DYNAMICAL INTEGRAL TRANSFORM ON FRACTAL SETS AND THE COMPUTATION OF ENTROPY

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Abstract : We introduce an integral transform of wavelet type, which we call Dynamical Integral Transform, and we show that it can be used to compute the 2-nd Renyi entropy for a large class of invariant measures. The method is then generalized to the whole spectrum of the Renyi entropies and establishes a correspondance between thermodynamic formalism and the Dynamical Integral Transform of expanding strange sets. Numerical examples are presented.

July 1992
CPT-92/P.2663 (revised version)
To appear in Physica D

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[3] Supported by Contrat CEE n° SC1*0281.

1 - Introduction

The wavelet transform and the methods derived from it have revealed to be powerful tools to analyze fractal sets. In particular they give a complete description of multifractal measures in two different ways : by a local analysis of the scaling exponents of the measure [1-3] or through the computation of the spectrum of the generalized dimensions [4-6].

The same methods have been succesfully extended to extract histograms of scaling exponents from fully developed turbulence data [7] .In this article we show that they can be extended to compute the whole spectrum of the Renyi entropies. We recall in section 2 that the Renyi entropies completely characterize the dynamical properties of an ergodic measure; we also indicate in section 5 how to perform a local analysis with a suitable integral transform in order to detect the entropy at a point x as given by the Brin-Katok theorem.

The approach we follow is taken from our previous works [4][5]; we defined there the following integral transform:

$$T_p(a,\mu) = a^{-p} \int\int_{J\times J} g\left(\frac{||x-y||}{a}\right) d\mu(x)d\mu(y) \tag{1}$$

where : μ is a probability non-atomic measure, a and p are positive numbers, J is the set supporting μ and $||.||$ is some distance on J. We called (1) Integrated Wavelet Transform for the close analogy with the usual wavelet transform. Our principal assumption for the function g [2][4][5] was that it is in $C^1(\mathbf{R})$ and rapidly decreasing (more than any power) on the positive semi-axis, in particular : $\lim_{a\to 0^+} a^{-p} g\left(\frac{r}{a}\right) = 0$ pointwise for $r>0$ and $p \geq 0$.

When rigorous proofs are carried out on invariant sets J, we showed that it is sufficient to assume g monotone on $\mathbf{R}^+$; we never need admissible analyzing functions g of zero mean (wavelets): we discussed this question in details in the introduction of [4][5]. There are some other advantages using our test functions: first, we are able to get some asymptotic local properties (see section 5 in [5] and 5 in this paper) which are not immediately recovered with admissible wavelets. Then we showed, in the context of the generalized dimensions [5], that the monotone analyzing functions work numerically better than the usual wavelets like the mexican-hat: they are also used in [8] [1]. Moreover the choice for monotone wavelets was explicitly assumed by Falconer [9] [2], in connection with order-two densities of certain fractal measures. Despite these facts and to avoid ambiguities, we think that the name *wavelets* could be reserved to the admissible wavelets introduced in the study of signals [10]. Therefore we call our g *analyzing functions* or *test functions* and the integral (1) *Fractal Integral Transform* or *FIT*.

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[1] "As recently addressed in various theoretical studies (...), the wavelet analysis of singular measures do not require the analyzing wavelet g to be of zero mean. In the present study, we will use a Gaussian function $g(r) = e^{-r^2}$." [8], p.4

[2] "Choice of a suitable (wavelet) w depends on the purpose for which the wavelet transform is used. There is considerable divergence between authors as to the conditions that w ought to satisfy-for example, some require certain moments of w to vanish, while others specify rapid decrease at infinity."[9], p.781

A basic concept in our method is that of adapted analyzing function. We say that g is *p-adapted to μ for the FIT* if $\lim_{a\to 0^+} \binom{\sup}{\inf} |T_p(a,\mu)|$ are different from 0 and $+\infty$.

In this case, we proved in [4] that, for the class of Gibbs measures μ_β on disconnected conformal mixing repellers, p is equal to the generalized dimension of order 2 of the measure μ_β, denoted with $D_2(\mu_\beta)$, defined by the usual partition function approach (as a root of the topological pressure or free energy). This result has then been generalized to the whole spectrum of the generalized dimensions $D_q(\mu_\beta)$ [5]. In [4][5] we checked the adaptedness for smooth sets in $\mathbf{R}^n$, for the ternary Cantor set and we presented an argument, relating the FIT to the correlation integral introduced by Grassberger, Hentschel and Procaccia [11][12], which formally establishes the adaptedness of the FIT for a large class of multifractal sets. This last argument is also illustrated in section 2 for the dynamical integral transform we are going to introduce. It is useful to remark that if the adaptedness partially fails, in the sense that at least one, but not both, of the previous limits is zero or infinity, the equalities in the theorems in [4] and [5] and in those presented in this paper, must be replaced with bounds, whose direction can be easily deduced from the proofs.

The paper is organized as follows. In section 2 we collect the definition and some basic properties of the Renyi entropies and we introduce the Dynamical Integral Transform. In section 3 we show how to use it to compute the 2-nd Renyi entropy in the case of mixing repellers (Theorem 1) and then we generalize this result to any q-Renyi entropy (Theorem 2). In section 4 we present some examples. In section 5, we suggest a local analysis to detect the local entropy. The conclusions are in section 6.

2 - The Renyi entropies and the Dynamical Integral Transform

We now briefly recall the definition of the Renyi entropies for invariant sets J (see for example [13-16]). Let $\mathcal{P}$ be a μ-measurable partition of J and $\mathcal{P}^{(n)}$ the dynamical partition obtained intersecting all the sets of the form $P_{\alpha_0}, T^{-1}P_{\alpha_1}, ..., T^{-n}P_{\alpha_n}$, where P_{α_i} is an element of $\mathcal{P}$ and T^{-i} denotes the preimage of order i of the mapping T generating the invariant set J. If the diameter of any element $P_i^{(n)} \in \mathcal{P}^{(n)}$ goes to zero when $n \to \infty$, we say that the partition $\mathcal{P}$ is generating and we define the q-order Renyi entropy of any invariant probability measure μ on J as :

$$K_q(\mu) = -\lim_{n\to\infty} \frac{1}{n}\frac{1}{(q-1)} \log \sum_{P_\alpha^{(n)} \in \mathcal{P}^{(n)}} \left[\mu(P_\alpha^{(n)})\right]^q \tag{2}$$

