Poincaré normal forms and Lie-point symmetries

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Abstract

We study Poincaré normal forms of vector fields in the presence of symmetry under general - i.e. not necessarily linear - diffeomorphisms. We show that it is possible to reduce both the vector field and the symmetry diffeomorphism to normal form by mean of an algorithmic procedure similar to the usual one for Poincaré normal forms without symmetry; this double normal form can be given a simple geometric characterization.

Introduction

The Poincaré-Dulac theory [1-3] of analytic normal forms (NF) of an analytic Ordinary Differential Equation (equivalently, vector field (VF)) in the vicinity of an isolated fixed point is a venerable topic, but also a powerful tool in the study of dynamical systems. Here we limit ourselves to the (properly speaking, Poincaré) case in which the linear operator A, giving the linearization of the ODE at the fixed point (see below), commutes with its adjoint, i.e. $[A, A^+] = 0$; in other words, we treat the case A does not contain Jordan blocks, or still the algebraic and geometric multeplicities of its eigenvalues are equal (we denote this condition as "Assumption A").

Some of the results we will obtain remain valid (in some cases, if suitably modified) also if this assumption is not verified, as we shall occasionally indicate: the extension goes along the lines of the extension of the Poincaré to the Poincaré-Dulac theory, see e.g. [1].

In the case of generic Poincaré NF, these are nicely characterized [2,4,5] by the fact that the linear part of, and the full evolution operator, do commute; in other words, the resonant vectors are those which commute with the linear operator A. In the case of linearly equivariant Poincaré NF, i.e. if the equations admit a linear symmetry, the general form of the Poincaré NF unfolding is restricted by another commutation relation, i.e. only the equivariant resonant terms can appear [4].

Here we generalize this result to the case of nonlinear symmetries. We also obtain some result concerning the properties of nonlinear symmetries admitted by dynamical systems in NF, and the connections existing between these symmetries and the Poincaré procedure for transforming the system into NF.

We would like to warmly thank prof. H. Duistermaat for convincing us normal forms are not only useful, but also a very nice subject in itself, and for introducing us to the particularly convenient geometrical approach to NF we use here.

1. Geometrical setting and notation

Let us consider the space $M \subseteq \mathbf{R}^\mathbf{N}$, and let $\mathcal{M}$ be the space of analytical vector fields in M. Elements of $\mathcal{M}$ are in correspondence with elements of V, the space of analytical functions $f : M \to \mathbf{R}^\mathbf{N}$ such that $f(x) \in T_x M$; with $x \in M$ we write in component expansion (we will use greek letters for elements of $\mathcal{M}$, latin ones for elements of V)

$$\varphi = f(x)\partial_x \equiv f^i(x)\frac{\partial}{\partial x^i} \tag{1.1}$$

Let us define $V_k \subset V$ as the space of homogeneous polynomial functions of order k in V, and let $\mathcal{M}_k$ be the corresponding subset of $\mathcal{M}$.

In the set $\mathcal{M}$ is naturally defined a bilinear antisymmetric operation $[.,.]$, the Lie commutator of vector fields, with which $\mathcal{M}$ becomes a Lie algebra. This induces a corresponding Lie-Poisson bracket $\{.,.\} : V \times V \to V$. Indeed, if $\varphi = f(x)\partial_x$, $\psi = g(x)\partial_x$, then

$$[\varphi, \psi] = \{f, g\}\partial_x \tag{1.2}$$

$$\{f, g\}^i = f^j \partial_j g^i - g^j \partial_j f^i \tag{1.3}$$

By means of these we define the (linear) *adjoint action* of $\varphi \in \mathcal{M}$ on $\mathcal{M}$ itself, respectively of $f \in V$ on V, by

$$\begin{aligned} \mathrm{ad}_\varphi(.) =& [\varphi, .] \equiv L_\varphi(.) \\ \mathrm{ad}_f(.) =& \{f, .\} \equiv L_f(.) \end{aligned} \tag{1.4}$$

It is clear that $\mathcal{M}$, V can be decomposed as

$$\mathcal{M} = \sum_{k=0}^{\infty}{}^{\oplus} \mathcal{M}_k \quad ; \quad V = \sum_{k=0}^{\infty}{}^{\oplus} V_k \tag{1.5}$$

and it is equally clear that

$$\begin{aligned} \varphi \in \mathcal{M}_m \Longrightarrow & L_\varphi : \mathcal{M}_k \to \mathcal{M}_{k+m-1} \\ f \in V_m \Longrightarrow & L_f : V_k \to V_{k+m-1} \end{aligned} \tag{1.6}$$

In particular, for $m = 1$ this shows that the decomposition (1.5) is a decomposition in invariant spaces under ad_φ, ad_f for $\varphi \in \mathcal{M}_1$, $f \in V_1$.

Let us now consider the flow induced in M by the vector field φ given by (1.1); this is described by the equation

$$\dot{x} = f(x) \ = \varphi \cdot x \quad ; \quad x \in M \ , \ f : M \to TM \tag{1.7}$$

A VF $\sigma \in \mathcal{M}$ will be called a (time independent) Lie-point (LP) symmetry of φ if and only if the flows of σ and φ commute; that is,

$$\begin{aligned} [\sigma, \varphi] = 0 \ &; \ L_\varphi(\sigma) = 0 = L_\sigma(\varphi) \\ \{f, s\} = 0 \ &; \ L_f(s) = 0 = L_s(f) \end{aligned} \tag{1.8}$$

where the second line is in component notation and we write $\sigma = s(x)\partial_x$, here and in the following.

The Lie algebra of LP symmetries of φ (respectively of f) will be denoted by $\mathcal{G}_\varphi \subseteq \mathcal{M}$ (respectively $\mathcal{G}_f \subseteq V$); notice that

$$\mathcal{G}_\varphi = \mathrm{Ker}(\mathrm{ad}_\varphi) \quad ; \quad \mathcal{G}_g = \mathrm{Ker}(\mathrm{ad}_f) \tag{1.9}$$

$$\sigma \in \mathcal{G}_\varphi \Longleftrightarrow \varphi \in \mathcal{G}_\sigma \quad ; \quad s \in \mathcal{G}_f \Longleftrightarrow f \in \mathcal{G}_s \tag{1.10}$$

Remark 1. Notice that if $x = x_0$ is an isolated fixed point for φ (an isolated zero for f), then it must also be a fixed point for σ (a zero for s), on the account of (1.8). From now on we will assume this to be the case, and set $x_0 = 0$.

