LIMIT THEOREM FOR THE LOCAL TIME OF BOND-TRUE SELF-AVOIDING WALK ON $\mathbb{Z}$

Bálint Tóth

Mathematical Institute of the
Hungarian Academy of Sciences

Abstract. The bond-true self-avoiding walk is a nearest neighbour random walk on $\mathbb{Z}$, for which the probability of jumping along a bond of the lattice is proportional to $\exp(-g \cdot \text{number of previous jumps along that bond})$. We prove a limit theorem for the distribution of the local time process of this walk. A consequence of the main theorem is a limit law for $n^{-3/2}T_n$ where T_n is the first hitting time of the lattice site at distance n from the origin.

1. Introduction and Result

In the present paper we consider a 'bond-true' self-avoiding walk (BTSAW) X_i on the one-dimensional integer lattice $\mathbb{Z}$, defined as follows: the walk starts from the origin of the lattice and at time $i+1$ it jumps to one of the two neighbouring sites of X_i, so that the probability of jumping along a bond of the lattice is proportional to

$$\exp(-g \cdot \text{number of previous jumps along that bond}).$$

More formally, for a nearest neighbour walk $\underline{x}_0^i = (x_0, x_1, \dots, x_i)$ and a lattice site $y \in \mathbb{Z}$ we define

$$r(y|\underline{x}_0^i) = \#\{\, 0 \le j < i \,:\, x_j = y-1,\, x_{j+1} = y \,\}, \tag{1.1}$$

$$l(y|\underline{x}_0^i) = \#\{\, 0 \le j < i \,:\, x_j = y,\, x_{j+1} = y-1 \,\}, \tag{1.2}$$

$$v(y|\underline{x}_0^i) = r(y|\underline{x}_0^i) + l(y|\underline{x}_0^i). \tag{1.3}$$

Denoting $\mathrm{e}^{-g} = \lambda \in (0,1)$, the walk is governed by the law:

$$\mathbf{P}\big(\, X_{i+1} = x_i + 1 \,\big|\, \underline{X}_0^i = \underline{x}_0^i \,\big) = \frac{\lambda^{v(x_i+1|\underline{x}_0^i)}}{\lambda^{v(x_i+1|\underline{x}_0^i)} + \lambda^{v(x_i|\underline{x}_0^i)}}, \tag{1.4}$$

$$\mathbf{P}\big(\, X_{i+1} = x_i - 1 \,\big|\, \underline{X}_0^i = \underline{x}_0^i \,\big) = \frac{\lambda^{v(x_i|\underline{x}_0^i)}}{\lambda^{v(x_i+1|\underline{x}_0^i)} + \lambda^{v(x_i|\underline{x}_0^i)}}. \tag{1.5}$$

Work supported by the Hungarian National Foundation for Scientific Research, grant No. 1902
Submitted for publication to *The Annals of Probability*, 17 May 1993

The only difference from the 'orthodox' site-true self-avoiding walk (STSAW), (see [APP]), is that here we count the local time spent on edges, while the jump probabilities of STSAW are determined by the local time spent on sites. We expect that the physical phenomena should be very similar in the two cases. Based on a renormalization group argument it has been conjectured in [APP] that the upper critical dimension of STSAW is $d_c = 2$. That is: in more than two dimensions the STSAW behaves diffusively, like an ordinary random walk, with logarithmic corrections in two dimensions. Computer simulations of the same authors seem to agree with this conjecture. It is natural to expect over-diffusive behaviour below the critical dimension, i.e. in $d = 1$. Unfortunately we do not know about any reliable computer simulations of the one dimensional problem. We do not know about the existence of any rigorous results concerning true self avoiding walks either. However, let us mention here two related problems: (1) the self-avoiding walk problem (which is, strictly speaking, not a random walk in the sense of stochastic process evolving in time), has a long history and there are deep and technically difficult results concerning it (see e.g. [BS], [HS], [Bo] and references cited in these papers), and (2) on the other end of the spectrum are the problems related to self-attracting rather than self-repelling walks, a typical example being the so-called reinforced random walk (see e.g. [P], or for a one-dimensional problem closer to our present paper [D]).

In this paper we prove a limit theorem for the local time process of the BTSAW considered. Denote by T_n the first hitting time of the site $n \in \mathbb{Z} \cap (0, \infty)$:

$$T_n = \min\{\, i \geq 0 \, : \, X_i = n \,\}. \tag{1.6}$$

The (bond) local time process considered is the following:

$$S_n(t) = l(n - t|\underline{X}_0^{T_n}), \quad t = 0, 1, 2, \dots \tag{1.7}$$

Looking at the formal definitions only, in principle, some of the $S_n(t)$, $t \geq 0$ or the hitting time T_n might be infinite, but we shall see soon that, with probability one, this is *not* the case. Clearly

$$v(n - t|\underline{X}_0^{T_n}) = \begin{cases} 2S_n(t) + 1 & \text{for} \quad 0 \leq t < n \\ 2S_n(t) & \text{for} \quad n \leq t. \end{cases} \tag{1.8}$$

Let

$$\omega_n = \inf\{t > n : S_n(t) = 0\}. \tag{1.9}$$

¿From the definition of $S_n(\cdot)$ follows that $S_n(t) = 0$ for $t \geq \omega_n$ and

$$T_n = \sum_{t \geq 0} v(n - t|\underline{X}_0^{T_n}) = 2\sum_{t=0}^{\omega_n} S_n(t) + n. \tag{1.10}$$

Proposition 1.

$$\omega_n < \infty \tag{1.11}$$

$$T_n < \infty \tag{1.12}$$

almost surely.

Let W_s, $s \in \mathbb{R}_+$ be a standard one-dimensional Wiener process and ω the following stopping time:

$$\omega = \inf\{\, s > 1 \, : \, W_s = 0 \,\}. \tag{1.13}$$

Our main result is the following theorem

Theorem 1.

$$\left(\frac{\omega_n}{n}, \frac{S_n([ns])}{\sigma\sqrt{n}} : 0 \le s \le \frac{\omega_n}{n}\right) \Rightarrow (\omega, |W_s| : 0 \le s \le \omega) \tag{1.14}$$

in $\mathbb{R}_+ \times D[0,\infty)$*, where*

$$\sigma^2 = \frac{\sum_{x\in\mathbb{Z}} x^2 \lambda^{x^2}}{\sum_{x\in\mathbb{Z}} \lambda^{x^2}}. \tag{1.15}$$

Straightforward consequence of Proposition 1 and Theorem 1 is the recurrence of the random walk considered:

Corollary 1. *The BTSAW visits infinitely often every lattice site, almost surely.*

¿From (1.10) and (1.14) directly follows the limit theorem for the first hitting times:

Corollary 2.

$$\frac{T_n}{2\sigma n^{3/2}} \Rightarrow \int_0^\omega |W_s|\, ds. \tag{1.16}$$

This last limit law shows that the BTSAW behaves *over-diffusively* indeed, the suggested rate of diffusion being $X_t \sim t^{2/3}$.

