Almost periodicity of some Jacobi matrices

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Abstract

In this paper we show that random Jacobi matrices are almost periodic whenever they have purely absolutely continuous spectrum having finitely many bands.

1. INTRODUCTION:

The study of random Schrödinger operators aquired importance in view of their usefulness in understanding the properties of condensed matter systems. The theory is well developed both in the continuous and in the discrete settings and there are excellent reviews of Simon [1], Spencer [2] , Carmona [3] on this subject, in addition the theory also appears in the book of Cycone et al [4] . In one dimension the theory is much sharper as the spectral proerties of such operators are shown to have consequences on the nature of the random potential, see for example the Kotani theory on the determinicity of potentials haveing some absolutely continuous spectrum ([5] for the continuous case and Simon [6] for the discrete case .) Continuing this further it was shown by Kotani,Krishna [7] and Craig [8] that some random Schrödinger operators with purely absolutely continuous spectrum of a certain type are almost periodic of necessity. The route for showing such a result was through inverse spectral theory. Showing the almost periodicity of such random potentials involved seting up and solving the Dubrovin equation for some spectral parameters, Jacobi inversion on a Riemann surface etc., All this was done for the continuous case by B.M. Levitan [9] McKean and Moerbeke [10], McKean and Trubowitz [11].

In the discrete case (i.e. for Jacobi matrices) the inverse spectral theory exists, see Kac, Moerbeke[12], Moerbeke[13], Toda[14] for the periodic case and Carmona,Kotani[15] for the random case and the references therein. However while the existence of solutions for the inverse spectral problem is given in the above works it was only in [12]-[13] that the nature of the matrices so constructed is presented. Of necessity these turnout to be periodic matrices. In this paper we show that in the discrete setting given a random potential, with finite band absolutely continuous spectrum, it is almost periodic, in the sense that the support of the random potential consists only of almost periodic sequences . The theory extends to the case of infinitely many bands and will appear in [16].

We have this paper in four sections. In the first we set up the direct spectral theory and identify the spectral functions that will play a role in the subsequent sections. In the second section we do the inverse theory and finally in section 3 the Jacobi inversion is presented.

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2. DIRECT-THEORY:

We consider $\Omega = l^2_{\mathcal{R}}(\mathbf{Z}, \frac{1}{1+|n|^2})$ and $\mathcal{B}$ the associated Borel σ- algebra. We consider a bimeasurable invertible transformation T on Ω whose action is $T\omega(n) = \omega(n+1)$ and consider a probability measure $\mathcal{P}$ on $(\Omega, \mathcal{B})$ such that it is invariant and ergodic with respect to T. Given $\omega \in$ Supp $\mathcal{P}$ we consider the operator q^ω of multiplication by ω on $l^2(\mathbf{Z})$ and consider the family of operators $H^\omega = \Delta + q^\omega, \Delta$ the discrete Laplacian. We assume further that $\omega \in$ Supp $\mathcal{P}$ implies that ω is bounded as a sequence in which case $\omega \to H^\omega$ will be a measurable self-adjoint operator valued map. Then the general theory [3] of such operators shows that there exist constant sets $\Sigma, \Sigma_{ac}, \Sigma_{sc}$ and Σ_{pp} such that the spectrum , the absolutely continuous, singularly continuous and purepoint spectra are the above sets respectively for the operators H^ω a. e. ω.

The main theorem of the paper is the following.

Theorem. 2.1 Suppose $(\Omega, \mathcal{B}, \mathcal{P})$ is as above with $\mathcal{P}$ erogodic with respect to the shift T acting on Ω. Suppose that for the associated self-adjiont operators H^ω the spectrum is purely absolutely continuous and consists of finitely many bands. Then, any ω in Supp $\mathcal{P}$ is an almost periodic sequence.

We present the proof of this theorem at the end of section 3.

Before we proceed further let us outline the strategy employed in proving the theorem. The information on the absolutely coninuous spectrum , together with the zeros of the green function for a given potential q gives us an expression for the green function. From this we obtain an expression for the sum of the Weyl m-functions. Using the additional property that the imaginary parts of the m-functions agree on the absolutely continuous spectrum, we obtain expressions for the difference of the Wely m-functions. The Weyl m-functions thus obtained are identified as the values, on the two sheets of a Riemann surface,of a meromorphic function and the poles of this meromorphic function are related , via the trace formula, to the potential q. These poles are then shown to be related to theta funcnctions , from which almost periodicity follows.

We would like to emphasize here that unlike the continuous case, [7,8,9], where the reflectionless property of the potentials is sufficient [8], here it is not sufficient. The reason is that the Dubrovin equation, that came for free in the continuous case ,was the heart of the matter there and its analog is missing in the discrete case.

Henceforth we fix a ω and consider H^ω on $l^2(\mathbf{Z})$ and drop the superscript also, refering to q^ω as q and the associated sequence ω(n) as q(n). We shall also write the limits $\lim_{\epsilon\to 0} g_{\lambda+i\epsilon}$ as simply g_λ for ease of writing and the appropriate definition should be clear from the context. Then form Weyl theory it is known that for $\lambda \in \mathbf{C}^+$, the difference equation

$$(H-\lambda)u_\lambda = 0$$

has two independent solutions u_1 ,u_2 such that $u_1(0)$=1,$u_1(1)$=0, $u_2(0)$=0, $u_2(1)$=1. One also has unique solutions $u_{\pm,\lambda}$ in $l^2(\mathbf{Z}^\pm)$ and there exist holomorphic functions $m_\pm(\lambda)$ such that $u_{\pm,\lambda}(n) = u_1(n) + m_\pm(\lambda)u_2(n)$. We also have that

