A Center-Stable Manifold Theorem for Differential Equations in Banach Spaces

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Abstract. We prove a center-stable manifold theorem for a class of differential equations in (infinite-dimensional) Banach spaces.

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1. Introduction

The center-stable manifold theorem is a standard tool in analyzing the behavior of a differentiable dynamical system in the vicinity of a stationary point. In its usual formulation, this theorem applies to smooth maps or flows in (finite or infinite-dimensional) Banach spaces (see for example Ruelle [1]). This framework is, however, too restrictive for many interesting applications, especially in the realm of partial differential equations. Indeed, even in the simple example of the heat equation $\partial_t u = \Delta u$, the solution curves do not define a flow in the function space, but only a semiflow, and no general theorem seems to be available in such cases. Moreover, in some elliptic differential problems where the spectrum of the linear operator is unbounded in the unstable direction, it is not even possible to associate a semiflow with the equation, since arbitrarily small initial data may diverge in arbitrarily short times. Nevertheless, center manifold techniques have been successfully applied to such problems, see Mielke [2].

It is thus important to formulate a center-stable manifold theorem directly for the differential equation itself, with no reference to any flow possibly associated with it. In this paper, we prove such a theorem for a class of equations characterized by weak assumptions on the linear operator in the right-hand side, but imposing relatively restrictive conditions on the non-linear terms (smoothness). The main motivation for this paper was to provide the mathematical apparatus for the companion paper, written jointly with J.-P. Eckmann [9], where the results presented here are applied to the problem of constructing front solutions for the Ginzburg-Landau equation with complex amplitudes. Our approach follows closely Eckmann and Wayne [3], but provides additional information on the regularity in time of the solutions. Analogous results for more general non-linearities and for non-autonomous equations can be found in Mielke [2], [4], [5]. For a recent review of center manifold theory in infinite dimensions, see Vanderbauwhede and Iooss [6].

Let $\mathcal{E}$ be a (real or complex) Banach space, A a linear operator in $\mathcal{E}$, and $f : \mathcal{E} \to \mathcal{E}$ a smooth map vanishing at the origin. We consider the autonomous differential equation in $\mathcal{E}$,

$$\frac{d}{dt} z(t) = Az(t) + f(z(t)) , \qquad t \geq 0 . \tag{1.1}$$

We are interested in the behavior of the solutions in a sufficiently small neighborhood of the fixed point $z = 0$. Our assumptions are:

A1) (*On the linear operator*) The Banach space $\mathcal{E}$ is the direct sum of two closed, A-invariant subspaces $\mathcal{E}^s, \mathcal{E}^u$, and the corresponding restrictions $A^s = A|_{\mathcal{E}^s}$, $A^u = A|_{\mathcal{E}^u}$ generate strongly continuous semigroups $e^{A^s t}$, $e^{-A^u t}$ for $t \geq 0$. Furthermore, there are real numbers $\lambda^s < \lambda^u$ such that

$$\sup_{t\geq 0} \|e^{A^s t}\| \, e^{-\lambda^s t} < \infty , \qquad \sup_{t\geq 0} \|e^{-A^u t}\| \, e^{\lambda^u t} < \infty .$$

A2) (*On the non-linear term*) The map f is of class C^k for some (not necessarily integer) $k > 1$, and verifies $f(0) = 0$, $Df(0) = 0$.

A3) (*On the spectral gap*) If $\lambda^s \geq 0$, we also assume that $\lambda^u > k\lambda^s$ and that $\mathcal{E}^s$ has the C^k extension property.