The paper contains three more sections and an Appendix. In section 2 we describe the local time process $S_n(\cdot)$ as a random walk on $\mathbb{Z}_+$. In section 3 we investigate in more detail an auxiliary Markov chain arising naturally in the previous description. Finally in section 4 we prove the results stated above, by coupling the process $S_n(\cdot)$ to an appropriately constructed homogeneous reflected random walk. The Appendix is devoted to the proof of a technical lemma.

2. The Local Time Process as a Markov Chain

The clue of the proof is the observation that, with n fixed, the local time process $S_n(\cdot)$ defined in (1.7) is a Markov chain on $\mathbb{Z}_+ = \mathbb{Z} \cap [0,\infty)$. Apparently this trick has its origin in F.B. Knight's paper [K] and has been rediscovered several times since then (see e.g. [KKS]). However, as opposed to the previous applications of this trick, the Markov process arising in our case will be more complicated than a branching process [K], or a branching process with random offspring distribution [KKS].

A finite walk which hits n for the first time at time i

$$0 = x_0, x_1, \dots, x_{i-1}, x_i = n, \qquad \min\{j : x_j = n\} = i \tag{2.1}$$

determines uniquely a finite sequence

$$L(0) = 0$$

$$\underline{l}(t) = \begin{cases} \big(l_0(t), \dots, l_{L(t-1)}(t)\big) & \text{if} \quad 1 \le t \le n \\[2ex] \big(l_1(t), \dots, l_{L(t-1)}(t)\big) & \text{if} \qquad t > n \end{cases} \tag{2.2}$$

$$L(t) = \sum_{k=0(1)}^{L(t-1)} l_k(t)$$

where $l_k(t)$ is the number of steps $(n-t) \to (n-t-1)$ between the k-th and $(k+1)$-th steps $(n-t) \to (n-t+1)$ performed by the walk. We shall refer to the sequence (2.2) as the system of left steps of the walk (2.1). Finiteness of the system of left steps means that there exists a $t_{\max} > n$ such that $L(t_{\max}) = 0$. This correspondence between finite walks hitting n and finite systems of left steps is *one-to-one*: given the sequence (2.2) one can reconstruct the complete walk (2.1) univocally.

Let (2.1) be a finite walk hitting n and (2.2) the corresponding system of left steps. Since T_n is a stopping time, the probability that the BTSAW coincides with (2.1) till T_n is

$$\mathbf{P}\Big(\underline{X}_0^{T_n} = \underline{x}_0^i\Big) = \prod_{j=0}^{i-1} \frac{\lambda^{v((x_j+x_{j+1}+1)/2|\underline{x}_0^j)}}{\lambda^{v(x_j|\underline{x}_0^j)} + \lambda^{v(x_j+1|\underline{x}_0^j)}}, \tag{2.3}$$

cf (1.4), (1.5). Written in terms of the $\underline{l}(t)$-s

$$\mathbf{P}\Big(\underline{X}_0^{T_n} = \underline{x}_0^i\Big) = \left[\prod_{t=1}^{n-1} \mathcal{P}_-\big(L(t-1); \underline{l}(t)\big)\right] \mathcal{P}_0\big(L(n-1); \underline{l}(n)\big) \left[\prod_{t>n} \mathcal{P}_+\big(L(t-1); \underline{l}(t)\big)\right], \tag{2.4}$$

with the transition probabilities:

$$\mathcal{P}_-\big(L; \underline{l}'\big) = \prod_{k=0}^{L} \left[\prod_{i=1}^{l'_k} \frac{\lambda^{2(\sum_{m=0}^{k-1} l'_m + i - 1)+1}}{\lambda^{2k} + \lambda^{2(\sum_{m=0}^{k-1} l'_m + i - 1)+1}}\right] \frac{\lambda^{2k}}{\lambda^{2k} + \lambda^{2\sum_{m=0}^{k} l'_m + 1}} \tag{2.5}$$

$$\mathcal{P}_0\big(L; \underline{l}'\big) = \prod_{k=0}^{L} \left[\prod_{i=1}^{l'_k} \frac{\lambda^{2(\sum_{m=0}^{k-1} l'_m + i - 1)}}{\lambda^{2k} + \lambda^{2(\sum_{m=0}^{k-1} l'_m + i - 1)}}\right] \frac{\lambda^{2k}}{\lambda^{2k} + \lambda^{2\sum_{m=0}^{k} l'_m}} \tag{2.6}$$

$$\mathcal{P}_+\big(L; \underline{l}'\big) = \prod_{k=1}^{L} \left[\prod_{i=1}^{l'_k} \frac{\lambda^{2(\sum_{m=1}^{k-1} l'_m + i - 1)}}{\lambda^{2k-1} + \lambda^{2(\sum_{m=1}^{k-1} l'_m + i - 1)}}\right] \frac{\lambda^{2k-1}}{\lambda^{2k-1} + \lambda^{2\sum_{m=1}^{k} l'_m}} \tag{2.7}$$

Thus, the process $S_n(\cdot)$ is indeed a Markov chain on $\mathbb{Z}_+$, homogeneous in the intervals $0 \le t < n$ and $t > n$.

Remark: In case of STSAW this rearrangement of the product, i.e. the transcription of (2.3) to (2.4) can not be performed. This is the step, where the proof of a similar result for the STSAW fails.

The transition probabilities $\mathcal{P}_{\pm,0}$ look quite threatening at the first sight, but we shall see soon, that they have a transparent interpretation and can be tamed. In order to see this we define an auxiliary Markov chain on $\mathbb{Z}$ which will help understanding better the random walk $S_n(\cdot)$.

For $z \in \mathbb{Z}$ let

$$p(z) = \frac{\lambda^{2z+1}}{1 + \lambda^{2z+1}}, \quad q(z) = \frac{1}{1 + \lambda^{2z+1}}, \tag{2.8}$$

and for $x, y \in \mathbb{Z}$

$$P(x, y) = \begin{cases} \displaystyle\prod_{z=x}^{y} p(z) q(y+1) & \text{if} \quad x - 1 \le y \\ 0 & \text{if} \quad x - 1 > y \end{cases} \tag{2.9}$$

(When $x = y+1$, the empty product is by definition equal to 1.) ¿From $\prod_{z=x}^{\infty} p(z) = 0$ follows that P is the transition matrix of a Markov chain η_k, $k = 0, 1, 2, \dots$ on $\mathbb{Z}$. On the right hand side of (2.5) (resp. (2.7)) we have exactly the probability distribution of the first $L+1$ steps (resp. the first L steps) of the Markov chain η. starting form 0 (resp. starting from -1). More precisely

$$\mathcal{P}_-(L; \underline{l}) = \mathbf{P}\Big(\eta_{k+1} - \eta_k = l_k - 1, \;\; k = 0, 1, \dots, L \Big\| \eta_0 = 0\Big) \tag{2.10}$$

$$\mathcal{P}_+(L; \underline{l}) = \mathbf{P}\Big(\eta_k - \eta_{k-1} = l_k - 1, \;\; k = 1, \dots, L \Big\| \eta_0 = -1\Big) \tag{2.11}$$