It is therefore natural to consider the linearization of (1.7) at $x = x_0$; this is given by

$$\dot{x} = Ax = f_0(x) \qquad\qquad A = (Df)(x_0) \tag{1.7$'$}$$

The linear operator A will play a central role in the following; we will make a fundamental assumption on it to simplify our work:

Assumption A. *The linear operator $A = (Df)(x_0)$ commutes with its adjoint.*

In order to avoid unnecessary duplication of equations, from now on we will use only the setting in V, and leave to the reader the translation of our statements to the setting in $\mathcal{M}$.

Let us now expand f, s in terms of the decomposition (1.5); we write

$$\begin{aligned} f(x) &= \sum_{k=0}^{\infty} f_k(x) \quad ; \quad f_k \in V_{k+1} \\ s(x) &= \sum_{k=0}^{\infty} s_k(x) \quad ; \quad s_k \in V_{k+1} \end{aligned} \tag{1.11}$$

so that by (1.6)

$$\mathrm{ad}_{f_k} : V_m \to V_{m+k} \quad ; \quad \mathrm{ad}_{f_k} : V_m \to V_{m+k} \tag{1.12}$$

We will consider in particular the linear operator

$$\mathrm{ad}_{f_0} \equiv L_{f_0} \equiv \mathcal{L} \tag{1.13}$$

which is now decomposed as

$$\mathcal{L} = \sum_{k=0}^{\infty}{}^{\oplus} \mathcal{L}_{(k)} \quad ; \quad \mathcal{L}_{(k)} : V_k \to V_k \quad ; \quad \mathcal{L}_{(k)} = \mathcal{L}|_{V_k} \tag{1.14}$$

so that in particular

$$\mathrm{Ker}(\mathcal{L}) = \sum_{k=0}^{\infty}{}^{\oplus} \mathrm{Ker}(\mathcal{L}_{(k)}) \quad ; \quad \mathrm{Ker}(\mathcal{L}_{(k)}) = \mathrm{Ker}(\mathcal{L}) \cap V_k \tag{1.14$'$}$$

Definition 1. A function $w \in V_k$ is called a *k-resonant vector* if and only if $w \in \mathrm{Ker}(\mathcal{L}_{(k)}) \subseteq V_k$.

Remark 2. Written explicitly, the condition $w \in \mathrm{Ker}(\mathcal{L})$ becomes

$$0 = \{Ax, w\}^i = A^{jk} x_k \partial_j w^i - A^{ij} w_j = (Ax)^j \partial_j w^i - (Aw)^i \equiv D_A w^i$$

where $D_A \equiv (Ax) \cdot \partial - A$ is the well known homological operator associated to A. If A is diagonalized (thanks to Assumption A) with eigenvalues $\alpha_1, \dots, \alpha_n$, and w^i is a monomial

$$w^i = x_1^{m_1} x_2^{m_2} \cdot \ldots \cdot x_n^{m_n}$$

the above condition acquires the familiar form [1] $D_A w^i = m_j \alpha_j - \alpha_i = 0$.

Notice that under Assumption A, one has [1]

$$\begin{aligned} V =& \mathrm{Ker}(\mathcal{L}) \oplus \mathrm{Ran}(\mathcal{L}) \\ V_k =& \mathrm{Ker}(\mathcal{L}_{(k)}) \oplus \mathrm{Ran}(\mathcal{L}_{(k)}) \end{aligned} \tag{1.15}$$

Let us now consider $h \in V$, and $f, s \in V$ such that (1.8) is satisfied; by the Jacobi identity, we then have

$$\{s, \{f, h\}\} = \{f, \{s, h\}\} \tag{1.16}$$

which also reads

$$L_s \cdot L_f = L_f \cdot L_s \tag{1.16$'$}$$

so that (1.8) implies in particular

$$\mathrm{ad}_s : \mathrm{Ker}(\mathrm{ad}_f) \to \mathrm{Ker}(\mathrm{ad}_f) \tag{1.17}$$

Notice also that

$$\{f, s\} = 0 \Longrightarrow \{f_0, s_0\} = 0 \tag{1.18}$$

so that when (1.8) is satisfied,

$$\mathcal{S} \equiv \mathrm{ad}_{s_0} : \mathrm{Ker}(\mathcal{L}_{(k)}) \to \mathrm{Ker}(\mathcal{L}_{(k)}) \tag{1.19}$$

In terms of the expansion (1.11), the condition (1.8) reads

$$\sum_{j=0}^{\infty} \{f_j, s_{k-j}\} = 0 \quad \forall k \geq 0 \tag{1.20}$$

[1] These decompositions can be easily verified on the basis of the monomials w^i introduced in Remark 2; one could also introduce a scalar product in the space V, see [4], but this is not necessary for our present purposes.

2. Poincaré normal forms

In the (Poincaré) theory of normal forms, one considers dynamical systems of the form (1.7) and proves that, if Assumption A is satisfied, by means of formal changes of coordinates they can be taken to the form

$$\dot{x} = g(x) \quad = \sum_{k=0}^{\infty} g_k(x) \tag{2.1}$$

where $g_k \in V_{k+1}$ and

$$g_0 = f_0 \quad ; \quad g_k \in \mathrm{Ker}(\mathcal{L}_{(k+1)}) \qquad k < k^* \tag{2.2}$$

for k^* arbitrarily large.

Remark 3. Notice that $f_0 \in \mathrm{Ker}(\mathcal{L}_{(1)})$ by definition.

Remark 4. If Assumption A is not satisfied, $\mathcal{L}$ should be substituted by its adjoint, and the above Remark would fail; this is actually the main reason to consider Assumption A.

We will take formally $k^* = \infty$, so that the Poincaré - Dulac theorem will read

Theorem 1. (Poincaré - Dulac) *By means of formal changes of coordinates, it is possible to take the system* (1.7) *to the form* (2.1)*, where* $g \in \mathrm{Ker}(\mathcal{L}) = \mathcal{G}_{f_0}$.

