QUANTUM MECHANICAL PROPAGATORS IN TERMS OF HIDA DISTRIBUTIONS

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ABSTRACT

We review some basic notions and results of White Noise Analysis that are used in the construction of the Feynman integrand as a generalized White Noise functional. After sketching this construction for a large class of potentials we show that the resulting Feynman integrals solve the Schrödinger equation .

1. Introduction

The idea of realizing Feynman integrals within the White Noise framework goes back to [15]. The "average over all paths" is performed with a Hida distribution as the weight. The existence of such Hida distributions corresponding to Feynman integrands has been established in [1]. In [6] Khandekar and Streit moved beyond the existence theorem by giving an explicit construction for a large class of potentials including singular ones. Here we generalize their construction by allowing time-dependent potentials of noncompact support. Also we prove that the expectation of the thus constructed Feynman integrand, i.e. the Feynman integral, does indeed solve the integral equation for quantum mechanical propagators, which corresponds to the Schrödinger equation. Furthermore we elaborate an idea suggested in [1] concerning the physical interpretation of T-transforms as propagators corresponding to systems with an additional "source " term in the potential.

2. White Noise Analysis

We start from the fundamental Gel'fand triple:

$$S(R) \subset L^2(R) \subset S'(R),$$

where $S'(R)$ denotes the Schwartz space. Using Minlos' theorem we construct a measure space $(S'(R), B, \mu)$ called the White Noise space by fixing the characteristic functional in the following way:

$$C(\xi) = \int_{S'(R)} \exp i \langle \omega, \xi \rangle \; d\mu(\omega) = \exp\left(-\frac{1}{2} |\xi|_0^2\right) \;, \; \xi \in S(R);$$

here $\langle ., \; . \rangle$ denotes the pairing between $S'(R)$ and $S(R)$ and $|.|_0$ the norm on $L^2(R)$.

Within this formalism a version of Wiener's Brownian motion is given by:

$$B(t) := \left\langle \omega, 1_{[0,t]} \right\rangle = \int_0^t \omega(s) \; ds.$$

We now consider the space (L^2), which is defined to be the complex Hilbert space $L^2(S'(R), B, \mu)$. For applications the space (L^2) is often too small. Here we enlarge the space (L^2) by first choosing a special subspace (S) of test functionals. Then the corresponding Gel'fand triple is:

$$(S) \subset \left(L^2\right) \subset (S)^* .$$

Elements of the space $(S)^*$ are called Hida distributions and its explicit construction is given in [4]. Instead of reproducing this construction here we shall completely characterize Hida distributions by their S- or T-transforms $(\Phi \in (S)^*, \; \xi \in S(R))$:

$$T\Phi(\xi) \equiv \langle\langle \Phi, \exp \; i \langle ., \xi \rangle \; \rangle\rangle = \int_{S'(R)} \exp\left(i \langle \omega, \xi \rangle\right) \Phi(\omega) \, d\mu(\omega), \tag{1}$$

$$S\Phi(\xi) \equiv \langle\langle \Phi, : \exp \langle ., \xi \rangle : \rangle\rangle,$$

here $\langle\langle ., . \rangle\rangle$ denotes the dual pairing between (S) and $(S)^*$ and we have used the traditional notation:

$$: \exp \langle ., \xi \rangle : \; \equiv \exp\left(\langle ., \xi \rangle - \frac{1}{2} |\xi|_0^2\right), \; \xi \in S(R) \tag{2}$$

As S- or T-transforms of Hida distributions possess analytic continuations, we have the relation:

$$T\Phi(\xi) = C(\xi)\ \ S\Phi(i\xi).$$

Let us now quote the above mentioned characterization theorem, which is due to Potthoff and Streit [13] and has been generalized in various ways (see eg.[8],[11],[16]) .

Theorem 2.1: The following statements are equivalent:

1. F: $S(R) \to C$ such that:

   (A) "Ray-Analyticity": For all $\zeta, \xi \in S(R)$ the mapping $\lambda \mapsto F(\lambda\xi+\zeta)$, $\lambda \in R$, extends to an entire function $z \mapsto f(z,\xi,\zeta)$, $z \in C$.

