# Some Remarks on Pathologies of Renormalization-Group Transformations for the Ising Model

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**Abstract.** The results recently obtained by van Enter, Fernandez and Sokal [EFS] on non-Gibbsianness of the measure $\nu = T_b\,\mu_{\beta,h}$ arising from the application of a single decimation transformation $T_b$, with spacing $b$, to the Gibbs measure $\mu_{\beta,h}$ of the Ising model, for suitably chosen large inverse temperature $\beta$ and non zero external field $h$, are critically analyzed. In particular we show that if, keeping fixed the same values of $\beta$, $h$ and $b$, one iterates a sufficiently large number of times $n$ the transfomation $T_b$, one obtains a new measure $\nu' = (T_b)^n\,\mu_{\beta,h}$ which is Gibbsian and moreover very weakly coupled.

Work partially supported by grant SC1-CT91-0695 of the Commission of European Communities

## § 1 **Introduction and Results**.

This note is motivated by a series of discussions with many colleagues and, in particular, with Giovanni Gallavotti and Joel Lebowitz, about the relationships between:

i) some recent results by van Enter, Fernandez and Sokal (EFS) concerning non-Gibbsianness of some measures $T\nu$ obtained by applying a renormalization group transformation $T$ to a Gibbsian measure $\nu$ (see [EFS]) and

ii) some recent results obtained by the authors (see [MO1],[MO2],[MO3]), on the application of finite size conditions, of the form originally introduced in [O] and [OP], to the study of equilibrium and non equilibrium properties of lattice spin systems near to a first order phase transition.

Joel Lebowitz suggested us to write a note to clarify these relationships in some example.

We will consider a simple case: the Ising model in dimensions $d \geq 3$ at large inverse temperature $\beta$ and non zero external magnetic field $h$ and we will denote by $\mu_{\beta,h}$ the associated unique Gibbs state.

In this case EFS prove that the *decimation transformation* $T_b$, on a scale $b$, gives rise, for suitable values of $\beta$ and $h$, depending on $b$, to a non-Gibbsian distribution.

We prove here that, as an immediate consequence of the results in [O],[OP] and [MO1], if, for *exactly the same* thermodynamical parameters $\beta$ and $h$ , we apply the decimation transformation $T_{b'}$, on any sufficiently large scale $b'$, we obtain a measure which not only is Gibbsian but is also weakly coupled (high temperature) . In particular one can take $b' = b^n$ for all sufficiently large $n$; namely one can iterate the EFS transformation to come back, in this way, to the set of Gibbs measures. Moreover, as a corollary, we obtain that $T_{b^n}\mu_{\beta,h}$ converges, at the level of the interaction, for $n$ tending to infinity, to the trivial fixed point corresponding to a free system with the appropriate magnetization.

Let us now give some definitions.

The configuration space of the system is $\Omega = \{-1,1\}^{\mathbf{Z^d}}$. The formal hamiltonian is

$$H(\sigma) = -\frac{1}{2}\sum_{\langle i,j\rangle}\sigma_i\sigma_j - \frac{1}{2}h\sum_i \sigma_i \tag{1.1}$$

where $\langle i,j\rangle$ stands for a pair of nearest neighbours in $\mathbf{Z^d}$ and $h > 0$.

We use $\Omega_\Lambda = \{-1,1\}^\Lambda$ to denote the configuration space in $\Lambda \subset \mathbf{Z^d}$.

Consider a *finite* region $\Lambda$ in $\mathbf{Z^d}$ (in this case we write : $\Lambda \subset\subset \mathbf{Z^d}$) and an arbitrary boundary condition $\tau$ outside $\Lambda$ ($\tau \in \Omega_{\Lambda^c}$). The energy of a configuration $\sigma$ in $\Lambda$ is given by :

$$H^\tau_\Lambda(\sigma) \;=\; -\frac{1}{2}\sum_{x,y\in\Lambda:\;|x-y|\,=\,1}\sigma_x\sigma_y \;-\; \frac{1}{2}\sum_{x\in\Lambda}[\;h\;+\sum_{y\notin\Lambda:\;|x-y|\,=\,1}\tau_y\;]\sigma_x \tag{1.2}$$

The finite volume Gibbs measure in $\Lambda$, with $\tau$ boundary conditions has the expression:

$$\mu^\tau_\Lambda(\sigma) = \exp(-\beta H^\tau_\Lambda(\sigma))/\text{normalization}$$

Notice that EFS use a different notation : they call magnetic field and denote by $h$ our quantity $\beta h$.