In this sense, the Poincaré - Dulac procedures makes explicit the symmetry of the dynamical system.

Remark 5. Since $g_0 = f_0$, the operator $\mathcal{L}$ is well defined and independent of the form, (2.1) or (1.7), of the system.

Remark 6. If the system (2.1) satisfies (2.2) with $k^* = n$, we say that it is in Poincaré normal form up to order n; when taking the formal limit $n \to \infty$, we speak of Poincaré normal form, tout court.

The changes of coordinates needed to transform the system to NF are of the form

$$x = y + h_k(y) + R_{k+1}(y) \tag{2.3}$$

where $h_k \in V_{k+1}$ and $R_k \in \sum_{m=k+2}^{\oplus\infty} V_m$; notice that this can be seen as corresponding to the (time-one) flow under the VF $\chi = h_k(x)\partial_x$.

Under (2.3), the system

$$\dot{x} = \sum_m f_m(x) \quad ; \quad f_m \in V_{m+1} \tag{2.4}$$

is changed into

$$\dot{y} = \sum_m \widetilde{f}_m(y) \quad ; \quad \widetilde{f}_m = f_m \qquad m < k \tag{2.4$'$}$$

where

$$\widetilde{f}_k = f_k - \{f_0, h_k\} \equiv f_k - \mathcal{L}(h_k) \tag{2.5}$$

so that if π is the projection $\pi : V \to \mathrm{Ran}(\mathcal{L})$, πf_k can be eliminated by judicious choice of h_k. The appropriate h_k for this, can be determined by solving the equation (also called "homological equation")

$$\mathcal{L} h_k = \pi f_k \tag{2.6}$$

Remark 7. Notice that h_k is only defined up to elements of $\mathrm{Ker}(\mathcal{L}_{(k+1)})$; in other words, the changes of coordinates determined by h_k and by

$$h'_k = h_k + \delta h_k \quad ; \quad \delta h_k \in \mathrm{Ker}(\mathcal{L}_{(k+1)}) \tag{2.7}$$

lead to the same $\widetilde{f}_k$.

By this Remark and by (1.15) we can, under Assumption A, decide to choose

$$h_k \in \mathrm{Ran}(\mathcal{L}_{(k+1)}) \tag{2.8}$$

Once h_k has been fixed, any $s(x) = \sum_m s_m(x)$ will be changed according to the same (2.4),(2.5) above; i.e.

$$\widetilde{s}_m = s_m \qquad m < k \quad ; \quad \widetilde{s}_k = s_k - \{s_0, h_k\} \tag{2.9}$$

It is maybe worth stressing that geometrical objects, such as $\varphi, \sigma \in \mathcal{M}$, are not changed by (2.3), which affects only their coordinate representation. In particular, $\mathcal{G}_\varphi$ remains unchanged, so that since $\varphi = f(x)\partial_x = g(y)\partial_y$ and $f_0 \in \mathcal{G}_g$, then there must be a VF $\sigma = f_0(y)\partial_y \in \mathcal{G}_\varphi$, which will be represented as $\sigma = s(x)\partial_x$ in the x coordinates; if $g(y) \neq f_0(y) \equiv g_0(y)$, then $\sigma \neq \varphi$, and the VF φ has at least a nontrivial symmetry.

Given a linear VF φ_0, one can ask to classify (the local flow of) all the VF which admit φ_0 as linear part; in terms of the dynamical system (1.7), this amount to classify (the local behaviour of solutions of) all the systems f which have the same linearization $(Df)(x_0) = A$ at the fixed point x_0, $f_0(x) = Ax$.

The problem of classifying all the f as above up to formal analytic transformations, reduces to the problem of classifying the most general $f(x)$ with linear part $f_0(x)$, upon reduction to Poincaré NF.

If the above classification is meant up to equivalence by formal analytic transformations, the Poincaré NF is a convenient tool; it should be stressed that if one is satisfied with a classification up to transformation in a different class, e.g. up to topological or C^k equivalence, this would lead to different kind of NF and NF reduction [1]. In the present paper, by NF we will always mean the Poincaré NF.

3. Symmetries and normal forms

We want now to consider the relations between symmetry properties of eq.(1.7) and its (reduction to) NF (2.1). The symmetry properties of systems in NF have already been considered by some authors, see e.g. [4,6]; in particular Elphick et al.[2] [4] have characterized the NF by means of the commutation properties between the full VF describing time evolution and its linear part at the fixed point x_0. Indeed, with Assumption A we have from [4]:

Theorem 2. *Let the VF $\Phi \in \mathcal{M}$ be written in the x coordinates as $\Phi = f(x)\partial_x = \Phi_0 + \Phi_1$, where $\Phi_0 = f_0(x)\partial_x$, $\Phi_1(x) = [f(x) - f_0(x)]\partial_x$, and $f_m \in V_{m+1}$. Let Assumption* A *be satisfied. Then Φ is in Poincaré NF if and only if $[\Phi, \Phi_0] = 0$*

Corollary 1. *Φ is in Poincaré NF if and only if the following conditions, all equivalent, are verified:*
i) $\{f, f_0\} = 0$; *ii)* $\Phi \in \mathrm{Ker}(\mathrm{ad}_{\Phi_0}) = \mathcal{G}_{\Phi_0}$; *iii)* $\Phi_0 \in \mathrm{Ker}(\mathrm{ad}_{\Phi}) = \mathcal{G}_{\Phi}$.

The proof can easily be obtained from the discussion of sects.1 and 2; indeed, in the present notation this amounts to a corollary of the Poincaré - Dulac theorem as given in sect.2.

Remark 8. From the point of view of symmetry properties, an interesting result comes from *iii)* of the Corollary above, which states that the linear part $f_0(x) = Ax$ determines a linear LP symmetry $\Phi_0 = Ax\partial_x$ for the full problem $\dot{x} = f(x)$ [4,6].

Remark 9. We notice that if Assumption A is not satisfied, the above theorem would be stated with the commutator condition $[\Phi_1, \Phi_0^+] = 0$ (and correspondingly modified conditions in the corollary). In this case, the statement of Remark 8, would be substituted by the weaker result that the linear operator $\Phi_0^+ = A^+ x\partial_x$ is a linear symmetry for the nonlinear part $\dot{x} = \Phi_1 \cdot x$ (and not for the full problem) .