   (B) "Bound": f is uniformly of order two, i.e. there exist $p \in N_0$ and constants $P, Q > 0$ so that for all $z \in C$, $\xi \in S(R)$,[1]

   $$|f(z,\xi,0)| \leq P \exp\left(Q\,|z|^2\,|\xi|_p^2\right).$$

2. F is the S- transform of a Hida distribution $\Phi \in (S)^*$.

3. F is the T- transform of a Hida distribution $\overset{\wedge}{\Phi} \in (S)^*$.

A functional satisfying (A) and (B) is usually called U-functional.
Since the space of U-functionals forms an algebra under pointwise multiplication and S- as well as T-transform are injective, we can introduce two algebraic structures in the space $(S)^*$, namely the Wick product $\diamond$ and a convolution $*$, which are defined as follows: Let $\Phi, \Psi \in (S)^*$, then

$$S(\Phi \diamond \Psi) = S\Phi \cdot S\Psi, \quad T(\Phi * \Psi) = T\Phi \cdot T\Psi.$$

Now we want to mention two other important consequences of theorem 2.1. The first one concerns the convergence of sequences of Hida distributions and can be found in [4], [13].

Theorem 2.2: Let $\{F_n\}_{n\in N}$ denote a sequence of U-functionals with the following properties:

1. For all $\xi \in S(R)$, $\{F_n(\xi)\}_{n\in N}$ is a Cauchy sequence,

[1] $\{|.|\,p, p \in N\}$ is a countable system of norms topologizing $S(R)$. Any of these norms on $S(R)$ may be used here.

2. There exist $P, Q > 0$ and $p \in N_0$, such that the bound

$$|F_n (z\xi)| \leq P \exp \left(Q \, |z|^2 \, |\xi|_p^2 \right) \ , \ \xi \in S(R) \, , \ z \in C$$

holds uniformly in $n \in N$ (here F_n denotes the entire analytic extension).

Then there is a unique $\Phi \in (S)^*$ such that $T^{-1} F_n$ converges strongly to Φ.

This theorem is also valid for S-transforms. The second corollary of theorem 2.1 deals with the integration of Hida distributions which depend on an additional parameter (see[4], [6]).

Theorem 2.3: Let (Ω, B, m) denote a measure space and $\lambda \mapsto \Phi(\lambda)$ a mapping from Ω to $(S)^*$. Let $F(\lambda)$ denote the T-transform of $\Phi(\lambda)$ which satisfies the following conditions for all $\lambda \in \Omega$:

1. $\lambda \mapsto F(\lambda, \xi)$ is a measurable function for all $\xi \in S(R)$,

2. There exists $p \in N_0$ such that

$$|F \ (\lambda, z\xi)| \leq P(\lambda) \exp \left(Q(\lambda) \, |z|^2 \, |\xi|_p^2 \right), \ \xi \in S(R) \, , \ z \in C$$

with $Q \in L^\infty (\Omega, m)$ and $P \in L^1 (\Omega, m)$.

Then Φ is Bochner integrable[2] and

$$\int_\Omega \Phi(\lambda) \ dm(\lambda) \in (S)^*$$

Let $\varphi \in (S)$, then

$$\left\langle\!\!\left\langle \int_\Omega \Phi(\lambda) \ dm(\lambda) \, , \varphi \right\rangle\!\!\right\rangle = \int_\Omega \langle\langle \Phi(\lambda) \, , \varphi \rangle\rangle \ dm(\lambda) \ .$$

The last equation allows us to intertwine T-transform and integration

$$T \left(\int_\Omega \Phi(\lambda) \ dm(\lambda) \right) (\xi) = \int_\Omega T(\Phi(\lambda)) (\xi) \ dm(\lambda) \, .$$

Again the same theorem holds for the S-transform.

[2]with respect to one of the Hilbertian norms topologizing (S)*

Before we close this section we would like to give two examples of Hida distributions.

Example 2.4: *Donsker's δ-function*

Now we study the following informal expression:

$$\Phi = \delta\left(B\left(t\right) - a\right)$$

$$\Phi = \delta\left(\left\langle ., 1_{\Delta}\right\rangle - a\right), \;\; a \in R, \;\; \Delta = [0, t) \tag{3}$$

The S- transform of Φ is calculated to be [4]:

$$S\Phi\left(\xi\right) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{1}{2t}\left(\int\limits_0^t \xi\left(s\right)\; ds - a\right)^{\mathbf{2}}\right)$$

and theorem 2.1 gives immediately that Φ is a well defined element in $(S)^*$.