$$m_\pm = -\frac{u_\pm(\pm 1)}{u_\pm(0)} \tag{1}$$

which can equivalently be taken as their definition. Weyl theory also gives the expression for the green kernel of H ,for m $\leq$ n in terms of the wronskian $[u_+, u_-]$, as

$$g_\lambda(n,m) = (H-\lambda)^{-1}(n,m) = \frac{u_+(n)u_-(m)}{[u_+,u_-]} \tag{2}$$

and the definition extended by symmetry to m $\geq$ n. An evaluation of the wronskian leads to the expression

$$(H-\lambda)^{-1}(n,n) = \frac{(-1)}{m_{+,n}(\lambda) + m_{-,n}(\lambda) + \lambda - q(n)} \tag{3}$$

where $m_{\pm,n}(\lambda)$ are defined as in (1) by taking T^nq instead of q. Let us recall the following relations for $m_\pm$ from [6].

$$m_{\pm,n}(\lambda) = -\frac{u_\pm(n\pm 1)}{u_\pm(n)} \tag{4}$$

and

$$m_{\pm,n}(\lambda) = q(n) - \lambda - (m_{\pm,(n\mp 1)})^{-1} \tag{5}$$

and in terms of the operators $H_{\pm,n}$ of H restricted to the subspaces $\{u \in l^2(\mathbf{Z}^\pm) : u(n) = 0\}$, we have

$$m_{\pm,n} = (H_{\pm,n} - \lambda)^{-1}(\delta_{n\pm 1}, \delta_{n\pm 1}) \tag{6}$$

¿From now on we retain the superscript ω and consider the set

$$A = \{\lambda \in \mathbf{R} : \mathbf{E}\{\mathbf{ln}\ \left|\mathbf{m}_+^\omega(\lambda + \mathbf{i0})\right|\} = \mathbf{0}\}$$

. Then we have the following theorem from the theory of random Jacobi matrices. See Simon [1] (proofs of Theorems 1 and 2)for the proofs.

Proposition. 2.2 Suppose A is as above with $|A|$ >0, then $\mathbf{E}^{\mathbf{H}}_{\mathbf{ac}}(\mathbf{A}) \neq \mathbf{0}$
If further A is an open interval , then $\mathbf{E}^{\mathbf{H}}_{\mathbf{sing}}(\mathbf{A}) = \mathbf{0}$.

Theorem. 2.3 Suppose the spectrum of H^ω contains the set A , then

$$\lim \; Re g_{\lambda+i\epsilon}(n,n) = 0 \text{ a.e. } \omega \text{ a.e. } \lambda \in \mathrm{A}.$$

For a.e. pair $\{(\omega,\lambda)\}$ we have

$$Im \; m_+^\omega(\lambda) = Im \; m_-^\omega(\lambda) \text{ and } Re[\; m_+^\omega(\lambda) + m_-^\omega(\lambda) + \lambda - q(0)] = 0.$$

Further, $\{m_+^\omega(\lambda), q(0)\}$, a.e. $\lambda \in$A determines $\{q^\omega(n), n \in \mathbf{Z}\}$ uniquely a.e. ω.
Infact we can also deduce as in [5] or [7] that

Corollory. 2.4 Suppose the spectrum of the Jacobi matrix H^ω is purely absolutely continuous and is the union of closed intervals (finitely or infinitely many),then we have

$$Re \; g_{\lambda+i0}^\omega(n,n) = 0 \;\; \forall\omega \text{ and } \forall \mathrm{n} \in \mathbf{Z}, \lambda \in \sigma(\mathrm{H}^\omega).$$

Also,

$$Im \; g_{\lambda+i0}^\omega(n,n) = 0 \;\; \forall\omega \text{ and } \forall \mathrm{n} \in \mathbf{Z}, \lambda \in \mathbf{R}\backslash\sigma(\mathrm{H}^\omega).$$

3. INVERSE THEORY:

In this section we specify the spectral data that is uniqely associated to a given potential q. The necessary spectral data are the band edges λ_i of the absolutely continuous spectrum , the Dirichlet eigenvalues ξ_i of an appropriate half space problem and the $\pm$ valued variables σ_i specifying as to which half space problem the Dirichlet eigen values ξ_i correspond.

Definition. 3.1 A set A is called reflectionless whenever, $Re \; g_{\lambda+i0}(n,n) = 0, \forall n \in \mathbf{Z}$ and a.e. $\lambda \in$ A. Then we have,

Lemma. 3.2 Suppose the spectrum Σ of H is a reflectionless set and is also the union of closed sets $[\lambda_{2i}, \lambda_{2i+1}], i = 0,..,N$ Then, the following are valid for every n $\in \mathbf{Z}$.

$$Re \; g_{\lambda+i0}(n,n) = 0 \text{ on } \Sigma \text{ and } Im \; g_{\lambda+i0}(n,n) = 0 \text{ on } \mathbf{R}\backslash\Sigma.$$

There is a unique zero $\xi_i(n)$ of $g_\lambda(n,n)$ in each of the intervals $[\lambda_{2i-1}, \lambda_{2i}], i = 1,..,N$.

Proof. The first part of the lemma follows from the assumption that the spectrum is a reflectionless set and the vanishing of the imaginary part on the complement of the spectrum in $\mathbf{R}$ is easy to verify. As for the zeros of the green function we note that if a zero exists it is unique, since the green function is real analytic and is strictly increasing as a function of λ in each connected componant of $\mathbf{R}\backslash\Sigma$. If there is a zero in $(\lambda_{2i-1}, \lambda_{2i})$ call it $\xi_i(n)$. Otherwise if g_λ is positive there then set $\xi_i(n)$ to be λ_{2i-1} and if it is negative set $\xi_i(n)$ to be λ_{2i}. This proves the lemma.