For comments on these assumptions, see the remarks below. In view of A1, we can write any $z \in \mathcal{E}$ as a pair (z^s, z^u) with $z^s \in \mathcal{E}^s$ and $z^u \in \mathcal{E}^u$. Going to an equivalent norm, we may (and do) assume that $\|z\| = \max(\|z^s\|, \|z^u\|)$ for all $z \in \mathcal{E}$, and that $\|e^{A^s t}\| \leq e^{\lambda^s t}$, $\|e^{-A^u t}\| \leq e^{-\lambda^u t}$ for all $t \in \mathbf{R}_+$ (see Pazy [3], Sect. 1.5). For all $r > 0$, we denote by B_r^s, B_r^u, B_r the balls of radius r around the origin in $\mathcal{E}^s, \mathcal{E}^u, \mathcal{E}$ respectively, and by $\mathcal{D}(A^s), \mathcal{D}(A^u), \mathcal{D}(A)$ the domains of the operators A^s, A^u, A. Finally, if $\beta \in (\lambda^s, \lambda^u)$ and if $z : \mathbf{R}_+ \to \mathcal{E}$ is continuous, we define $\|z\|_\beta = \sup_{t \geq 0} \|z(t)\| e^{-\beta t}$. With these notations, we can formulate our main result:

Theorem 1.1. (Local center-stable manifold theorem)
Assume that the conditions A1, A2, A3 above are fulfilled.

Stable case $(\lambda^s < 0)$ **:** *Let $\beta \in (\lambda^s, \lambda^u)$, $\beta \leq 0$. Then, for sufficiently small $r > 0$, there is a (unique) C^k map $h : B_r^s \to B_r^u$ with $h(0) = 0$, $Dh(0) = 0$, whose graph $\mathcal{V} \subset B_r$* (the local stable manifold) *has the following properties:*

a) *(Invariance) For all $z_0 \in \mathcal{V}$ such that $z_0^s \in \mathcal{D}(A^s)$, there exists a unique solution $z(t)$ of Eq.(1.1) such that $z(0) = z_0$, $z(t) \in \mathcal{V}$ for all $t \in \mathbf{R}_+$ and $\|z\|_\beta < \infty$.*

b) *(Uniqueness) If $z(t)$ is any solution of Eq.(1.1) such that $z(t) \in B_r$ for all $t \in \mathbf{R}_+$ and $\|z\|_\beta < \infty$, then $z(t) \in \mathcal{V}$ for all $t \in \mathbf{R}_+$.*

Center-stable case $(\lambda^s \geq 0)$ **:** *For sufficiently small $r > 0$, there is a C^k map $h : B_r^s \to B_r^u$ with $h(0) = 0$, $Dh(0) = 0$, whose graph $\mathcal{V} \subset B_r$* (the local center-stable manifold) *has the following properties:*

a) *(Invariance) For all $z_0 \in \mathcal{V}$ such that $z_0^s \in \mathcal{D}(A^s)$, there exists a C^1 curve $z : [0, \infty) \to \mathcal{E}$ with $z(0) = z_0$ such that, as long as $z(t) \in B_r$, then $z(t) \in \mathcal{V}$ and Eq.(1.1) holds. If moreover $z(t) \in B_r$ for all $t \in \mathbf{R}_+$, then $z(t)$ is the unique solution of Eq.(1.1) with these properties.*

b) *(Uniqueness) If $z(t)$ is any solution of Eq.(1.1) such that $z(t) \in B_r$ for all $t \in \mathbf{R}_+$, then $z(t) \in \mathcal{V}$ for all $t \in \mathbf{R}_+$.*

Remarks

1) By a solution of Eq.(1.1), we always mean a *classical solution*, that is, a continuous function $z : [0, \infty) \to \mathcal{E}$ such that, for all $t > 0$, $z(t)$ is continuously differentiable, $z(t) \in \mathcal{D}(A)$ and Eq.(1.1) is verified.

2) The Assumption A1 implies that A^s, A^u are closed, densely defined linear operators in $\mathcal{E}^s, \mathcal{E}^u$, whose spectra are contained in the half-planes $\{w \in \mathbf{C} \,|\, \mathrm{Re}\,(w) \leq \lambda^s\}$,

$\{w \in \mathbf{C} \,|\, \mathrm{Re}\,(w) \geq \lambda^u\}$ respectively (Hille-Yosida theorem). Note that we do not assume that A itself generates a semigroup. As a consequence, the Cauchy problem for Eq.(1.1) is very awkward: in general, this equation does not define a semiflow at all, even in a neighborhood of 0 and for small values of t. Thus, the existence of solutions with initial condition on the manifold $\mathcal{V}$ is part of the assertion of the theorem. Note also that we do not suppose the semigroups $e^{A^s t}$, $e^{-A^u t}$ to be analytic.

