WHITE NOISE ANALYSIS AND FUNCTIONAL INTEGRALS

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ABSTRACT. We review some of the recent developments of White Noise Analysis as preparation for a discussion of a recent, mathematically rigorous and effective reformulation of the Feynman integral as one involving generalized functions of infinite dimensional analysis. In a last section we indicate a first step towards a similar treatment of Schwinger's functional formulation for the expectation values and his functional differential equations for quantum field theory.

1. Introduction

Modeling of fluctuating systems with infinitely many independent variables requires an infinite dimensional calculus. In particular, averaging over fluctuations of such systems is often attempted through functional integrals. Recently considerable progress has been made in the development of these mathematical tools, and our goal here is to argue this point. We shall do so essentially in two steps - first by presenting the mathematics, and then its applicability. Results have grown in such a way that any review attempting a reasonable measure of completeness would assume the size of a monograph [1]. Hence we shall limit the first part of our discussion to the necessary minimum, and the second to essentially only one type of application as an example for the effectiveness of the framework, and shall give a number of references for further study, for the reader who judges our case favorably and wants to learn more.

The mathematics in question is that of stochastic resp. infinite dimensional analysis. Here WNA is beginning to be considered as a manageable and useful structure [1]. Development has gone in two large directions. The first, obvious one has been to develop intensively this infinite dimensional analysis. Concepts such as spaces of test and generalized functions and the study of their respective properties, differential calculus for test functions and distributions, integration, Fourier analysis [3], the study of Dirichlet forms [4][5] and the corresponding Markov processes [6], rotation

[1] see e.g. the remarks in [2]

group [7], etc. have been set up and discussed for infinitely many variables within the framework of WNA. Secondly, as results began to accumulate and structures to emerge, it became increasingly interesting to generalize these: to other Gelfand triples [8], to other Gaussian spaces [9], and to non Gaussian settings [10].

At the same time applications have multiplied. There is the field of stochastic differential equations and of fixed point methods for their solution [11][12], with applications to turbulent transport and to the flow through random media [13] , polymer interactions have been modeled in terms of self intersection local times using generalized White Noise functionals [14], states and dynamics of constructive quantum field theory have been reformulated in terms of positive White Noise distributions, resp. through the corresponding Dirichlet forms [15]. Finally there is the vast field of "functional integrals" in various domains of physics for which WNA provides a well-defined and manageable framework. It is this topic that will serve as our example; more specifically we shall focus on the Feynman "integral". For physics this concept is one of central interest, mathematically it is a kind of worst case because of its oscillatory nature [16][17][18][19].

2. **Basic Concepts of WNA**

The point of departure is the Gaussian white noise measure μ on (the Borel algebra over cylinder sets of) tempered distributions $\omega \in S^*(R^d)$, with characteristic function

$$C(f) = E(e^{i<\omega,f>}) = \int_{S^*} d\mu\,[\omega]\, e^{i<\omega,f>} = e^{-\frac{1}{2}\int f^2(t)dt},\ f \in S(R^d). \tag{1}$$

(We shall mainly focus on d=1 to keep notation simple).

The construction of suitable classes of white noise functionals starts from the Hilbert space

$$(L^2) \equiv L^2(S^*, d\mu). \tag{2}$$

Polynomials in $< \omega, f >$, $f \in S$, span a dense subspace P of (L^2), and by the Gaussian nature of the measure μ one can construct a base of Hermite polynomials. In fact there are vector valued distributions, denoted by $:\omega^{\otimes n}:$, i.e. such that $\phi_n(\omega) =<: \omega^{\otimes n}:, F_n >$ defines an element ϕ_n of (L^2) for any symmetric test function $F_n \in Sy\, S(R^n)$, and furthermore the (L^2) scalar product of such ϕ is equal to

$$(\phi_n, \phi_m) = \delta_{m,n} n! \int F_n^* F_m d^m t \tag{3}$$

This allows extension by continuity to all square integrable kernel functions F, and gives rise to the canonical isomorphism between (L^2) and a symmetric Fock space

$$(L^2) \simeq \bigoplus_n SyL^2(R^n, n!dt), \tag{4}$$

generated by the correspondence $\phi_n \simeq F_n$.