■

Definition. 3.3 We consider Ω as in the previous section. We call a point q of Ω a reflectionless potential if the spectrum of Δ + q is a reflectionless set.

Definition. 3.4 We define the hull $\mathcal{H}$(q) of a reflectionless potential as the closure in the topology of Ω of $\{q(.+n) : n \in \mathbf{Z} \text{ and } \mathrm{q} \in \Omega\}$

Proposition. 3.5 If q is a reflectionless potential with spectrum Σ of the type mentioned in the above lemma. Then each q in the Hull $\mathcal{H}$(q) is also a reflectionless potential with the same spectrum.

Proof. It is known that the functions $m_{\pm}^{(k)}(\lambda)$ converge compact uniformly in $\mathbf{C}^{+}$ whenever a sequence of potentials q_k converges to q in the topology of Ω, see [5]. The rest of the proof follows as in [7].

Remark. 3.6 We note that recalling the equations (2.3,2.4,2.5) for $g_\lambda(n,n)$, $m_{+,n}(\lambda)$,$m_{-,n}(\lambda)$, any zero $\xi_i(n)$ of $g_\lambda(n,n)$ in $[\lambda_{2i-1}, \lambda_{2i}]$ corresponds to a pole of $m_{+,n}(\lambda)$ (or $m_{-,n}(\lambda)$. Equivalently to an eigen value of $H_{+,n}$ (or $H_{-,n}$).

It is possible to reconstruct the green function of the Jacobi matrix satisfying the conditions of Lemma (3.2) from the spectrum and the zeros of the green function.

Theorem. 3.7 Suppose the spectrum of H satisfies the conditions of Lemma 3.2. Then, the data $\{\lambda_i\}$ and $\{\xi_i(n)\}$ is sufficient to recover the green function $g_\lambda(n,n)$ uniquely.

Proof. Since $g_z(n,n)$ is Herglotz, its logarithm $\ln g_z(n,n)$ is also Herglotz. Further $0 \leq Im \ln g_\lambda \leq \pi$ in $\bar{\mathbf{C}}^{+}$ and the Herglotz representation theorem gives the expression for $\ln g_z(n,n)$ as

$$\ln g_z(n,n) = a + bz + \frac{1}{\pi}\int_{\mathbf{R}} (\frac{1}{(x-z)} - \frac{x}{1+x^2}) Im \ \ \ln g_{x+i0}(n,n) \ dx \tag{1}$$

Since we have that

$$Im \ \ \ln g_{x+i0}(n,n) = \frac{1}{2}\pi \ \text{ on } \Sigma \text{ and } \mathrm{Im} \ \ \ln \mathrm{g_{x+i0}(n,n)} = 0 \text{ or } \pi \text{ for } \mathrm{g_{x+i0}} > 0 \text{ or } < 0.$$

the equation (1) becomes, after dropping the zero contribution from the integrand,

$$\ln g_z(n,n) = a + bz + \frac{1}{\pi}\{\sum_{i=0} \int_{\lambda_{2i}}^{\lambda_{2i+1}} + \sum_{i=1} \int_{\lambda_{2i-1}}^{\xi_i(n)} + \int_{\lambda_{2N+1}}^{\infty} \}(\frac{1}{(x-z)} - \frac{x}{1+x^2}).$$

$$.Im \ln g_{x+io}(n,n) dx \tag{2}$$

Now using the values of the Imaginary part of the logarithm from the equation (2) and performing the integration we have that

$$\ln g_z(n,n) = a + bz + \{\sum_{i=0} \int_{\lambda_{2i}}^{\lambda_{2i+1}} + \frac{1}{2}\sum_{i=1} \int_{\lambda_{2i-1}}^{\xi_i(n)} \}(\frac{1}{(x-z)} - \frac{x}{1+x^2})dx$$

$$+\ln(\lambda_{2N+1} - z) \quad + C(\lambda_{2N+1}) \tag{3}$$

where $C(\lambda_{2N+1})$ is a constant. Now the asysmptotic expansion for $(g_z(n,n))^2$requires that it behave like $\frac{1}{z^2}$ at ∞ since the spectrum of H is bounded. Hence ,we see that collecting the constant terms in the above equation together , the expression reduces to

$$\ln g_z(n,n) = \frac{1}{2}\sum_{i=0}^{2N+1} \ln\frac{(\lambda_{2i+1} - z)}{(\lambda_{2i} - \lambda)} + \sum_{i=1}^{N} \ln\frac{(\xi_i(n) - z)}{(\lambda_{2i-1} - \lambda)} \tag{4}$$

■

Proposition. 3.8 Suppose H satisfies the conditions of Lemma 3.2, then the potential has the following expression in terms of $\{\lambda_i, \xi_i(n)\}$.

$$q(n) = \frac{1}{2}\lambda_0 + \frac{1}{2}\sum_{i=1}^{N}(\lambda_{2i-1} + \lambda_{2i} - 2\xi_i(n)) \tag{5}$$

Proof. Using the asysmptotic expansion of $g_\lambda(n,n)$ as a power series and comparing the $\frac{1}{\lambda^2}$ terms from equation (2.2) on the one hand and equation (4) on the other we get the above formula.