3) In the Assumption A2, we denote by C^k (with $k = n + \alpha$, $n \in \mathbf{N}^*$, $\alpha \in (0,1]$) the class of n times differentiable functions whose n^{th} derivative is Hölder continuous with exponent α. For example, we mean by C^2 the space of once differentiable functions with Lipschitz derivative. By $\mathcal{C}^k \subset C^k$, we mean the class of functions with bounded derivatives up to order k.
4) In the Assumption A3, the Banach space $\mathcal{E}^s$ is said to have the C^k extension property if there is a $\mathcal{C}^k$ function $\chi : \mathcal{E}^s \to [0,1]$ equal to 1 in the unit ball B^s_1 and vanishing outside the ball B^s_R for some $R > 1$ (see Ruelle [1], Bonic and Frampton [8]). Any Hilbert space or finite-dimensional Banach space has the C^k extension property for all k.

We shall give a complete proof of Theorem 1.1 in the case $k \in (1,2]$ only; higher order differentiability of the manifold $\mathcal{V}$ can be proved recursively along the same lines. Using additional assumptions on the non-linearity f, we first (Section 2) state a global version of our result (Theorem 2.1), and prove it by successive applications of the Contraction Mapping Principle. Then (Section 3), we show how Theorem 1.1 follows from Theorem 2.1 by restricting the system to the ball B_r (stable case) and by "cutting off" the non-linear terms (center-stable case). In conclusion (Section 4), we state the corresponding version of the center manifold theorem (Theorem 4.1). Except for some basic facts in semigroup theory, the proofs are elementary and completely self-contained.

2. The Global Center-Stable Manifold Theorem

In this section, we prove a global version of Theorem 1.1 in the case $k \in (1,2]$. For convenience, we assume from the outset that we are given two Banach spaces $(\mathcal{E}^s, \|\cdot\|_s)$, $(\mathcal{E}^u, \|\cdot\|_u)$, and we define $\mathcal{E}$ as the direct sum $\mathcal{E}^s \oplus \mathcal{E}^u$ equipped with the norm $\|z\| = \max(\|z^s\|_s, \|z^u\|_u)$ for all $z = (z^s, z^u) \in \mathcal{E}$. Rewriting Eq.(1.1) in the form

$$\begin{aligned} \frac{dz^s}{dt} &= A^s z^s + f^s(z^s, z^u)\,, \\ \frac{dz^u}{dt} &= A^u z^u + f^u(z^s, z^u)\,, \end{aligned} \tag{2.1}$$

we express our hypotheses as follows:

H1) The linear operators $A^s : \mathcal{E}^s \to \mathcal{E}^s$ and $-A^u : \mathcal{E}^u \to \mathcal{E}^u$ define strongly continuous semigroups $e^{A^s t}$, $e^{-A^u t}$ for $t \geq 0$. Moreover, there exist real numbers $\lambda^s < \lambda^u$ and $D^s, D^u \geq 1$ such that

$$\|e^{A^s t}\|_s \leq D^s e^{\lambda^s t}\,, \qquad \|e^{-A^u t}\|_u \leq D^u e^{-\lambda^u t}\,,$$

for all $t \geq 0$.

H2) The functions $f^s : \mathcal{E} \to \mathcal{E}^s$ and $f^u : \mathcal{E} \to \mathcal{E}^u$ are globally Lipschitz and vanish at the origin: $f^s(0) = 0$, $f^u(0) = 0$, and

$$\|f^s(z) - f^s(\tilde{z})\|_s \leq \ell^s \|z - \tilde{z}\|\,, \qquad \|f^u(z) - f^u(\tilde{z})\|_u \leq \ell^u \|z - \tilde{z}\|\,,$$

for all $z, \tilde{z} \in \mathcal{E}$.