Example 2.5:

Let us consider the following informal expression for complex $c \neq \frac{1}{2}$:

$$\exp\left(c\int_a^b \omega^2\left(s\right)\; ds\right).$$

Calculation of its S- transform produces a U-functional "up to an infinite constant". So as a renormalization we omit this divergent factor and get a well defined U-functional:

$$F\left(\xi\right) = \exp\left(\frac{c}{1-2c}\int_a^b \xi^2\left(s\right)\; ds\right)$$

Hence we may define $N\exp\left(c\int_a^b \omega^2\left(s\right)\; ds\right) \equiv S^{-1}F$. Then roughly speaking

$$N\exp\left(c\int_a^b \omega^2\left(s\right)\; ds\right) = \frac{\exp\left(c\int_a^b \omega^2\left(s\right)\; ds\right)}{E\left(\exp\left(c\int_a^b \omega^2\left(s\right)\; ds\right)\right)}.$$

See [4] for all details.

3. The Feynman integrand as a Hida distribution

We follow Ref.[1] and [15] in viewing the Feynman integral as a weighted average over Brownian paths. These paths are modeled within the White Noise framework according to

$$x(t) \equiv x_0 + \sqrt{\frac{\hbar}{m}} \int_{t_0}^{t} \omega(\tau) \; d\tau,$$

in the sequel we set $\hbar = m = 1$.

In Ref.[1] the (distribution-valued) weight for the free motion from $x(t_0) = x_0$ to $x(t) = x$ is constructed from a kinetic energy factor $N \exp\left(\frac{i}{2}\int_{t_0}^{t} \omega^2(\tau) \; d\tau\right)$ and a Donsker delta function $\delta(x(t) - x)$. Furthermore a factor $\exp\left(\frac{1}{2}\int_{t_0}^{t} \omega^2(\tau) \; d\tau\right)$ is introduced to compensate the Gaussian fall-off of the White Noise measure in order to mimic Feynman's non-existing "flat" measure $D^\infty x$. Thus in Ref.[1] the Feynman integrand for the free motion reads:

$$J = N \exp\left(\frac{i+1}{2} \int_{t_0}^{t} \omega^2(\tau) \; d\tau\right) \delta(x(t) - x).$$

However the distribution

$$I_0 = N \exp\left(\frac{i+1}{2} \int_{R} \omega^2(\tau) \; d\tau\right)$$

has recently been seen to be particulary useful in this context because of its relation to complex scaling transformation (see Ref.[14]). It turns out that it is unnecessary to use the time interval $[t_0, t]$ in the kinetic energy factor, because the delta function introduces the interval into the resulting distribution $I_0\delta$. Indeed it will be shown that $I_0\delta$ produces the correct physical results (see below especially theorem 3.4). As the choice of $I_0\delta$ rather than J as a starting point produces only minor modifications in calculations and formulae, all the pertinent results in Ref.[1] and [6] can be established in a completely analogous manner. We give just a brief account. As in Ref.[1] $I_0\delta$ is a Hida distribution, with T- transform given by

$$TI_0\delta(\xi) = \frac{\theta(t-t_0)}{\sqrt{2\pi i\,|t-t_0|}} \exp\left(-\frac{i}{2}|\xi|^2 + \frac{i}{2\,|t-t_0|}\left(\int_{t_0}^{t} \xi(\tau)\,d\tau + x - x_0\right)^2\right),$$

where we have introduced the Heaviside function to ensure causality. Furthermore the Feynman integral $E(I_0\delta) = TI_0\delta(0)$ is indeed the (causal) free particle propagator $\frac{\theta(t-t_0)}{\sqrt{2\pi i|t-t_0|}} \exp\left[\frac{i}{2|t-t_0|} \; (x-x_0)^2\right]$. Not only the expectation but also the T- transform has a physical meaning. By a formal integration by parts

$$TI_0\delta(\xi) = E\left(I_0\delta \; e^{-i\int_{t_0}^{t} x(\tau)\dot{\xi}(\tau)\,d\tau}\right) \; e^{\,ix\xi(t) - ix_0\xi(t_0)} \; e^{-\frac{i}{2}\left|\xi_{[t_0,t]^c}\right|^2}.$$