The Kolmogorov-Sinai metric entropy $K_1(\mu)$ is recovered in the limit $q \to 1$, while the topological entropy is recovered in the limit $q \to 0$. For the class of Gibbs measures on the invariant sets of Markov maps and for the hyperbolic iterated function systems (IFS), the q-entropies can be expressed in terms of the topological pressure [14] [17]. For the first class of systems we have [17] :

$$K_q(\mu_\beta)(q-1) = qP(\beta) - P(q\beta) \tag{3}$$

where μ_β and $P(\beta)$ are respectively the Gibbs measure and the topological pressure corresponding to the potential : $-\beta \log ||DT(x)||$ [18], being DT the derivative of the mapping generating the invariant set. For the IFS generated by linear mappings with contraction rates $\lambda_1 \dots \lambda_s$ equipped with a balanced measure μ of weights $p_1 \dots p_s, p_1 + \dots + p_s = 1$, we have [14] [17] :

$$K_q(\mu)(q-1) = -\log(p_1^q + \dots + p_s^q) \tag{4}$$

These relations are important because they are the key of the proofs we give below; moreover they can be generalized to larger classes of hyperbolic dynamical systems. The possibility to reconstruct the topological pressure from the Renyi entropies shows the importance of the latters in the investigations of the dynamical properties of strange sets. From the pressure one can extract the Lyapunov exponents and the metric entropy of any Gibbs measure [18]; moreover the Legendre transform of the pressure is the deviation function of the sum of the positive Lyapunov exponents. At this regard, the connection between the Renyi entropies and the large deviations for the Lyapunov exponents and the Kolmogorov entropy are made explicit in [19]. We also recall that in [15] a Legendre transform relates the set of $K_q(\mu)$ to a scaling function $S(\gamma)$ which is the topological entropy of the set of points for which the (positive) Lyapunov exponents converge to γ. A similar Legendre transform has also been proposed by Paladin and Vulpiani in [19] with a different interpretation. The corresponding function $\hat{S}(\gamma)$ was in fact related to a set of Local Expansion Parameters (LEP) which are nothing but the local entropies given by the Brin-Katok theorem. We will return to this point in section 5.

There are at least two formal useful characterizations for the 2-nd entropy $K_2(\mu)$. The first is in terms of the so-called correlation integral introduced by Grassberger and Procaccia in [13], namely :

$$C_d(\ell) = \int\int_{J\times J} \Theta(\ell - ||x-y||_d) d\mu(x) d\mu(y)$$

where Θ is the Heaviside function and $||x-y||_d$ is the "dynamic norm" :

$$||x-y||_d = \left[\sum_{j=0}^{d-1} ||T^j x - T^j y||^2\right]^{1/2} \tag{5a}$$

It is also possible to use the norm

$$|||x-y|||_d = \max_{0\le j\le d-1} ||T^j x - T^j y|| \tag{5b}$$

which is equivalent to (5a) and even simpler to handle with in the proofs of the following theorems. We will use the norm (5a) and we will return to (5b) in the last section. With $d=0$, we sometimes mean the usual norm.

According to Grassberger and Procaccia [13], if $C_0(\ell)$ scales like $C_0(\ell) \sim \ell^\nu$, where $\nu = D_2(\mu)$ is the correlation dimension, then $C_d(\ell)$ vanishes like $C_d(\ell) \sim \ell^\nu f_d(\ell)$, where $f_d(0) \sim e^{-dK_2(\mu)}$ for $d \to +\infty$.

The second approach is in term of the d-order energy integral $\phi_d(\alpha)$, defined as :

$$\phi_d(\alpha) = \int\int_{J\times J} ||x-y||_d^{-\alpha} d\mu(x)d\mu(y)$$

It was proved in [20][21] that, for some expanding sets, $\phi_d(\alpha)$ is a meromorphic function with the smallest positive pole located at $D_2(\mu)$; moreover the residue at this pole behaves like $D_2(\mu)e^{-dK_2(\mu)}$ for large d.

We now introduce the *dynamical integral transform* (DIT), defined as :

$$T_p(a,\mu,d) = a^{-p} \int\int_{J\times J} g\left(\frac{||x-y||_d}{a}\right) d\mu(x)d\mu(y) \tag{6}$$

where $||x-y||_d$ is the dynamic norm (5) : clearly a dynamic T must be defined on the set J and μ is meant to be invariant with respect to T.

We say that the function g is *p-adapted to μ for the DIT* if for any d :

$$\lim_{a\to 0^+} \binom{\sup}{\inf} |T_p(a,\mu,d)| \neq (0,+\infty) \ .$$

It is easy to relate the DIT to the correlation integral $C_d(\ell)$ defined above; we follow here the same method as presented in section 4 of [4]. A direct verification (an integration by parts), shows that :

$$T_p(a,\mu,d) = a^{-p}\left[g\left(\frac{r}{a}\right)C_d(\ell)\right]_0^\Delta - a^{-(p+1)}\int_0^\Delta C_d(\ell)g'\left(\frac{\ell}{a}\right)d\ell \tag{7}$$

where Δ is the diameter of J. We neglect the first term in the right hand side since it vanishes in the limit $a\to 0^+$, by the rapid decay of g at infinity. Assuming for $C_d(\ell)$ the scaling [13] :

$$C_d(\ell) \underset{\substack{\ell\to 0^+ \\ d\to+\infty}}{\sim} \ell^\nu e^{-dK_2(\mu)}$$

where $\nu = D_2(\mu)$ is the correlation dimension, and substituting in (7), we immediately get:

$$|T_p(a,\mu,d)| \sim a^{\nu-p}e^{-dK_2(\mu)} \int_0^{\Delta/a} \ell^\nu |g'(\ell)| d\ell$$

Since the integral in the r.h.s. surely converges by the fast decay of g at infinity, we finally have

$$|T_p(a,\mu,d)| \underset{\substack{a\to 0^+ \\ d\to+\infty}}{\sim} a^{\nu-p} \quad e^{-dK_2(\mu)} \tag{8}$$

.

This heuristic argument shows that the class of test functions considered in this paper is adapted for the measures satisfying the preceding scaling for the correlation integral and therefore gives the correct value for the 2-nd Renyi entropy. We give now a rigorous proof of the scaling (8).