Thus, denoting by $\xi_n(t)$ the t-th step of the random walk $S_n(\cdot)$, i.e.

$$\xi_n(t) = S_n(t) - S_n(t-1), \qquad t \geq 1 \tag{2.12}$$

we get

$$\mathbf{P}\Big(\, \xi_n(t) = x \,\Big\|\, S_n(t-1) = k \,\Big) = \begin{cases} P^{k+1}(0, x-1) & \text{for} \quad t < n, \\ \\ P^k(-1, x-1) & \text{for} \quad t > n. \end{cases} \tag{2.13}$$

The transition probabilities for $t = n$ are a little different, but this single step will not have any effect on the limit law of the local time process.

Proof of Proposition 1, first part. As $S_n(0) = 0$ and (2.13) are bona fide transition probabilities, i.e. $\mathbf{P}\left(\xi_n(t) \in [-k, \infty) \| S_n(t-1) = k < \infty\right) = 1$, the local times $S_n(t) \in \mathbb{Z}_+$ are almost surely finite. $\omega_n < \infty$ will be proved in Proposition 2, section 4.

3. The Auxiliary Markov chain: Technical Lemmas

In this section we summarize the properties of the Markov chain η_i needed for the proof of Theorem 1. Denote by π the following probability distribution on $\mathbb{Z}$:

$$\pi(x) = \frac{\lambda^{x^2}}{\sum_{x \in \mathbb{Z}} \lambda^{x^2}}, \qquad x \in \mathbb{Z} \tag{3.1}$$

Lemma 1. *The unique stationary distribution of the Markov chain η is*

$$\rho(x) = \pi(x+1), \qquad x \in \mathbb{Z}. \tag{3.2}$$

There exist constants $C_1 < \infty$ and $C_2 > 0$ such that for $x = 0$ or $x = -1$ the following exponential bound holds:

$$\sum_{y \in \mathbb{Z}} \left| P^k(x, y) - \rho(y) \right| < C_1 e^{-C_2 k}, \tag{3.3}$$

Proof. ($\imath$) The identity

$$P(x, y) = \begin{cases} \dfrac{1}{\rho(x)} \left[p(x) \displaystyle\prod_{z=x+1}^{y+1} q(z) \right] \rho(y) & \text{for} \quad x - 1 \leq y \\ \qquad\qquad 0 & \text{for} \quad x - 1 > y \end{cases} \tag{3.4}$$

is straightforward. Hence

$$\sum_{x\in\mathbb{Z}}\rho(x)P(x,y)=\left[\sum_{x\le y+1}p(x)\prod_{z=x+1}^{y+1}q(z)\right]\rho(y)=\rho(y) \tag{3.5}$$

which proves that ρ is a stationary distribution indeed.

The proof of uniqueness and exponential convergence (3.3) is a bit lengthy, but consists of standard procedures. First of all notice that due to stationarity of the distribution ρ inequality

$$P^k(x,y)\le\frac{\rho(y)}{\rho(x)}. \tag{3.6}$$

holds, which provides an over-exponential bound, uniform in k, on the tails of the distributions $P^k(x,\cdot)$. In consequence all the expectations below make sense.

Denote by $\theta_\pm$ and σ the following stopping times

$$\theta_+ = \min\{k\ge 0 : \eta_k\ge 0\} \tag{3.7}$$

$$\theta_- = \min\{k\ge 0 : \eta_k\le 0\} \tag{3.8}$$

$$\sigma\ = \min\{k\ge 1 : \eta_k= 0\} \tag{3.9}$$

According to Theorem 6.14 and Example 5.5(a) of [NU] the uniqueness of the stationary distribution and the exponential convergence (3.3) follow from

$$\mathbf{E}\Big(\exp(C_2\sigma)\Big\|\eta_0=0\Big)<\infty, \tag{3.10}$$

with $C_2>0$. So, our goal is to prove (3.10). In the following expression we use the fact that the Markov chain $\eta.$ doesn't jump more than one lattice step to the left:

$$\begin{aligned}\mathbf{E}\Big(\mathrm{e}^{C_2\sigma}\Big|\eta_0=0\Big)&=\mathrm{e}^{C_2}\sum_{y\ge0}P(0,y)\mathbf{E}\Big(\mathrm{e}^{C_2\theta_-}\Big|\eta_0=y\Big)\\&+\mathrm{e}^{C_2}P(0,-1)\sum_{y\ge0}\mathbf{E}\Big(\mathrm{e}^{C_2\theta_+}1\!\!1(\eta_{\theta_+}=y)\Big|\eta_0=-1\Big)\mathbf{E}\Big(\mathrm{e}^{C_2\theta_-}\Big|\eta_0=y\Big).\end{aligned} \tag{3.11}$$

Next we use a special consequence of the structure (2.9) of the transition kernel P. Namely, it is an easy computation to check that given $\eta_0=x<0$ the random variables θ_+ and η_{θ_+} are independent and, for $y\ge0$

$$\mathbf{E}\Big(\mathrm{e}^{C_2\theta_+}1\!\!1(\eta_{\theta_+}=y)\Big|\eta_0=-1\Big)=\frac{P(0,y)}{1-P(0,-1)}\mathbf{E}\Big(\mathrm{e}^{C_2\theta_+}\Big|\eta_0=-1\Big). \tag{3.12}$$

¿From (3.11), (3.12):

$$\begin{aligned}\mathbf{E}\Big(\mathrm{e}^{C_2\sigma}\Big\|\eta_0=0\Big)&=\mathrm{e}^{C_2}\sum_{y\ge0}P(0,y)\mathbf{E}\Big(\mathrm{e}^{C_2\theta_-}\Big|\eta_0=y\Big)\\&\times\left[1+\frac{P(0,-1)}{1-P(0,-1)}\mathbf{E}\Big(\mathrm{e}^{C_2\theta_+}\Big|\eta_0=-1\Big)\right].\end{aligned} \tag{3.13}$$

¿From a dominated convergence argument we can see that

$$P(-\infty, y) = \lim_{x\to-\infty} P(x,y) = \prod_{z=-\infty}^{y+1} p(z)q(y+1) \tag{3.14}$$

is a probability distribution on $\mathbb{Z}$. Let $x_1 \le x_2$, then for $y \ge x_2 - 1$

$$P(x_1, y) = \left[\sum_{z\ge x_2-1} P(x_1,z)\right] P(x_2,y) \tag{3.15}$$

which implies that the distributions $P(x,\cdot)$ are stochastically ordered: for $x_1 < x_2$ and arbitrary y_0

$$\sum_{y\ge y_0} P(-\infty,y) < \sum_{y\ge y_0} P(x_1,y) < \sum_{y\ge y_0} P(x_2,y) \tag{3.16}$$

¿From (3.16) follows that

$$\mathbf{P}\Big(\theta_+ > k \Big\| \eta_0 = -1\Big) \le r^k, \qquad \text{where} \qquad r = \sum_{y<0} P(-\infty,y) < 1. \tag{3.17}$$

Hence

$$\mathbf{E}\Big(\exp(C_3\theta_+)\Big\|\eta_0 = -1\Big) < \infty \tag{3.18}$$

for $C_3 < -\log r$.