We want to consider here the symmetries of the original system (1.7), and how these are reflected into the NF coordinates. The motivation for this comes from the following obvious but interesting fact (see later discussion). Let $\varphi, \sigma \in \mathcal{M}$, expressed in two systems of coordinates x and y in M as

$$\begin{aligned} \varphi = f(x)\partial_x &= \widetilde{f}(y)\partial_y \equiv g(y)\partial_y \\ \sigma = s(x)\partial_x &= \widetilde{s}(y)\partial_y \equiv t(y)\partial_y \end{aligned} \tag{3.1}$$

[2] In [2], p.67, this theorem is quoted from [5]; unfortunately this book is not available (to our knowledge) in the western literature.

The relation $[\sigma, \varphi] = 0$, equivalent to $\varphi \in \mathcal{G}_\sigma$, $\sigma \in \mathcal{G}_\varphi$, is independent of the coordinate choice, so that

$$\{f, s\} = 0 \Longleftrightarrow \{\widetilde{f}, \widetilde{s}\} = 0 \tag{3.2}$$

Therefore, the presence of a symmetry for eq.(1.7) will pose some restriction to the NF (2.1): while in general the NF satisfies only

$$\widetilde{f} \in \mathrm{Ker}(\mathrm{ad}_{f_0}) \equiv \mathrm{Ker}(\mathcal{L}) \tag{3.3}$$

in the presence of the symmetry we will also have from (3.2) that

$$\widetilde{f} \in \mathrm{Ker}(\mathrm{ad}_{\widetilde{s}}) \tag{3.4}$$

The combination of (3.3) and (3.4) can lead to a simplification of the NF unfolding; see sections 5,6.

It could be worth checking explicitely (3.2) in the following way: let us rewrite (1.20) at order k as

$$\begin{aligned} &\mathcal{C}_k = u_k \qquad (k = 0, 1, 2, \ldots) \qquad \text{where} \\ &\mathcal{C}_k \equiv \{f_0, s_k\} + \{f_k, s_0\} \quad \text{and} \quad u_k \equiv - \sum_{j=1}^{k-1} \{f_j, s_{k-j}\} \quad , \quad \text{with} \quad u_0 = u_1 = 0 \end{aligned} \tag{3.5}$$

and let us consider a transformation (2.3): under this

$$\mathcal{C}_k \to \widetilde{\mathcal{C}}_k = \mathcal{C}_k - \{f_0, \{s_0, h_k\}\} - \{\{f_0, h_k\}, s_0\}$$

or, using Jacobi identity,

$$\widetilde{\mathcal{C}}_k = \mathcal{C}_k + \{\{s_0, f_0\}, h_k\} \tag{3.6}$$

so that $\{s_0, f_0\} = 0 \Longrightarrow \widetilde{\mathcal{C}}_k = \mathcal{C}_k$; the r.h.s. of (3.5) remains unchanged in the change of coordinates, since it contains only terms of degree smaller than k, and so the whole equation (3.5) is invariant.

Remark 10. Notice that if S is a linear operator such that $[A, S] = 0$ and if A satisfies Assumption A, then also $[A, S^+] = 0$; which implies that $S + S^+$ and $S - S^+$ commute with A. Then, we can assume that the linear part $S \equiv (Ds)(x_0)$ of the symmetry σ satisfies Assumption A. In the following we will use freely the fact that both S and A satisfy Assumption A, and denote ad_{s_0} by $\mathcal{S}$. It should be stressed that if Assumption A is not verified, the results stated in Theorem 3 below fail to be true, in general.

We will find it useful to have the following Lemma, which easily follows from Jacobi identity.

Lemma 1. *Let $v, w \in V$ such that $v, w \in \mathrm{Ker}(\mathcal{L})$. Then $\{v, w\} \in \mathrm{Ker}(\mathcal{L})$.*

We also have the following

Theorem 3. *Let $\Phi \in \mathcal{M}$ be expressed in Poincaré NF as $\Phi = f(x)\partial_x$; then any σ such that $[\sigma, \Phi] = 0$ is expressed in the x coordinates as $\sigma = s(x)\partial_x$ where $s \in \mathrm{Ker}(\mathcal{L})$. In other words, all LP symmetries σ of a dynamical system in NF, which are obtained as formal series expansion, are also LP symmetries of the linearized system $\dot{x} = f_0(x) = Ax$.*

Proof. We can proceed recursively using the set of equations (3.5). For $k = 0$, $\{f_0, s_0\} = 0$ is satisfied, see (1.18); for $k = 1$ we have

$$\{f_0, s_1\} + \{f_1, s_0\} = 0 \tag{3.7}$$

Applying $\mathcal{L}$ to this equation, we obtain

$$\mathcal{L}(\mathcal{L}(s_1)) \equiv \{f_0, \{f_0, s_1\}\} = 0 \tag{3.8}$$

being $\{f_0, \{f_1, s_0\}\} = 0$. Using (1.15) one has

$$\mathcal{L}(s_1) = 0 \qquad \text{or} \qquad s_1 \in \mathrm{Ker}(\mathcal{L}) \tag{3.9}$$

The argument can be repeated recursively for each k: indeed, $\{f_k, s_0\} \in \mathrm{Ker}(\mathcal{L})$ for Lemma 1, and similarly if $s_j \in \mathrm{Ker}(\mathcal{L})$, $\forall j < k$, then also $u_k \in \mathrm{Ker}(\mathcal{L})$. Therefore, using again (1.15), eq. (3.5) can be solved only with

$$\mathcal{L}(s_k) = 0 \qquad \text{and} \qquad \mathcal{S}(f_k) = u_k \tag{3.10}$$

and this completes the proof.