3 - Rigorous results

In this section we restrict ourselves to disconnected conformal mixing repellers endowed with Gibbs measures and to the disconneected attractors of hyperbolic iterated function systems endowed with balanced measures (see [4] for a review of the properties of these systems). For these dynamical systems, we can prove the following result :

Theorem 1:

If the function g is p-adapted to μ for the FIT, in which case $p = D_2(\mu)$, then it is p-adapted to μ for the DIT and moreover :

$$\lim_{d\to+\infty} -\frac{1}{d}\log \limsup_{a\to 0^+} |T_p(a,\mu,d)| = \lim_{d\to+\infty} -\frac{1}{d}\log \liminf_{a\to 0^+} |T_p(a,\mu,d)| = K_2(\mu)\ . \quad (9)$$

Proof:

The p-adaptedness of g for the DIT follows from the equivalence of the dynamical norm with the original one.

We prove the rest of the theorem in the particular case of a linear Cantor set with two scales λ_1 and λ_2 equipped with a balanced measure of different weights p_1 and p_2; the general proof can be carried out quite easily using the techniques developped in [4][5] and the characterization of the Renyi entropies in terms of the pressure. We will indicate at the end of the proof how to perform these generalizations. Applying the balanced property of the measure, we can rewrite the DIT as (note that, by the previous assumptions, g is either positive or negative on the positive semi axis):

$T_p(a,\mu,d) =$

$$a^{-p}\left\{\sum_{k_1,\ldots,k_{d-1}=1}^{2} p_{k_1}^2\ldots p_{k_{d-1}}^2 \int\int_{J\times J} g\left(\frac{\Lambda_{k_1\ldots k_{d-1}}||x-y||}{a}\right) d\mu(x)d\mu(y)\right\} + a^{-p}\phi(a) \quad (10)$$

where we have put for simplicity:

$$\Lambda_{k_1\ldots k_{d-1}} = \left(\lambda_{k_1}^2\ldots\lambda_{k_{d-1}}^2 + \lambda_{k_2}^2\ldots\lambda_{k_{d-1}}^2 + \ldots + \lambda_{k_{d-1}}^2 + 1\right)^{1/2} \quad (11)$$

and $\phi(a)$ contains the integrals whose g have an argument strictly positive since x and y are iterated backward respectively on two sets at a finite distance (we use here the disconnectedness of J). Therefore, when $a \to 0^+$, the term $a^{-p}\phi(a)$ goes to zero by the rapid decrease of g at infinity and we neglect it.

Then we can write the DIT as :

$$T_p(a,\mu,d) = \sum_{k_1,\dots k_{d-1}=1}^{2} \left(p_{k_1}^2 \dots p_{k_{d-1}}^2\right) \Lambda_{k_1\dots k_{d-1}}^{-p} T_p(a_{k_1\dots k_{d-1}},\mu) \tag{12}$$

where $a_{k_1\dots k_{d-1}} = a\Lambda_{k_1\dots k_{d-1}}^{-1}$.

By the adaptedness of g for the FIT for $p = D_2(\mu)$, we have :

$$\lim_{a\to 0^+} \binom{\sup}{\inf} |T_p(a,\mu)| = \binom{s_1}{s_2} \neq (0,+\infty)$$

and therefore there are two positive constants ρ_1 and ρ_2 for which :

$$0 < s_2 - \rho_2 \leq |T_p(a,\mu)| \leq s_1 + \rho_1 < \infty$$

for a sufficiently small.

Then we can take the $\lim_{a\to 0^+} \binom{\sup}{\inf}$ in (12) and get:

$$\begin{aligned}
&(s_2-\rho_2) \sum_{k_1\dots k_{d-1}=1}^{2} \left(p_{k_1}^2 \dots p_{k_{d-1}}^2\right) \Lambda_{k_1\dots k_{d-1}}^{-p} \\
\leq &\lim_{a\to 0^+} \binom{\sup}{\inf} |T_p(\mu,a,d)| \\
\leq &(s_1+\rho_1) \sum_{k_1\dots k_{d-1}=1}^{2} \left(p_{k_1}^2 \dots p_{k_{d-1}}^2\right) \Lambda_{k_1\dots k_{d-1}}^{-p}
\end{aligned} \tag{13}$$

Taking the logarithm, dividing by d and finally sending d to infinity, we get (note that p is still equal to $D_2(\mu)$):

$$\begin{aligned}
&\lim_{d\to+\infty} -\frac{1}{d} \log \lim_{a\to 0^+} \binom{\sup}{\inf} |T_p(\mu,a,d)| \\
= &\lim_{d\to+\infty} -\frac{1}{d} \log \sum_{k_1\dots k_{d-1}=1}^{2} \left(p_{k_1}^2 \dots p_{k_{d-1}}^2\right) \Lambda_{k_1\dots k_{d-1}}^{-p}
\end{aligned} \tag{14}$$

Since :

$$1 \leq \Lambda \leq d^{1/2} \quad \text{and} \quad \sum_{k_1\dots k_{d-1}=1}^{2} \left(p_{k_1}^2 \dots p_{k_{d-1}}^2\right) = \left(p_1^2+p_2^2\right)^{d-1} \tag{15}$$

we finally get :

$$\lim_{d\to+\infty} -\frac{1}{d} \log \lim_{a\to 0^+} \binom{\sup}{\inf} T_p(\mu,a,d) = -\log\left(p_1^2+p_2^2\right), \tag{16}$$

where the r.h.s. is exactly the 2-nd Renyi entropy of the balanced measure of weights p_1 and p_2 (cf. (4)).

To handle with the general case of a disconnected conformal mixing repeller endowed with a Gibbs measure μ_β, we have to use the machinery developed in [4], that allows us to replace the left hand side of (14) with :

$$\lim_{d\to\infty} -\frac{1}{d}\log\sum_{k_1\ldots k_{d-1}} \frac{e^{-2(d-1)P(\beta)}}{||DT^{d-1}(\eta_{k_1\ldots k_{d-1}})||^{2\beta}}\Lambda^{-p}_{k_1\ldots k_{d-1}} \tag{17}$$

where $\eta_{k_1\ldots k_{d-1}}$ is an arbitrary point in the corresponding element of the Markov partition of J obtained iterating J backward $(d-1)$ times with the inverse branches of T. Besides $\Lambda_{k_1\ldots k_{d-1}}$ has the structure

$$\Lambda_{k_1\ldots k_{d-1}} = \left[||DT^{d-1}(\eta_{k_1\ldots k_{d-1}})||^{-2} + \ldots + ||DT(\eta_{k_1})||^{-2} + 1\right]^{1/2}$$

and can be bounded uniformly as in (15) by hyperbolicity.