On the other hand, since $p(z) > p(z+1)$, (see (2.8)), the distributions $P(x, x+\cdot)$ are stochastically ordered in the opposite sense: for $x_1 < x_2$ and arbitrary y_0

$$\sum_{y\ge y_0} P(x_1, x_1+y) = \prod_{z=0}^{y_0} p(x_1+z) > \prod_{z=0}^{y_0} p(x_2+z) = \sum_{y\ge y_0} P(x_2,x_2+y). \tag{3.19}$$

Hence, for $y \ge 0$

$$\mathbf{P}\Big(\min_{0\le j\le k}\eta_j \le 0\Big\|\eta_0 = y\Big) \ge \mathbf{P}\Big(\min_{0\le j\le k}\sum_{i=1}^{j}\zeta_i \le -y\Big), \tag{3.20}$$

where the ζ_i-s are i.i.d. random variables with distribution:

$$\mathbf{P}\Big(\zeta_i = x\Big) = P(0,x). \tag{3.21}$$

Now using the explicit form of the distribution $P(0,x)$ we find that

$$\forall\alpha\in\mathbb{R}: \qquad \sum_{x\in\mathbb{Z}} P(0,x)\mathrm{e}^{\alpha x} < \infty, \tag{3.22}$$

and

$$\sum_{y\in\mathbb{Z}} P(0,x)x < 0 \tag{3.23}$$

for any $\lambda \in (0,1)$. From (3.20), (3.21), (3.22) and (3.23) follows that, if $y \geq 0$

$$\mathbf{E}\Big(\mathrm{e}^{C_4\theta_-}\Big|\eta_0 = y\Big) \leq \mathrm{e}^{\beta y} \tag{3.24}$$

with some $C_4 > 0$ and $\beta < \infty$.

(3.10), with $C_2 = \min\{C_3, C_4\}$, follows from (3.13), (3.18), (3.22) and (3.24). □

The next lemma establishes an over-exponential bound on the rate of decay of the tails of the distributions $P^k(0,\cdot)$, uniform in k. This bound is much stronger than what we actually need in the proof of the Theorem.

Lemma 2. *There exists a constant $C_5 < \infty$ such that for any $k \geq 0$ and $x \geq 0$*

$$P^k(0,x+1) \leq C_5\lambda^x P^k(0,x). \tag{3.25}$$

Remark: A similar bound can be established for the left tail of the distributions, too, but we do not need it in the proof of the Theorem.

Proof. We prove the lemma by induction on k and x. (3.25) clearly holds for $k = 0$ and any $x \geq 0$. As, due to (3.1), (3.2) and (3.3) $\lim_{k\to\infty} P^k(0,1)/P^k(0,0) = \lambda^3 < \infty$,

$$C_5 = \sup_{k\geq 0} \frac{P^k(0,1)}{P^k(0,0)} < \infty. \tag{3.26}$$

Thus, (3.25) holds for $\forall k \geq 0$ and $x = 0$. We proceed now by induction. Given an arbitrary probability distribution r on $\mathbb{Z}$ the following identity holds:

$$\big[rP\big](x+1) = \frac{p(x+1)}{q(x+1)} q(x+2)\big[rP\big](x) + q(x+2)r(x+2). \tag{3.27}$$

Assume (3.25) holds for $(k, x+1)$ and for $(k+1, x)$. Using (3.27) we get

$$\begin{aligned}
P^{k+1}(0,x+2) &= \frac{p(x+2)}{q(x+2)} q(x+3)P^{k+1}(0,x+1) + q(x+3)P^k(0,x+3) \\
&\leq \frac{p(x+2)}{q(x+2)} q(x+3)C_5\lambda^x P^{k+1}(0,x) + q(x+3)C_5\lambda^{x+2}P^k(0,x+2) \\
&= C_5\lambda^{x+1}\frac{\lambda q(x+3)}{q(x+2)}\left[\frac{p(x+1)}{q(x+1)} q(x+2)P^{k+1}(0,x) + q(x+2)P^k(0,x+2)\right] \\
&= C_5\lambda^{x+1}\frac{\lambda q(x+3)}{q(x+2)} P^{k+1}(0,x+1).
\end{aligned} \tag{3.28}$$

Since

$$\frac{\lambda q(x+3)}{q(x+2)} = \frac{\lambda(1+\lambda^{2x+5})}{1+\lambda^{2x+7}} < 1, \tag{3.29}$$

(3.28) yields (3.25) for $(k+1, x+1)$. □

4. General Formulation and Proofs

In the proofs we use only some of the qualitative features (formulated in the lemmas of the previous section) and not the explicit form of the transition probabilities of the random walk $S_n(\cdot)$. We are going to formulate our main Theorem in these more general terms.

Let

$$S_n(t) = \sum_{s=1}^{t} \xi_n(s) \tag{4.1}$$

be a space-time inhomogeneous random walk on $\mathbb{Z}_+$ with the following law:

$$\mathbf{P}\Big(\xi_n(t+1) = x \Big\| S_n(t) = k\Big) = \pi_n(x|k,t) \tag{4.2}$$

$$\sum_{x \geq -k} \pi_n(x|k,t) = 1. \tag{4.3}$$

Remark: Compare (4.2), (4.3) with (2.13). The time inhomogeneity and dependence on n of the law of the random walk is just a nuisance we have to live with. As all the estimates below will be uniform in t and n this will cause no real trouble. All the constants arising in various inequalities below are absolute constants not depending on n or t.