It was remarked in sect.2 that the h_k identifying the normalizing transformation are identified by solutions of the homological equation (2.6) only modulo $\mathrm{Ker}(\mathcal{L})$, see Remark 7. This means that once the system is in NF we can still apply changes of coordinates of the form

$$x = y + \delta h_k(y) \qquad \delta h_k \in \mathrm{Ker}(\mathcal{L}_{(k+1)}) \tag{3.11}$$

without modifying $f(x)$. Under this change of coordinates, s_k will be changed to

$$\widetilde{s}_k = s_k - \{s_0, \delta h_k\} \equiv s_k - \mathcal{S}(\delta h_k) \tag{3.12}$$

Notice that if $s_k \in \mathrm{Ker}(\mathcal{L})$, also $\widetilde{s}_k$ does, see (1.19). By opportunely choosing $\delta h_k \in \mathrm{Ker}(\mathcal{L}_{(k+1)})$, we can therefore eliminate the component of s_k in $\mathrm{Ker}(\mathcal{L}) \cap \mathrm{Ran}(\mathcal{S})$. We have therefore proved the

Proposition 1. *Let $f(x)$ be in NF, and let $s(x)$ be a symmetry of $f(x)$. Then it is possible to choose coordinates in which $f(x)$ is still in NF and such that $s \in \mathrm{Ker}(\mathcal{L}) \cap \mathrm{Ker}(\mathcal{S})$.*

Remark 12. Just as in the reduction to Poincaré NF, this change of coordinates will in general be purely formal.

Remark 13. We always have $f_0 \in \mathrm{Ker}(\mathcal{L}) \cap \mathrm{Ker}(\mathcal{S})$; notice that for $s = f$ we have $\mathcal{S} = \mathcal{L}$, and indeed $s = f \in \mathrm{Ker}(\mathcal{L}) \cap \mathrm{Ker}(\mathcal{S}) \equiv \mathrm{Ker}(\mathcal{L})$; the symmetry f will be called trivial.

The above Proposition suggests the

Definition 2. A VF $\Phi = f(x)\partial_x$ is in NF if $f \in \mathrm{Ker}(\mathcal{L})$, with $\mathcal{L} = \mathrm{ad}_{f_0}$; its symmetry VF $\sigma = s(x)\partial_x$ is in NF if $s \in \mathrm{Ker}(\mathcal{L}) \cap \mathrm{Ker}(\mathcal{S})$, with $\mathcal{S} = \mathrm{ad}_{s_0}$.

4. Determining equations for vector fields in normal forms

Let us consider f given and try to determine its symmetries s; in order to do this we have to consider again the relation (1.20), to be regarded as the determining equation for s. We will assume $f(x)$ is in NF, and look for solutions $s(x)$ which are in NF as well, i.e. $s \in \mathrm{Ker}(\mathcal{L}) \cap \mathrm{Ker}(\mathcal{S})$; the Proposition 1 above ensures that we are legitimate to restrict to such s.

At order k the determining equations are, see (3.5),

$$\mathcal{L}(s_k) - \mathcal{S}(f_k) = u_k \qquad ; \qquad u_0 = u_1 = 0 \tag{4.1}$$

Remark 14. If for all $j < k$ it happens that $f_j \in \mathrm{Ker}(\mathcal{S})$, and necessarily then $f_j \in \mathrm{Ker}(\mathcal{L}) \cap \mathrm{Ker}(\mathcal{S})$ since f is in NF, then also $u_k \in \mathrm{Ker}(\mathcal{L}) \cap \mathrm{Ker}(\mathcal{S})$. Indeed, this just follows from lemma 1 of the previous section.

Lemma 2. *If $f(x)$ and its symmetry $s(x)$ are in NF according to the above definition, then*

$$f \in \mathrm{Ker}(\mathcal{L}) \cap \mathrm{Ker}(\mathcal{S}) \qquad s \in \mathrm{Ker}(\mathcal{L}) \cap \mathrm{Ker}(\mathcal{S})$$

Proof. The result for s comes directly from Def. 2. If f is in NF, then $f \in \mathrm{Ker}(\mathcal{L})$; it follows from (4.1) that if there is a m such that $f_j \in \mathrm{Ker}(\mathcal{S})$ for all $j < m$, then $f \in \mathrm{Ker}(\mathcal{S})$. Indeed, if $u_k \in \mathrm{Ker}(\mathcal{S})$, then $\mathcal{S}(f_k) = 0$. It suffices now to check that $f_1 \in \mathrm{Ker}(\mathcal{S})$, as it follows from (4.1) for $k = 1$.

Remark 15. The above lemma restores the symmetry of roles between f and s.

Remark 16. If f and s are in NF, we can still make a change

Date: Tue, 11 May 1993 14:38:27 +0200 From: gaeta@orphee.polytechnique.fr (Giuseppe Gaeta) To: cicogna@ipifidpt.difi.unipi.it of coordinates (2.3) generated by $\Delta h_k \in \mathrm{Ker}(\mathcal{L}) \cap \mathrm{Ker}(\mathcal{S})$; such a transformation will not change f_k neither s_k.

It is remarkable that equations (4.1) are of the same form as the full determining equations (1.20) for the fields $(f - f_0)$ and $(s - s_0)$; we could therefore repeat the above discussion and continue iteratively. With the notation $(r \geq 1)$

$$f^{[r]} = \sum_{m=r}^{\infty} f_m \equiv f - \sum_{m=0}^{r-1} f_m \tag{4.2}$$

$$\mathcal{L}^{[r]} = \mathrm{ad}_{f^{[r]}} \quad , \quad \mathcal{L} \equiv \mathcal{L}^{[0]} \quad ; \quad \mathcal{S}^{[r]} = \mathrm{ad}_{s^{[r]}} \quad , \quad \mathcal{S} \equiv \mathcal{S}^{[0]} \tag{4.3}$$

(so that $f^{[0]} = f$, $s^{[0]} = s$) we would arrive to the conclusion that if $\Phi, \sigma \in \mathcal{M}$ satisfy $[\Phi, \sigma] = 0$ and are in NF when expressed as $\Phi = f(x)\partial_x$, $\sigma = s(x)\partial_x$, then for any $r \geq 0$ both $f^{[r]}$ and $s^{[r]}$ are in $\mathrm{Ker}(\mathcal{S}^{[r]}) \cap \mathrm{Ker}(\mathcal{L}^{[r]})$.