The Dobrushin-Lanford-Ruelle (DLR) theory of Gibbs measures is based on the conditional probabilities $\pi_\Lambda$ for the behaviour of the system in a finite box $\Lambda \subset\subset \mathbf{Z^d}$ subject to a specific configuration in the complement of $\Lambda$. According to [EFS] a probability measure whose conditional probabilities for finite subsets $\Lambda \subset\subset \mathbf{Z^d}$ : $(\pi_\Lambda)_{\Lambda\subset\subset\mathbf{Z^d}}$ satisfy

$$\lim_{\Lambda'\uparrow\mathbf{Z^d}}\;\sup_{\omega_1,\omega_2\in\Omega:(\omega_1)_{\Lambda'}=(\omega_2)_{\Lambda'}}|\pi_\Lambda f(\omega_1)-\pi_\Lambda f(\omega_2)| = 0 \tag{1.3}$$

(namely the conditional expectations in $\Lambda$ of any cylindrical function $f$ corresponding to different boundary conditions $\omega_1,\omega_2$, coinciding in $\Lambda' \supset \Lambda$, tend to coincide as $\Lambda'$ tends to $\mathbf{Z^d}$ ), is called *quasilocal*.

A quasilocal probability measure on $\Omega$ satisfying also a so called *nonnullity* condition, i.e. a sort of absence of hard core exclusion, is called *Gibbsian* (see [EFS] for more details).

In [EFS] it is shown that the above notion of Gibbsianness of a measure is equivalent to the usual notion based on absolute summability properties of the interaction which gives sense to DLR equations.

The following Theorem is proved in [EFS] (see Theorem 4.7.therein).

**Theorem 1.**

*For each $d \geq 3$ and $b \geq 2$ there is a $\bar\beta$ and a function $\bar h(\beta)$ with $\bar h(\beta) > 0$ if $\beta > \bar\beta$ such that for all $\beta > \bar\beta$ and $h < \bar h$ the following is true: Let $\mu$ be a Gibbs measure for the d-dimensional Ising model described by the hamiltonian (1.1) with inverse temperature $\beta$ and magnetic field $h$. Then the renormalized measure $T_b\mu$, arising from a decimation transformation with spacing $b$, is not consistent with any quasilocal specification. In particular it is not the Gibbs measure for any uniformly convergent interaction.*

(we refer to Definitions 2.1,2.2,2.3,2.4 in [EFS] for precise definitions concerning interactions)

Let us now state our result.

**Theorem 2.**

*For each $d \geq 3$, $h > 0$ there is a $b_0$ and a $\beta_0$ such that for all $\beta > \beta_0$ and $b' > b_0$ the following is true: Let $\mu$ be the Gibbs measure for the d-dimensional Ising model described by the hamiltonian (1.1) with inverse temperature $\beta$ and magnetic field h. Then the renormalized measure $T_{b'}\mu$ arising from a decimation transformation with spacing $b'$ is Gibbsian; moreover the corresponding interaction is absolutely summable and the sum of all but the one body terms tends to zero (in the norm $\mathcal{B}^1$ defined in [EFS]) as $b'$ tends to infinity.*

## § 2 **Proof of Theorem2.**

We will use definitions and notation of [MO1] to which we refer for details.

Let us first recall the notion of finite volume strong mixing condition (in its simplest form) that has been introduced in [MO1].

