Density Operators as an Arena for Differential Geometry

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What I am going to describe may be called an interplay between concepts of differential geometry and the superposition principle of quantum physics. In particular it concerns a metrical distance introduced by Bures [14] as a non-commutative version of a construction of Kakutani [24] on the one hand, and on the other hand the purifications of mixed states in physically larger systems, including the problem of geometric phases associated with a distinguished class of such extensions. The Bures distance and the general transition probability [15], [27] are discussed in [10], [11], [30], and further papers.

For the sake of clarity, and to avoid technicalities, I will be concerned with finite dimensional objects. Let $\mathcal{H}$ denote an Hilbert space with complex dimension n. The set of density operators defined on it is

$$\Omega = \{\varrho \geq 0, \quad \operatorname{trace} \varrho = 1\} \tag{1}$$

which in turn containes the pure states as its extremal set. This extremal set is isomorphic to the complex projective space $\mathbf{P}(\mathcal{H})$, the points of which are conveniently described by the projection operators of rank one:

$$\mathbf{P}(\mathcal{H}) = \{P \in \Omega : P^2 = P\} \tag{2}$$

There are natural mappings

$$\psi \mapsto \frac{\psi}{\sqrt{< \psi . \psi >}} \mapsto P_\psi := \frac{|\psi >< \psi|}{< \psi, \psi >}, \quad \psi \not\equiv 0 \tag{3}$$

from $(\mathcal{H}) - \{0\}$ onto the unit sphere of the Hilbert space

$$\mathbf{S}(\mathcal{H}) = \{\psi \in \mathcal{H} :< \psi . \psi >= 1\} \tag{4}$$

and from that unit sphere onto $\mathbf{P}(\mathcal{H})$.
Further, referring to an arbitrarily choosen orthogonal base, the sets (1) and (2) may be considered as sets of hermitian matrices. This gives embeddings

$$\mathbf{P}(\mathcal{H}) \subset \Omega \subset \mathcal{R}^{n^2} \tag{5}$$

Pure States

This section is a short review of some selected geometrical properties of $\mathcal{H}$, $\mathbf{S}\mathcal{H}$, and $\mathbf{P}\mathcal{H}$. Their study goes back to the work on adiabatic approximations of Born, Oppenheimer, and Fock [13] in the late twenties (see also [20], appendix). In the last decade the interest on this topic has been triggered by the ideas of Berry [12], of Simon [26], and many others, in particular [37], [2], [8], [6]. See also [1].
The following consideration should prepare the study of similar problems for density operators. Let

$$t \mapsto \psi_t \in \mathbf{S}(\mathcal{H}), \quad 0 \leq t \leq 1 \tag{6}$$

be a path on the unit sphere of $\mathcal{H}$. Its length is given by

$$\int_0^1 \sqrt{< \dot{\psi}, \dot{\psi} >}\, dt \tag{7}$$

where here and later the dot notation indicates differentiation. Hence $\dot{\psi}$ is an element of the appropriate tangent space at ψ.
The length (7) does *not* remain invariant under phase changes

$$\psi_t \mapsto \epsilon(t)\psi_t, \quad |\epsilon| = 1 \tag{8}$$

which can be understood as a $U(1)$ gauge transformation of the path (6). This opens the possibility of a canonical gauge fixing by demanding a minimal length of the path: The gauge is fixed if a replacement (8) never can result in a shorter path.
What can be achieved by the desired gauge fixing is easily seen by the help of the inequality

$$< \dot{\psi}, \dot{\psi} > \geq < \dot{\psi}, \dot{\psi} > - < \psi, \dot{\psi} >< \dot{\psi}, \psi > \geq 0 \tag{9}$$

showing that minimality of the length is reached if and only if

$$< \psi, \dot{\psi} > = 0, \tag{10}$$

i.e. if the Berry - Simon parallel transport condition is fulfilled. The gauge is fixed by (10) up to a parameter independent gauge (a global gauge). Therefore, if (10) is fulfilled for the path (6)

$$|\psi_0 >< \psi_1| \quad \text{and} \quad A \mapsto < \psi_0, A\,\psi_1 > \tag{11}$$

are gauge invariant quantities. They depend only on the image in $\mathbf{P}\mathcal{H}$ of the curve (6) under the map (3). If this image is a closed curve then $\psi_1 = \epsilon\psi_0$, and ϵ is Berry's geometric phase factor.