We recall the important role played by $F_1 = 1_{[0,\tau]} \in L^2(R, dt)$. The corresponding ϕ_1 is a version of the Wiener Brownian motion process, and we shall denote this white noise functional ϕ_1 by $B(\tau)$. Formally we would write

$$B(\tau)(\omega) =< \omega, 1_{[0,\tau]} >= \int_0^\tau \omega(t)dt \text{ , resp. } \omega(t) = \frac{d}{dt}B(t)(\omega), \tag{5}$$

i.e., in the sense of distributions, white noise is the derivative of Brownian motion, and Brownian functionals are functions of white noise.

Often such functions will arise heuristically in applications, and it will not always be evident whether they are in (L^2). By the isomorphism (4) it suffices to represent such a function as a convergent sum

$$\phi(\omega) = \sum_{n=0}^{\infty} \phi_n(\omega) = \sum_{n=0}^{\infty} <: \omega^{\otimes n} :, F_n > \tag{6}$$

with kernel functions F in Fock space. To obtain such an expansion the following definition is of help.

We consider first the two white noise functions $\exp(i < \omega, f >)$ as well as

$$: \exp(< \omega, f >) :\equiv \sum \frac{1}{n!} <: \omega^{\otimes n} :, f^{\otimes n} >= C(f) \exp(< \omega, f >) \text{ for } f \in S(R) \tag{7}$$

Then we can make the

Definition 1. *The transformation defined for all test functions f and (L^2) functions ϕ by the scalar product*

$$(S\phi)(f) \equiv (\overline{\phi}, : \exp(< ., f >) :). \tag{8}$$

is called the S-transform of ϕ at f . Likewise the T-transform is defined by the scalar product

$$(T\phi)(f) \equiv (\overline{\phi}, \exp(i < ., f >) \tag{9}$$

We note that both transformations are well-defined since both exponential functions are in (L^2). They are linear, note the complex conjugation in the definition.

Furthermore they are related through $(S\phi)(f) = C(f).(T\phi)(-if)$. With regard to the kernel functions F it is very useful to remark their appearance in the S-transform; from (8), (6) and (3) one finds immediately that they appear in the multilinear expansion of the S-transform:

$$(S\phi)\ (f) = \sum_{n=0}^{\infty} \int_{R^n} F_n(t_1, ..., t_n)\ f(t_1)...f(t_n)\ d^n t \tag{10}$$

Knowledge of the S-transform in the form (10) reduces the question of the (L^2) property of ϕ to that of whether $\oplus F_n$ is in Fock space.

Here are some counterexamples:

(1) "local Wick powers"

$$\phi(\omega) =: \omega^m(t) : \qquad (S\phi)(f) = f^m(t) \tag{11}$$

Comparison with (10) shows that the F_n are all zero, except for n=m. F_m is a generalized, non-L^2 kernel, hence ϕ cannot be in (L^2) .

(2) "Donsker's δ - function":

$$\phi = \delta(B(t) - a) \qquad (S\phi)(f) = (2\pi t)^{-\frac{1}{2}} e^{-\frac{1}{2t}(F(t)-a)^2}, \tag{12}$$

with $F(t) = \int_0^t f(s)ds$. Here one observes by expanding the S-transform as in (10) that in fact all the kernels are square integrable, yet ϕ of course fails to be in (L^2) . This is reflected in the divergence of the Fock space norm, $\sum_n \| F_n \|^2 = \infty$. Consequently an enlargement of our function space to include generalized white noise functionals such as in the two examples above should relax the two conditions that define Fock space kernels: square integrability of the kernel functions and also the square summability of their norms.