($\xi_{[t_0,t]}c$ denotes the restriction of ξ to the complement of $[t_0,t]$). The term $e^{-i\int_{t_0}^{t} x(\tau)\dot{\xi}(\tau)\,d\tau}$ would thus arise from a time-dependent potential $W(x,t) = \dot{\xi}(\tau)\,x$. And indeed it is straightforward to verify that

$$TI_0\delta(\xi) = K_0^{\left(\dot{\xi}\right)}(x,t|x_0,t_0)\ e^{\,ix\xi(t)-ix_0\xi(t_0)}\ e^{-\frac{i}{2}\left|\xi_{[t_0,t]}c\right|^2}, \tag{4}$$

where

$$K_0^{\left(\dot{\xi}\right)}(x,t|x_0,t_0) = \frac{\theta(t-t_0)}{\sqrt{2\pi i\,|t-t_0|}} \times$$
$$\exp\left(ix_0\xi(t_0) - ix\xi(t) - \frac{i}{2}\left|\xi_{[t_0,t]}\right|^2 + \frac{i}{2\,|t-t_0|}\left(\int_{t_0}^{t}\xi(\tau)\,d\tau + x - x_0\right)^2\right) \tag{5}$$

is the Green's function corresponding to the potential W, i.e. $K_0^{\left(\dot{\xi}\right)}$ obeys the Schrödinger equation

$$\left(i\partial_t + \frac{1}{2}\partial_x^2 - \dot{\xi}(t)\,x\right) K_0^{\left(\dot{\xi}\right)}(x,t|x_0,t_0) = i\,\delta(t-t_0)\ \delta(x-x_0)\,.$$

More generally one calculates

$$T\left(I_0\prod_{i=1}^{n+1}\delta\left(x(t_i)-x_i\right)\right)(\xi) = e^{-\frac{i}{2}\left|\xi_{[t_0,t]}c\right|^2} e^{\,ix\xi(t)-ix_0\xi(t_0)} \prod_{i=1}^{n+1} K_0^{\left(\dot{\xi}\right)}(x_i,t_i|x_{i-1},t_{i-1})\,. \tag{6}$$

Here $t_0 < t_1 < ... < t_n < t_{n+1} \equiv t$ and $x_{n+1} \equiv x$.

In order to pass from the free motion to more general situations, one has to give a rigorous definition of the heuristic expression

$$I = I_0\delta\exp\left(-i\int_{t_0}^{t} V(x(\tau))\ d\tau\right).$$

In Ref.[6] Khandekar and Streit accomplished this by pertubative methods in case V is a finite signed Borel measure with compact support. Here we give a brief summary of the construction taking into account the afore-mentioned modification. Also we generalize the construction by allowing time-dependent potentials and a Gaussian fall-off instead of a bounded support. Let $\Delta \equiv [\mathsf{T}_0, \mathsf{T}] \supset [t_0,t]$ and let v be a finite signed Borel measure on $R\times\Delta$. Let v_x denote the marginal measure

$$v_x(A \subset B(R)) \equiv v(A\times\Delta)$$

similary

$$v_t\left(B \subset B\left(\Delta\right)\right) \equiv v\left(R \times B\right).$$

We assume that v_x and v_t satisfy:
i) $\exists R > 0 \; \forall r > R : |v_x| \left(\{x : \;\; |x| > r\}\right) < e^{-\beta r^2}$ for some $\beta > 0$,
ii) $|v_t|$ has a L^∞density.

Let us first describe heuristically the construction by treating v as an ordinary function V before stating the rigorous result 3.1. The starting point is a power series expansion of $\exp\left(-i \int_{t_0}^t V\left(x\left(\tau\right), \tau\right) d\tau\right)$ using $V\left(x\left(\tau\right), \tau\right) = \int dx \, V\left(x, \tau\right) \; \delta\left(x\left(\tau\right) - x\right)$:

$$\exp\left(-i \int_{t_0}^t V\left(x\left(\tau\right), \tau\right) d\tau\right) = \sum_{n=0}^{\infty} \left(-i\right)^n \int_{\Delta_n} d^n t \prod_{i=1}^{n} \int dx_i \, V\left(x_i, t_i\right) \delta\left(x\left(t_i\right) - x_i\right)$$

where $\Delta_n = \{(t_1, ..., t_n) \,|\, t_0 < t_1 < ... < t_n < t\}$.

Theorem 3.1:

$$I = I_0 \delta\left(x\left(t\right) - x\right) + \sum_{n=1}^{\infty} \left(-i\right)^n \int_{R^n} \int_{\Delta_n} \prod_{i=1}^{n} v\left(dx_i, dt_i\right) \; I_0 \delta\left(x\left(t\right) - x\right) \; \prod_{j=1}^{n} \delta\left(x\left(t_j\right) - x_j\right)$$

exists as a Hida distribution in case V obeys i) and ii).