Remark 17. With this notation, and recalling (1.18), it is immediate to check that $\{f, s\} = 0$ enforces $\{f^{[1]}, s^{[1]}\} = 0$: indeed, $\{f, s\} = \{f_0 + f^{[1]}, s_0 + s^{[1]}\}$ and due to $f, s \in \mathrm{Ker}(\mathcal{L}) \cap \mathrm{Ker}(\mathcal{S})$, we just have $\{f, s\} = \{f^{[1]}, s^{[1]}\}$.

Notice that now we can define

$$u_k^{[r]} = - \sum_{j=1+r}^{k-r-1} \{f_j, s_k\} \qquad u_k^{[r]} \equiv 0 \ \text{ for } k \leq r$$

(so that $u_k^{[0]} \equiv u_k$) and $\{f^{[1]}, s^{[1]}\} = 0$ reads now

$$\mathcal{L}^{[r]}(s_k) - \mathcal{S}^{[r]}(f_k) = u_k^{[r]} \qquad k \geq r \tag{4.4}$$

with $r = 1$. By repeating the discussion iteratively, we get the equation for generic $r \geq 0$.

We will summarize our discussion by stating the following

Theorem 4. *Let $\Phi = g(y)\partial_y$, $\sigma = t(y)\partial_y$, where $g(y), t(y) \in V$ satisfy $\{s, f\} = 0$. Then by means of formal changes of coordinates* (1.3) *we can take them to the form $\Phi = f(x)\partial_x$, $\sigma = s(x)\partial_x$, where $g_0 = f_0$, $t_0 = s_0$ and, with $\mathcal{L} = \mathrm{ad}_{f_0}$, $\mathcal{S} = \mathrm{ad}_{s_0}$,*

$$\begin{aligned} f &\in \mathrm{Ker}(\mathcal{S}) \cap \mathrm{Ker}(\mathcal{L}) \\ s &\in \mathrm{Ker}(\mathcal{S}) \cap \mathrm{Ker}(\mathcal{L}) \end{aligned}$$

Moreover, with the notation (4.3),(4.4)*, in the same coordinates we also have*

$$\begin{aligned} f^{[r]} &\in \mathrm{Ker}(\mathcal{S}^{[r]}) \cap \mathrm{Ker}(\mathcal{L}^{[r]}) \\ s^{[r]} &\in \mathrm{Ker}(\mathcal{S}^{[r]}) \cap \mathrm{Ker}(\mathcal{L}^{[r]}) \end{aligned}$$

Remark 18. This theorem can be seen as a generalisation of theorem 4 of [4] to the case of nonlinear symmetries; see this also for the generalisation to the case in which f_0 does not meet Assumption A.

Remark 19. After the completion of the present work, prof. Duistermaat pointed out that this result can be obtained in an alternative way based on the remark that the changes of variables (2.3) amount to the adjoint action of $h \in V$ on V, so that the kernels considered in the above Theorem 4 are necessarily invariant and provide a classification of NF equations. This can also be seen as a consequence of arguments concerning filtration of Lie algebras applied to NF reduction, which are contained in the thesis by Broer [7]. Our present discussion and statement of results has nevertheless the advantage of being completely elementary and explicit.

5. Unfoldings of equivariant normal forms

We want now to discuss how the above Theorem 4 is of help in the determination of equivariant normal forms unfoldings, i.e. in the classification of systems $\dot{x} = f(x)$ and symmetries $s(x)$ of these with given (necessarily commuting) linear parts f_0, s_0.

In this respect, it is useful to remark that in view of the discussion in the previous section, we can restate our Theorem 4 as

Proposition 2. *Let $f(x)$ be in NF; the necessary and sufficient condition for $s(x)$ to be a symmetry of f in NF is that, with the notation introduced above, the equation*

$$\mathcal{L}^{[r]}(s_k) - \mathcal{S}^{[r]}(f_k) = u_k^{[r]}$$

is satisfied for all r and for all $k \geq r$.

In many (and most of) concrete applications, one has to deal with symmetry vectors that are linear or quadratic; it is therefore worth discussing briefly these special cases, which will also make clear the general procedure.

If the symmetry is linear,

$$s = s_0 \in \mathrm{Ker}(\mathcal{L}) \tag{5.1}$$

so that $s \in \mathrm{Ker}(\mathcal{L}) \cap \mathrm{Ker}(\mathcal{S})$ is trivially satisfied. The determining equations, or equivalently the condition $f \in \mathrm{Ker}(\mathcal{L}) \cap \mathrm{Ker}(\mathcal{S})$, imply now that

$$\{f_k, s_0\} = 0 \quad \forall k \geq 0 \tag{5.2}$$

Correspondingly, the equivariant NF unfolding for (5.1) can be determined order by order.

Notice that, writing $s_0(x) = Sx$, eq.(5.2) is equivalent to

$$f_k(Sx) = S f_k(x) \quad \forall k \geq 0 \tag{5.3}$$

In the same vein, writing $f_0(x) = Ax$, the condition $f \in \mathrm{Ker}(\mathcal{L})$ reads

$$f_k(Ax) = A f_k(x) \quad \forall k \geq 0 \tag{5.4}$$

Notice that

$$\{f_0, s_0\} = 0 \Longleftrightarrow [A, S] = 0 \tag{5.5}$$

so that we fully recover the setting of [4].

In the case of given quadratic symmetries (already in NF), $s = s_0 + s_1$, we can proceed in a similar way. First of all, $f \in \mathrm{Ker}(\mathcal{L})$ ensures that (5.4) does apply, and $f \in \mathrm{Ker}(\mathcal{L}) \cap \mathrm{Ker}(\mathcal{S})$ yields again (5.3). Now we do also have $f^{[1]} \in \mathrm{Ker}(\mathcal{S}_1)$, which enforces

$$\{f_k, s_1\} = 0 \quad \forall k \geq 1 \tag{5.6}$$

We can therefore still determine the equivariant normal form unfolding order by order.