To do so we will embed (L^2) in a Gel'fand triple, by first constructing a space of test functions (S) and then its dual $(S)^*$, the space of generalized functions ("Hida distributions"):

$$(S) \subset (L^2) \subset (S)^* \tag{13}$$

Recall that the Schwartz test functions space $S(R)$ can be constructed as the intersection (topologically: the projective limit) of the domains of all positive powers of the operator

$$A = -d^2/dt^2 + t^2 + 1, \tag{14}$$

self adjoint in $L^2(R, dt)$. Here we proceed similarly, using the "second quantization" [20] $\Gamma(A)$ of A,

$$(S) \equiv \lim_{\overleftarrow{n}} (S)_n, \text{ with } (S)_n \equiv D(\Gamma(A^n)). \tag{15}$$

We should list here at least some of the useful properties of these new spaces:

- (S) is nuclear and an algebra under pointwise multiplication.
- Complex scaling given by

$$\sigma_z : \phi \to \phi_z, \text{with } \phi_z(\omega) = \phi(z\omega), \tag{16}$$

  extends to a continuous linear transformation on (S) for all complex z.
- Elements of (S) have Gateaux derivatives

$$D_h\phi(\omega) = \lim_{e \to 0} \frac{1}{e}(\phi(\omega + h) - \phi(\omega)) \tag{17}$$

in (S) , for any distribution valued direction $h \in S^*(R)$. Choosing in particular

- for h the Dirac distribution δ_t, the corresponding derivative operator is commonly denoted by ∂_t ("Hida derivative").
- Further, (S) is Fréchet differentiable. $\nabla : (S) \to S(R) \otimes (S)$ is given by $(\nabla\phi)(t, \omega) = \partial_t\phi(\omega)$. We have $D_h\phi =< h, \nabla\phi >$ and $\| \nabla\phi \|^2_{L^2} \in (S)$ for all Hida test functions ϕ.
- Every Hida test function ϕ has a version

$$\phi(\omega) = \sum_{n=0}^{\infty} <: \omega^{\otimes n} :, F_n > \tag{18}$$

with smooth kernels $F^{(n)} \in SyS(R^n)$, extending continuously to all $\omega \in S^*(R)$[21].

For the dual space $(S)^*$ of Hida distributions we note that T$-$ and S-transform extend to $(S)^*$, as application of the distribution Φ to exponential functions which are in (S)

$$(S\Phi)(f) =< \Phi, : \exp(< ., f >) :> \tag{19}$$

$$(T\Phi)(f) =< \Phi, \exp(i < ., f >) > . \tag{20}$$

In fact the images of Hida distributions are conveniently characterized by the fact that they are "U-functionals", as defined in

Definition 2. *A mapping G from Schwartz test function space $S(R)$ to C is called a "U-functional" if*

(1) $G(\lambda f_1 + f_2)$ has an analytic extension in λ for any pair f_i of test functions, and

(2) a bound

$$\mid G(zf) \mid \leq C_1 \exp(C_2 \mid z \mid^2 \| A^p f \|_2^2) \tag{21}$$

holds for some positive C_i and p, and all complex z.

With this definition one has a complete characterization of generalized white noise functionals in

Theorem 1. [**22**] *The following are equivalent:*
(1) G is a U-functional
(2) G is the S-transform of a Hida distribution $\Phi \in (S)^*$
(3) G is the T-transform of a Hida distribution $\overset{\wedge}{\Phi} \in (S)^*$.

Corollary 1. [**1**] *If a sequence of U-functionals converges pointwise and obeys a common bound as in (21), then the corresponding Hida distributions converge strongly.*

Remarks:
(1) For generalizations see [2][8][9]

(2) Note that U-functionals form an algebra under pointwise multiplication; this implies two algebraic structures on $(S)^*$ [1]

(3) The linear map $T^{-1}S : \Phi \to \overset{\wedge}{\Phi}$ generalizes Fourier transforms to the infinite dimensional setting ([1]. Ch.9).