We assume the following behaviour of the step distributions $\pi_n(\cdot|k,t)$:
(1) The step distributions $\pi_n(\cdot|k,t)$ converge in $\ell_1(\mathbb{Z})$, uniformly in n and t, exponentially fast as $k \to \infty$, to an asymptotic distribution π:

$$\sum_{x \in \mathbb{Z}} |\pi_n(x|k,t) - \pi(x)| < C_1 \exp(-C_2 k). \tag{4.4}$$

The asymptotic distribution is symmetric

$$\pi(-x) = \pi(x), \tag{4.5}$$

and its moments of any order are assumed finite. We denote by σ^2 its variance:

$$\sum_{x \in \mathbb{Z}} x^2 \pi(x) = \sigma^2 \in (0, \infty). \tag{4.6}$$

Remark: (4.4), (4.5) and the moment conditions should be compared with (3.3) and the explicit form (3.1). The symmetry condition (4.5) is not really needed, $\sum_{x\in\mathbb{Z}} x\pi(x) = 0$ would be sufficient. We assume (4.5) only for shortening the argument (see the observation after (4.28) and the rightmost inequality in (A.2)).
(2) The tails of the distributions $\pi_n(\cdot|k,t)$ are exponentially bounded, uniformly in k, t and n, i. e. there are two constants $C_6 < \infty$, $C_7 > 0$ such that

$$\pi_n(x|k,t) \leq C_6 \exp(-C_7|x|). \tag{4.7}$$

For technical purposes we assume even more about the decay of the right tail of the distributions $\pi_n(\cdot|k,t)$: there is an $x_0 \in \mathbb{Z}$ independent of k, t or n such that for $x \geq x_0$:

$$\pi_n(x+1|k,t) < \mathrm{e}^{-C_7}\pi_n(x|k,t). \tag{4.8}$$

Remark: Compare (4.7) with (3.6) and (4.8) with (3.25).

(3) The random walk is not trapped in a bounded domain or in a domain away from the origin. There is a constant $C_8 > 0$ independent of t or n such that

$$\forall\, k \geq 0: \quad \sum_{x>0} \pi_n(x|k,t) > C_8 > 0. \tag{4.9}$$

$$\forall\, k > 0: \quad \sum_{x<0} \pi_n(x|k,t) > C_8 > 0. \tag{4.10}$$

The ω-stopping-time of the random walk $S_n(\cdot)$ is

$$\omega_n = \inf\{t \geq n : S_n(t) = 0\}. \tag{4.11}$$

Denote by θ_I the exit time from the interval $I \in \mathbb{Z}_+$

$$\theta_I = \min\{t \geq 0 : S_n(t) \notin I\}. \tag{4.12}$$

Lemma 3. *(i.) There exists a constant $C_9 > 0$, such that for any $0 \leq a < k < b$ in $\mathbb{Z}_+$*

$$\mathbf{P}\Big(S_n(\theta_{(a,b)}) \geq b \Big\| S_n(0) = k\Big) > C_9 \frac{k-a}{b-a}, \tag{4.13}$$

and for any $0 \leq a < k < b$ in $\mathbb{Z}_+$

$$\mathbf{P}\Big(S_n(\theta_{(a,b)}) \leq a \Big\| S_n(0) = k\Big) > C_9 \frac{b-k}{b-a}. \tag{4.14}$$

(ii.) There exists a constant $C_{10} > 0$, such that for any $0 \leq k < b$ in $\mathbb{Z}_+$

$$\mathbf{E}\Big(\theta_{[0,b)} \Big\| S_n(0) = k\Big) < C_{10} b^3. \tag{4.15}$$

Since the proof of this lemma is a rather standard application of submartingale techniques and the optional sampling theorem, (see [Br]), and has no relevance to the rest of the proofs, we postpone its sketch to the Appendix. We should remark here, that in (4.15) a better bound $\sim b^2$ can be proved with some more work. However this bound is sufficient for our purposes.

Proposition 2.

$$\omega_n < \infty \tag{4.16}$$

almost surely.

Proof. Given Lemma 3 (*i.*), (4.14), this is evident. This statement also completes the proof of Proposition 1. ☐

The following theorem is the natural general formulation of Theorem 1:

Theorem 2. *Under the conditions 1, 2 and 3*

$$\left(\frac{\omega_n}{n}, \frac{S_n([ns])}{\sigma\sqrt{n}} : 0 \le s \le \infty\right) \Rightarrow (\omega, |W_s| : 0 \le s \le \infty) \tag{1.17}$$

in $\mathbb{R}_+ \times D[0,\infty)$*, where* σ^2 *is the variance of the asymptotic step distribution given in (4.6).*

The outline of the proof is quite simple: due to the exponential convergence of the step distributions, in the domain above the threshold

$$b_n = [n^\epsilon] \tag{4.18}$$

the trajectory will be that of a homogeneous random walk with step distribution π, with very high probability. On the other hand, with help of Lemma 3 (*ii.*), we prove that the total amount of time spent below the threshold b_n will be rather small, of order $o(n^{1/2+5\epsilon})$ in a time interval of order $n^{1+\epsilon}$, with overwhelming probability. Joining these two arguments we couple a reflected homogeneous random walk, $Y_n(\cdot)$, with step distribution π to our original random walk $S_n(\cdot)$ so that the difference of their ω-stopping-times is of order $o(n^{1/2+5\epsilon})$ and the supremum distance of the two processes is of order $o(n^{1/4+3\epsilon})$, with probability converging to 1.

Proof of Theorem 2. (*i.*) Due to (4.4) we can couple to our random walk $S_n(\cdot)$ a sequence of i.i.d. random variables $\zeta_n(t)$, $t = 1, 2, \dots$, with distribution π, so that

$$\mathbf{P}\Big(\zeta_n(t+1) \ne \xi_n(t+1) \Big\| S_n(t) = k\Big) < C_1 \exp(-C_2 k). \tag{4.19}$$

We denote by $\sigma_n(t)$ the amount of time spent above the threshold level b_n, before t:

$$\sigma_n(t) = \#\{0 \le s < t : S_n(s) > b_n\}, \tag{4.20}$$

and define the following sequence of sampling times:

$$\begin{aligned} \tau_n(-1) &= -1 \\ \tau_n(t) &= \min\{s > \tau_n(t-1) : S_n(s) > b_n\}, \qquad 0 \le t < \infty. \end{aligned} \tag{4.21}$$

τ_n is the inverse function of σ_n in the sense that

$$\sigma_n(\tau_n(t)) = t \quad \text{for} \quad 0 \le t < \infty. \tag{4.22}$$

Let

$$\tilde{\xi}_n(t) = \xi_n(\tau_n(t-1)+1) \tag{4.23}$$

and

$$\tilde{\zeta}_n(t) = \zeta_n(\tau_n(t-1)+1). \tag{4.24}$$

As the $\tilde{\zeta}_n$-s are selected according to an increasing sequence of sampling times they are still i.i.d. random variables with the common distribution π. On the other hand, due to (4.19)

$$\forall t \in \mathbb{N} : \qquad \mathbf{P}\Big(\tilde{\xi}_n(t) \ne \tilde{\zeta}_n(t)\Big) < C_1 \exp(-C_2 n^\epsilon). \tag{4.25}$$