Since, for $d \to +\infty$, the term $d^{-1}\log\sum_{k_1\ldots k_{d-1}} ||DT^{d-1}(\eta_{k_1\ldots k_{d-1}})||^{-2\beta}$ converges to $P(2\beta)$ [14], we get from the limit (17) :

$$\lim_{d\to+\infty} -\frac{1}{d}\log \lim_{a\to 0^+} \begin{pmatrix}\sup\\ \inf\end{pmatrix} |T_p(\mu, a, d)| = 2P(\beta) - P(2\beta)$$

which agrees with the 2-nd Renyi entropy for the Gibbs measure μ_β computed with the thermodynamic formalism (cf.(3)). $\diamondsuit$

We now generalize the DIT in order to get all the Renyi entropies; we follow the same idea that leaded us to generalize the FIT to get the generalized dimensions, that is we define *the q-Dynamical Integral Transform*, $q-$DIT, as :

$$T_p(a, \mu, d, q) = a^{-p}\left\{\int_J d\mu(x)\left|\int_J d\mu(y) g\left(\frac{||x-y||_d}{a}\right)\right|^{q-1}\right\}^{\frac{1}{q-1}}$$

If the function g is adapted for a certain p for the corresponding $q-$FIT, $T_p(a, \mu, q)$, (that is obtained setting $d = 0$)[3], we proved that p is equal to the generalized dimension $D_q(\mu)$ when $q > 1$ and is a lower bound to it (eventually equal) for $q < 1$. The $q-$DIT behaves in a similar way; we still formulate the following theorem for the dynamical systems considered in theorem 1.

[3] In this case we also say that g is $(p-q)$-adapted to μ

Theorem 2:

If g is p-adapted to μ for the q-FIT, then it is p-adapted to μ for the q-DIT and moreover :

$$\begin{aligned} &\lim_{d\to+\infty} -\frac{1}{d}\log \lim_{a\to 0^+} \binom{\sup}{\inf} T_p(a,\mu,d,q) = K_q(\mu) \quad \text{for} \quad q>1 \\ &\text{and} \\ &\limsup_{d\to+\infty} -\frac{1}{d}\log \lim_{a\to 0^+} \binom{\sup}{\inf} T_p(a,\mu,d,q) \le K_q(\mu) \quad \text{for} \quad q<1. \end{aligned} \tag{18}$$

Proof:

The p-adaptedness of g for the q-DIT follows from the equivalence of the dynamical norm with the usual norm.

We specialize again the proof to the linear Cantor set with two scales λ_1, λ_2 and two weights $p_1 \neq p_2$. The generalization is straightfoward using the same suggestions as at the end of the proof of theorem 1.

Applying the balancement of the measure $(d-1)$ times we get :

$$\begin{aligned} T_p^{q-1}(a,\mu,d,q) = \sum_{k_1\dots k_{d-1}=1}^{2} & p_{k_1}\dots p_{k_{d-1}} \int_J d\mu(x) \Big[p_{k_1\dots k_{d-1}} \\ & a^{-p}\int_J |g\left(\Lambda_{k_1\dots k_{d-1}}\frac{||x-y||}{a}\right)|d\mu(y) + \phi_a(x)\Big]^{q-1} \end{aligned} \tag{19}$$

where $\phi_a(x)$ is a positive function uniformly bounded in x away from zero and going to 0 when $a \to 0^+$; in fact it collects the integrals which g has the argument bounded away from zero; we also call $F_{k_1\dots k_{d-1}}(a,x)$ the first term in the square bracket in (19). Note that we cannot neglect $\phi_a(x)$ at this point since it can be of the same order as $F_{k_1\dots k_{d-1}}(a,x)$; therefore we need to integrate before. As in the proof of Theorem 2 in [5], we can bound the integral in the sum in (19) by Minkowski's inequality from above and below for $q>1$ by an expression of the type :

$$\int_J d\mu(x) F_{k_1\dots k_{d-1}}^{q-1}(a,x)\left[1+\left\{\frac{\int_J d\mu(x)\phi_a^{q-1}(x)}{\int_J d\mu(x) F_{k_1\dots k_{d-1}}^{q-1}(a,x)}\right\}^{\beta}\right]^{\beta^{-1}} \tag{20}$$

where β is equal to 1 or $\frac{1}{q-1}$.

Whenever $q<1$, we can only keep the upper bound corresponding to the case $\beta=1$ and this is the reason of the inequality in the statement of the theorem; in the following we consider the case $q>1$.

We observe that :

$$\int_J d\mu(x) F_{k_1\dots k_{d-1}}^{q-1}(a,x) = p_{k_1}^{q-1}\dots p_{k_{d-1}}^{q-1}\ \Lambda_{k_1\dots k_{d-1}}^{-p(q-1)}\ T_p^{q-1}(\mu, a_{k_1\dots k_{d-1}}, q)$$

where $a_{k_1 \ldots k_{d-1}} = a\Lambda^{-1}_{k_1 \ldots k_{d-1}}$. The second factor in (20) goes to 1 when $a \to 0^+$, since the integral in the numerator converges to zero and the denominator is bounded away from zero and infinity by the adaptedness of g. Therefore we can bound $T_p^{q-1}(a, \mu, d, q)$ from above and below by

$$\sum_{k_1 \ldots k_{d-1}=1}^{2} p^q_{k_1} \ldots p^q_{k_{d-1}} \Lambda^{-p(q-1)}_{k_1 \ldots k_{d-1}} \; T_p^{q-1}(\mu, a_{k_1 \ldots k_{d-1}}, q)$$

Using the same arguments as in the proof of Theorem 1, we thus get :

$$\lim_{d \to +\infty} -\frac{1}{d} \log \lim_{a \to 0^+} \begin{pmatrix} \sup \\ \inf \end{pmatrix} T_p(a, \mu, d, q) = -\frac{1}{(q-1)} \log(p_1^q + p_2^q) \tag{21}$$

which is just the q-Renyi entropy $K_q(\mu)$ (cf. (4)). $\diamondsuit$

Under the hypothesis of adaptedness for the q-FIT and for more general systems, we can define:

$$\begin{pmatrix} \overline{K_q^+}(\mu) \\ \overline{K_q^-}(\mu) \end{pmatrix} = \limsup_{d \to +\infty} -\frac{1}{d} \log \lim_{a \to 0^+} \begin{pmatrix} \sup \\ \inf \end{pmatrix} T_p(a, \mu, d, q)$$

where the $\limsup_{d \to +\infty}$ is taken in view of the special but important case of Theorem 2. Apart the obvious bound $\overline{K_q^+}(\mu) \leq \overline{K_q^-}(\mu)$, we can easily prove the following monotonicity property of the indices $\overline{K^\pm}_q(\mu)$ as functions of q, which is consistent with the analogous property for the K_q solutions of eqs (3) and (4).