H3) The functions f^s, f^u are (Fréchet) differentiable, and the derivatives Df^s, Df^u are globally Hölder (for some exponent $\alpha \in (0,1]$) and vanish at the origin: $Df^s(0) = 0$, $Df^u(0) = 0$, and

$$\|Df^s(z) - Df^s(\tilde{z})\| \leq L^s \|z - \tilde{z}\|^\alpha\,, \qquad \|Df^u(z) - Df^u(\tilde{z})\| \leq L^u \|z - \tilde{z}\|^\alpha\,,$$

for all $z, \tilde{z} \in \mathcal{E}$.

Theorem 2.1. (Global center-stable manifold theorem)
Given A^s, A^u, f^s, f^u verifying the hypotheses H1, H2, H3 above, assume that there exists a $\beta \in (\lambda^s, \lambda^u)$ such that the conditions C1, C2 below are fulfilled. Then there exist a (unique) map $h : \mathcal{E}^s \to \mathcal{E}^u$ and a (unique) semiflow $\phi : \mathbf{R}_+ \times \mathcal{E}^s \to \mathcal{E}^s$ with the following properties:

i) h *is* $C^{1+\alpha}$, $h(0) = 0$, $Dh(0) = 0$.

ii) $\phi_t(\xi)$ *is* C^0 *in* $t \in \mathbf{R}_+$ *and* $C^{1+\alpha}$ *in* $\xi \in \mathcal{E}^s$*; also,* $\phi_t(0) = 0$, $D\phi_t(0) = e^{A^s t}$.

iii) $h(\mathcal{D}(A^s)) \subset \mathcal{D}(A^u)$ *and* $\phi_t(\mathcal{D}(A^s)) \subset \mathcal{D}(A^s)$ *for all* $t \geq 0$.

iv) For all $\xi \in \mathcal{D}(A^s)$, $\phi_t(\xi)$ *is* C^1 *in* t *and* $z(t) = (\phi_t(\xi), h(\phi_t(\xi)))$ *is a solution of Eq.(2.1) satisfying* $\|z\|_\beta < \infty$.

v) If $z(t)$ *is any solution of Eq.(2.1) such that* $\|z\|_\beta < \infty$, *then* $z(t) = (\phi_t(\xi), h(\phi_t(\xi)))$ *for some* $\xi \in \mathcal{E}^s$.

Remarks. We recall that $\mathcal{D}(A^s), \mathcal{D}(A^u)$ are the (dense) domains of the operators A^s, A^u, and that $\|z\|_\beta = \sup_{t \geq 0} \|z(t)\| e^{-\beta t}$. The conditions C1, C2, are defined in Lemma 2.8 and Lemma 2.10 below; they are fulfilled if $\lambda^u > (1+\alpha)\lambda^s$ and if the Lipschitz constants ℓ^s, ℓ^u are sufficiently small.

Since the proof of Theorem 2.1 is somewhat lengthy, we shall divide it into several pieces. In a first stage (Sect. 2.1), we show the existence of solutions $z(t)$ of Eq.(2.1) with $\|z\|_\beta < \infty$. As we shall see, all these solutions lie on the graph of some map $h : \mathcal{E}^s \to \mathcal{E}^u$. In Sect. 2.2, we prove the differentiability of this map and of the semiflow defined on its graph by Eq.(2.1).

2.1. Existence of Solutions

We want to prove the existence of solutions $z(t)$ of Eq.(2.1) with $\|z\|_\beta < \infty$ for some $\beta \in (\lambda^s, \lambda^u)$. The basic observation is that all these solutions (if they exist) must satisfy an integral equation.