Sketch of Proof:

1) $I_n = \int_{R^n} \int_{\Delta_n} \prod_{i=1}^{n} v\left(dx_i, dt_i\right) \; I_0 \delta\left(x\left(t\right) - x\right) \; \prod_{j=1}^{n} \delta\left(x\left(t_j\right) - x_j\right)$ is a Hida distribution for $n \geq 1$. This is shown by applying theorem 2.3. Choose $q > 2$ and 0$< \gamma < \beta / q$ and p such that $\frac{1}{p} + \frac{1}{q} = 1$. Formulas (**??**) (6) yield the estimate

$$\left| T\left(I_0 \prod_{j=1}^{n+1} \delta\left(x\left(t_j\right) - x_j\right)\right)\left(z\xi\right) \right|$$

$$\leq \left(\prod_{i=1}^{n+1} \frac{\Theta(t_i - t_{i-1})}{\sqrt{2\pi \left|t_i - t_{i-1}\right|}}\right) \exp\left(\gamma \left(\sup_{0 \leq j \leq n+1} |x_i|\right)^2\right) \exp\left(\frac{1}{2}(1 + \frac{2}{\gamma} + |\Delta|)|z|^2 \left|\xi\right|_s^2\right),$$

where s is such that $\sup_{t \in \Delta} \left|\xi\left(t\right)\right| < \left|\xi\right|_s$. The property i) of v yields that $e^{\gamma x^2} \in L^q(R \times \Delta, |v|)$. Let $Q \equiv \left(\int_R |v_x| \left(dx\right) e^{\gamma q x^2}\right)^{\frac{1}{q}}$, then

$$\left(\int_{R^n} \int_{\Delta^n} \prod_{i=1}^{n} |v| \left(dx_i, dt_i\right) e^{\gamma q \left(\sup_{0 \leq i \leq n+1} |x_i|\right)^2}\right)^{\frac{1}{q}} \leq e^{\gamma |x_0|^2} e^{\gamma |x|^2} Q^n.$$

Using the property ii) of v and the formula

$$\int_{\Delta_n} d^n t \prod_{j=1}^{n+1} \frac{1}{(2\pi\,|t_j - t_{j-1}|)^\alpha} = \left(\frac{\Gamma(1-\alpha)}{(2\pi)^\alpha}\right)^{n+1} \frac{|t-t_0|^{n(1-\alpha)-\alpha}}{\Gamma\left((n+1)(1-\alpha)\right)}, \quad \alpha < 1$$

we obtain the following estimate:

$$\left(\int_{R^n}\int_{\Delta_n} \prod_{i=1}^{n} |v|\,(dx_i, dt_i) \prod_{j=1}^{n+1} \left(\frac{1}{\sqrt{2\pi\,|t_j - t_{j-1}|}}\right)^p\right)^{\frac{1}{p}} \leq |v_t|_\infty^{\frac{n}{p}} \frac{\Gamma(\frac{2-p}{2})^{\frac{n+1}{p}}}{(2\pi)^{\frac{n+1}{2}}} \frac{|\Delta|^{\frac{n}{p}-\frac{1}{2}(n+1)}}{\Gamma\left((n+1)(\frac{2-p}{2})\right)^{\frac{1}{p}}}$$

Let

$$C_n(x, |\Delta|) \equiv e^{\gamma|x_0|^2} e^{\gamma|x|^2} Q^n\, |v_t|_\infty^{\frac{n}{p}} \frac{\Gamma(\frac{2-p}{2})^{\frac{n+1}{p}}}{(2\pi)^{\frac{n+1}{2}}} \frac{|\Delta|^{\frac{n}{p}-\frac{1}{2}(n+1)}}{\Gamma\left((n+1)(\frac{2-p}{2})\right)^{\frac{1}{p}}}\,.$$

Hölder's inequality yields the following estimate:

$$\left|\int_{R^n}\int_{\Delta_n} \prod_{i=1}^{n} v\,(dx_i, dt_i)\, T\left(I_0 \prod_{j=1}^{n+1} \delta\left(x\left(t_j\right) - x_j\right)\right)(z\xi)\right|$$
$$\leq C_n \exp\left(\frac{1}{2}(1 + \frac{2}{\gamma} + |\Delta|)|\mathrm{z}|^2\, |\xi|_s^2\right) \tag{7}$$

This establishes the bound required for the application of theorem 2.3 and hence I_n exists as a Bochner integral in $(S)^*$.