We define two 'truncated' processes $\tilde{S}_n(\cdot)$ and $\tilde{Y}_n(\cdot)$:

$$\tilde{S}_n(0) = 0, \quad \tilde{S}_n(t) = \left(\tilde{S}_n(t-1) + \tilde{\xi}_n(t)\right) \bigvee 0, \tag{4.26}$$

$$\tilde{Y}_n(0) = 0, \quad \tilde{Y}_n(t) = \left(\tilde{Y}_n(t-1) + \tilde{\zeta}_n(t)\right) \bigvee 0. \tag{4.27}$$

And finally, let the random walk Y_n be defined as follows:

$$Y_n(0) = 0, \quad Y_n(t) = \left| Y_n(t-1) + \tilde{\zeta}_n(t) \right|. \tag{4.28}$$

Due to the symmetry of the distribution π, $Y_n(\cdot)$ is a homogeneous random walk reflected at the origin, and $\tilde{Y}_n(\cdot)$ differs from $Y_n(\cdot)$ only by the cutoffs of the overshootings of the origin: using the constructions (4.27) and (4.28) one can easily check

$$0 \le Y_n(t) - \tilde{Y}_n(t) \le \max_{0 \le s \le t} \left| \tilde{\zeta}_n(s) \right|. \tag{4.29}$$

We also need the ω-stopping times of the processes $\tilde{Y}_n(\cdot)$ and $Y_n(\cdot)$:

$$\omega_n' = \inf\{t \ge n : \tilde{Y}_n(t) = 0\}, \tag{4.30}$$

$$\omega_n'' = \inf\{t \ge n : Y_n(t) = 0\}. \tag{4.31}$$

The standard invariance principle holds for $Y_n(\cdot)$ (see [Bi], [L]), and consequently:

$$\left(\frac{\omega''}{n}, \frac{Y_n([ns])}{\sigma\sqrt{n}} : 0 \le s < \infty \right) \Rightarrow (\omega, |W_s| : 0 \le s < \infty) \quad \text{in} \quad \mathbb{R} \times D[0,\infty). \tag{4.32}$$

Due to the closeness of the $\tilde{Y}_n(\cdot)$ and $Y_n(\cdot)$ paths, (4.29), the same convergence in distribution is easily proved for the process $\tilde{Y}_n(\cdot)$:

$$\left(\frac{\omega'}{n}, \frac{\tilde{Y}_n([ns])}{\sigma\sqrt{n}} : 0 \le s < \infty \right) \Rightarrow (\omega, |W_s| : 0 \le s < \infty) \quad \text{in} \quad \mathbb{R} \times D[0,\infty). \tag{4.33}$$

We omit the details of this straightforward step.

We shall prove the Theorem by proving

$$n^{-1/2} \max_{0 \le t \le n^{1+\epsilon}} \left| S_n(t) - \tilde{Y}_n(t) \right| \xrightarrow{\mathbf{P}} 0 \tag{4.34}$$

and

$$n^{-1} \left| \omega_n - \omega_n' \right| \xrightarrow{\mathbf{P}} 0. \tag{4.35}$$

Consider first (4.34):

$$\max_{0 \le t \le n^{1+\epsilon}} \left| S_n(t) - \tilde{Y}_n(t) \right| \le \max_{0 \le t \le n^{1+\epsilon}} \left| S_n(t) - \tilde{S}_n(\sigma_n(t)) \right| + \max_{0 \le t \le n^{1+\epsilon}} \left| \tilde{S}_n(\sigma_n(t)) - \tilde{Y}_n(\sigma_n(t)) \right| + \max_{0 \le t \le n^{1+\epsilon}} \left| \tilde{Y}_n(\sigma_n(t)) - \tilde{Y}_n(t) \right|. \tag{4.36}$$

In order to estimate the three terms on the right hand side of (4.36) we define the following events:

$$\mathcal{A}_n = \Big\{ \max_{1\le t\le n^{1+\epsilon}} |\xi_n(t)| < n^{\epsilon} \Big\} \tag{4.37}$$

$$\mathcal{B}_n = \Big\{ \tilde{\xi}_n(t) = \tilde{\zeta}_n(t), \;\; 0 \le t \le n^{1+\epsilon} \Big\} \tag{4.38}$$

$$\mathcal{C}_n = \Big\{ n^{1+\epsilon} - \sigma_n(n^{1+\epsilon}) < n^{\frac{1}{2}+5\epsilon} \Big\} \tag{4.39}$$

$$\mathcal{D}_n = \Big\{ \max\Big\{ \Big|\tilde{Y}_n(s) - \tilde{Y}_n(t)\Big| : 0 \le s,t \le n^{1+\epsilon}, |s-t| < n^{\frac{1}{2}+5\epsilon} \Big\} < n^{\frac{1}{4}+3\epsilon} \Big\} \tag{4.40}$$

$$\mathcal{E}_n = \Big\{ \# \Big\{ t : 0 \le t \le n^{1+\epsilon}, \;\; \tilde{Y}_n(t) = 0 \Big\} < n^{\frac{1}{2}+\epsilon} \Big\}. \tag{4.41}$$

By construction of the process $\tilde{S}_n(\cdot)$ we have

$$0 \le S_n(t) - \tilde{S}_n(\sigma_n(t)) \le b_n + \max_{0\le s\le t} \xi_n(s). \tag{4.42}$$

So the first term on the right hand side of (4.36) is easily evaluated:

$$\mathbf{P}\Big(\max_{0\le t\le n^{1+\epsilon}} \Big| S(t) - \tilde{S}_n(\sigma_n(t)) \Big| < 2n^{\epsilon} \Big) \ge \mathbf{P}\Big(\mathcal{A}_n\Big) \to 1 \tag{4.43}$$

as $n \to \infty$. The convergence on the right hand side of (4.43) is due to the uniform bound (4.7) on the decay of the tails of step distributions.

The second term on the right hand side of (4.36) is even simpler:

$$\mathbf{P}\Big(\max_{0\le t\le n^{1+\epsilon}} \Big| \tilde{S}_n(\sigma_n(t)) - \tilde{Y}_n(\sigma_n(t)) \Big| = 0 \Big) \ge \mathbf{P}\Big(\mathcal{B}_n\Big) \to 1 \tag{4.44}$$

as $n \to \infty$, due to the exponential closeness (4.25).

To estimate the last term, notice that

$$\mathcal{C}_n \cap \mathcal{D}_n \subset \Big\{ \max_{0\le t\le n^{1+\epsilon}} \Big| \tilde{Y}_n(\sigma_n(t)) - \tilde{Y}_n(t) \Big| < n^{\frac{1}{4}+3\epsilon} \Big\}, \tag{4.45}$$

so the estimates

$$\mathbf{P}\Big(\mathcal{C}_n\Big) \to 1 \tag{4.46}$$

$$\mathbf{P}\Big(\mathcal{D}_n\Big) \to 1 \tag{4.47}$$

as $n \to \infty$, is still wanted. (4.47) and

$$\mathbf{P}\Big(\mathcal{E}_n\Big) \to 1 \tag{4.48}$$

as $n \to \infty$, are well established facts about homogeneous random walks. The proof of (4.47) is based on an adaptation of the method of proof of Theorem 8.3 of [Bi] (the finiteness of high moments of the distribution π is needed here). The estimate (4.48) of the number of visits to the origin of the random walk $\tilde{Y}_n(\cdot)$ is an easy exercise. As the proofs of (4.47) and (4.48) are quite standard, we do not give the details here.

