}'_q$ given by

$$\begin{aligned} [H, w] &= -\frac{1}{2}(q^2 - 2q)(1-u)w \ , \\ [H, w^*] &= \frac{1}{2}(q^2 - 2q)(1-u)w^* \ , \end{aligned} \tag{15}$$

The corresponding solutions for the endomorphism e^{itad_H} are

$$\begin{aligned} w(t) &= e^{-it\frac{1}{2}(q^2-2q)(1-u)}w \ , \\ w^*(t) &= e^{it\frac{1}{2}(q^2-2q)(1-u)}w^* \ . \end{aligned} \tag{16}$$

Passing to the semiclassical limit we get the corresponding vector field on S^2,

$$\tilde{\Gamma} = (1-u)(z\frac{\partial}{\partial v} - v\frac{\partial}{\partial z}) \ , \tag{17}$$

which is really the reduction of Γ, since

$$\begin{aligned} &\Gamma \cdot u = 0 \ , \\ &\Gamma \cdot v = 2(y_1^2 + y_2^2)(y_4 y_1 - y_3 y_2) = (1-u)z \ , \\ &\Gamma \cdot z = -2(y_1^2 + y_2^2)(y_3 y_1 + y_4 y_2) = -(1-u)v \ . \end{aligned} \tag{18}$$

By using the stereographic projection : $S^2 \to \mathbf{R}^2$, $(x, y) = \frac{1}{1-u}(v, z)$, we obtain the following vector field on $\mathbf{R}^2$:

$$\Gamma(x, y) = \frac{2}{x^2 + y^2 + 1}(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}) \ . \tag{19}$$

8 Conclusions

The guiding idea of this paper is that completely integrable systems should arise from reduction of simple ones. In particular, many of them arise from reduction of free systems.

Knowing that these systems admit alternative Lagrangian or Hamiltonian descriptions, we have considered the behaviour of these additional structures with respect to the generalized reduction procedure. We have not addressed the problem of showing how the recursion operator, known to exist for many completely integrable systems, arises in the present approach; however this is not difficult to analyze.

Taking advantage of the duality between a manifold M and the ring $\mathcal{F}(M)$ of functions defined on it, we have cast our reduction procedure in an algebraic framework. As an application we have given a simple example of reduction in non-commutative geometry. We may benefit of this approach to deal with the reduction of quantum systems in the operatorial approach. These aspects will be addressed in future papers.

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