2) $I = \sum_{n=0}^{\infty} I_n$ exists in $(S)^*$.

As the C_n are rapidly decreasing in n the hypotheses of theorem 2.2 are fulfilled and hence the convergence in $(S)^*$ is established. □

As an example of the class of admissible potentials take any finite signed Borel measure v satisfying i) on R. This can be as singular as desired, e.g. a sum of Delta's such as $\sum_{n\in N} e^{-n^2}\delta_n$ or a devil' s staircase. Now take two bounded measurable functions f and g on Δ. Use one to move the potential around and the other one to vary its strength: $v(x,t) = f(t)\,v(x - g(t))$.

As in the case of the free motion we expect

$$K^{\left(\xi\right)}(x, t|x_0, t_0) \equiv\ e^{\,+\frac{i}{2}\left|\xi_{[t_0,t]^c}\right|^2}\ e^{\,-ix\xi(t)+ix_0\xi(t_0)}\, TI\left(x, t|x_0, t_0\right)(\xi) \tag{8}$$

to be the propagator corresponding to the potential $W\left(x,t\right) = V\left(x,t\right) + \dot{\xi}\ (t)\,x$. More precisely we have to use the measure $\omega\left(dx, dt\right) = v\left(dx, dt\right) + \dot{\xi}\ (t)\,x\ dx\ dt$.

We now proceed to show some properties of $K^{(\dot{\xi})}$. As the propagators $K_0^{(\dot{\xi})}$ are continuous on $\mathrm{R}^2 \times \Delta_2$ (see (5)), the product $\prod_{j=1}^{n+1} K_0^{(\dot{\xi})}(x_j, t_j | x_{j-1}, t_{j-1})$ is continuous on $R^{n+1} \times \Delta_{n+1}$. Set

$$K^{(\dot{\xi})}(x,t|x_0,t_0) = \sum_{n=0}^{\infty} K_n^{(\dot{\xi})}(x,t|x_0,t_0) \tag{9}$$

where

$$K_n^{(\dot{\xi})}(x,t \mid x_0,t_0) = (-i)^n \int_{R^n} \int_{\Delta_n} \prod_{i=1}^{n} v\,(dx_i, dt_i) \prod_{j=1}^{n+1} K_0^{(\dot{\xi})}(x_j, t_j | x_{j-1}, t_{j-1})\,.$$

As the test functions ξ are real the explicit formula (5) yields

$$|K_0^{(\dot{\xi})}(x,t|x_0,t_0)\,| = \frac{\Theta(t-t_0)}{\sqrt{2\pi |t-t_0|}} \equiv M_0 \tag{10}$$

and for n≥ 1 the bounds

$$\left| K_n^{(\dot{\xi})}(x,t|x_0,t_0) \right| \leq \int_{R^n} \int_{\Delta_n} \prod_{i=1}^{n} |v|\,(dx_i, dt_i) \prod_{j=1}^{n+1} \frac{1}{\sqrt{2\pi\,|t_j - t_{j-1}|}} \tag{11}$$

$$\leq |v_t|_\infty^n \frac{|t-t_0|^{\frac{n-1}{2}}}{2^{\frac{n+1}{2}}\Gamma(\frac{n+1}{2})} \leq |v_t|_\infty^n \frac{|\Delta|^{\frac{n-1}{2}}}{2^{\frac{n+1}{2}}\Gamma(\frac{n+1}{2})} \equiv M_n.$$

Hence $\prod_{j=1}^{n+1} K_0^{(\dot{\xi})}(x_j, t_j | x_{j-1}, t_{j-1})$ is integrable on $R^n \times \Delta_n$ with respect to v^n. (This is also established in the course of a detailed proof of theorem 3.1 and we have reproduced the argument here for the convenience of the reader.) Thus we can apply Fubini's theorem to change the order of integration in $K_n^{(\dot{\xi})}$ to obtain

$$K_n^{(\dot{\xi})}(x,t|x_0,t_0) = -i \iint v\,(dx_n, dt_n)\, K_0^{(\dot{\xi})}(x,t|x_n,t_n) \;\times$$

$$(-i)^{n-1} \int_{R^{n-1}} \int_{\Delta'_{n-1}} \prod_{i=1}^{n-1} v\,(dx_i, dt_i) \prod_{j=1}^{n} K_0^{(\dot{\xi})}(x_j, t_j | x_{j-1}, t_{j-1})$$