Finally

$$\mathbf{P}\Big(\mathcal{C}_n \Big\| \mathcal{E}_n \cap \mathcal{B}_n\Big) \to 1 \qquad \text{as} \qquad n \to \infty. \tag{4.49}$$

is a consequence of Lemma 3 (*ii.*), (4.15): notice that in $\mathcal{E}_n \cap \mathcal{B}_n$ there are altogether less than $n^{\frac{1}{2}+\epsilon}$ visits of $S_n(t)$, $0 \le t \le n^{1+\epsilon}$ to the interval $[0, b_n)$ and Lemma 3 (*ii.*) provides an upper bound on the expectation of the duration of each of these visits. In consequence, the conditional expectation of the total time spent below the threshold level b_n is bounded like

$$\mathbf{E}\Big(n^{1+\epsilon} - \sigma_n(n^{1+\epsilon})\Big\| \mathcal{E}_n \cap \mathcal{B}_n\Big) \le n^{\frac{1}{2}+\epsilon} \max_{0\le k\le b_n} \mathbf{E}\Big(\theta_{[0,b_n]}\Big\| S(0) = k\Big) \le C_{10} n^{\frac{1}{2}+4\epsilon}. \tag{4.50}$$

Hence (4.49), via Markov's inequality.

(4.43), (4.44), (4.46), (4.47), (4.48) and (4.49) imply (4.34).

In order to prove (4.35) we have to define two more events:

$$\mathcal{F}_n = \Big\{\omega_n' < \frac{n^{1+\epsilon}}{2}\Big\} \tag{4.51}$$

$$\mathcal{G}_n = \Big\{\omega_n' < \omega_n\Big\}. \tag{4.52}$$

¿From the weak convergence of (4.33) of ω_n'/n clearly follows

$$\mathbf{P}\Big(\mathcal{F}_n\Big) \to 1, \tag{4.53}$$

as $n \to \infty$. To prove

$$\mathbf{P}\Big(\mathcal{G}_n\Big) \to 1 \tag{4.54}$$

as $n \to \infty$, notice that, by construction of $\tilde{S}_n(\cdot)$, $\tilde{S}(\sigma_n(\omega_n)) = 0$ and consequently, on $\mathcal{B}_n$ $\tilde{Y}(\sigma_n(\omega_n)) = 0$, too. Thus, either $\omega_n' \le \sigma_n(\omega_n) < \omega_n$ or $\sigma_n(\omega_n) \le n < \omega_n$. The first of these alternatives implies $\mathcal{G}_n$, from the second alternative follows

$$\mathcal{G}_n^{\mathrm{c}} \cap \mathcal{B}_n \cap \mathcal{C}_n \cap \mathcal{D}_n \cap \mathcal{F}_n \subset \Big\{\tilde{Y}_n(n) < n^{\frac{1}{4}+3\epsilon}\Big\}. \tag{4.55}$$

But, due to the weak convergence (4.33), the probability of this event clearly goes to zero, as $n \to \infty$. Hence (4.54).

Note that, using identity (4.22)

$$S(\tau_n(\omega_n')) = S_n(\tau_n(\omega_n')) - \tilde{Y}_n(\omega_n') = S_n(\tau_n(\omega_n')) - \tilde{Y}_n(\sigma_n(\tau_n(\omega_n'))). \tag{4.56}$$

Thus, due to (4.43) and (4.44), on $\mathcal{A}_n \cap \mathcal{B}_n \cap \mathcal{C}_n \cap \mathcal{F}_n$ we have

$$S_n(\tau_n(\omega_n')) \le 2n^{\epsilon}. \tag{4.57}$$

¿From Lemma 3 (*i.*) and (*ii.*), (4.14) and (4.15), easily follows, that

$$\mathbf{P}\Big(\theta_{(0,\infty)} < n^{7\epsilon} \Big\| S_n(0) \leq 2n^{\epsilon}\Big) \to 1 \tag{4.58}$$

as $n \to \infty$. Note that $\theta_{(0,\infty)}$ is the hitting time of $0 \in \mathbb{Z}_+$. We conclude, that

$$\mathbf{P}\Big(\omega_n - \omega_n' < 2n^{\frac{1}{2}+5\epsilon} \Big\| \mathcal{A}_n \cap \mathcal{B}_n \cap \mathcal{C}_n \cap \mathcal{F}_n \cap \mathcal{G}_n\Big) \to 1 \tag{4.59}$$

as $n \to \infty$, which, given (4.43), (4.44), (4.48), (4.49), (4.53) and (4.54), proves (4.35). □

Given the explicit form (2.13) of the transition probabilities and Lemmas 1 and 2 of section 3, Theorem 1 becomes simply a concrete case of Theorem 2.

Appendix: Proof of Lemma 3.

The proof of (4.13), (4.14) and (4.15) is standard application of submartingale technique and the optional sampling theorem (see [Br]), we only sketch the main steps.
(*i.*) Conditions (4.4), (4.5) and (4.7) imply

$$\Big|\sum_{x\in\mathbb{Z}} \pi_k(x)x\Big| < C_{11}\exp(-C_{12}k) \tag{A.1}$$

with $C_{11} < \infty$ and $C_{12} = C_2/2$, and

$$\liminf_{k\to\infty} \sum_{x\in\mathbb{Z}} \pi_k(x)\mathrm{e}^{-C_{12}x} \geq \sum_{x\in\mathbb{Z}} \pi(x)\mathrm{e}^{-C_{12}x} > 1. \tag{A.2}$$

Here once again we used the symmetry (4.5) of the distribution π. Due to (A.2) we can choose $a_1 \in \mathbb{Z}_+$ and a finite constant $C_{13} < \infty$ such that for $k \geq a_1$

$$C_{13}\left(\sum_{x\in\mathbb{Z}} \pi_k(x)\mathrm{e}^{-C_2x} - 1\right) \geq C_{11}. \tag{A.3}$$

It is straightforward to check that from (A.3) and (A.1) follows that both

$$\pm S(t) + C_{13}\exp(-C_{12}S(t)), \qquad 0 \leq t \leq \theta_{(a_1,\infty)} \tag{A.4}$$

are submartingales.