Lemma 2.2. *If $z(t)$ is a solution of Eq.(2.1) such that $\lim_{t\to+\infty} \|z^u(t)\| e^{-\lambda^u t} = 0$, then $z(t)$ verifies the integral equation*

$$\begin{aligned} z^s(t) &= e^{A^s t}\xi + \int_0^t e^{A^s(t-\tau)} f^s(z(\tau))\,d\tau\,, \\ z^u(t) &= -\int_0^\infty e^{-A^u\tau} f^u(z(t+\tau))\,d\tau\,, \end{aligned} \tag{2.2}$$

with $\xi = z^s(0) \in \mathcal{E}^s$.

Proof. Fix $t > 0$, and define $x(\tau) = e^{A^s(t-\tau)} z^s(\tau)$, for $0 \le \tau \le t$. The function $x(\tau)$ is continuous on $[0,t]$ and since $z^s(\tau) \in \mathcal{D}(A^s)$ for all $\tau > 0$, $x(\tau)$ is also differentiable on $(0,t)$ with derivative given by $x'(\tau) = e^{A^s(t-\tau)}(dz^s/dt - A^s z^s)(\tau) = e^{A^s(t-\tau)} f^s(z(\tau))$. Thus, integrating over $\tau \in [0,t]$ and noting that $x(t) = z^s(t)$, $x(0) = e^{A^s t}\xi$, we obtain the first line of Eq.(2.2).

Similarly, define $y(\tau) = e^{-A^u(\tau-t)} z^u(\tau)$, for $\tau \ge t \ge 0$. As above, $y'(\tau) = e^{-A^u(\tau-t)} f^u(z(\tau))$ for all $\tau > t$. Integrating over $\tau \in [t,T]$, we find for all $T > t$,

$$z^u(t) = e^{-A^u(T-t)} z^u(T) - \int_t^T e^{-A^u(\tau-t)} f^u(z(\tau))\,d\tau\ .$$

In view of H1, the first term in the right-hand side goes to zero by assumption as $T \to \infty$, and we obtain the second line of Eq.(2.2). □

Definition. For all $\beta \in (\lambda^s, \lambda^u)$, we define

$$R_\beta = \max\left(\frac{D^s \ell^s}{\beta - \lambda^s}\,,\, \frac{D^u \ell^u}{\lambda^u - \beta}\right)\ .$$

The main result of this subsection is:

Proposition 2.3. *Suppose that the hypotheses H1, H2, H3 are fulfilled, and assume that there exists a $\beta \in (\lambda^s, \lambda^u)$ such that $R_\beta < 1$. Then, for all $\xi \in \mathcal{D}(A^s)$, Eq.(2.1) has a unique solution $z(t)$ such that $z^s(0) = \xi$ and $\|z\|_\beta < \infty$. Moreover, $z(t)$ is the unique continuous solution of Eq.(2.2) such that $\|z\|_\beta < \infty$.*

The first step in proving this proposition is to show that Eq.(2.2) has a unique solution for all $\xi \in \mathcal{E}^s$.

Lemma 2.4. *Assume that there exists a* $\beta \in (\lambda^s, \lambda^u)$ *such that* $R_\beta < 1$. *Then, for all* $\xi \in \mathcal{E}^s$, *the integral equation Eq.(2.2) has a unique continuous solution* $z(t)$ *such that* $\|z\|_\beta < \infty$; *moreover,* $\|z\|_\beta \le D^s \|\xi\|_s$.

Proof. Fix $\xi \in \mathcal{E}^s$, and consider the Banach space $L_\beta = \{z \in C^0(\mathbf{R}_+, \mathcal{E}) \,|\, \|z\|_\beta < \infty\}$ equipped with the norm $\|z\|_\beta$. For all $z \in L_\beta$, denote by Tz the right-hand side of Eq.(2.2) (as a function of t). We shall show that T is a contraction in L_β and maps the ball $\|z\|_\beta \le D^s \|\xi\|_s$ into itself.