(4) Returning to our examples (11) and (12) we now recognize them immediately as Hida distributions, since the S-transforms obviously are U-functionals. The same is true for "normalized Gaussians" $\phi(\omega) = N \exp < \omega, K\omega >$

$$\phi(\omega) = \frac{e^{<\omega, K\omega>}}{E(e^{<\omega, K\omega>})} \text{ with } (S\phi)(f) = e^{<f \frac{K}{1-2K} f>}; \tag{22}$$

we note in passing that applying the inverse S-transform to the rhs allows to define the normalized exponential ϕ for a large class of kernels K for which the direct definition of ϕ on the left would break down.

Another useful result concerns the integration of families of Hida distributions.

Theorem 2. [1] *Let (Ω, B, m) be a measure space, and Φ_λ a mapping defined on Ω with values in $(S)^*$. We assume that the T-transform of Φ_λ*

(1) is an m-measurable function of λ for any test function f

(2) obeys an estimate as in (21) for some fixed p and for $C_1(\lambda) \in L^1(m), C_2 \in L^\infty(m)$

Then Φ_λ is (Bochner-) integrable in the Hilbert space $(S)_{-q}$ for q large enough,

$$\int_\Omega dm(\lambda)\Phi_\lambda \in (S)^* \tag{23}$$

and we may interchange T-transform and integration:

$$T(\int_\Omega dm(\lambda)\Phi_\lambda)(f) = \int_\Omega dm(\lambda)(T(\Phi_\lambda)(f)). \tag{24}$$

3. The Feynman Integral.

Since its inception almost fifty years ago the Feynman integral has been the prototype of a "functional integral", extremely inspiring for the most varied branches of quantum physics whenever a transition from classical to quantum dynamics was to be implemented, but extremely tough to tackle as a mathematical object. We learn that it's not really an integral, but rather a certain limit of finite dimensional integrals, or the analytic continuation of an integral, or an integral w.r. to a pro-measure, etc. In view of these complications progress has been slow with regard to the two essential mathematical problems of Feynman's approach to quantization - the still very limited class of admissible interactions, and the justification of those formal manipulations which the Feynman integral suggests. It is these latter that bring out the full physical significance of the functional integral ansatz, hence it would be extremely desirable to make them reliable.

In view of these deficits it may be worth while to take another look - in the light of WNA - at objects like

$$\int d^\infty x(\tau) \exp(\frac{i}{h}\int L(\dot{x}, x, \tau)d\tau)F[x] \tag{25}$$

L is the Lagrangian from classical mechanics; we shall be considering the standard case where

$$S[x] = \int L(\dot{x}, x, \tau)d\tau = S_0[x] + S_1[x] = \int \frac{\dot{x}^2(\tau)}{2m}d\tau - \int V(x)d\tau. \tag{26}$$

Given the necessity of generalizing the concept of integration it is natural to ask whether we might not proceed as in finite dimensional analysis and interpret equation (25) as the action of a (Hida) distribution on the test functional F.

Of course, the expression in equation (25) - at least formally - contains more than just this; it suggests an explicit form for this distribution, and we shall take Feynman's hints for its construction.

To begin with we shall make an ansatz for the trajectories $x(.)$ in terms of Brownian paths starting at an initial point x_0 :

$$x(t) \equiv x_0 + \sqrt{\frac{\hbar}{m}} \int_0^t \omega(\tau)\ d\tau. \tag{27}$$

As a consequence, the exponential in the Feynman integral (25) would - at least formally - read

$$e^{\frac{i}{\hbar}S_0} = e^{\frac{i}{2}\int \omega^2(\tau)d\tau} \tag{28}$$

We should note, however, that implicit in the heuristic equation (25) there is an integration with respect to a (nonexisting) "flat" measure and not w.r. to the Gaussian measure of Brownian paths or of White Noise, formally

$$d\mu = N^{-1} e^{\frac{-1}{2}\int \omega^2(\tau)d\tau} d^\infty \omega.$$

Hence we shall try to cancel this Gaussian "density" by including its inverse in the Feynman integrand, and defining