It is sometimes necessary not to restrict oneself to the unit sphere and to allow more general gauges

$$\psi_t \;\mapsto\; \lambda(t)\psi_t, \quad \lambda \neq 0, \text{ complex} \tag{12}$$

then the inequaltiy (9) is usefully rewritten as

$$\frac{< \dot\psi, \dot\psi >}{< \psi, \psi >} \geq \frac{< \dot\psi, \dot\psi >}{< \psi, \psi >} - \frac{< \psi, \dot\psi >< \dot\psi, \psi >}{< \psi, \psi >^2} = (\frac{ds}{dt})^2_{SF} \tag{13}$$

where the right hand side is the Study Fubini metric which is invariant with respect of (12). Hence we can consider this line element as a lift of a metric of $\mathbf{P}(\mathcal{H})$. Indeed, because of (3) a short calculation yields

$$(\frac{ds}{dt})^2_{SF} = \frac{1}{2}\,\mathrm{tr}\,(\dot P_\psi)^2 \tag{14}$$

Further: In leaving the unit sphere one has to replace the crucial condition (10) by

$$\frac{1}{2}\,\frac{< \psi, \dot\psi > - < \dot\psi, \psi >}{< \psi, \psi >} \;=\; 0, \tag{15}$$

where the left hand side is a $U(1)$-connection form which remains unchanged under a rescaling of the vector length. (Remark that its curvature defines the Kähler Hodge 2-form of $\mathbf{P}(\mathcal{H})$ which also defines the Chern character of the bundle $\mathbf{S}(\mathcal{H})$ over $\mathbf{P}(\mathcal{H})$.) A solution of (15) is given by

$$\dot\psi = G\psi, \qquad G = \dot P + (\frac{d}{dt}\ln\sqrt{< \psi, \psi >})\,P, \tag{16}$$

so that $G = \dot P$ at the unit sphere. Because G is hermitian (15) is fulfilled.

Real Planes and Circles.

For later use I need some elementary geometric facts. A real 2-dimensional subspace of a Hilbert space is called a real plane. Every such plane can be given as

$$\{\psi \,:\, \psi = \lambda_1 \psi_1 + \lambda_2 \psi_2, \quad \lambda_j \quad \text{reell}\,\} \tag{17}$$

Every pair of vectors that generates the plane as in (17) is called a base of the plane. The intersection of a real plane with the unit sphere $\mathbf{S}(\mathcal{H})$ is a large circle. Its length in the Hilbert space metric equals 2π. Every geodesic closes on the unit sphere. Every closed geodesic on the sphere is a large circle and vice versa. Let us now assume that in (17)

$$< \psi_1, \psi_1 >=< \psi_2, \psi_2 >= 1, \quad < \psi_1, \psi_2 >= a + ib \tag{18}$$

with reals a, b. Then the intersection of the plane (17) with the sphere is parametrized by an angle α as

$$\lambda_1 = \cos\alpha - \frac{a}{\sqrt{1-a^2}} \sin\alpha, \qquad \lambda_2 = \frac{\sin\alpha}{\sqrt{1-a^2}} \tag{19}$$