Proposition 1:

$$\overline{K_q^\pm}(\mu) \leq \overline{K_r^\pm}(\mu) \textbf{ for } r \leq q$$

Proof:

The proof is similar to that of Proposition 1(iii) in [5] and relies on the structure of the $q-$DIT as a L^{q-1}-norm and on Jensen's inequality. $\diamondsuit$

Remarks:

A] To get the Kolmogorov-Sinai entropy corresponding to $q = 1$, we have to take the logarithm of the q-DIT and derive it at the point $q = 1$. We are thus led to conjecture that :

$$\lim_{d \to +\infty} -\frac{1}{d} \lim_{a \to 0^+} \int_J d\mu(x) \log \left| a^{-p} \int_J g\left(\frac{||x-y||_d}{a}\right) d\mu(y) \right| = K_1(\mu) \tag{22}$$

whenever $p = D_1(\mu)$ (the information dimension): this conjecture is numerically verified in example 3 below.

B] Another interesting question is how the limit (18) approximates the topological entropy, that is obtained setting $q = 0$ in the definition (2) : by theorem 2, we expect in general this limit to be smaller than the topological entropy and this is not surprising since the topological entropy is defined independently of any measure.

4 - Examples

Example 1: Ternary Cantor set

In the case of the q-FIT, we considered in [5] the usual ternary Cantor set with equal scales $\lambda_1 = \lambda_2 = \lambda, \quad 0 < \lambda \leq 1/3$ endowed with the Gibbs measures $\mu_\beta, \quad \beta \in \mathbf{R}$; assuming g $(p-q)$-adapted to μ_β for all q, we proved that $p = \log\ 2/\log\ \lambda^{-1}$ for all q. A result of the same type holds for the q-DIT. We first observe that the left hand side of (18) is equal to : $k_q(\mu_\beta) = \log\ 2$ for $q > 1$ and is smaller or equal to $\log\ 2$ for $q < 1$ (this easily follows from the expression for the pressure : $P(\beta) = \log(2\lambda^\beta))$. Then, by Proposition 1 and passing to the limit for $p = \log\ 2/\log\ \lambda^{-1}$, we have for $r < 1 < q$:

$$\begin{aligned}\log\ 2 \geq & \limsup_{d\to+\infty} -\frac{1}{d}\log \lim_{a\to 0^+} \begin{pmatrix}\sup\\ \inf\end{pmatrix} T_p(a,\mu,d,r)\\ \geq & \lim_{d\to+\infty} -\frac{1}{d}\log \lim_{a\to 0^+} \begin{pmatrix}\sup\\ \inf\end{pmatrix} T_p(a,\mu,d,q) = \log\ 2\end{aligned}$$

The same argument holds by replacing $\limsup$ with $\lim\ \ \inf$ in the first limit and this shows that the $\lim\limits_{d\to+\infty}$ exists for $q < 1$ and therefore the limit in (18) recovers the same entropy for all $q \in \mathbf{R}$.

Example 2: Irrational rotation

We showed in [4] that, for the unit hypercube in $\mathbf{R}^n$, the adaptedness of g to the Lebesgue measure implies $p = n$; moreover we checked the adaptedness of the function $g(r) = e^{-r}$. We now identify the unit interval [0,1) with the torus $\mathbf{T}$ and put on it the irrational rotation : $T(x) = x + \alpha \ mod\ 1, \quad x \in \mathbf{T}$ and $\alpha \in \mathbf{R}\backslash\mathbf{Q}$; T leaves invariant the Lebesgue measure. Then we define on $\mathbf{T}$ the norm : $||z|| = \inf\limits_{k\in\mathbf{Z}} \mid z+k \mid,\ z \in \mathbf{R}$, being $|\cdot|$ the Euclidean distance on $\mathbf{R}$. It is easily seen that :

$$|||x-y|||_d = \begin{cases} \mid x-y \mid & \text{for} \quad \mid x-y \mid\leq 1/2 \\ 1- \mid x-y \mid & \text{for} \quad \mid x-y \mid\geq 1/2, \end{cases}$$

so that the dynamic norm is independent of d. This and the preceding argument on the adaptedness imply that the 2-nd Renyi entropy is zero, which is consistent with the well known fact that the irrational rotations have zero entropy.

Example 3: Numerical

The relations (9), (18) and (22) give a precise method to compute numerically the Renyi entropies once the corresponding generalized dimensions are known, and these can just be computed by means of the q-FIT (see [5] for the examples).

We now present a few examples, others being in preparation [22]. First of all, we compute the 2-nd Renyi entropy by means of formula (9) in two cases : (i) the ternary Cantor set with scale $\lambda = 1/3$ and equal weights $p_1 = p_2 = 1/2$; (ii) the same Cantor set with different weights $p_1 = 1/4$ and $p_2 = 3/4$; these examples will also show the adaptedness of the test function $g(r) = e^{-r^2}$. In the former case $D_2 = \log\ 2/\log\ 3$ and $K_2 = \log\ 2 = 0.6931...$; in the latter $D_2 = \log(8/5)/\log\ 3$ and $K_2 = \log(16/10) = 0.470004...$, as given by (4). The integrals have been computed by averaging over the predecessors of order n of an arbitrary point of the unit interval as explained in [4] : the value of n was taken equal to 14. We did not take into account the values of a comparable in size with the scales of the Cantor set determined by the n-th order of iteration, that is $\left(\frac{1}{3}\right)^{n-d}$: in fact, for values of a smaller than this threshold, the DIT goes down abruptly.

The values of the DIT for fixed d are given extrapolating the data as a function of a and are reported in figs. 1 and 2. Morover a non-linear fitting was performed on these data with a function of type :

$$f(d) = C_1 + C_2\ d^{-C_3}$$

where C_1, C_2 and C_3 are constants and C_1 is just the expected value of the DIT in the limit $d \to +\infty$.

For the Cantor set with equal weights we found : $C_1 = 0.6914$ and for the Cantor with different weights : $C_1 = 0.4669$, in excellent agreement with the theoretical values. We note that, up to numerical errors, the convergence is of type $\frac{1}{d}$.