We say that the Gibbs measures $\mu_\Lambda^\tau$ in $\Lambda$, with boundary condition $\tau$ outside $\Lambda$, satisfy the *strong mixing condition* in $\Lambda$, with parameters $C > 0$, $\gamma > 0$ and denote it by $SMC(\Lambda, C, \gamma)$, if, for all $x, y \in \Lambda$:

$$\sup_{\tau \in \Omega_{\Lambda^c}} |\mu_\Lambda^\tau(\sigma_x \sigma_y) - \mu_\Lambda^\tau(\sigma_x)\mu_\Lambda^\tau(\sigma_y)| \leq C \exp(-\gamma|x-y|) \tag{1.4}$$

In [MO1] we have shown that, given $C, \gamma$, if $SMC(., C, \gamma)$ is verified for a sufficiently large cube $Q_L(C, \gamma)$ of side $L$ then there are $C' > 0$, $\gamma' > 0$ such that $SMC(\Lambda, C', \gamma')$ is verified for all arbitrarily large regions $\Lambda$ which are *multiples* of the basic cube $Q_L$ ; where, given the odd integer $L$, a set $\Lambda$ is said to be a multiple of the basic cube $Q_L(0)$ (of edge $L$ centered at the origin):

$$Q_L(0) = \{y \in \mathbf{Z^d}; |y_i| \leq \frac{L-1}{2},\ i = 1\ldots, d\},$$

if it is a union of translated cubes $Q_L(x) \equiv Q_L(0) + x$, $x \in \mathbf{Z^d}$, with disjoint interior :

$$\Lambda = \cup_{y \in Y} Q_L(L\ y)$$

for some $Y \subset \mathbf{Z^d}$

This property, namely the propagation to all larger scales of a finite volume strong mixing condition is called *effectiveness*.

**Remark**

Notice that in [MO1] different notions of strong mixing were defined in a much more general set-up. The possibility of using the particularly simple form given in (1.4) is a consequence of the peculiarities of the standard Ising model. In [MO1] this condition was called $SMT(\Lambda, 1, C, \gamma)$

It was shown in [MO1] that the following Proposition holds true:

**Proposition 1.**

*For all $d \geq 2$, $h > 0$, there exists $L_0 = L_0(d,h)$ and $\beta_0 = \beta_0(d,h,L)$ such that $SMC(Q_L, C, \gamma)$ holds for all $L \geq L_0(d,h)$ provided $\beta > \beta_0(d,h,L)$*

**Proof.**

Let us give here a proof of the above statement less sketchy than the one given in Section 5 of [MO1].

Consider a cube $\Lambda = Q_L$ in $\mathbf{Z^d}$.

By F.K.G. inequalities ( see [FKG], [H] ) and by taking the limit $\beta \to \infty$ of $\mu_\Lambda^{-\underline{1}}$, where $-\underline{1}$ is the configuration identically equal to $-1$, it follows that, if the ground state configuration of $H_\Lambda^{-\underline{1}}(\sigma)$ with minus boundary conditions is identically equal to $+1$ for all $x \in \Lambda$, then the same holds for the ground state configurations of $H_\Lambda^\tau(\sigma)$ with arbitrary boundary conditions $\tau$.

We want now to prove that if $L > \frac{2d}{h}$:

$$\min_\sigma H_\Lambda^{-\underline{1}}(\sigma) = H_\Lambda^{-\underline{1}}(+\underline{1}); \quad H_\Lambda^{-\underline{1}}(\sigma) > H_\Lambda^{-\underline{1}}(+\underline{1}) \; \forall \, \sigma \neq +\underline{1} \tag{1.5}$$

namely that the configuration with all spins $+1$ in $\Lambda$ is the unique ground state for $-\underline{1}$ boundary conditions.

Indeed for every configuration $\sigma \in \Omega_\Lambda$ consider the union $C(\sigma)$ of all the closed unit cubes centered at each site $x \in \Lambda : \sigma_x = +1$. Consider, also, the union $D(\sigma)$ of the closed unit cubes centered at sites $x \in \mathbf{Z^d} : \sigma_x = -1$ (we recall that we set $\sigma_x = -1 \; \forall \; x \in \mathbf{Z^d} \setminus \Lambda$) and call $D^* = D^*(\sigma)$ the unique infinite connected component of $D(\sigma)$. $C(\sigma)$ splits into maximal connected components $C_1, \ldots, C_k$. Among $C_1, \ldots, C_k$ we select the subset $\bar{C}_1, \ldots, \bar{C}_j$ of components touching $D^*$. We call them *outer components* and denote by $\gamma_1, \ldots, \gamma_j$ their exterior boundaries (i.e. $\bar{\gamma}_i = \bar{C}_i \cap D^*$). We call $|\gamma_i|$ the measure of their boundaries $\gamma_i$ and $|\theta(\gamma_i)|$ the measure (cardinality) of the interior $\theta(\gamma_i)$ of $\gamma_i$, namely the set of points that are separated from the boundary $\partial\Lambda$ by $\gamma_i$.