This implies $< \dot{\psi}, \dot{\psi} >= 1$, and the differential $d\alpha$ is the line element on the circle. The (oriented) arc length $\alpha_{1,2}$ between ψ_1 and ψ_2 is given by $\cos\alpha_{1,2} = a$.
If one varies the relative phases of the two vectors, the arc length becomes minimal iff $b = 0$. Then the transition amplitude becomes real, and *every* curve on the plane satisfies the parallel transport condition (15).
On a general circle it follows from (18) and (19)

$$< \dot{\psi}, \psi >= -i \frac{b}{\sqrt{1-a^2}}, \tag{20}$$

and the invariant (11) is given by $\epsilon \, |\psi_2 >< \psi_1|$ with $\epsilon = \exp < \dot{\psi}, \psi > \alpha_{1,2}$. Further, introducing (20) into the Fubini Study metric (13) one gets along the circle the line element

$$ds = \frac{1}{2} \sqrt{\frac{1 - (a^2 + b^2)}{1 - a^2}} \, d\check{\alpha}, \qquad \check{\alpha} = 2\alpha \tag{21}$$

The manipulation with the factor 2 has been done because after projecting down the Hilbert space circle according to (3) into $\mathbf{P}(\mathcal{H})$, it appears as a double covering of the resulting base curve. This is a reminiscent of the Hopf bifurcation: ψ

changes it sign if the circle is rotated by π. Therefore, the factor before $d\breve{\alpha}$ becomes the radius of a circle if $\mathbf{P}(\mathcal{H})$ is embedded into an Euclidean space (5) with a suitable choosen Euclidean metric. This radius becomes minimal, namely $\frac{1}{2}$, iff $b = 0$. On a circle with this condition the equality sign holds in (13). (14) then shows (3) to be locally isometric. Thus a geodesic on the unit sphere gives a geodesic on the projective space of pure states iff $b = 0$ is fulfillled. As a further consequence, every geodesic closes in $\mathbf{P}(\mathcal{H})$ with length π.
It has been shown above that a unit circle in a plane with $b = 0$ contains with ψ also $\dot{\psi}$, and both vectors form an orthoframe. The arc connecting both vectors is of arc length $\pi/2$. It has been already remarked that on a real plane with $b = 0$ *every* curve fulfills the parallell transport condition (15). Let us call such a plane *horizontal.*

Density Operators

While in the pure state case the starting point is a Hilbert space $\mathcal{H}$ which is projected down to $\mathbf{P}(\mathcal{H})$ (with the exception of its zero vector), the situation with the density operators is somewhat reversed. We have to start with the space of density operators, Ω, and to look for a covering by an Hilbert space, say $\mathcal{H}^{\mathrm{ext}}$, which carries a representation of the operators acting on the Hilbert space $\mathcal{H}$ which defines Ω. One may use any "large enough" representation, the results are the same. Physically, the construction of $\mathcal{H}^{\mathrm{ext}}$ is an embedding of the original physical system into a larger one so that it becomes a subsystem. (This procedure is also heavily used in the so-called thermo field theory.) The vectors of the extended system, i.e. the pure states of the extended system, can be reduced to the subsystem. The result of the reduction is a density operator (a mixed state). The extension should be large enough, so that every density operator should be reachable by reducing a vector state of $\mathcal{H}^{\mathrm{ext}}$.
If a density operator ϱ can be gained by reducing a vector W of $\mathcal{H}^{\mathrm{ext}}$ then W is called a *purification* of ϱ. While a reduction gives a unique result, there is some arbitrariness in the procedure of purification: There are many vectors in the extended systems which purify a given density operator.

A good choice to realize $\mathcal{H}^{\mathrm{ext}}$ is by the set of all those operators W acting on $\mathcal{H}$ for

which WW^* (and hence W^*W) has finite trace, i.e. we consider Hilbert - Schmidt operators. Of course, the trace condition is trivially fulfilled for finite dimensional Hilbert spaces with which we are dealing with.
The scalar product of $\mathcal{H}^{\text{ext}}$ is given by

$$(W_1, W_2) = \text{tr } W_1^* W_2 = \text{tr } W_2 W_1^* \tag{22}$$

The mapping that replaces (3) is

$$W \mapsto WW^* \mapsto \varrho_w := \frac{WW^*}{\text{tr } WW^*}, \quad W \not\equiv 0 \tag{23}$$