In figs. 3 and 4, we report on the computation of the DIT for the same sets but for different values of a. As explained before, when a becomes comparable with $\left(\frac{1}{3}\right)^{n-d}$, the DIT goes to zero and this is evident from the figures : this means that we have to keep the structure of the set enough fine (by taking n sufficiently large) when it is explored by the integral transform at decreasing values of a.

The same procedure has then been applied to the computation of the other Renyi entropies for the Cantor set with weights $p_1 = 1/4$ and $p_2 = 3/4$, for which the entropies are different and given by eq. (4). In fig. 5, we report on the numerical and analytical curves of the K_q in the range $q \in [-11, 11]$; the precise values are quoted in table 1. Note that the entropy K_1, as given by formula (22), is in excellent agreement with the theoretical value and that, when $q < 1$, the numerical entropies are smaller than the theoretical ones, as predicted by Theorem 2. We also report in fig. 6 on the Legendre transform of the quantity $K_q(q-1)$. The interpretation of the corresponding curve will be given at the end of section 5.

We conclude this section by computing the entropies for a non-hyperbolic invariant set, precisely the Hénon attractor generated by the mapping: $x' = 1 - ax^2 + y; y' = bx$, with $a = 1.4$ and $b = 0.3$. The q-DIT was computed as usual by approximating the integral with the ergodic mean with respect to the physical measure. In figs. 7 and 8, we show the results for the entropies K_0 and K_2: the extrapolated values are 0.443049 for K_0 and 0.298903 for K_2, in agreement with the same entropies given, for example, in [13] and [24], where $K_0 = 0.445$ and $K_2 = 0.325 \pm 0.02$.

5 - A "multientropy" local analysis

Given the transform at the point $x \in J$:

$$T_p(a, \mu, x) = a^{-p} \int_J g\left(\frac{||x-y||}{a}\right) d\mu(y)$$

and defined the functions of p : $T_p^{\pm}(\mu, x) = \lim_{a \to 0^+} \binom{\sup}{\inf} |T_p(a, \mu, x)|$, we showed in [1][5] that:

i) there are two transition points $p^{\pm}(x)$ such that $T_p^{\pm}(\mu, x) = 0, p < p^{\pm}(x)$ and $T_p^{\pm}(\mu, x) = +\infty, p > p^{\pm}(x)$; ii) these points are related to the local exponents of the measure μ, that is :

$$\beta_-(x) \leq p^+(x) \leq p^-(x) \leq \beta_+(x) \tag{23}$$

where

$$\beta_{\pm}(x) = \lim_{r \to 0^+} \binom{\sup}{\inf} \frac{\log \mu(B(x,r))}{\log r} \tag{24}$$

being $B(x, r)$ a ball of center x and radius r in the metric $||.||$. We now introduce the *local dynamical integral transform* at x defined as :

$$T_p(a, \mu, d, x) = a^{-p} \int_J g\left(\frac{|||x-y|||_d}{a}\right) d\mu(y)$$

where the metric $|||.|||_d$ is defined by (5b). By the equivalence of the norms, the transition points of the functions :

$$T_p^{\pm}(\mu, d, x) = \lim_{a \to 0^+} \binom{\sup}{\inf} |T_p(a, \mu, d, x)|$$

will be again $p^{\pm}(x)$. We guess that if the following limits exist (note that the functions $T_p^{\pm}$ are computed in their transition points):

$$T^+(x) = \lim_{d \to +\infty} -\frac{1}{d} \log T^+_{p^+(x)}(\mu, d, x)$$

$$T^-(x) = \lim_{d \to +\infty} -\frac{1}{d} \log T^-_{p^-(x)}(\mu, d, x)$$

then they coincide respectively with the local entropies $h^{\pm}(x)$ defined by the Brin-Katok theorem [23] as:

$$\binom{h^+(x)}{h^-(x)} = \lim_{r \to 0^+} \lim_{d \to +\infty} \binom{\sup}{\inf} \left(-\frac{1}{d} \log \mu(B(x, d, r))\right) \tag{25}$$

where $B(x, d, r)$ is a ball of radius r in the metric $|||.|||_d$. For μ-almost every $x \in J, h^+(x) = h^-(x)$ and, for an ergodic measure μ, the local entropies are constant almost everywhere and equal to the metric entropy $K_1(\mu)$. Our conjecture is suggested by the formal analogy of the limits (24) and (25) and by a local extension of Theorem 1. A local analysis could be carried out at a point $x \in J$ by analyzing the behavior of the function $-\frac{1}{d}\log|T_p(a,\mu,d,x)|$ in the variables (a,d), for small values of a and large values of d and keeping p in the interval given by (23). In this way, the asymptotic oscillations of the function $-\frac{1}{d}\log|T_p(a,\mu,d,x)|$ at different points $x \in J$ should be proportional to the different strength of the local entropies $h^{\pm}(x)$. An investigation of these questions is in progress, but we already present here a numerical study of the above conjecture for the ternary Cantor set and we also give an interpretation of the Legendre transform of the spectrum of the K_q in terms of the local entropies given by the Brin-Katok theorem.

5.1 - We performed a numerical analysis of the Brin-Katok formula for the ternary Cantor set endowed with the balanced measure μ of equal weights and we observed two facts: first, the limit for $r \to 0^+$ is inessential, being the correct value for the entropy already reached when $d \to +\infty$, at fixed r. This is not surprising if one considers the following relations among the measures of a ball in the different metrics:

$$\mu(B(x,(\tfrac{1}{3})^{n+d-1}r)) \leq \mu(B(x,d,(\tfrac{1}{3})^{n}r)) \leq \mu(B(x,(\tfrac{1}{3})^{n+d-1})) \tag{26}$$

for $r \leq 1$ and $n \in \mathbf{N}$.

By suitably bounding r with power of $\frac{1}{3}$ and recalling that the measure of the intervals generating the Cantor set at the m-th step is 2^{-m}, we immediately recover the entropy $\log 2$ in the limit $d \to +\infty$. We show the numerical computation in figs. 9,10 and 11.

The second and not yet understood fact appears in figs. 10 and 11 and consists in the periodic oscillations with affect, for a particular choice of fixed r, the convergence of the limit for $d \to +\infty$.

In fig. 12, we report on the computation of the local entropy with the local DIT (the choice of the point x does not change the numerical results), after having taken the limit for $a \to 0^+$; what we found is in agreement with the expected value $\log 2$. We want to point out that a rigorous proof of this result for the ternary Cantor set seems difficult to get. For example, the relation (26), which shifts the problem from the DIT to the FIT by replacing the measure of the dynamical ball with the measure of a ball in the ordinary metric, is apparently not sufficient.