To prove (4.13) notice first that due to the strong exponential bound (4.8) on the decay rate of the right tails, for any $k \in (a,b) \subset \mathbb{Z}_+$

$$\mathbf{E}\Big(S(\theta_{(a,b)})\Big\|[S(0)=k] \wedge [S(\theta_{(a,b)}) \geq b]\Big) \leq (b + C_{13}) \tag{A.5}$$

with some $C_{13} < \infty$. Let $a_2 = \max\{a_1, C_{13}\}$. For $a_2 \leq a < b$ consider the submartingale

$$S(t) + C_{13}\exp(-C_{12}S(t)) - [a + C_{13}\exp(-C_{12}a)], \qquad 0 \leq t \leq \theta_{(a,b)}, \tag{A.6}$$

then

$$\begin{aligned}
&k + C_{13}\mathrm{e}^{-C_{12}k} - [a + C_{13}\exp(-C_{12}a)] \leq \\
&\mathbf{E}\Big(S(\theta_{(a,b)}) + C_{13}\mathrm{e}^{-C_{12}S(\theta_{(a,b)})} - [a + C_{13}\exp(-C_{12}a)] \Big\| S(0) = k\Big) \leq \\
&\qquad (b + C_{13} + C_{13} - a)\mathbf{P}\Big(S(\theta_{(a,b)}) > b \Big\| S(0) = k\Big) \qquad \text{(A.7)}
\end{aligned}$$

which implies (4.13) with some constant $C_{15} > 0$ for $a_2 \leq a < k < b$. Changing the constant C_{15} to a smaller one

$$C_9 = \min_{0 \leq a' < a_2} \min_{a' < k' \leq a_2} \mathbf{P}\Big(S(\theta_{(a',a_2]}) > a_2 \Big\| S(0) = k'\Big) C_{15} > 0 \qquad \text{(A.8)}$$

the inequality extends to any $0 \leq a < k < b$ and (4.13) is proved.

(4.14) is proved in a very similar way considering now the submartingale

$$-S(t) + C_{13}\exp(-C_{12}S(t)) - [-b + C_{13}\exp(-C_{12}b)], \qquad 0 \leq t \leq \theta_{(a,b)}, \qquad \text{(A.9)}$$

with $a_1 \leq a < b \leq \infty$. We do not repeat the same arguments.
($ii.$) (4.4) and (4.9) imply

$$0 < \underline{\sigma}^2 \leq \sum_{x \in \mathbb{Z}} \pi_k(x)|x|^2 \qquad \text{(A.10)}$$

Choose now $a_4 \in \mathbb{Z}_+$ so that for $k \geq a_4$

$$2kC_{11}\exp(-C_{12}k) < \frac{\underline{\sigma}^2}{2}. \qquad \text{(A.11)}$$

Then, in the domain $k > a_4$ we have

$$\mathbf{E}\Big(S^2(t+1) - \frac{\underline{\sigma}^2}{2}(t+1) \Big\| S(t) = k\Big) \geq k^2 - \frac{\underline{\sigma}^2}{2}t \qquad \text{(A.12)}$$

and consequently

$$S^2(t) - \frac{\underline{\sigma}^2 t}{2}, \qquad t = 0, 1, \ldots, \theta_{(a_4,b)} \qquad \text{(A.13)}$$

is a submartingale. Hence

$$\mathbf{E}\Big(S^2(\theta_{(a_4,b)}) - \frac{\underline{\sigma}^2}{2}\theta_{(a_4,b)} \Big\| S(0) = k\Big) \geq k^2. \qquad \text{(A.14)}$$

¿From (4.8), similarly to (A.5), we get

$$\mathbf{E}\Big(S^2(\theta_{(a,b)}) \Big\| [S(0) = k] \wedge [S(\theta_{(a,b)}) > b]\Big) \leq (b + C_{16})^2. \qquad \text{(A.15)}$$

(A.14) and (A.15) imply

$$\frac{\underline{\sigma}^2}{2}\mathbf{E}\Big(\theta_{(a_4,b)} \Big\| S(0) = k\Big) \leq (b + C_{16})^2, \qquad \text{(A.16)}$$

and, finally (4.13) and (A.16) lead directly to

$$\mathbf{E}\Big(\theta_{[0,b)} \Big\| S(0) = k\Big) \leq C_9^{-1} b \left(\max_{0 \leq k' \leq a_4} \mathbf{E}\Big(\theta_{[0,a_4]} \Big\| S(0) = k'\Big) + \frac{4}{\underline{\sigma}^2}(b + C_{16})^2 \right). \qquad \text{(A.17)}$$

Due to the non-trapping condition (4.9) $\max_{0 \leq k' \leq a_4} \mathbf{E}\Big(\theta_{[0,a_4]} \Big\| S(0) = k'\Big)$ is finite. Thus (A.17) proves (4.15). □

Acknowledgement: A very pleasant and instructive discussion with Erwin Bolthausen, about self-repelling random walks, lead me to the formulation of the problem. Helpful conversations with Endre Csáki and Péter Major during the preparation of this work are also gratefully acknowledged.

References

[APP] Amit, D., Parisi, G., Peliti, L.: Asymptotic behaviour of the 'true' self-avoiding walk. *Phys. Rev. B* **27** 1635-1645, (1983).

[Bi] Billingsley, P.: *Convergence of Probability Measures.* J. Wiley & Sons, New York, 1968.

[Bo] Bolthausen, E.: On self-repellent one-dimensional random walks. *Probab. Th. Rel. Fields* **86** 423-441, 1990

[Br] Breiman., L.: *Probability.*, Addison-Wesley, Reading, MA, 1968.

[BS] Brydges, D., Spencer, T.: Self-avoiding walk in 5 or more dimensions. *Commun. Math. Phys.* **97** 125-148 (1985).

[CM] Csáki, E., Mohanty, S.G.: Excursion and meander in random walk. *The Canadian J. of Stat.* **9** 57-70, (1981)

[D] Davis, B.: Reinforced random walk. *Probab. Th. Rel. Fields.* **84** 203-229 (1990).

[HS] Hara, T., Slade, G.: The lace expansion for self-avoiding walk five and more dimensions. *Reviews in Math. Phys.* **4** 235-327 (1992).

[KKS] Kesten, H., Kozlov, M.V., Spitzer, F.: A limit law for random walk in random environment. *Compositio Mathematica* **30** 145-168, 1975.

[K] Knight, F.B.: Random walks and a sojourn density process of Brownian motion. *Transactions of AMS* **109** 56-86, (1963)

[L] Lindwall, T: Weak convergence of probability measures and random functions in the function space $D[0,\infty)$. *J. Appl. Probab.* **10** 109, 1973.

[N] Nummelin, E.: *General Irreducible Markov Chains and Non-Negative Operators.* Cambridge Univ. Press, 1984.

[P] Pemantle, R.: Phase transition in reinforced random walk and RWRE on trees. *The Annals of Probab.* **16** 1229-1242 (1988).

B. Tóth
Mathematical Institute of the
Hungarian Academy of Sciences
POB 127
H-1364 Budapest, Hungary
e-mail: h1222tot@huella.bitnet