It is easy to prove the isoperimetric estimate:

$$\sum_i |\theta(\gamma_i)| \leq (\sum_i \frac{|\gamma_i|}{2d})^{\frac{d}{d-1}} \tag{1.6}$$

(see, for instance, Theorem 1.1 in [T])

¿From (1.6) we get, for every $\sigma \in \Omega_\Lambda$:

$$H_\Lambda^{-1}(\sigma) - H_\Lambda^{-1}(-\underline{1}) \geq -h\sum_i |\theta(\gamma_i)| + \sum_i |\gamma_i| \geq -h\sum_i |\theta(\gamma_i)| + 2d(\sum_i |\theta(\gamma_i)|)^{\frac{d-1}{d}}. \tag{1.7}$$

¿From (1.7) we get, for $L > \frac{2d}{h}$:

$$H_\Lambda^{-1}(\sigma) - H_\Lambda^{-1}(-\underline{1}) \geq -hL^d + 2dL^{d-1} \equiv H_\Lambda^{-1}(+\underline{1}) - H_\Lambda^{-1}(-\underline{1}) \tag{1.8}$$

and the first equality in (1.5) is proven ; the uniqueness of the minimum also follows from (1.7).

As we already said, from (1.5) we also get that $\forall\, \tau$, $+\underline{1}$ is the unique minimum for the energy.

Now, for every $L > \frac{2d}{h}$, $C > 0$, $\gamma > 0$ given, if we choose a sufficiently large $\beta h$ it is easy to get Condition $SMC(Q_L, C, \gamma)$ (simply because $\mu_\Lambda^\tau(\sigma_x = -1$ for some $x \in \Lambda) \to 0$ as $\beta \to \infty$ so that the Gibbs measure in $\Lambda$ is, for every $\tau$, a small perturbation of a $\delta$-measure concentrated on the unique ground state $+\underline{1}$).

Now, from the effectiveness of $SMC(Q_L, C, \gamma)$ for $L$ large enough, which has been proven in [MO1], we are able to deduce properties of the renormalized interaction obtained by applying a block decimation transformation on a scale $L$. Before stating the result in Proposition 2 below we need some more definitions.

Let $b = 2L$ and call $A(x)$ the cubic block $Q_L(bx)$ and $\alpha_x \in \Omega_{A(x)}$ the corresponding spin configuration. We call $A$ the set of all the $A(x)$'s and we identify it with the subset of $\mathbf{Z^d}$ given by the union of the cubes $A(x)$.

For $\alpha \in \Omega_A$ let $H_A^{(r)}(\alpha)$ be the (formal) renormalized hamiltonian obtained by integrating out the spins in $\mathbf{Z^d} \setminus A$. To be more precise consider a big cube $\bar\Lambda \equiv Q_{\bar L}(0)$ centered at the origin with side $\bar L = (2p+1)L$, $p$ integer. Choose free (empty) boundary conditions outside $\bar\Lambda$ . For every $\sigma_{\bar\Lambda} \in \Omega_{\bar\Lambda}$ call

$$\alpha_{\bar\Lambda} = \sigma_{A\cap\bar\Lambda}\ ;\ \eta_{\bar\Lambda} = \sigma_{\bar\Lambda\setminus A}$$

Let $H(\sigma_{\bar\Lambda}) \equiv H(\alpha_{\bar\Lambda}, \eta_{\bar\Lambda})$ be given by (1.1) and consider the renormalized hamiltonian $H^{(r)}_A(\alpha_{\bar\Lambda})$ given by:

$$\exp(-H^{(r)}_A(\alpha_{\bar\Lambda})) = \sum_{\eta_{\bar\Lambda}} \exp(-\beta H(\alpha_{\bar\Lambda}, \eta_{\bar\Lambda}))$$

