Overlap Distributions for Deterministic Systems with Many Pure States

A. C. D. van Enter
Institute for Theoretical Physics, P. O. Box 800
University of Groningen
Groningen, The Netherlands

A. Hof
Departement Mathematik
ETH-Zentrum
8092 Zürich, Switzerland

Jacek Miękisz
Institut de Physique Théorique
Université Catholique de Louvain
B-1348 Louvain-la-Neuve, Belgium

April 25, 1997

Abstract. We discuss the Parisi overlap distribution function for various deterministic systems with uncountably many pure ground states. We show examples of trivial, countably discrete, and continuous distributions.

In Parisi's proposed solution for the Sherrington-Kirkpatrick spin-glass model [1, 2, 3] there occurs an overlap distribution $p(q)$ which is non-trivial in the sense that it has a continuous part and two delta functions. The suggestion is that such a non-trivial $p(q)$ represents the presence of infinitely many pure extremal Gibbs or ground states. As the mathematical status of Parisi's theory is yet ill-understood, it seems of interest to study its various aspects in simpler models. For instance, Fisher and Huse [4] have studied the behaviour of the overlap distribution and discussed its strong dependence on boundary conditions in various examples with only two pure states. Their work was partly motivated by their conjecture that short-ranged spin-glasses have only two pure states [5, 6, 7].

We present here some results from a complementary point of view and consider what might happen in deterministic systems with infinitely many pure states. We suspect that some spin glass models do indeed have infinitely many ground states [8, 9], although we consider this matter unsettled at present. The scenario with many states was recently considered by Newman and Stein [10]. In fact, our paper originated from discussions with C. Newman. Despite somewhat different physical motivations, our conclusions support those of Fisher and Huse: the overlap distribution does not describe the number of states well.

The reason for this conclusion differs between our examples and theirs.

Whereas Fisher and Huse show that standard boundary conditions (free, periodic, antiperiodic) can either suppress some pure state (as happens for example in the random field Ising model) or give rise to a continuous overlap distribution due to floating defects (as happens for example in the nearest neighbour ferromagnet with antiperiodic boundary conditions), in our examples we work with states which are mixtures of uncountably many pure states that are free of defects. Hence no pure state is suppressed, and floating defects do not occur. As we work directly with the infinite-volume measures, we need not consider boundary conditions. Our systems are deterministic in the sense that their configurations are generated by deterministic rules: standard substitution rules producing Thue-Morse and Fibonacci sequences. Also, our configurations are ground state configurations of deterministic translation-invariant interactions.

More specifically, in our models at every site i of the one-dimensional lattice $\mathbf{Z}$ there is a spin variable σ_i which can attain the values ± 1. An infinite lattice configuration is an assignment of spin orientations to lattice sites, that is an element of $\Omega = \{-1,+1\}^{Z}$. We are concerned with nonperiodic configurations which have nevertheless uniformly defined frequencies for all finite patterns. These are examples of the so-called similar but incongruent pure phases discuss ed in [4]. More precisely, to find a frequency of a finite pattern in a given configuration we first count the number of times it appears in a segment of size l and centered at the origin of the lattice, divide it by l, and then take the limit $l \to \infty$. If the convergence is uniform with respect to the position of the segments then we say that the configuration has a uniformly defined frequency of this pattern. The closure of the orbit under translation of any such configuration supports exactly one ergodic translation-invariant measure on Ω, say μ, which is uniquely specified by the frequencies of all finite patterns. Such systems are called strictly ergodic if every finite pattern that occurs in the configuration occurs with a uniformly defined frequency that is strictly larger than zero. The measures we consider are strictly ergodic. Strictly ergodic measures can be considered to be the typical ground states for translation-invariant interactions [11, 12, 13, 14].

Let us denote by q_{XY} the overlap between two configurations X and Y in the support of μ. It is defined by

$$q_{XY} = lim_{N \to \infty}(1/N) \sum_{i=1}^{N} \sigma_i(X)\sigma_i().$$

Then the Parisi overlap distribution $p(q)$ is the distribution of q_{XY} with respect to the product measure $\mu \otimes \mu$.

Our first result is a simple application of a well known result from ergodic theory. It concerns the so called weakly-mixing measures. Let us recall that a measure μ is weakly mixing if $\mu(fT^{k(n)}f) \to [\mu(f)]^2$ for all f square-integrable with respect to μ, where T is a shift operator, and $k(n)$ the sequence of natural numbers, possibly excluding a set of zero density (depending on f). This property is equivalent to T having a continuous spectrum [15].

Theorem 1 *If μ is weakly mixing then $p(q)$ is a point distribution concentrated on $[\mu(\sigma_0)]^2$.*

Proof: If μ is weakly mixing then $\mu \otimes \mu$ is ergodic [15] and by the ergodic theorem $(1/N)\sum_{i=1}^{N} \sigma_i(X)\sigma_i(Y)$ converges with probability one to $[\mu(\sigma_0]^2$. $\square$

A specific weakly mixing example of a 3-dimensional ferromagnetic Ising model wi th uncountably many Gibbs states and a trivial overlap distribution was already given in [16]. However, in that example all pure states are related by a global symmetry of the system. This is not the case in the models considered here.