■

Now we define two functions R and P_n as follows

$$R(\lambda) = \prod_{i=0}^{2N+1} (\lambda_{2i} - \lambda)(\lambda_{2i+1} - \lambda) \text{ and } \mathrm{P_n}(\lambda) = \prod_{\mathrm{i=1}}^{\mathrm{N}}(\xi_{\mathrm{i}}(\mathrm{n}) - \lambda) \tag{6}$$

Then in terms of R and P_n the expression for $g_\lambda(n,n)$ becomes in view of the equation (4)

$$g_\lambda(n,n) = \frac{P_n(\lambda)}{\sqrt{R(\lambda)}} \tag{7}$$

The sign of the square root is determined and fixed according to the requirement that $g_\lambda(n,n)$ is positive in $(-\infty, \lambda_0)$.

At this stage we see that the absolutely continuous spectrum together with the Dirichlet eigenvalues gives the green function uniquely. The knowledge of the green function $g_\lambda(0,0)$ is actually sufficient to recover the potential when it is symmetric about zero. However when the potential is not symmetric we need to know either of $m_\pm(\lambda)$ and q(0) is sufficient to recover the potential uniquely. We proceed to find the parameters that will determine $m_+(\lambda)$ uniquely.

We already know from equation (2.4) that the green funtion is written in terms of $m_\pm(\lambda)$. We observe that the funciton

$$F(\lambda) = -g_\lambda(0,0)^{-1} = m_+(\lambda) + m_-(\lambda) + \lambda - q(0) \tag{8}$$

is Herglotz in $\mathbf{C}^+$ and has purely imaginary boundary values on Σ and is analytic except for poles in the complement of Σ. The poles are located precisely at the zeros of $g_\lambda(0,0)$. Since the function $m_+(\lambda), m_-(\lambda)$ are also Herglotz as can be seen from equation (2.6), it is clear that if we can find an expression for $m_+(\lambda) - m_-(\lambda)$ we can find m_+ and m$_-$. To this end we have

Proposition. 3.9 Suppose q is a reflectionless potential with spectrum of the type given in Lemma 3.2, suppose further that m_+ and m$_-$ have the same imaginary parts on the spectrum Σ of q and ξ_i is a pole of $m_\pm$ according as σ_i is ± 1. Then $m_\pm$ can be uniquely recovered from the knowledge of Σ, $\{\xi_i(0)$ and $\sigma_\mathrm{i}\}$.

Proof. From equation (7) it is clear that $m_+(\lambda) + m_-(\lambda) + \lambda - q(0)$ is Herglotz and has boundary values a.e. on $\mathbf{R}$ and poles at $\xi_i(0)$. Also m_+, m_- both are analytic in $\mathbf{C}^+$ and have non zero imaginary parts on Σ and zero imaginary parts on $\mathbf{R}\backslash\Sigma$. Therefore the function $G(\lambda) = (m_+ - m_-)(\lambda)$ is analytic in $\mathbf{C}^\pm$. Also G has zero imaginary part on $\mathbf{R}$. Further from equation (7) it is clear that the poles of $F(\lambda)$ are simple and come precisely from those of m_+ or m$_-$. Therefore $G(\lambda)$ has the same poles on $\mathbf{R}$. Therefore it is a meromorpic function in $\mathbf{C}$ with simple poles at ξ_i. Then we use the relation (7) to compute the residues of the function G at the poles ξ_i we find that

$$G(\lambda) = \sum_{i \in I_+} \frac{C_i}{(\xi_i - \lambda)} - \sum_{i \in I_-} \frac{C_i}{(\xi_i - \lambda)} \tag{9}$$

where $I_\pm = \{i : \xi_i \text{ is a pole of } m_\pm\}$ and C_i are the residues of $F(\lambda)$ at the points ξ_i. Explicitly C_i's turnout to be

$$C_i = \frac{\sqrt{R(\xi_i(0))}}{P'(\xi_i(0))}$$

prime denoting the derivative with respect to λ. Now defining the function σ from $I_+ \cup I_-$ to $\{+,-\}$ we can write the expression for $G(\lambda)$ as

$$G(\lambda) = \sum_{i=1}^{N} \frac{\sigma_i C_i}{(\xi_i - \lambda)} \tag{10}$$

since the difference of the left and right hand sides of the above equation is an entire function bounded on $\mathbf{C}$ and has the value 0 at infinity, hence vanishes identically by

Liouville's theorem. ¿From equations (9) and (8) we can write the expressions for $m_\pm$ as

$$m_\pm(\lambda) = \frac{1}{2}\{[\frac{\sqrt{R(\lambda)}}{P_0(\lambda)} - \lambda + q(0)] \ \pm \ \sum_{i=1}^{N} \frac{\sigma_i C_i}{(\xi_i(0) - \lambda)}\} \tag{11}$$

■

Clearly since the poles of $m_\pm$ in **R** are the eigenvalues of $H_\pm$, we can collect all the above results into the following

Theorem. 3.10 Suppose we have the operator H = Δ + q with spectrum Σ , a reflectionless set of the type assumed in Lemma 3.2. Further suppose on Σ , Im $m_+ =$ Im m_- a.e. Then corresponding to each set of points $\{\xi_i(0), \sigma_i(0)\}$, there is a unique potential q such that the spectrum of the associated H is purely absolutely continuous and equals Σ and further $\{\xi_i(0) : \sigma_i(0) = \pm\}$ are precisely the eigen values of $H_\pm$. In this case the value of the potential at 0 is given by equation (5).

Remark. 3.11 The above theorem is valid if q,$\xi_i(0)$,$\sigma_i(0)$,$m_\pm$ are replaced by T^nq,$\xi_i(n)$,$\sigma_i(n)$, $m_{\pm,n}$ for each fixed n ∈ **Z**.