5.2 - In [19], Paladin and Vulpiani considered the Legendre transform of the spectrum of the $K_q(\mu) = K_q$, that is

$$K_q(q-1) = \min_{\gamma}(q\gamma - \hat{S}(\gamma)) \tag{27}$$

and they interpreted $\hat{S}(\gamma)$ as the topological entropy of the set $\Omega(\gamma)$, where $x \in \Omega(\gamma)$ when, with our notation:

$$\mu(B(T^k x, d, r)) \sim e^{-d\gamma} \tag{28}$$

for $r \to 0^+$ and $k \geq 0$. They called γ Local Expansion Parameter (LEP): see also Eckmann and Procaccia [19] and [25] for similar interpretations.

When $k = 0$, the scaling (28) is the physical way to write down mathematically the Brin-Katok formula. We think that the correct way of interpreting the function $\hat{S}(\gamma)$ is the following: let $\Omega^{\pm}(\gamma)$ the sets of points x for which respectively $h^{\pm}(x) = \gamma$. Then $\hat{S}(\gamma)$ is the common topological entropy for the sets $\Omega^{\pm}(\gamma)$. This assertions deserve to be proved analytically, probably using the large deviation techniques employed in the rigorous derivation of the $\alpha - f(\alpha)$ theory for the generalized dimensions [26] and the Bowen characterization of the topological entropy in terms of the $(n-\epsilon)$ spanning sets [27]. Apparently, the interpretation of the LEP's as the local Brin-Katok entropies, although implicit in (28), was at our knowledge not given before. Note that, according to this interpretation, the maximum of the concave curve $\hat{S}(\gamma)$ is the topological entropy K_0, while the same curve intersects the bissectrix at the Kolmogorov entropy K_1. This is evident in fig. 6 for the linear Cantor set and follows also from the Legendre transform of (4).

6 - Conclusions

We showed in this paper that a suitable integral transform of wavelet type, that we called Dynamical Integral Transform (DIT), allows us to compute the spectrum of Renyi entropies. For mixing repellers, we gave rigorous results, that we can extend to non-hyperbolic invariant sets.

Our method can be numerically implemented quite easily and compared to other techniques, like the correlation integral and the energy integral, shows some universality in the choice of the test functions that satisfy all the same asymptotic scalings. Moreover, our technique is intrinsically dynamic, that is we extract the entropies by (ergodically) averaging over orbits instead of partitionning the invariant sets as prescribed by formula (2), which is the most commonly used for the computation of the Renyi entropies.

Finally, the local version of the DIT allows us to explore the local entropies of strange sets which topological distribution we claimed is given by the Legendre transform of the Renyi entropies. This analysis is the natural extension to entropies of the capability of the integral transform of wavelet type to capture the local dimensions of fractal measures and should give a "multientropy" description of strange sets.

Acknowledgments: we want to thank S. Siboni for useful discussions concerning section 5.

Figures Captions

Fig. 1: Dynamical integral transform as a function of d, after having extrapolated on a, for the ternary Cantor set with equal weights. Fitting the data with the function : $f(d) = C_1 + C_2\ d^{-C_3}$, we found : $C_1 = 0.6914$ (giving K_2) ; $C_2 = -0.5156$; $C_3 = 1.022$.

Fig. 2: Dynamical integral transform as a function of d, after having extrapolated on a, for the ternary Cantor set with weights $p_1 = 1/4$ and $p_2 = 3/4$. Fitting the data with the function : $f(d) = C_1 + C_2\ d^{-C_3}$, we found : $C_1 = 0.4669$ (giving K_2) ; $C_2 = -0.3305$; $C_3 = 1.060$.

Fig. 3: Dynamical integral transform as a function of d for different values of a, for the ternary Cantor set with equal weights.

Fig. 4: Dynamical integral transform as a function of d for different values of a, for the ternary Cantor set with weights $p_1 = 1/4$; $p_2 = 3/4$.

Fig. 5: Numerical and analytical spectra of the Renyi entropies K_q for the ternary Cantor set with weights $p_1 = 1/4$; $p_2 = 3/4$.

Fig. 6: Legendre transform $\hat{S}(\gamma)$ of $K_q(q-1)$ for the ternary Cantor set with weights $p_1 = 1/4$; $p_2 = 3/4$. The maximum of this curve is the topological entropy K_0 and the curve intersects the bissectrix at the Kolmogorov entropy K_1.

Fig. 7: 0-Dynamical integral transform as a function of d, after having extrapolated on a, for the Hénon attractor. Fitting the data with the function : $f(d) = C_1 + C_2\ d^{-C_3}$, we found : $C_1 = 0.4430$ (giving K_0) ; $C_2 = 0.7300$; $C_3 = 1.3885$.

Fig. 8: Dynamical integral transform as a function of d, after having extrapolated on a, for the Hénon attractor. Fitting the data with the function : $f(d) = C_1 + C_2\ d^{-C_3}$, we found : $C_1 = 0.2989$ (giving K_2) ; $C_2 = 0.5773$; $C_3 = 0.6682$.

Fig. 9: Local entropy given by Brin-Katok formula for the ternary Cantor set with equal weights. $-\frac{1}{d}\mu(B(x,d,r))$ is plotted vs. d for fixed $r = 0.012048$. Fitting the data with the function : $f(d) = C_1 + C_2\ d^{-C_3}$, we found : $C_1 = 0.6931$ (giving the local entropy) ; $C_2 = 1.3863$; $C_3 = 1.0000$.

Fig. 10: Local entropy given by Brin-Katok formula for the ternary Cantor set with equal weights. $-\frac{1}{d}\mu(B(x,d,r))$ is plotted vs. d for fixed $r = 0.162105$. Note the oscillations which affect the convergence.

Fig. 11: Local entropy given by Brin-Katok formula for the ternary Cantor set with equal weights. $-\frac{1}{d}\mu(B(x,d,r))$ is plotted vs. d for fixed $r = 0.044194$. Note the oscillations which affect the convergence.

Fig. 12: Local dynamical integral transform as a function of d, after having extrapolated on a, for the ternary Cantor set with equal weights. Fitting the data with the function : $f(d) = C_1 + C_2\ d^{-C_3}$, we found : $C_1 = 0.7001$ (giving the local entropy) ; $C_2 = -0.5660$; $C_3 = 0.9122$.