$$I_0 \equiv N \exp\left(\frac{i+1}{2}\int \omega^2(\tau)d\tau\right) \tag{29}$$

where Nexp is the normalized Gaussian introduced in equation (22), with K=$\frac{1+i}{2}1$, i.e. with T-transform

$$TI_0(f) \equiv \exp\left(\frac{-i}{2}\int f^2(\tau)d\tau\right).$$

We may fix the final point x of paths at time t through a Donsker δ-function, i.e. by considering $I_0\delta(x(t)-x)$. It is worth noting here that this confronts us with the problem of having to justify multiplication of two distributions. We shall postpone this discussion, for now it is sufficient to note that it can be done in various equivalent ways by regularizations of the $\delta-$function and limit-taking; for the moment it is sufficient to state the result - in terms of T-transforms -

$$T(I_0\delta)(f) = \frac{1}{\sqrt{2\pi i t}} \exp\left(\frac{-i}{2}\int f^2(s)ds + \frac{i}{2t}\left(\int_0^t f(s)ds + x - x_o\right)^2\right) \tag{30}$$

and to observe that Theorem 1 assures us of the fact that this expression does indeed define a unique Hida distribution $\mathrm{I}_0\delta$.

While this takes care of the mathematical side we must still verify that our construct makes sense in terms of physics. We do so by observing

- The expectation of $I_0\delta$, given by $\mathrm{E}(I_0\delta) \equiv T(I_0\delta)(0)$ is evidently equal to the propagator for a free quantum mechanical particle
$$K_0(x,t;x_0,0) = \frac{1}{\sqrt{2\pi i t}} \exp(\frac{i}{2t}(x-x_o)^2)$$
- In fact, even the T-transform for $f \neq 0$ has a quantum mechanical interpretation; it is - up to trivial factors - the propagator $\mathrm{K}_0^{(\dot{f})}$ for a particle in the field of a time dependent linear potential $V_t(x) = \dot{f}(t)x$:
$$T(I_0\delta)(f) = E(I_0\delta \cdot e^{i\int x(t)\dot{f}(t)dt}) = K_0^{(\dot{f})}(x,t;x_0,o)e^{ixf(t)-ix_0 f(o)}e^{-\frac{i}{2}\int_{\Delta^c} f^2(t)dt} \tag{31}$$
(where Δ^c is the complement of the interval $\Delta = [0,t].$). Mathematically one verifies this simply by substituting the rhs of (30) into equation (31) and checking that the resulting expression for K obeys the appropriate Schrödinger equation
$$i\partial_t K_0^{(\dot{f})}(x,t;x_0,o) = (H_0 + \dot{f}(t)x)K_0^{(\dot{f})}(x,t;x_0,o)$$
and boundary condition
$$K_0^{(\dot{f})}(x,t;x_0,o)\mid_{t=+0} = \delta(x-x_0),$$
in physics terms we recognize the well known correspondence between generating functionals (T-transforms) and the occurrence of an additional linear ("external source") term in the Hamiltonian [25].

The next step to tackle is the introduction of interactions. In accordance with the ansatz of (25,26) this should be done through multiplication with another Brownian functional:
$$I \equiv I_0\delta \cdot e^{-i\int_0^t V(x(s))ds}. \tag{32}$$

Some examples may be studied through closed form evaluation, such as potentials V that are polynomials of no more than second order (see e.g.[17]). Another large class can be handled by expansion of I into a power series , with the ansatz
$$V(x(t)) = \int dv(x)\delta(x(t)-x). \tag{33}$$

In particular one can show the following

Theorem 3. *[18] For any finite signed Borel measure v with compact support*

$$I=\sum_{n=0}^{\infty}(-i)^n\int_{\Delta_n}d^nt\prod_{l=1}^{n}(\int dv(x_l)\delta(x(t_l)-x_l))\cdot I_o\cdot\delta(x(t)-x) \tag{34}$$

is a Hida distribution.