First, it is not difficult to see that, for all $z \in L_\beta$, $(Tz)(t)$ is a continuous function of t. Next, using H1, H2, we find for all $t \in \mathbf{R}_+$,

$$\begin{aligned}
\|(Tz)^s(t)\|_s &\le D^s e^{\lambda^s t} \|\xi\|_s + \int_0^t D^s e^{\lambda^s (t-\tau)} \ell^s \|z\|_\beta e^{\beta\tau}\, d\tau \\
&\le D^s e^{\lambda^s t} \|\xi\|_s + \frac{D^s \ell^s}{\beta - \lambda^s} \|z\|_\beta (e^{\beta t} - e^{\lambda^s t}) \\
&\le e^{\beta t} \max\left(D^s \|\xi\|_s \,,\, \frac{D^s \ell^s}{\beta - \lambda^s} \|z\|_\beta \right) , \\
\|(Tz)^u(t)\|_u &\le \int_0^\infty D^u e^{-\lambda^u \tau} \ell^u \|z\|_\beta e^{\beta(t+\tau)}\, d\tau \;=\; e^{\beta t} \frac{D^u \ell^u}{\lambda^u - \beta} \|z\|_\beta \,,
\end{aligned}$$

so that $\|Tz\|_\beta \le \max(D^s \|\xi\|_s \,,\, R_\beta \|z\|_\beta) < \infty$. This means that T maps L_β into itself. Furthermore, if $\|z\|_\beta \le D^s \|\xi\|_s$, then also $\|Tz\|_\beta \le D^s \|\xi\|_s$, since $R_\beta < 1$.

Finally, we bound the difference $(Tz)(t) - (T\tilde{z})(t)$ for all $z, \tilde{z} \in L_\beta$. The "inhomogeneous term" $e^{A^s t}\xi$ of Eq.(2.2) drops in this calculation, and we obtain as above

$$\|(Tz)(t) - (T\tilde{z})(t)\| \le e^{\beta t} \max\left(\frac{D^s \ell^s}{\beta - \lambda^s} \,,\, \frac{D^u \ell^u}{\lambda^u - \beta} \right) \|z - \tilde{z}\|_\beta \,,$$

so that $\|Tz - T\tilde{z}\|_\beta \le R_\beta \|z - \tilde{z}\|_\beta$. This means that $T : L_\beta \to L_\beta$ is a contraction. □

We next observe that the solution $z(t)$ of Eq.(2.2) is also a solution of Eq.(2.1) as soon as it is continuously differentiable.

Lemma 2.5. *Let* $z(t)$ *be the solution of Eq.(2.2) given by Lemma 2.4, for some* $\xi \in \mathcal{E}^s$. *The following assertions are equivalent:*

i) $z(t)$ *is continuously differentiable for all* $t > 0$.

ii) For all $t > 0$, $z(t) \in \mathcal{D}(A)$ *and* $t \to Az(t)$ *is continuous.*

iii) $z(t)$ *is a solution of Eq.(2.1).*

Proof. For all $0 < \epsilon < t$, we have the identities

$$\begin{aligned}\frac{1}{\epsilon}(z^s(t+\epsilon) - z^s(t)) &= \frac{1}{\epsilon}\left(e^{A^s\epsilon} - 1\right) z^s(t) + \frac{1}{\epsilon}\int_0^\epsilon e^{A^s(\epsilon-\tau)} f^s(z(t+\tau))\, d\tau \;, \\ \frac{1}{\epsilon}(z^u(t) - z^u(t-\epsilon)) &= \frac{1}{\epsilon}\left(1 - e^{-A^u\epsilon}\right) z^u(t) + \frac{1}{\epsilon}\int_0^\epsilon e^{-A^u(\epsilon-\tau)} f^u(z(t-\tau))\, d\tau \;,\end{aligned}$$