This corresponds to applying to the Gibbs measure $\mu_{\bar\Lambda}$ a kind of decimation in $\bar\Lambda \setminus A$, obtained by integrating out the spins in $\bar\Lambda \setminus A$, that is to construct the *relativization* of $\mu_{\bar\Lambda}$ to $\Omega_{\bar\Lambda\cap A}$. Call $\mu^{(r)}_{\bar\Lambda}$ the renormalized measure on $\Omega_{\bar\Lambda\cap A}$ obtained in this way :

$$\mu^{(r)}_{\bar\Lambda} = \frac{\exp(-H^{(r)}_A(\alpha_{\bar\Lambda}))}{Z_\Lambda}, \quad Z_\Lambda = \sum_{\alpha_{\bar\Lambda}} \exp(-H^{(r)}_A(\alpha_{\bar\Lambda})) \equiv \sum_{\sigma_{\bar\Lambda}} \exp(-H(\sigma_{\bar\Lambda}))$$

One can repeat the same construction for any boundary condition $\tau \in \Omega_{\bar\Lambda^c}$ and get, in this way, the renormalized measure $\mu^{(r),\tau}_{\bar\Lambda}$.

**Proposition 2.**

*For $L$ large enough we have:*

*i)*

$$\lim_{\bar\Lambda\uparrow \mathbf{Z^d}} \mu^{(r),\tau}_{\bar\Lambda} = \mu^{(r)}$$

*independently of (the sequence of) boundary conditions $\tau$*

*ii) $\mu^{(r)}$ is Gibbsian; there exists a corresponding interaction $(\Phi_V)_{V\subset A}$ (see [EFS]) which is absolutely summable and:*

*iii)*

$$\sum_{V\ni A(0):|V|>1} \|\Phi_V\| = o(L)$$

**Proof.**

Take any finite cube $\bar\Lambda$ with side $\bar L = (2p+1)L$ and $\tau$ boundary conditions outside $\bar\Lambda$ . It is sufficient to notice that $\bar\Lambda \setminus A$ is a multiple ol $Q_L$ ; so, by Proposition 1, for $L$ sufficiently large the Gibbs measure $\mu^{\tau,\alpha}_{\bar\Lambda\setminus A}$ satisfies $SMC(\bar\Lambda, C', \gamma')$ for suitable $C', \gamma'$, uniformly in $\bar\Lambda$. The same is true for any ( not necessarily cubic) region $\bar V$ multiple of $Q_L$ ( see [MO1] for more details).

i), ii) immediately follow from effectiveness. Indeed let $\mu^{(r)}(\alpha_x|\alpha_y)$ be the conditional probability, with respect to the measure $\mu^{(r)}$ on $\Omega_A$, of the configuration $\alpha_x$ in $A(x)$ given $\alpha_y$ in $A(y)$. Gibbsianness follows from nonnullity and quasilocality which, in turn, follows from

$$|\mu^{(r)}(\alpha_x|\alpha_y) \; - \; \mu^{(r)}(\alpha_x)| \leq C" \exp(-\gamma"|x-y|) \tag{1.9}$$

for suitable $C", \gamma"$, uniformly in $\bar{\Lambda}, \tau, \alpha_x, \alpha_y$.

(1.9) is a direct consequence of the strong mixing condition valid uniformly in $\bar{\Lambda}$ ( effectiveness).

To get iii) we need more detailed estimates; it easily follows from the arguments developed in [O], [OP], based on the cluster expansion, from Proposition 1 and Appendix 2 in [MO1].

Let us now conclude the proof of Theorem 2. Let us use $\omega_x \in \{-1,+1\}$ to denote the value of the original spin variable $\sigma_{bx}$ at the center $bx$ of the cube $A(x)$ . We set $\alpha_x = (\omega_x, \bar{\alpha}_x)$ ; $\bar{\alpha}_x \in \{-1,+1\}^{A(x)\setminus bx}$ is the restriction of $\alpha_x$ to $A(x) \setminus bx$ .

Let $B = \{y = bx, x \in \mathbf{Z^d}\}$ be the sublattice of $\mathbf{Z^d}$ of spacing $b$. Consider the measure $\nu = T_b\mu$ obtained by applying the usual decimation transformation in $\mathbf{Z^d}\setminus B$ ( relativization to $\Omega_B$ of the original Gibbs measure $\mu$ in $\Omega_{\mathbf{Z^d}}$).