($\Delta'_{n-1} = \{(t_1, ..., t_{n-1}) \mid t_0 < t_1 < ... < t_{n-1} < t_n\}$). This establishes the following recursion relation for $K_n^{(\dot{\xi})}$

$$K_n^{(\dot{\xi})}(x, t|x_0, t_0) = (-i) \iint v(dy, d\tau)\, K_0^{(\dot{\xi})}(x, t|y, \tau)\, K_{n-1}^{(\dot{\xi})}(y, \tau|x_0, t_0)\,. \tag{12}$$

We now claim that the series $K^{(\dot{\xi})}(y, \tau|x_0, t_0) = \sum_n K_n^{(\dot{\xi})}(y, \tau|x_0, t_0)$ converges uniformly in y, τ on $R \times (t_0, \mathsf{T})$. To see this recall the above estimate (11) which is uniform in y, τ. Because the M_n are rapidly decreasing it follows that

$$\sum_{n=1}^{\infty} \sup \left\{ \left| K_n^{(\dot{\xi})}(y, \tau|x_0, t_0) \right| ; (y,\ \tau) \in R \times (t_0, \mathsf{T}) \right\} \le \sum_{n=1}^{\infty} M_n < \infty\ .$$

Due to the uniform convergence we may interchange summation and integration in the following expression

$$\begin{aligned} &-i \iint v(dy, d\tau)\, K_0^{(\dot{\xi})}(x, t|y, \tau)\, K^{(\dot{\xi})}(y, \tau|x_0, t_0) \\ &= -i \iint v(dy, d\tau)\, K_0^{(\dot{\xi})}(x, t|y, \tau) \sum_n K_n^{(\dot{\xi})}(y, \tau|x_0, t_0) \\ &= \sum_n -i \iint v(dy, d\tau)\, K_0^{(\dot{\xi})}(x, t|y, \tau)\, K_n^{(\dot{\xi})}(y, \tau|x_0, t_0)\,. \end{aligned}$$

By the above recursion relation (12) for $K_n^{(\dot{\xi})}$ this equals

$$\sum_n K_{n+1}^{(\dot{\xi})}(x, t|x_0, t_0) = K^{(\dot{\xi})}(x, t|x_0, t_0) - K_0^{(\dot{\xi})}(x, t|x_0, t_0)\,.$$

Hence we obtain the following

Theorem 3.2:

$K^{(\dot{\xi})}$ as defined in (8) obeys the following integral equation:

$$K^{(\dot{\xi})}(x, t|x_0, t_0) = K_0^{(\dot{\xi})}(x, t|x_0, t_0) - i \iint v(dy, d\tau)\, K_0^{(\dot{\xi})}(x, t|y, \tau)\, K^{(\dot{\xi})}(y, \tau|x_0, t_0)\,.$$

In particular the Feynman integral $E(I) \equiv K$ obeys the well-known propagator equation:

$$K(x,t|x_0,t_0) = K_0(x,t|x_0,t_0) - i\iint v(dy,d\tau)\,K_0(x,t|y,\tau)\,K(y,\tau|x_0,t_0)\,.$$

We now proceed to show that this corresponds to the Schrödinger equation. To prove this we first prepare the following

Lemma 3.3:

The mapping $(x,t) \mapsto K^{\left(\dot{\xi}\right)}(x,t|x_0,t_0)$ is continuous on $R \times (t_0, \mathsf{T})$.

Proof:

Because the series (9) converges uniformly it is sufficient to show the continuity of $K_n^{\left(\dot{\xi}\right)}$. For $n = 0, 1$ this is straightforward from the explicit formula (5). For $n > 1$ we use (12) and the estimate (11) to obtain

$$\left|K_n^{\left(\dot{\xi}\right)}(x',t'|x_0,t_0) - K_n^{\left(\dot{\xi}\right)}(x,t|x_0,t_0)\right|$$

$$\le M_{n-1}\int_R\int_\Delta |v|\,(dx_n,dt_n)\left|K_0^{\left(\dot{\xi}\right)}(x',t'|x_n,t_n) - K_0^{\left(\dot{\xi}\right)}(x,t|x_n,t_n)\right|$$