Tables Captions

Tab. 1: Spectrum of the Renyi entropies K_q for the ternary Cantor set with weights $p_1 = 1/4$; $p_2 = 3/4$. Theoretical values are computed according to eq. (4). Numerical values are computed according to Theorem 2.

References

[1] J.-M. Ghez, S. Vaienti, J. Stat. Phys. **57** (1989) 415.

[2] S. Vaienti, Nonlinearity **4** (1991) 1241.
J.-M. Ghez, S. Vaienti, in Proceed. *Large Scale Structures in Nonlinear Physics*, J.-D. Fournier, P.L. Sulem, Eds., Lect. Notes in Phys., Springer-Verlag (Berlin, Heidelberg, New-York) **392** (1991) 320.

[3] A. Arneodo, G. Grasseau, M. Holschneider, Phys. Rev. Lett. **61** (1988) 2281.
M. Holschneider, J. Stat. Phys. **50** (1988) 963.
F. Argoul, A. Arneodo, J. Elezgaray, G. Grasseau, R. Murenzi, Phys. Lett. **A135** (1989) 327.

[4] J.-M. Ghez, S. Vaienti, Nonlinearity **5** (1992) 777.

[5] J.-M. Ghez, S. Vaienti, Nonlinearity **5** (1992) 791.

[6] J.-F. Muzy, E. Bacry, A. Arneodo, Phys. Rev. Lett. **67** (1992) 3515.
M. Holschneider, Thesis, Marseille, (1988).
E. Bacry, J.-F. Muzy, A. Arneodo, *Singularity spectrum of fractal signals from wavelet analysis: exact results*, Preprint CRPP (March 1992).

[7] E. Bacry, A. Arneodo, U. Frisch, Y. Gagne, E.J. Hopfinger, in *Turbulence and Coherent Structures*, O. Métais, M. Lesieur, Eds., Kluwer Academic Publishers (1990) 203.
F. Argoul, A. Arneodo, G. Grasseau, Y. Gagne, E.J. Hopfinger, U. Frisch, Nature **338** (1989) 51.
R. Benzi, M. Vergassola, *The Optimal Wavelet Transform and its Application to Two-Dimensional Turbulence*, Fluid. Dyn. Res. (1991), in press.
M. Vergassola, U. Frisch, Physica **D54** (1991) 58.
M. Vergassola, R. Benzi, L. Biferale, D. Pisarenko, *Wavelet Analysis of a Gaussian Kolmogorov Signal*, preprint (September 1991).

[8] A. Arneodo, F. Argoul, E. Bacry, J.-F. Muzy, M. Tabard, *Golden Mean Arithmetic in the Fractal Branching of Diffusion-Limited Aggregates*, Preprint CRPP (November 1991).

[9] K.J. Falconer, J. Stat. Phys. **67** (1992) 781.

[10] A. Grossmann, J. Morlet, SIAM J. Math. Anal. **15** (1984) 723
J.-M. Combes, A. Grossmann, Ph. Tchamitchian, Eds., *Wavelets*, Springer-Verlag (Berlin, Heidelberg, New-York) (1989).
R.R. Coifman, Y. Meyer, *Ondelettes, Opérateurs et Analyse non linéaire*, Hermann (Paris) (1989).

[11] H. Hentschel, I. Procaccia, Physica **D8** (1983) 435.

[12] P. Grassberger, I. Procaccia, Physica **D9** (1983) 189.
A. Cohen, I. Procaccia, Phys. Rev. **A31** (1985) 1872.

[13] P. Grassberger, I. Procaccia, Phys. Rev. **28** (1985) 2591.

[14] D. Bessis, G. Paladin, G. Turchetti, S. Vaienti, J. Stat. Phys. **51** (1988) 109.

[15] T. Bohr, D. Rand, Physica **D25** (1987) 387.

[16] J.-P. Eckmann, D. Ruelle, Rev. Mod. Phys. **57** (1985) 617.

[17] G. Servizi, G. Turchetti, S. Vaienti, Il Nuovo Cimento **B101** (1988) 285.

[18] D. Ruelle, *Thermodynamic Formalism*, Addison-Wesley (New York) (1978).

[19] G. Paladin, S. Vaienti, J. Phys. **A21** (1988) 4609.
J.-P. Eckmann, I. Procaccia, Phys. Rev. **A34** (1986) 659.
P. Szepfalusy, T. Tel, Phys. Rev. **A35** (1987) 477.
G. Paladin, A. Vulpiani, Phys. Rep. **156** (1987) 147.
G. Paladin, L. Peliti, A. Vulpiani, J. Phys. , Math. Gen. **A19** (1986) L991.
S. Vaienti, J. Stat. Phys. **56** (1989) 403.
R. Badii, P. Grassberger, A. Politi : J. Stat. Phys. **51** (1988), 135.

[20] D. Bessis, G. Servizi, G. Turchetti, S. Vaienti, Phys. Lett. **A119** (1987) 345.
J.-D. Fournier, G. Turchetti, S. Vaienti, Phys. Lett. **A140** (1989) 331.
D. Bessis, J.-D. Fournier, G. Servizi, G. Turchetti, S. Vaienti, Phys. Rev. **A36** (1987) 920.

[21] E. Orlandini, G. Servizi, M.C. Tesi, G. Turchetti, Il Nuovo Cimento **B106** (1991) 1221.
E. Orlandini, M.C. Tesi, G. Turchetti, J. Stat. Phys. **66** (1992) 515.
G. Turchetti, S. Vaienti, *Generalized Dimensions of Strange Sets and Cantorian Approximation*, to appear in Proc. Math. Soc. Egypt.
G. Turchetti, in *Nonlinear Dynamics*, G. Turchetti Ed., World Scientific (Singapore) (1989), 388.

[22] J.-M. Ghez, E. Orlandini, M.-C. Tesi, S. Vaienti : in preparation.

[23] M. Brin, A. Katok, in *Geometric Dynamics*, Lect. Not. in Math., Springer-Verlag (Berlin, Heidelberg, New-York) **1007** (1983) 30.

[24] S. Vaienti, in *Nonlinear Dynamics*, G. Turchetti, Ed., World Scientific (Singapore) (1989) 397.

[25] M. Sano, S. Sato, Y. Sawada, Prog. Theor. Phys. **76** (1986) 945.

[26] P. Collet, J. Lebowitz, A. Porzio, J. Stat. Phys. **47** (1987) 609.

[27] R. Bowen, in *Proc. Symposium Pure Math.* **14**, Amer. Math. Soc (Providence, R.I) (1970).