Our next result answers a question of C. Newman about the Thue-Morse system. To define the Thue-Morse system we start by taking a sequence of all $+1$ spins. At the first step we flip every second spin. At the *nth* step we flip all blocks of 2^{n-1} spins within the previous $n-1st$ configuration from the site $(2k+1)2^{n-1}+1$ to $(2k+2)2^{n-1}$ for every k. A cluster point of this sequence of periodic configurations of period $2 \cdot 2^n$ is a nonperiodic sequence called a Thue-Morse sequence [17, 18, 19, 20, 21, 22]. The closure of its orbit under translation supports exactly one translation-invariant measure μ_{TM} which is in fact strictly ergodic [17]. Thue-Morse sequences can also be obtained by iterating the following substitution rule: $1 \to 1 - 1$, $-1 \to -1\ 1$. The Thue-Morse measure μ_{TM} has been shown to be the unique ground sta te for arbitrary rapidly decaying 4-spin interactions [20].

Theorem 2 *The overlap distribution $p(q)$ for μ_{TM} is a point measure concentrated on $q = 0$.*

Proof: $L^2(\mu_{TM})$, the space of functions which are square-integrable with respect to μ_{TM} can be decomposed into the direct sum of the two spaces spanned by the odd and the even functions with respect to the spin-flip operator $\sigma_i \to -\sigma_i$. The shift operator acting on the space of odd functions has a singular continuous spectrum [17]. Therefore when you consider only odd observables, like σ_i, then μ_{TM} behaves as if it were weakly mixing and $\mu_{TM} \otimes \mu_{TM}$ as if it were ergodic with respect to these observables and so the conclusion follows as in the proof of the Theorem 1. □

Now, let X be any Thue-Morse sequence and let $Y(i) = X(i) \cdot X(i + 1)$. The closure of the orbit of Y obviously supports exactly one ergodic translation-invariant measure and the resulting strictly ergodic system is called a Toeplitz system [23, 24]. Every Toeplitz sequence can be constructed in the following way. First choose a sublattice L_1 of period 2 and put a -1 on every site in L_1. Next, choose a sublattice L_2 of period 4 that is disjunct from L_1 and put a +1 on every site in L_2. In this way one continues: L_j is a sublattice of period 2^j that is disjunct from $L_1, \ldots, L_{j-1}$ and the spins in L_j are $(-1)^j$. In the interpretation of [22] the Toeplitz sequence describes the molecules of the Thue-Morse system.

Theorem 3 *The overlap distribution for the Toeplitz system μ_T contains countably many points.*

Proof: We will fix one Toeplitz configuration Y and calculate its overlaps with all Toeplitz sequences grouped with respect to the constant overlap. First consider all Toeplitz configurations such that the minuses of the first sublattice are exactly off the first sublattice of Y. This gives rise to a point measure of mass $1/2$ concentrated on $q = -1/2+1/4-1/8+... = -1/3$. Now consider all Toeplitz configurations such that minuses of the first sublattice are on the first sublattice of Y and the pluses of the second sublattice are off the second sublattice of Y. This gives us a point measure of mass $1/4$ concentrated on $q = 1/2 - 1/2 \cdot 1/3 = 1/3$. In the next step the first two sublattices are coinciding and the third ones miss each other, giving rise to a point measure with mass $1/8$ and concentrated on $q = 1/2 + 1/4 - 1/4 \cdot 1/3 = 2/3$. Repeating this procedure infinitely many times we obtain $p(q) = \sum_{n=0}^{\infty}(1/2^{n+1})\delta\Big(q - (3 \cdot 2^{n-2} - 1)/(3 \cdot 2^{n-2})\Big)$. $\square$

Note that this construction resembles the "ultrametric" structure that occurs in Parisi's theory [2].

Our last example is the Fibonacci system. It too is strictly ergodic. We will show that it has a continuous part in its overlap distribution. A Fibonacci sequence can be obtained using the following substitution rules: $1 \to -1$, $-1 \to -1\ 1$.

Theorem 4 *The overlap distribution of the Fibonacci system μ_F has a continuous part.*

Proof: We will use an equivalent construction of Fibonacci sequences by rotations T over the circle by an amount $2\pi\gamma$ with $\gamma = 2/(1+\sqrt{5})$ being the golden ratio (see e.g. [25]). To every angle $2\pi\phi \in [0, 2\pi)$ there corresponds a Fibonacci sequence X in the following way. If $T^n\phi$ is in the arc segment $[0, 2\pi\gamma)$ then $X(n) = 1$, otherwise $X(n) = -1$. Now, because of the irrationality of γ, T is ergodic with respect to the Lebesgue measure μ_L on the circle. Hence the overlap between sequences corresponding to $2\pi\phi_1$ and $2\pi\phi_2$ depends only on $\alpha = \phi_1 - \phi_2$. Namely, $q(\alpha) = \mu_L(A) - \mu_L(A^C)$, where A is the event where the two line segments that define the angle $2\pi\alpha$ are both in the same arc segment $[0, 2\pi\gamma)$ or $[2\pi\gamma, 2\pi)$. Hence

$$q(\alpha) = \begin{cases} 1 - 4\alpha & \text{if } 0 \leq \alpha < 1-\gamma \\ 1 - 4(1-\gamma) & \text{if } 1-\gamma \leq \alpha < \gamma \\ 1 - 4(1-\alpha) & \text{if } \gamma \leq \alpha < 1 \end{cases}$$