It goes from $(\mathcal{H}^{\text{ext}}) - \{0\}$ onto the unit sphere of the extended Hilbert space, $\mathbf{S}(\mathcal{H})^{\text{ext}}$, and from that unit sphere onto Ω.
If $\varrho = WW^*$ then W is called a *standard purification* of ϱ. Let $\mu_1, \mu_2, \ldots$ be the set of non-zero eigenvectors, and $\psi_1, \psi_2, \ldots$ an orthoframe of corresponding eigenvectors of ϱ. If now W is a standard purification of ϱ, there exists a unique orthoframe, $\varphi_1, \varphi_2, \ldots$, of the same length such that

$$W = \sum \mu_j^{1/2} \, |\psi_j >< \varphi_j| \tag{24}$$

Clearly, every W, given by an expression (24), is a purification of ϱ. Given ϱ the fibre (or leaf) of all purifications sitting on the unit sphere of the extended Hilbert space is the Stiefel manifold of orthoframes of $\mathcal{H}$ of rank k, $k = \text{rank}(\varrho)$. The fibre admits the unitaries of $\mathcal{H}$ as right multipliers,

$$W \mapsto WU, \qquad U \,\text{unitary} \tag{25}$$

If the rank, k, equals $n = \dim \mathcal{H}$, then this action is free. Otherwise there is a stable subgroup $\mathcal{U}(n-k)$.
If A is an operator acting on $\mathcal{H}$ then one defines

$$A \mapsto AW := L_A W \quad \text{and} \quad A \mapsto WA := R_A W. \tag{26}$$

The first expression defines the representation of the operators of $\mathcal{H}$ as operators acting on the Hilbert space $\mathcal{H}^{\text{ext}}$. The second expression defines its commutant. Hence the *gauge group* given by (25) is build from the unitaries of the commutant. After these preliminaries it is possible to mimic the pure state case in the domain of density operators.

Starting with a curve of density operators and one of its purifications,

$$t \mapsto \varrho_t, \quad t \mapsto W_t, \qquad \varrho_t = W_t W_t^*, \tag{27}$$

the length of the latter in the extended Hilbert space is by no means invariant against gauge transformations (25). To get a gauge fixing, and at the same time a length to the original curve of density operators, we consider the variational problem [30], [33]

$$\text{length}(\, t \mapsto \varrho_t \,) = \inf \int \sqrt{(\dot W, \dot W)}\, dt \tag{28}$$

where the infimum is running over *all* purifications, or, what is the same, over all curves $t \mapsto W_t U_t$ with unitaries U_t. The Euler equations of (28) read

$$\dot W^* \, W \; = \; W^* \, \dot W, \tag{29}$$

a set of equations which I call the *(extended) parallel condition*, see [29]. It is a family of Berry conditions which is stable under the commutant of the representation $A \mapsto L_A$. Indeed, (29) is equivalent with

$$(R_A \dot W, R_A W) = (R_A W, R_A \dot W) \quad \text{for all} \quad A \tag{30}$$

If W_1 is the endpoint and W_0 the beginning of a parallel lift fullfilling (29), $W_0 W_1^*$ is important for the definition of the geometric phase attached to curves of density operators. It generalizes the invariant (11). However, in this paper I shall not go into this branch of the game, which can be successfully considered even in von Neumann and C*-algebras [4], [5].

The parallel condition can be solved by an ansatz [17], [31]

$$\dot W = G W, \qquad G = G^* \tag{31}$$

After inserting this into the differentiated relation $\varrho = W W^*$ one gets G as the solution of a Bloch like equation

$$\dot\varrho = G\varrho + \varrho G \qquad \text{or} \qquad d\varrho = \varrho \mathbf{G} + \mathbf{G}\varrho \tag{32}$$

The last relation defines a matrix-valued 1-form, $\mathbf{G}$, the restriction of which to a given curve of density operators yields G. (31), (32) generalize (16). If in (32) ϱ and $\dot\varrho$ commute, then G is the logarithmic derivative of $\sqrt{\varrho}$. If no eigenvalue of ϱ equals zero, (32) has exactly one solution.