In this quadratic case, we can write

$$s_1 = \frac{1}{2} \Gamma_{\ell m} x^\ell x^m \quad ; \quad \gamma_{[\ell]} = \Gamma_{\ell m} x^m \tag{5.7}$$

with Γ a symmetric tensor; eq.(5.6) does now read

$$\{f_k, \gamma_{[\ell]}\} = 0 \quad \forall \ell \ , \ \forall k \geq 1 \tag{5.8}$$

It is clear that the same procedure can be applied for the determination of the equivariant NF unfolding under given arbitrary $s \in V$ (provided of course that $s \in \mathrm{Ker}(\mathcal{L}) \cap \mathrm{Ker}(\mathcal{S})$); at order k, we just have

$$f_k \in \mathcal{F}_k = \mathrm{Ker}(\mathcal{L}) \cap \mathrm{Ker}(\mathcal{S}) \cap \mathrm{Ker}(\mathcal{S}^{[1]}) \cap \ldots \cap \mathrm{Ker}(\mathcal{S}^{[k]}) \tag{5.9}$$

Notice that at any finite order we have to solve a finite number of algebraic equations of the form

$$\{f_j, s_m\} = 0 \tag{5.10}$$

For $k = r + 1$, eq.(4.4) reduces to

$$\mathcal{L}^{[r]}(s_{r+1}) = \mathcal{S}^{[r]}(f_{r+1}) \tag{5.11}$$

which also means, repeating previous considerations,

$$\begin{aligned} f_{r+1} \in & \mathrm{Ker}(\mathcal{L}^{[r]}) \cap \mathrm{Ker}(\mathcal{S}^{[r]}) \\ s_{r+1} \in & \mathrm{Ker}(\mathcal{L}^{[r]}) \cap \mathrm{Ker}(\mathcal{S}^{[r]}) \end{aligned} \tag{5.12}$$

In particular, if f_j, s_j have been determined for $j < k = r+1$, then (5.11) and $\{f_k, s_k\} = 0$ and $\{f_k, s_k\}$ determine f_k, s_k.

This means that we can just limit ourselves to solve recursively equations of the form

$$\{f_k, s_k\} = 0 \quad ; \quad \mathcal{L}^{[k]}(s_{k+1}) = \mathcal{S}^{[k]}(f_{k+1}) \tag{5.13}$$

This also permits to classify the commuting pairs (f, s) with given (necessarily commuting) linear parts (f_0, s_0), up to formal analytical equivalence, i.e. of solving the general problem of equivariant unfoldings of the NF. Indeed, our discussion can be summarized as follows.

Theorem 5. *Let $\Phi_0, \sigma_0 \in \mathcal{M}_1$ be linear VFs, $\Phi_0 = f_0(x)\partial_x$, $\sigma_0 = s_0(x)\partial_x$, with $f_0(x) = Ax$, $s_0(x) = S(x)$, A and S satisfying Assumption* A, *and $[\Phi_0, \sigma_0] = \{f_0, s_0\} = [A, S] = 0$. Then for any pair $\Phi, \sigma \in \mathcal{M}$ of VFs, $\Phi = g(y)\partial_y$, $\sigma = t(y)\partial_y$, such that $[\Phi, \sigma] = \{g, t\} = 0$ and $(Dg)(0) = A$, $(Dt)(0) = S$, there is a formal change of coordinates taking them to the form $\Phi = f(x)\partial_x$, $\sigma = s(x)\partial_x$, with $f_0 = g_0 = Ax$, $s_0 = t_0 = Sx$, $f_m, s_m \in \mathrm{Ker}(\mathcal{L}^{[k]}) \cap \mathrm{Ker}(\mathcal{S}^{[k]})$ for any $m \leq k$, and $\mathcal{L}^{[k]}(s_{k+1}) = \mathcal{S}^{[k]}(f_{k+1})$.*

Corollary 2. *For given commuting A, S, satisfying Assumption* A, *the equivariant normal form unfolding can be determined by solving recursively equations of the form $\mathcal{L}^{[k]}(s_{k+1}) = \mathcal{S}^{[k]}(f_{k+1})$ and $\mathcal{L}^{[k]}(s_k) = 0 = \mathcal{S}^{[k]}(f_k)$.*

6. Examples and discussion

The classical problem of finding the most general form of a dynamical system in NF once its linear part $f_0(x) = Ax$ is given, has been already examined from the point of view of the symmetry properties [2,4-8]: for a generic A (i.e. not necessarily satisfying Assumption A), the result can be written

$$\dot{x} = Ax + K(\kappa(x))x \tag{6.1}$$

where K is the most general matrix such that

$$KA^+ = A^+K \tag{6.2}$$

and the entries K_{ij} of K are functions of the constants of motions $\kappa = \kappa(x)$ of the *linear* system

$$\dot{x} = A^+x \ . \tag{6.3}$$

Some special cases are discussed in [8]. The fact that K satisfies (6.2) and its elements depend on the system (6.3) involving A^+ and not A, clarifies the relevance of Assumption A in the context of symmetry properties. In particular, it is now clear why, if $[A, A^+] \neq 0$, the linear symmetry $A^+ x\partial_x$ is a symmetry for the nonlinear part, and *not* for the full problem (see Remarks 8 and 9).

Remark 20. If the nonlinear terms were resonant with A^+ (not with A), then $\sigma = (Ax)\partial_x$ would be a linear symmetry for the full problem (for some other considerations on this situation, see [6]).

A very well known case of a reduction to NF concerns the classical 2-dimensional Hopf periodic bifurcation problem: the matrix A has eigenvalues $\pm i$, and the problem in NF exhibits an explicit rotation covariance. As a trivial application of our above results, let us notice that this problem in NF cannot possess, in agreement with Theorem 3, a scaling symmetry along one axis, i.e. a symmetry of the form

$$\sigma' = u\partial_u$$

where u is either x or y $\big(\text{with } (x,y) \in R^2\big)$. Instead, the presence of a scaling symmetry in the plane

$$\sigma = x\partial_x + y\partial_y \tag{6.4}$$

which in fact is admitted by Theorem 3, would imply that the problem is trivially a linear problem.