For use in the next section we note that the equations (11) and (2.6) imply that the following relations are valid

$$m_{+,n}(\lambda) = \frac{1}{2}\{[-\lambda + q(n) + \sum_{i=1}^{N} \frac{\sigma_i C_i}{(\xi_i(n) - \lambda)}] - \frac{\sqrt{R(\lambda)}}{P_n(\lambda)}\} \tag{12}$$

and

$$-\lambda + q(n) - m_{-,n}(\lambda) = \frac{1}{2}\{[-\lambda + q(n)] + \sum_{i=1}^{N} \frac{\sigma_i C_i}{(\xi_i(n) - \lambda)}] + \frac{\sqrt{R(\lambda)}}{P_n(\lambda)}\} \tag{13}$$

We note also the relation

$$\frac{P_n(\lambda)}{P_0(\lambda)} = \frac{g_\lambda(n,n)}{g_\lambda(0,0)} = \phi_{+,n}(\lambda)\phi_{-,n}(\lambda) \tag{14}$$

where

$$\phi_{+,n}(\lambda) = \prod_{i=0}^{n-1} m_{+,i}(\lambda) \ \text{ and } \ \phi_{-,\mathrm{n}}(\lambda) = \prod_{\mathrm{i=0}}^{\mathrm{n-1}} (\lambda - \mathrm{q(i)} + \mathrm{m}_{-,\mathrm{i}}(\lambda)) \tag{15}$$

Clearly then at ∞ the behaviour of $\phi_{\pm,n}(\lambda)$ is given by

$$\phi_{\pm,n}(\lambda) \approx (\lambda)^{\mp n} \tag{16}$$

4. ALMOST-PERIODICITY:

In the last section we showed that given the operator H $= \Delta + q$ with purely absolutely continuous spectrum we can recover the operator H uniquely under certain additional data given by theorem 3.10. Àpriori it is not clear taht there is any relation between the zeros $\xi_i(n)$ and $\xi_i(0)$ of the green funtions $g_\lambda(n,n)$ and $\mathrm{g}_\lambda(0,0)$. In this section however we show that actually these zeros cannot vary independently for each n but infact need to move in a way to make the funciton $\Phi(n) = \sum \xi_i(n)$ almost periodic as n varies. As a consequence the potential which necessarily satisfies the 'trace' formula equation (3.5) also becomes almost periodic in n.

We recall the relations (3.12,3.13 and 3.14), of the last section , for the functions $\phi_\pm$. It is clear that these can be thought of as the same function ϕ on the Riemann surface $\mathcal{R}$ associated with the function $\sqrt{R(\lambda)}$. We recall that $\mathcal{R}$ is constructed by taking two copies of the λ sphere slit along $[\lambda_{2i}, \lambda_{2i+1}]$, i = 1,..,N and joined appropriately to form a two sheeted branched cover , with branch points the λ_i's i = 2,..,2N+1. The points on the spheres corresponding to the points at infinity of the plane are denoted p_∞ and $p_{\infty'}$ respectively and fall on either sheet of $\mathcal{R}$. On $\mathcal{R}$ we consider closed paths α_i and β_i , i= 1,..,N forming crosscuts in such a way that α_i lies on the upper sheet and encloses $[\lambda_{2i}, \lambda_{2i+1}]$, i = 1,..,N and β_i starts at λ_1 goes to λ_{2i} , i = 1,..,N on the upper sheet crosses over to the lower sheet and returns to λ_1. These paths form a set of generators for the group of closed paths on $\mathcal{R}$. We also choose a point P_0 on $\mathcal{R}$, away from the branch points, the points p_∞ and $\mathrm{p}_{\infty'}$ and the paths α_i and β_i, as a reference point where the value of the function $\frac{1}{\sqrt{R(\lambda)}}$ is chosen and fixed. Then the family $\{\frac{\lambda^m d\lambda}{\sqrt{R(\lambda)}}\}$ m = 0,..,N-1 forms a collection of N holomorphic differentials in terms of which we choose a basis ,

$$d\omega_m = \sum_{k=1}^{N} c_{mk} \frac{\lambda^{k-1} d\lambda}{\sqrt{R(\lambda)}} \tag{1}$$

of holomorphic differentials for the vector space of holomorphic differentials on $\mathcal{R}$, which necessarily has dimension N. The basis $\{d\omega_m\}$ is normalized so that the matrix T with entries

$$T_{ij} = \int_{\alpha_j} d\omega_i, \;\; T_{i,j+N} = \int_{\beta_j} d\omega_i \;\; 1 \le j, i \le N$$

takes the form [I,τ] , where I is the identity matrix and $i\tau$, of necessity,a symmetric nonpositive definite matrix. T is called the period matrix associated with the basis α_i, β_i and $\mathrm{d}\omega_\mathrm{i}$.

Now we consider the single valued function $\phi_n(\lambda)$ on $\mathcal{R}$ which takes values $\phi_{\pm,n}(\lambda)$ defined in equation (3.15) on the upper and lower sheets of $\mathcal{R}$. Clearly a point ξ_i in $(\lambda_{2i-1}, \lambda_{2i})$ has unique point p_i over it in $\mathcal{R}$ for each value ± 1 of σ_i. Therefore the single valued function $\phi_n(\lambda)$ has poles at $p_{\infty'}$ of order n and zero of order n at p_∞ (by equation 3.16) and also has exactly N poles of order 1 at the points $p_i(0)$ and N zeros of order 1 at the points $p_i(n)$ by eqution (3.14). The exact location of these points in terms of which sheet they belong to is unimportant in the subsequent discussion.