Equation (34) can be viewed as a strong version of a Dyson expansion: not for the Feynman integral, but for the distribution itself. We shall not reproduce the proof here in detail, it seems worth while however to review its salient points as an illustration of White Noise Analysis methods:

- First we observe that I depends on White Noise through the product of I_o with Donsker $\delta-$functions. This constitutes a slight generalization of equation (30), again the T-transform can be calculated in closed form:

$$T(I_o\prod_{l=1}^{n}\delta(x(t_l)-x_l))(f)=e^{ix_nf(t_n)-ix_0f(o)}e^{-\frac{i}{2}\int_{\Delta^c}f^2(t)dt}\prod_{l=1}^{n}K_0^{(\dot{f})}(x_l,t_l;x_{l-1},t_{l-1}),$$

  with theorem 1 assuring us that this indeed defines a Hida distribution.

- Next the x- an t-integrations in (34) must be justified. Theorem 2 ensures their existence as Bochner integrals.

- Finally the convergence of the sum is controlled by Corollary 1.

With the existence of this Hida distribution established we should again check that the construction produces the right quantum mechanical object. As in equation (31) for the case V$\equiv 0$ we would expect that T(Iδ) should be related to a quantum mechanical propagator. We set

$$K^{(\dot{f})}(x,t;x_0,o)\equiv T(I\delta)(f)e^{-ixf(t)+ix_0f(o)}e^{\frac{i}{2}\int_{\Delta^c}f^2(t)dt}. \tag{35}$$

In view of the very singular potentials that we consider we replace the Schrödinger equation by the corresponding integral equation for the propagator and verify whether the above $K^{(\dot{f})}$ is indeed a solution. This is done in the following

Theorem 4. *[19] Under the conditions of theorem 3 the function $K^{(\dot{f})}$ of equation (35) is a solution to the quantum mechanical propagator equation*

$$\begin{aligned}&K^{(\dot{f})}(x,t;x_0,o)=\\&K_0^{(\dot{f})}(x,t;x_0,o)-i\textstyle\int_0^t ds\int dyK_o^{(\dot{f})}(x,t;y,s)V(y)K^{(\dot{f})}(y,s;x_0,o).\end{aligned} \tag{36}$$

In the above discussion the multiplication problem inherent in equation (32) was reduced to multiplication of I_0 by Donsker δ-functions through equation (33) and a power series expansion. While this allows for a large class of singular potentials, it is also desirable to go beyond the perturbative framework. To this end we must probe deeper into equation (32). Outside the regime of analytic perturbation theory we should expect to find e.g. real potentials V such that (32) is well-defined whereas this is no more the case if we change their sign, i.e. if we substitute $V \to -V$. The multiplication problem in (32) is clarified through the following observation. The scaling operator σ of equation (16) acting continuously on (S), its adjoint $\sigma^\dagger$ is well defined on $(S)^*$. In fact one calculates, using e.g. T-transforms, that

$$\sigma^\dagger_{\sqrt{i}}\, 1 = I_0.$$

Once this is established it is not hard to show which sequences of test functions from (S) produce convergent sequences of multipliers. We give the result in the following

Theorem 5. *[1][23] Let ϕ_n be a sequence of Hida test functions. Then the following are equivalent:*
(1) the sequence $I_o\phi_n$ converges to a distribution $I \in (S)^*$
(2) the sequence $\sigma_{\sqrt{i}}\, \phi_n$ converges to a distribution $I \in (S)^*$
(3) the sequence of expectations $E(\psi\cdot\, \sigma_{\sqrt{i}}\, \phi_n)$ converges for all $\psi \in (S)^*$.

If one of the conditions holds, the action of I is given by

$$< I, \psi > = \lim_n\ E(\sigma_{\sqrt{i}}(\ \phi_n \cdot \psi)).$$

Complex scaling is not new to the mathematical theory of Feynman integrals; see e.g. the work of Doss [24] and references there. Doss solves the Schrödinger equation with potential V and initial value f through the expectation

$$\psi(x,t) = E(f(x+\sqrt{i}B(t)) \exp(-i\int_0^t V(x+\sqrt{i}B(s))ds)),$$

if the functions f and V satisfy suitable analyticity and growth conditions.