We have:

$$\nu(\omega_x|\omega_y) \;\equiv\; \mu(\omega_x|\omega_y) = \sum_{\bar{\alpha}_x} \mu^{(r)}(\bar{\alpha}_x, \omega_x|\omega_y) \;\equiv\; \sum_{\bar{\alpha}_x} \mu(\bar{\alpha}_x, \omega_x|\omega_y) \tag{1.10}$$

On the other hand

$$\mu(\alpha_x|\omega_y) = \sum_{\bar{\alpha}_y} [\mu(\alpha_x|\bar{\alpha}_y, \omega_y) - \mu(\alpha_x|\bar{\alpha}^*_y, \omega^*_y)]\mu(\bar{\alpha}_y|\omega_y) + \mu(\alpha_x|\bar{\alpha}^*_y, \omega^*_y) \tag{1.11}$$

where $\alpha^*_y, \omega^*_y$ denote a reference configuration (e.g. equal to all $+1$ in $A_y$).

From (1.9),(1.10).(1.11) we get the quasilocality condition (1.3); the nonnullity condition is trivially satisfied so that we get the desired Gibbs property for $\nu$. Absolute summability of the renormalized interaction immediately follows from the arguments of [O], [OP], together with the estimate of the norm of the more than one body interaction, which estimate vanishes as L increases to infinity. This concludes the proof of Theorem2.

## § 3 **Conclusions.**

As it was noticed in [EFS] the non existence of the renormalized interaction is a consequence of the presence of a first order phase transition for the original model in $\mathbf{Z^d} \setminus B$ for particular values of $(\omega_x)_{x \in B}$ and suitable $h$ and $\beta$ ; for example $\omega_x = -1 \, \forall x$ and uniform positive $h$ , exponentially in $\beta$ near to the value $h^*(b)$ which is needed to compensate, in $\mathbf{Z^d} \setminus B$, the effect of the $-1$'s in $B$ and to give rise to a degeneracy in the ground state in $\mathbf{Z^d} \setminus B$ (see also [I]).

It seems clear, from the above analysis, that this pathology comes from the fact that, on a too short spatial scale $b$ (with respect to the thermodynamic parameters and mainly the magnetic field $h$), the system is reminiscent of the existence of a phase transition for $h = 0$ .

One needs to analyze the system on a large enough scale to put in evidence the uniqueness of the phase and the absence of long range order. This scale, on which bulk effects become dominant with respect to surface effects, corresponds to the formation of a critical droplet of the stable phase; in other words it is necessary to go to distances of this order to be sure that the boundary conditions have been screened out. The fact that on shorter distances the system is sensible to the boundary conditions and ordered is somehow related to the phenomenon of metastability taking place near to a first order phase transition.

The general philosophy suggested by the outcome of our Theorem 2 is that, when applying a renormalization group transformation, the system behaves as if it was weakly coupled, provided the scale of the transformation is chosen, depending on the thermodynamic parameters, in such a way that our strong mixing condition becomes effective; however it is important to stress that the relevant length scale near to a low temperature coexistence line is not the correlation length of the unique pure phase but, rather, the length of the critical droplet of the stable phase inside the metastable one.

Finally we want to underline the fact that Theorem 2 is based on a finite size condition related to a particularly simple shape: a cube.

As we discussed in [MO1] an *effective* condition à la Dobrushin and Shlosman, implying their Complete Analyticity ( see, for instance, [DS]), could not be verified in the region of thermodynamic parameters that we are considering here. Indeed Dobrushin-Shlosman's finite size condition involves the consideration of *arbitrary shapes*; it is clear that to exploit the presence of a positive magnetic field as a mechanism of screening we need, say, a plurirectangle with sufficiently large minimal edge. For not sufficiently "fat" and regular regions (as, for instance, pathological regions with many holes in the bulk) it is conceivable that not only a finite size condition coming from the screening effect of $h$ does not hold but, also, that , for special values of $h$ and $\beta$, the Dobrushin-Shlosman complete analyticity can even fail. This is actually what EFS prove, in some cases, as a direct consequence of their methods to show non-Gibbsianness of some renormalized measures.

At the same time the equivalent of complete analyticity, not stated for *all* regions but, rather, for arbitrarily large *but sufficiently regular* domains directly follows from [O],[OP] and the above described finite size condition on a suitable cube.

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