Having identified the function $\phi_n(\lambda)$ as having the appropriate behaviour we shall use it to construct the differential

$$d\omega(n) = \frac{d\ln\phi_n(\lambda)}{d\lambda}d\lambda \qquad (2)$$

with the property that it has poles at $p_\infty, p_{\infty'}$ with residues n and -n and simple poles at $p_i(n), p_i(0)$ i = 1,..,N with residues +1 and -1 respectively. This differential will be crucial for us to obtain a relation between the points $p_i(n)$ and $p_i(0)$. To start with we have the relations , which are consequences of Cauchy's integral formula.

$$\int_{\alpha_j} d\omega(n) = 2\pi i k_j \ \text{ and } \int_{\beta_j} \mathrm{d\omega(n) = 2\pi im_j} \qquad (3)$$

Now $d\omega(n)$ being a differential with poles we can write it in terms of the differentials of first, second and normalized differentials of the third kind as

$$d\omega(n) = D + n d\omega(p_\infty, p_{\infty'}) + \sum_{i=1}^{N} d\omega(p_i(n), p_i(0)) + \sum_{j=1}^{N} c_j d\omega_j \qquad (4)$$

where $d\omega(a, b)$ is a normalized differential of the third kind with residues +1 and -1 respectively at a and b . D is a differential of the second kind and $d\omega_j$ are defined in eqution (1). Since it is always possible to add differentials $d\omega_j$ to any $d\omega(a, b)$ to make the integral over the α_j vanish we obtain the relation that in view of the normalization (3) $c_j = 2\pi i k_j$ for some integer k_j for each j by integrating the above equation over α_j.

Lemma. 4.1 We have the following relation among the points $p_i(n)$, $p_i(0)$, p_∞ and $p_{\infty'}$

$$\sum_{j=1}^{N} \int_{P_0}^{P_j(n)} d\omega_i = \sum_{j=1}^{N} \int_{P_0}^{P_j(0)} d\omega_i - \sum_{j=1}^{N} k_j \tau_{ij} + m_i + \int_{p_\infty}^{p_{\infty'}} d\omega_i \qquad (5)$$

for each i = 1,..,N.

Proof. Integrating the equation (4) on both the sides with respect to β_i we have that

$$\int_{\beta_i} d\omega(n) = +n \int_{\beta_i} d\omega(p_\infty, p_{\infty'}) + \sum_{i=1}^{N} \int_{\beta_i} d\omega(p_i(n), p_i(0)) + \sum_{j=1}^{N} c_j \int_{\beta_i} d\omega_j \qquad (6)$$

By the comment before the lemma we know that the the sum of integrals w.r.t β_i of $d\omega_j$ gives the term, $2\pi i \sum_{i=1}^{N} k_j \tau_{ij}$ and the integral of the left hand side provides us with the term $2\pi i m_i$. The remaining sum is computed as follows. Suppose $d\omega(a, b)$ is a normalized differential of the third kind with residues +1 and -1 at p and q respectively. Then, we **claim** that

$$\int_{\beta_l} d\omega(p,q) = 2\pi i \int_q^p d\omega_l$$

To show the claim we note first that by addition of differentials of first kind we can always make $\int_{\alpha_i} d\omega(p,q)$ vanish for each i. Now consider the normalized differentials $d\omega_k$ and compute the integral $\int_C \omega_k d\omega(p,q)$,where ω_k is the integral of the differential $d\omega_k$. Then $\omega_k d\omega(p,q)$ is an Abelian differential , regular everywhere except for poles p,q. Therefore by Cauchy integral theorem the integral evaluates on the one hand to

$$\int_C \omega_k d\omega(p,q) = 2\pi i(\omega_k(p) - \omega_k(q)) \tag{7}$$

since the residues of $d\omega(p,q)$ at p and q are respectively +1 and -1. On the other hand going down to the polygonal region S corresponding to the normal form of the canonically dissected Riemann surface we obtain the relation (see Siegal[17],Chapter 4, Section 7)

$$\int_C \omega_k d\omega(p,q) = \sum_i \{\omega_k(A_i)\omega(p,q)(B_i) - \omega_k(B_i)\omega(p,q)(A_i) \tag{8}$$

where A_i, B_i are the sides of the polygon corresponding to the closed paths α_i, β_i. Now, we use the normalizations and their implications

$$\int_{\alpha_i} d\omega(p,q) = 0 \Leftrightarrow \omega(p,q)(A_i) = 0 \ \text{ and } \int_{\mathrm{\alpha_i}} \mathrm{d\omega_k} = \delta_{\mathrm{ik}} \Leftrightarrow \omega_{\mathrm{k}}(\mathrm{A_i}) = \delta_{\mathrm{ik}} \tag{9}$$

Hence the equations (7),(9) together yield,

$$\omega(p,q)(B_k) = \int_C \omega_k d\omega(p,q) = 2\pi i(\omega_k(p) - \omega_k(q)) \tag{10}$$

But the integral $\omega_k(p) - \omega_k(q)$ is precisely $\int_{P_0}^p d\omega_k - \int_{P_0}^q d\omega_k$ from which the claim follows.

∎

The equation (5) can be inverted to get a relation for $P_j(n)$'s in terms of the right hand side of (5). The existance of a unique inverse is assured by the Jacobi inversion. Then an explicit formula will be obtained for the inverse through Θ functions. Therefore we define the necessary quantities here.

Definition. 4.2 A divisor P is a formal product $\frac{P_1...P_m}{Q_1...Q_k}$ of points in $\mathcal{R}$. An integral divisor is the product of the type $P_1...P_m$. An integral divisor $P_1...P_N$ is called general if the matrix with entries $A_{ij} = \frac{d\omega_i}{d\lambda}(P_j)$,i,j = 1,..,N is nonsingular.

In the following theorem we identify the divisors which are general.