White Noise Analysis allows us to go beyond this framework, by considering *limits of analytic functions,* in accordance with theorem 5. Work on this is in progress, both with regard to larger classes of (non-analytic) potentials and regarding non-analytic wave functions. We note in passing that it is not hard to check that Donsker's δ-function may be extended to complex argument as a Hida distribution, i.e. propagators, with f=δ, can be discussed.

4. Quantum Field Theory - an Outlook

In quantum field theory there corresponds to the Feynman integral the "τ-functional", generating functional of the time-ordered vacuum expectation values. For a scalar field A this would take the form

$$\tau\{j\} =< 0|Te^{i\int A(x)j(x)dx}|0 >=\sum_n \frac{i^n}{n!}\int dx_1...dx_n j(x_1)...j(x_n)\tau(x_1...x_n)$$

with

$$\tau(x_1...x_n) =< 0|TA(x_1)...A(x_n)|0 >=< 0|A(x_{\sigma(1)})...A(x_{\sigma(n)})|0 > .$$

Here $x = (x^0...x^s) \in R^{s+1}$, with scalar product xy$=x^0y^0 - \sum_1^s x^i y^i$, and x_0 the time component. σ is the permutation which provides time-ordering, i.e. for which $\mathrm{x}^0_{\sigma(k)} \geq x^0_{\sigma(k-1)}$ for all k. Correspondingly we shall need to introduce s+1-parameter White Noise $\omega = \omega(x)$. Concepts such as e.g. the Hida derivative $\partial = \partial_x$, and results such as the characterization theorem 1 remain essentially unaltered [9], and one has e.g.

Theorem 6. *If A is a relativistic free field of mass m> 0 then $\tau\{\cdot\}$is a U-functional. I. e. there is a Hida distribution Θ such that$< \Theta, e^{i<\omega,j>} >= \tau\{j\}$*

Proof: From the explicit expression

$$\tau\{j\} = \exp(-\frac{i}{2}\int dx \int dy j(x)\Delta_F(x-y)j(y)) \tag{37}$$

with

$$\Delta_F(x) = \frac{1}{(2\pi)^4}\int d^{s+1}p \frac{e^{-ipx}}{p^2 - m^2 + i\epsilon}.$$

one sees that $\tau\{\lambda j_1 + j_2\}$ extends from real λ to the whole complex plane as an analytic function. An estimate for τ as in equation (21) follows from

$$|\int dx \int dy j(x)\Delta_F(x-y)j(y))| \leq |\langle \Delta_F, j * j\rangle| \leq c|A^p j|^2$$

for suitable constants c and p.□

For this simple case Schwinger's functional differential equation [25] is of course linear. On the level of the τ-functional one obtains directly from 37 that

$$(i(\Box + m^2)\frac{\delta}{\delta j(x)} + j(x))\tau\{j\} = 0,$$

which translates into the following equation for the Hida distribution $\Theta = \Theta_0$:

$$(\partial_x^\dagger - i(\Box + m^2)\omega(x))\Theta_0 = 0 \tag{38}$$

since the T-transform intertwines multiplication and derivatives as follows:

$$\omega(x) \overset{T}{\leftrightarrow} -i \cdot \frac{\delta}{\delta j(x)} \text{ and } \partial_x^\dagger \overset{T}{\leftrightarrow} i \cdot j(x).$$

Formally, or invoking regularizations, interactions would be introduced as in equation (32) through an additional action functional S_1 in

$$\Theta = \Theta_0 e^{iS_1}$$

so that

$$(\partial_x^\dagger - i(\Box + m^2)\omega(x) - i \cdot \partial_x S_1)\Theta = 0,$$

work is in progress regarding this circle of ideas.

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