It is easy to insert $\dot{W} = GW$ into the expression for the Hilbert space metric, and one gets, using (32),

$$(\dot{W}, \dot{W}) = (GW, GW) = \mathrm{tr}\, G^2 \varrho = \frac{1}{2} \mathrm{tr}\, G\, \dot{\varrho} \tag{33}$$

Now the minimal lenght (28) can be calculated as well on the unit sphere of the extended Hilbert space as on Ω. One can get rid of the norm condition and extend these expressions to $\mathcal{H}^{\mathrm{ext}} - \{0\}$ and to the cone of all positive (trace class) operators to obtain the analogue to (14), which may be called the projective Bures metric:

$$\frac{(W, G^2 W)}{(W, W)} - \frac{(W, GW)^2}{(W, W)^2} = \frac{1}{2} \frac{\mathrm{tr}\, G\, \dot{\varrho}}{\mathrm{tr}\ \varrho} - \frac{1}{4} (\frac{\mathrm{tr}\ \dot{\varrho}}{\mathrm{tr}\ \varrho})^2 \tag{34}$$

This metric is scale invariant and gauge invariant.

The next goal is to ask for the appropriate connection form in the extended Hilbert space. To this end let us use the differential 1-form $\mathbf{G}$ which is a gauge invariant. Hence $\mathbf{G}W$ behaves under gauge transformations (25) like W. If one therefore defines the 1-form $\mathbf{A}$ by

$$\mathrm{d}W - W\mathbf{A} = \mathbf{G}W \tag{35}$$

the result is an operator valued connection form:

$$\mathbf{A} \mapsto \tilde{\mathbf{A}} =: U^* \mathbf{A} U + U^* \mathrm{d}U \tag{36}$$

Reinserting (35) into (32) one finds compatibility iff

$$\mathbf{A} + \mathbf{A}^* = 0 \tag{37}$$

Finally, calculating the left hend side of the following equation by the help of the relations above, one easily finds

$$W^* \mathrm{d}W - (\mathrm{d}W^*) W = W^* W \cdot \mathbf{A} + \mathbf{A} \cdot W^* W \tag{38}$$

Originally, [32], I guessed this as the relevant definition of $\mathbf{A}$.
Unfortunately it is difficult to obtain explicite expressions for $\mathbf{G}$ and $\mathbf{A}$ with the exception of density operators of rank two, where various results have been obtained: [25], [16], [22], [23], [36]. See also [18], [19] and a forthcoming paper of J. Dittmann.

Ω- Horizontality, Geodesics

To get some insight into the geometric meaning of the Bures metric, we play again the game with real planes and circles, but this time within the extendet Hilbert space. In accordance with (17) we consider real planes

$$\{W \, : \, W = \lambda_1 W_1 + \lambda_2 W_2, \quad \lambda_j \quad \text{reell}, \quad W_1, W_2 \in \mathcal{H}^{\text{ext}}\} \tag{39}$$

Let us assume, the plane contains an invertible element. Then one may choose as a base two elements, W_1, W_2, which are both invertible and of Hilbert norm one. The determinant of $W_1 + \lambda W_2$ can be zero for at most n real values of λ. From (39) one sees $\lambda = \lambda_2 \lambda_1^{-1}$, and with W always $-W$ sits on the circle. Hence the number k of points on the unit circle of the plane which are not invertible is even and restricted by $k \leq 2n$. After projecting down the circle to Ω according to (23), these points and no others become boundary points of Ω. If there is one turn of the W-circle the projection on Ω runs through two turns.
Parametrizing the circle as indicated in (19), we may call its non-invertible W's and the corresponding boundary points