Lemma. 4.3 An integral divisor $p_1...p_N$ is general provided the factors p_i are distinct and no two of the p_i is in $\{p_\infty, p_{\infty'}\}$.

Proof. We note that the result follows by checking that the matrix J_{ij} with entries $\frac{d}{d\lambda}(\frac{\lambda^{(j-1)}}{\sqrt{R(\lambda)}})$ at P_j is nonsingular. Using the local parameters t = z at ordinary points and t = $\sqrt{(z-P_i)}$ at branch points the verification is easy.

∎

Definition. 4.4 The Θ funtion on $\mathbf{C^N}$ is defined by

$$\Theta(z) = \sum_{m \in \mathbf{Z}^N} exp(2\pi i < m, z >) exp(\pi i < m, \tau m >) \tag{11}$$

where τ is as defined after equation (1). The Θ function satisfies the periodicity relations as follows

$$\Theta(z + e_k) = \Theta(z) \text{ and } \Theta(\mathrm{z} + \tau_\mathrm{k}) = \mathrm{e}^{-2\pi \mathrm{i} \mathrm{z_k} - \pi \mathrm{i} \tau_{\mathrm{kk}}} \Theta(\mathrm{z}) \tag{12}$$

The Jacobi's imaginary transformation is given by

$$\Theta(u, \tau) = \frac{i^{N/2}}{\sqrt{|\tau|}} e^{-\pi i < u, \tau^{-1} u >} \Theta(\tau^{-1} u, -\tau^{-1}) \tag{13}$$

where $\Theta(u, \tau)$ is the theta function at u with the period matrix given by τ. Next we consider a divisor P = $P_1..P_N$ and define the Abel-Jacobi function A(P) as the function $A(P) = \sum_{i=1}^{N} \int_{P_0}^{P_i} d\omega$, taking the divisor to a point in $\mathbf{C^N}/\mathbf{\Pi}$. We recall the following theorems from Siegel [17],Section 10.

Theorem. 4.5 Consider A(P) defined above for an integral divisor P. Whenever P is general , then it is the unique integral divisor in the preimage of A(P).

While the above theorem guarantees a solution for the Jacobi-inversion, the the components of the point P are obtained as the zeros of an appropriate theta funciton acting on $\mathcal{R}$. Explicitly we consider on $\mathcal{R}$ the integrals $\omega(P) = \int_{P_0}^{P} d\omega$. Then we consider the function $\phi(P) = \Theta(\omega(P) - s + \tilde{c})$ where $\tilde{c}$ is a vector of Riemann constants depending upon τ and ω, s $\in \mathbf{C^N}/\mathbf{\Pi}$. It is also known that there are exactly N zeros for the function $\phi(P)$ in $\mathcal{R}$. We have the following theorem from [17](section 10).

Theorem. 4.6 Suppose $\phi(P)$ does ot vanish identically for a fixed s. Then , the zeros $q_k \in \mathcal{R}$ of $\phi(P)$ satisfy the relation $s = \sum_{k=1}^{N} \int_{P_0}^{q_k} d\omega$, the equation is to be understood component wise.

The upshot of the above theorems is that if we rewrite the equation (5) as

$$\sum_{j=1}^{N} \int_{P_0}^{P_j(n)} d\omega_i = nc_i + K_i \tag{14}$$

to obtain $P_j(n)$ in terms of n c + K , one needs only to check that the required inverse is a general point and consider the function

$$\Theta(\omega(P) - \sum_{i=1}^{N} \int_{P_0}^{P_i(n)} d\omega + \tilde{c}) = \Theta(\omega(P) - nc - K + \tilde{c}) \tag{15}$$

and find its zero. The Riemann constant $\tilde{c}$ is chosen so that the function $\phi(P)$ does not vanish identically.

Till now we are still on the Riemann surface and identified the function whose zeros are precisely the points above $\xi_i(n)$. Now we shall obtain an expression for the sum of $\xi_i(n)$ in terms of the theta function mentioned above. To this end we have the following Lemmas. We consider the map λ from $\mathcal{R}$ to $\mathbf{C}$. The map λ is a meromorphic function on $\mathcal{R}$ with poles precisely at the points p_∞ and $p_{\infty'}$.

Lemma. 4.7 We have the following relation between the theta function and the $\xi_i(n)$.

$$\sum_{i=1}^{N} \xi_i(n) = Const + \sum_{i=1}^{N} C_{iN} D_i \ln(\frac{\Theta(nc+d)}{\Theta((n+1)c+d)}) \tag{16}$$

where the constants $\{C_{iN}\}$ are the same as those which appear in equation (1) ,D_i is the partial derivative in the ith direction , $\mathbf{c}$ is a vector with purely imaginary entries and $\mathbf{d}$ some constant vector independent of $\mathbf{n}$.

Proof. We consider the meromorphic differential $\lambda d \ln(\phi(P))$. It has poles at $p_\infty, p_{\infty'}$ and at the zeros $P_i(n)$ of $\phi(P)$. Let C be a closed contour issueing from the reference point P_0 and enclosing all the poles and zeros of the function $\lambda d \ln(\phi(\lambda))$. Therefore a computation using the Cauchy's integral theorem gives that

$$\int_C \lambda d \ln \phi = \sum_{i=1}^{N} \lambda(P_i(n)) + \text{Residueat} p_\infty + \text{Residueat} p_{\infty'} \tag{17}$$