. Using the explicit form (5) of $K_0^{\left(\dot{\xi}\right)}$ it is now straightforward to check that

$$\int_R\int_\Delta |v|\,(dx_n,dt_n)\left|K_0^{\left(\dot{\xi}\right)}(x',t'|x_n,t_n) - K_0^{\left(\dot{\xi}\right)}(x,t|x_n,t_n)\right|$$

$$\le \ |\, x - x' \,|\; C(x,t) \ + \ |\, t - t' \,|^{\alpha}\; C_\alpha(x,t)$$

where $0 < \alpha < \frac{1}{2}$ and $x > x'$, $t > t'$. □

An application of lemma 3.3 combined with the estimate (10) shows that $K^{\left(\dot{\xi}\right)}(.,.|x_0,t_0)$ is locally integrable on $R \times (\mathsf{T}_0, \mathsf{T})$ with respect to both v and Lebesgues measure. We can thus regard $K^{\left(\dot{\xi}\right)}$ as a distribution on $D(\Omega) \equiv D(R \times (\mathsf{T}_0, \mathsf{T}))$:

$$\left\langle K^{\left(\dot{\xi}\right)}, \varphi \right\rangle = \iint dx\,dt\,K^{\left(\dot{\xi}\right)}(x,t|x_0,t_0)\,\varphi(x,t)\,,\ \ \varphi \in D(\Omega)\,.$$

And we can also define a distribution $vK^{(\dot{\xi})}$ by setting

$$\left\langle vK^{(\dot{\xi})}, \varphi \right\rangle = \iint v(dx,\, dt)\, K^{(\dot{\xi})}\,(x,t|x_0,t_0)\,\varphi\,(x,t)\,,\ \ \varphi \in D\,(\Omega)\,.$$

($K^{(\dot{\xi})}$ is locally integrable with respect to v, φ is bounded with compact support and v is finite, hence $\varphi K^{(\dot{\xi})}$ is integrable with respect to v).

We now proceed to show that $K^{(\dot{\xi})}$ solves the Schrödinger equation as a distribution. To abbreviate we set $\hat{L} = \left(i\partial_t + \frac{1}{2}\partial_x^2 - \dot{\xi}\,(t)\,x\right)$ and let $\hat{L}^*$ denote its adjoint. Let $\varphi \in D\,(\Omega)$. By theorem 3.2 we have

$$\left\langle \hat{L}K^{(\dot{\xi})}, \varphi \right\rangle = \left\langle K_0^{(\dot{\xi})}\,(x,t|x_0,t_0) - i\iint v\,(dy,d\tau)\,K_0^{(\dot{\xi})}\,(x,t|y,\tau)\,K^{(\dot{\xi})}\,(y,\tau|x_0,t_0)\,, \hat{L}^*\varphi \right\rangle.$$

By Fubini's theorem this equals

$$\left\langle K_0^{(\dot{\xi})}, \hat{L}^*\varphi \right\rangle - i\iint v\,(dy,d\tau)\left[\iint dx\,dt\,K_0^{(\dot{\xi})}\,(x,t|y,\tau)\,\hat{L}^*\varphi\,(x,t)\right]K^{(\dot{\xi})}\,(y,\tau|x_0,t_0)\,.$$

As $K_0^{(\dot{\xi})}$ is a Green's function of $\hat{L}$ we obtain

$$i\varphi\,(x_0,t_0) + \iint v\,(dy,d\tau)\,\varphi\,(y,\tau)\,K^{(\dot{\xi})}\,(y,\tau|x_0,t_0) = \langle i\delta_{x_0}\delta_{t_0}, \varphi\rangle + \left\langle vK^{(\dot{\xi})}, \varphi \right\rangle.$$

Hence we have the following

Theorem 3.4:

$K^{(\dot{\xi})}$ is a Green's function for the full Schrödinger equation, i.e.

$$\left(i\,\partial_t + \frac{1}{2}\partial_x^2 - \dot{\xi}\,(t)\,x - v\right)K^{(\dot{\xi})}\,(x,t|x_0,t_0) = i\,\delta_{x_0}\,\delta_{t_0}.$$

In particular the Feynman integral $E\,(I) = K$ solves the Schrödinger equation

$$i\,\partial_t\;K\,(x,t|x_0,t_0) = \left(-\frac{1}{2}\partial_x^2 + v\right)K\,(x,t|x_0,t_0)\,,\ \text{for } t > t_0.$$

Hence the construction proposed by Khandekar and Streit yields a (mathematically) rigorously defined Feynman integrand whose expectation is the correct quantum mechanical propagator.

Acknowledgements

We would like to thank A. Boukricha, M. de Faria and Y. Kondratiev for various helpful discussions. We are indebted to the University of Madeira for the hospitality and support extended to us. This work was made possible by financial support from STRIDE.

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