$$W(\alpha_1), W(\alpha_2), \dots, W(\alpha_k) \quad \text{and} \quad \varrho(\alpha_1), \varrho(\alpha_2), \dots, \varrho(\alpha_k) \tag{40}$$

ordered with increasing α-parameters. (Every ϱ appears twice.) If $k = 0$ the projected curve remains in the interior of Ω. If $k = 2$, the ϱ-curve starts at a boundary point and return to it. Generally, the projected curve may be started at $\varrho(\alpha_1)$, to go through inner points to $\varrho(\alpha_2)$. It then starts from this point to go to $\varrho(\alpha_3)$ through the interior of Ω, and so on. At last it goes from the projection $\varrho(\alpha_k)$ of $W(\alpha_k)$ to that of $W(\alpha_1)$, i.e. to $\varrho(\alpha_1)$. This shows that the projected curve generally goes through the interior of the set of density operators, and if it hits a boundary point, it will be reflected to start again into the interior region. We may image this by a polygon with $k/2$ vertices which are sitting on the boundary of Ω.
A plane (39) is called Ω-*horizontal* iff

$$(W_1)^* W_2 = (W_2)^* W_1 \tag{41}$$

is valid for one of its bases, or, what is the same, is valid for every base of the plane. It is also evident that a plane is Ω-horizontal iff every curve on it fulfills the extended parallelity condition (29).

A circle of an Ω-horizontal plane consists of peaces, which, after projecting them down according to (23) to Ω, become geodesics with respect to the metric of Bures. Returning to the "generic" case where there are inverible elements within in plane, let us choose a base of the plane consisting of normed elements W_1, W_2 which are sufficiently near neighbours. Then $W_1^* W_2$ is not only hermitian but also positve definite. In moving these two points continuously, the positive definiteness can be lost only if on eigenvalue becomes zero, and, consequently, the determinant of one of the two points becomes zero. This means at least one of the ϱ's goes into the boundary of Ω. The argument shows

$$W^*(\gamma) W(\beta) \,>\, 0 \qquad \beta, \gamma \in \{\alpha \,:\, \alpha_j < \alpha < \alpha_{j+1}\} \tag{42}$$

where $\alpha_{k+1} = \alpha_1$ is to be understood.
k is twice the number of real zeros of the determinant of $W_2 + \lambda W_1$ if λ varies. Multiplying by the non-singular W_1^* and exploiting Ω-horizontality one gets a characteristic equation of the form $\det(A + \lambda B) = 0$ with positive definite matrices A and B. Such an equation allows only for real, negative solutions. If a root is degenerated the corresponding density operator posesses as many zero eigenvalues as the degree of the relevant root. This means

$$\sum \text{number of zeros of } \varrho(\alpha_j) \,=\, n \,=\, \dim(\mathcal{H}), \quad 0 \le \alpha_j < \pi \tag{43}$$

¿From this and other observations I wonder (as a *working hypothesis*) wether there is a Riemann manifold of dimension $n^2 - 1$ which allows for a decomposition into n cells. The interior of every cell should be isometric to the interior of Ω, equipped with the metric of Bures. Let me call that hypothetical Riemann manifold Ω^{fancy}. A more stringent hypothesis for Ω^{fancy} is by demanding additional: A density operator ϱ is covered by Ω^{fancy} exactly $\text{rank}(\varrho)$ times. The hypothesis is true if $n = 2$ where the wanted manifold is a 3-sphere with radius 1/2.

Ω-horizontality obviously imlies horizontality as defined in the case of pure states. Consequently, the intersection of such a plan with the unit sphere $\mathbf{S}(\mathcal{H})$ is a large circle of this sphere, and at the same time one of its geodesics. If $\alpha \mapsto W(\alpha)$ parametrizes this circle as indicated in (19) then $W, \dot{W}$ is an orthonormal base of the plane for every $W = W(\alpha)$, and it is $\pm\dot{W} = W(\alpha \mp \pi/2)$.