The computation of the residues are done using the local parameter as follows. The local parameter at p_∞ is $\lambda = \zeta^{-1}$. Therefore we have

$$\lambda \frac{d}{d\lambda} \ln(\phi(\omega(\lambda) - nc + d))|_{p_\infty} = \sum_{i=1}^{N} \frac{d}{d\lambda}\omega(\lambda).\{D_i \ln(\phi(\omega(\lambda) - nc + d)))\}|_{p_\infty} \tag{18}$$

where ω is the integral of the differential $d\omega$ and D_i refers to the derivative of the argument of ϕ in the **i** th direction. The summand of the above eqution is evaluated as

$$\frac{d}{d\lambda}\omega(\lambda) = -\frac{1}{\zeta^2} \sum_{j=1}^{N} C_{ij} \frac{\zeta^{-j+1}}{\sqrt{R(\zeta^{-1})}} \tag{19}$$

Since p_∞ belongs to the upper sheet , the square root has a positive sign and also since $R(\lambda)$ goes likeλ^{N+1} at ∞,we have that the above eqution evaluates to $-C_{iN}$. Therefore we have

$$\textit{Residue} \text{ at } \mathrm{p}_\infty = -\sum_{\mathrm{i}=1}^{\mathrm{N}} \mathrm{C}_{\mathrm{iN}} \mathrm{D}_{\mathrm{i}} \ln(\Theta(\omega(\mathrm{p}_\infty)) - \mathrm{nc} + \mathrm{d}) \tag{20}$$

Similarly noting that at $p_{\infty'}$, the square root has a negative sign since $p_{\infty'}$ belongs to the lower sheet of $\mathcal{R}$, we have

$$\textit{Residue} \text{ at } \mathrm{p}_{\infty'} = \sum_{\mathrm{i}=1}^{\mathrm{N}} \mathrm{C}_{\mathrm{iN}} \mathrm{D}_{\mathrm{i}} \ln(\Theta(\omega(\mathrm{p}_{\infty'})) - \mathrm{nc} + \mathrm{d}) \tag{21}$$

Now, since $c = \int_{p_{\infty'}}^{p_\infty} d\omega$, we have $\omega(p_\infty) = c + \omega(p_{\infty'})$. Putting thiese equations together and absorbing the value$\omega(p_\infty)$ into **d** ,

$$\int_C \lambda d \ln \phi = \sum_{i=1}^{N} \xi_i(n) - \sum_{i=1}^{N} C_{iN} D_i \{\ln \frac{\Theta(nc + d)}{\Theta((n+1)c + d)}\} \tag{22}$$

This equation provides us with one expression for the left hand side. On the other hand going down to the Riemann region corresponding to the canonically dissected surface and taking the polygonal path $\{a_1 b_1 a_1^{-1} b_1^{-1} ... b_N^{-1}\}$, we can integrate the function explicitly using the relations, for each k ,

$$\frac{\mathrm{df}}{\mathrm{f}}(a_k^{-1}) = \frac{df}{f}(a_k) - 2\pi i d\omega_k \text{ and } \frac{\mathrm{df}}{\mathrm{f}}(\mathrm{b}_\mathrm{k}^{-1}) = \frac{\mathrm{df}}{\mathrm{f}}(\mathrm{b}_\mathrm{k}) \tag{23}$$

The equations are understood to mean equality of evaluation of the differentials at a point on the ark a_k, b_k and the corresponding point on a_k^{-1}, b_k^{-1} etc., Then we obtain that

$$\int_C \lambda d \ln \phi = \sum_{i=1}^{N} \int_{\alpha_i} \lambda d\omega_i \tag{24}$$

Putting equations (17) and (15) together we obtain the lemma.

■

Lemma. 4.8 The sum $\sum_{i=1}^{N} \xi_i(n)$ of the above lemma are almost periodic in n.

Proof. The Jacobi-imaginary transformation equation (13) shows that the right hand side of the equation (16) for the sum $\sum_{i=1}^{N} \xi_i(n)$ is indeed real and further the vector $\tau^{-1}c$ is a vector with real entries. Therefore the required almost periodicity is now immediate from that of the theta function with real argument.

■

Now we are ready to prove the Theorem 1 of section 1.

Proof. The assumptions of Theorem 3.10 are satisfied for each fixed ω in support of $\mathcal{P}$. Therefore for each fixed ω the potential q^ω is almost periodic as a sequence in view of Lemma 4.8 and the trace formula.

■

We would like to make a few comments regarding the theorems of this paper. Firstly if we replace Δ in H by non constant off diagonal entries (i.e.coming from a positive sequence a_n) , the theorems of the last two sections will go through with some modifications for the expressions for the m-functions and the trace formula. Secondly We have stated that the meromorphic function ϕ_n has exactly N poles. It might happen that if for some n , some of the zeors $\xi_i(n)$ coincide with $\xi_i(0)$, then this is not exactly correct. However the proof still go through as this phenomenon does not persist for the neighbouring values n-1 and n+1. Such coincidence will show only the periodicity of the appropriate $\xi_i(n)$ in n.

A forteriori using the first order eqations (2.4) for $m_+(\lambda)$, we can write a difference equation for the $\xi_i(n)$ in terms of $\xi_i(n-1)$ as follows. This equation shows clearly that the poles of $m_{+,n}$ are determined by the zeros of $m_{+,n-1}$ and similarly for m_-. Therefore by looking at the appropriate single valued function on the Riemann surface which agrees with m_+ on the upper and $\lambda - q + m_-$ on the lower sheets , we can write the following equation for the points $\overrightarrow{p(n)}$ above $\overrightarrow{\xi_i(n)}$.

$$\vec{p}(n) = V(\vec{p}(n), \vec{p}(n-1)) \tag{25}$$

The explicit form for V can be obtained by combining equations (2.4) and (3.12,3.13). This equation will provide the analogue of the Dubrovin equation of the continuous case.

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