It follows that $p(q) = (2\gamma - 1)\delta\Big(q - (1 - 4(1-\gamma)\Big) + \frac{1}{2}1_{[1-4(1-\gamma),1]}(q)dq$. $\square$

It is not known if there is a simple (e.g. finite-spin exponentially decaying) translation-invariant interaction with μ_F as its unique ground state, although by [13, 14] there are infinite-spin interactions for which μ_F is the unique ground state. Let us remark here that such a deterministic interaction has a continuous part in its overlap distribution, a property usually attributed to systems with random interactions like spin glasses.

Let us mention that overlaps have been studied in the literature on substitution dynamical systems [23, 26, 27] under the name "coincidence density". However, not much seems to be known about their distributions. The overlap between two finite sequences of ± 1 is a linear function of their Hamming distance.

We also remark that the Edwards-Anderson parameter as for example studied in [28] measures the maximal overlap, and hence would be 1 in our examples. This shows that there can be a big difference between a maximal and a typical overlap.

Concluding, we have shown that in various examples where one can compute the overlap distribution for systems with infinitely many states, various types of distributions occur. Thus overlap distributions do not provide a good description of the number of pure phases of the system. The fact that we worked at $T = 0$ should not matter too much as similar nonperiodic long range order and infinitely many pure Gibbs states can occur at positive temperatures [21, 29]. This conclusion fully supports what Fisher and Huse found in their examples with finitely many states and suggests for spin-glass models that the overlap distribution also there might not be a very useful quantity.

Acknowledgments. We thank Michel Dekking, Chuck Newman, and Marinus Winnink for discussions. The research of A. C. D. van Enter has been made possible by a fellowship of the Royal Netherlands Academy of Arts and Sciences and of J. Miękisz by Bourse de recherche UCL/FDS.

References

[1] Parisi G 1983 *Phys. Rev. Lett.* **50** 1946

[2] Mezard M, Parisi G, and Virasoro M A 1987 *Spin Glass Theory and Beyond* (Singapore: World Scientific)

[3] Sherrington D and Kirkpatrick S 1975 *Phys. Rev. Lett.* **35** 1792

[4] Huse D A and Fisher D S 1987 *J. Phys.* **A20** L997

[5] Fisher D S and Huse D A 1987 *J. Phys.* **A20** L1005

[6] Fisher D S and Huse D A 1986 *Phys. Rev. Lett.* **56** 1601

[7] Fisher D S and Huse D A 1988 *Phys. Rev.* **B30** 386

[8] Bovier A and Fröhlich J 1986 *J. Stat. Phys.* **44** 347

[9] Enter A C D van 1990 *J. Stat. Phys.* **60** 275

[10] Newman C M and Stein D L 1992 *Multiple States and Thermodynamic Limits in Short-Ranged Ising Spin Glass Models* To appear in Phys. Rev. B

[11] Aubry S 1983 *J. Phys. (Paris)* **44** 147

[12] Radin C 1986 *J. Stat. Phys.* **43** 707

[13] Aubry S 1989 *J. Phys. (Paris) Coll.* **50** 97

[14] Radin C 1991 *Rev. Math. Phys.* **3** 125

[15] Walters P 1982 *Introduction to Ergodic Theory* (New York: Springer)

[16] Miękisz J 1989 *J. Stat. Phys.* **55** 351

[17] Keane M 1968 *Z. Wahrs. verw. Geb.* **10** 335

[18] Aubry S 1989 *Weakly Periodic Structures with a Singular Continuous Spectrum* in Toledano J I (ed): *Proceedings of the NATO advanced research workshop on Common Problems in Quasicrystals, Liquid Crystals, and Incommensurate Insulators*, Preveza, Greece, 4th-8th September 1989

[19] Luck J M 1989 *Phys. Rev.* **B39** 5834

[20] Gardner C, Miękisz J, Radin C, and Enter A C D van 1989 *J. Phys.* **A22** L1019

[21] Enter A C D van and Miękisz J 1990 *Commun. Math. Phys.* **134** 647

[22] Enter A C D van and Miękisz J 1992 *J. Stat. Phys* **66** 1147

[23] Dekking F M 1980 *Substitutions* Thesis Nijmegen and private communication

[24] Jacobs K and Keane M 1969 *Z. Wahrs. verw. Geb.* **13** 123

[25] Schröder M 1991 *Fractals, Chaos, Power Laws* (New York: Freeman) p. 308

[26] Michel P 1987 *Z. Wahrs. verw. Geb.* **42** 205

[27] Queffélec M 1987 *Substitution Dynamical Systems* (Springer LNM 1294)

[28] Enter A C D van and Griffiths R B 1983 *Commun. Math. Phys.* **91** 319

[29] Israel R. B. Private communication