There is an intersting observation. Let W_1, W_2 denote a base of an Ω-horizontal plane (39), and Y an invertible operator. Then

$$\tilde{W}_1 = \frac{Y^* W_1}{(W_1, YY^* W_1)}, \qquad \tilde{W}_2 = \frac{Y^{-1} W_2}{(W_2, (YY^*)^{-1} W_2)} \tag{44}$$

generate an Ω-horizontal plane. Indeed, (41) remains valid after such a replacement. Inserting (44) into (22) alllows for the corollary:
If W_1, W_2 is an orthonormal base, so it is $\tilde{W}_1, \tilde{W}_2$.
Hence (44) induces, if applied to an orthonormal base, an isometric map from one Ω-horizontal plane onto another one. Remark, however, that the positions of the singular points (40) are changed by that transformation, and the same is with the regions (42).
As a next step we choose an element $W \in \mathbf{S}(\mathcal{H}^{\text{ext}})$ which is not singular and call $\mathbf{h}_p(W)$ the union of all those Ω-horzontal planes which contain W. The intersection of two different planes contributing to $\mathbf{h}_p(W)$ consists of W only. (Otherwise they would be identical.) $\mathbf{h}_p(W)$ is an Euclidean submanifold of $\mathcal{H}^{\text{ext}}$.
Proof: In order that X belongs to $\mathbf{h}_p(W)$ the expression $H = W^* X$ has to be hermitian. Thus $\mathbf{h}_p(W)$ is parametrized by $H \mapsto X = (W^*)^{-1} H$.
Two such spaces, $\mathbf{h}_p(W)$ and $\mathbf{h}_p(\tilde{W})$, are diffeomerphic, and, moreover, their constitutent planes are isometric. To see this one sets $W_1 = W$, $\tilde{W}_1 = \tilde{W}$ in (44), and for W_2, $\tilde{W}_2$ their orthonormal partners.
By (23) $\mathbf{h}_p(W)$ is mapped onto the cone of positive (trace class) operators

$$X \in \mathbf{h}_p(W) \;\mapsto\; XX^* = H(WW^*)^{-1} H \tag{45}$$

while H runs through the hermitian operators.
All this remains essentially valid if going from $\mathbf{h}_p(W)$ to $\mathbf{h}_c(W)$ which will be defined as the union of all horizontal unit circles containing W.
$\mathbf{h}_c(W)$ contains a cell $\mathbf{h}_c^0(W)$ diffeomorphic to Ω. The diffeomorphism can be realized by (23). The reason is, W belongs to exactly one segment (42) of every circle constituing the manifold $\mathbf{h}_c(W)$, and the union of this segments is called $\mathbf{h}_c^0(W)$.
Along the intersection of $\mathbf{h}_c^0(W)$ with a horizontal circle the diffeomorphism is an isometry from the induced Hilbert space geometry to the metric of Bures. Its

projections to Ω give all the Bures geodesics of Ω which cross ϱ_w. Transverse to the mentioned directions all this is not true.

How far away is now Ω^{fancy} ? Let us equip $\mathbf{h}_c(W)$ with the lifted Bures metric (33) or (34). Then, already for $n = 2$, there are points on $\mathbf{h}_c(W)$ where this metric degenerates.

Let $\varrho \in \Omega$. Because $\mathbf{h}_c(W)$ is mapped continuously onto Ω, the inverse mapping maps ϱ onto a closed subset of $\mathbf{h}_c(W)$ with (eventually) several connected components. Ω^{fancy} should be obtained after contracting these connected components to points for every ϱ.

But is the result a manifold? As already said, presently this is only known in the most simple case $\dim \mathcal{H} = 2$. It would be certainly very interesting to get complete control for the dimensions 3 and 4. And this seems to be not out of range.

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