A less trivial example is the following. With $(x, y, z) \in R^3$, let us consider the system

$$\begin{aligned} \dot{x} &= x - yr^2 \\ \dot{y} &= y + xr^2 \qquad\qquad (r^2 = x^2 + y^2) \\ \dot{z} &= (y + xr^2)z \end{aligned} \tag{6.5}$$

which corresponds, in the expansion in homogeneous terms $\varphi_k \in V_{k+1}$, to

$$\begin{aligned} \varphi_0 &= x\partial_x + y\partial_y \\ \varphi_1 &= yz\partial_z \\ \varphi_2 &= r^2(x\partial_y - y\partial_x) \\ \varphi_3 &= r^2 xz\partial_z \\ \varphi_j &= 0 \quad j \geq 4 \end{aligned} \tag{6.6}$$

This is symmetric under [9]

$$\sigma = s(x)\partial_x = s_0\partial_x + s_1\partial_x \equiv (x\partial_y - y\partial_x) + xz\partial_z \tag{6.7}$$

The above system is not in NF (actually, all nonlinear terms in (6.5) are nonresonant): accordingly, we have that $s_1 \notin \mathrm{Ker}(\mathcal{L})$ and $s \notin \mathrm{Ker}(\mathcal{L})$ (or, which is the same, σ is not a LP symmetry for the linear part of (6.5) - see Theorem 3).

Writing now a generic polynomial VF as

$$h = \begin{pmatrix} \alpha(x) \\ \beta(x) \\ \gamma(x) \end{pmatrix} \tag{6.8}$$

the action of $\mathcal{L}$ is given by

$$\mathcal{L}(h) = \begin{pmatrix} x\alpha_x + y\alpha_y - \alpha \\ x\beta_x + y\beta_y - \beta \\ x\gamma_x + y\gamma_y \end{pmatrix} \tag{6.9}$$

and it is immediate to check that

$$h \in \mathrm{Ker}(\mathcal{L}) \iff \begin{pmatrix} \alpha = & a(z)x + b(z)y \\ \beta = & c(z)x + d(z)y \\ \gamma = & e(z) \end{pmatrix} \tag{6.10}$$

As for $\mathcal{S} = \mathrm{ad}_{s_0}$, its action is given by

$$\mathcal{S}(h) = \begin{pmatrix} x\alpha_y - y\alpha_y + \beta \\ x\beta_y + y\beta_x - \alpha \\ x\gamma_y - y\gamma_x \end{pmatrix} \tag{6.11}$$

and one can check that

$$h \in \mathrm{Ker}(\mathcal{S}) \iff \begin{pmatrix} \alpha = & \widehat{a}(z,r^2)x + \widehat{b}(z,r^2)y \\ \beta = & -\widehat{b}(z,r^2)x + \widehat{a}(z,r^2)y \\ \gamma = & \widehat{e}(z,r^2) \end{pmatrix} \tag{6.12}$$

so that in particular

$$h \in \mathrm{Ker}(\mathcal{L}) \cap \mathrm{Ker}(\mathcal{S}) \iff \begin{pmatrix} \alpha = & a(z)x + b(z)y \\ \beta = & -b(z)x + a(z)y \\ \gamma = & e(z) \end{pmatrix} \tag{6.13}$$

Let us now proceed to the normalizing quadratic transformation; the homological equation is

$$\mathcal{L}(h_1) = \begin{pmatrix} 0 \\ 0 \\ yz \end{pmatrix} \tag{6.14}$$

which gives trivial equations for α and β, and

$$x\gamma_x + y\gamma_y = yz \tag{6.15}$$

so that

$$\alpha = \beta = 0; \qquad \gamma = yz + cz^2, \quad c \in R \tag{6.16}$$

i.e. x, y are not changed, and

$$z = (1+y)\widetilde{z}$$

With the h_1 given by (6.16),

$$\widetilde{f}_1 = f_1 - \{f_0, h_1\} = 0 \tag{6.17}$$

$$\widetilde{s}_1 = s_1 - \{s_0, h_1\} = s_1 - \mathcal{S}(h_1) \tag{6.17$'$}$$

Using (6.11) we immediately have

$$\mathcal{S}(h_1) = \begin{pmatrix} 0 \\ 0 \\ xz \end{pmatrix} \tag{6.18}$$

which means

$$\widetilde{s}_1 = 0 \ . \tag{6.19}$$

The above calculations can be extended to the higher orders, and one can see that the results (6.17) and (6.19) are true to all orders: indeed, as already remarked, all terms in (6.5) are nonresonant; once reduced to NF, the system becomes a linear system, and

$$\sigma = s_0 \partial_x = x\partial_y - y\partial_x \tag{6.20}$$

is (trivially) a symmetry for it.

We can also give examples containing resonant terms. Consider e.g., again with $(x, y, z) \in R^3$,

$$\begin{aligned} \dot{x} =& \lambda x - y(1+r^2) \\ \dot{y} =& \lambda y + x(1+r^2) \qquad\qquad (r^2 = x^2 + y^2) \\ \dot{z} =& z^2 + 2\lambda y^2 + 2xy(1+r^2) - 2y^2 z + y^4 \end{aligned} \tag{6.21}$$

If $\lambda = 0$, then yr^2, xr^2 and z^2 are resonant terms; if $\lambda \neq 0$, then only z^2 is resonant. In both cases a LP symmetry for the system is [9]

$$\sigma = x\partial_y - y\partial_y + 2xy\partial_z \tag{6.22}$$

Once the system (6.21) is reduced to NF, and all nonresonant terms are dropped, its symmetry becomes (both for $\lambda = 0$ and $\neq 0$) the rotation symmetry (6.20), in agreement with Theorem 3.

Let us point out finally that all our results concern symmetries σ which are obtained as series expansions. Other symmetries are actually possible, as this example shows. Consider the problem in R^2, in NF

$$\begin{aligned} \dot{x} =& -x^3 \\ \dot{y} =& -y \end{aligned} \tag{6.23}$$

One of the symmetries of this problem is $\sigma = \mathrm{e}^{-1/2x^2}\partial_y$, which cannot be obtained as a series expansion and which is *not* a symmetry for the linearized problem (or equivalently $s \notin \mathrm{Ker}(\mathcal{L})$): this is not in contrast with the conclusion of Theorem 3; in fact, the series expansion of the VF defining this symmetry would be identically zero. Other symmetries of the above dynamical system, e.g. $x^3\partial_x$ or $y\partial_y$ do actually satisfy the hypotheses (and the conclusions as well) of the theorems given above.

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