which follow easily from Eq.(2.2). Let us consider the first equation. As $\epsilon \to 0$, the last term in the right-hand side converges to $f^s(z(t))$, the first one to $A^s z^s(t)$ *provided that* $z^s(t) \in \mathcal{D}(A^s)$, and the left-hand side to $D_+ z^s(t)$ (the right-hand derivative of $z^s(t)$) *provided that* z^s is differentiable from the right at t. Thus, if $z^s(t)$ is C^1 for $t > 0$, then $z^s(t) \in \mathcal{D}(A^s)$ for all $t > 0$ and $(dz^s/dt)(t) = A^s z^s(t) + f^s(z(t))$; in particular, $t \to A^s z^s(t)$ is continuous. Conversely, if $z^s(t) \in \mathcal{D}(A^s)$ and $t \to A^s z^s(t)$ is continuous for $t > 0$, then $z^s(t)$ is differentiable from the right for all $t > 0$ and $D_+ z^s(t) = A^s z^s(t) + f^s(z(t))$ is continuous, so that z^s is C^1. In both cases, $z^s(t)$ verifies the first line of Eq.(2.1). Now, repeating the argument with $z^u(t)$ (second equation), we conclude the proof of Lemma 2.5. □

According to this result, the proof of Proposition 2.3 will be complete if we show that the solution $z(t)$ of Eq.(2.2) is C^1 when $\xi \equiv z^s(0) \in \mathcal{D}(A^s)$. We shall first show that $z(t)$ is Lipschitz in t (Lemma 2.6). Then, using this result, we shall prove the differentiability (Lemma 2.7).

Lemma 2.6. *Let $z(t)$ be the solution of Eq.(2.2) given by Lemma 2.4. For $\epsilon > 0$, define*

$$\gamma_\epsilon = \sup_{t \ge 0} \left(e^{-\beta t}\|z(t+\epsilon) - z(t)\|\right) \;.$$

If $\xi \in \mathcal{D}(A^s)$, then $\gamma_\epsilon = \mathcal{O}(\epsilon)$ as $\epsilon \to 0$.

Proof. First of all, we note that $\gamma_\epsilon \le \|z\|_\beta(1 + e^{\beta\epsilon}) < \infty$. Next, using Eq.(2.2), we obtain for all $\epsilon > 0$,

$$\begin{aligned} z^s(t+\epsilon) - z^s(t) &= e^{A^s t}\left\{(e^{A^s\epsilon} - 1)\xi + \int_0^\epsilon e^{A^s(\epsilon-\tau)} f^s(z(\tau))\, d\tau\right\} \\ &\quad + \int_0^t e^{A^s(t-\tau)}(f^s(z(\tau+\epsilon)) - f^s(z(\tau)))\, d\tau \;, \\ z^u(t+\epsilon) - z^u(t) &= -\int_0^\infty e^{-A^u\tau}(f^u(z(t+\tau+\epsilon)) - f^u(z(t+\tau)))\, d\tau \;. \end{aligned}$$

If $\xi \in \mathcal{D}(A^s)$, the quantity in brackets $\{\cdot\}$ behaves like $\epsilon(A^s\xi + f^s(z(0))) + o(\epsilon)$ as $\epsilon \to 0$.

We thus find

$$
\begin{aligned}
\|z^s(t+\epsilon)-z^s(t)\|_s &\le D^s e^{\lambda^s t}\{\epsilon\|A^s\xi+f^s(z(0))\|_s+o(\epsilon)\}+\int_0^t D^s e^{\lambda^s(t-\tau)}\ell^s\gamma_\epsilon e^{\beta\tau}\,d\tau \\
&\le e^{\beta t}\left(\epsilon D^s\|A^s\xi+f^s(z(0))\|_s+o(\epsilon)+\frac{D^s\ell^s}{\beta-\lambda^s}\gamma_\epsilon\right)\ , \\
\|z^u(t+\epsilon)-z^u(t)\|_u &\le \int_0^\infty D^u e^{-\lambda^u\tau}\ell^u\gamma_\epsilon e^{\beta(t+\tau)}\,d\tau \;=\; e^{\beta t}\frac{D^u\ell^u}{\lambda^u-\beta}\gamma_\epsilon\ .
\end{aligned}
$$

As a consequence, we have $\gamma_\epsilon \le R_\beta\gamma_\epsilon+\epsilon D^s\|A^s\xi+f^s(z(0))\|_s+o(\epsilon)$. Since $R_\beta<1$, this means that $\gamma_\epsilon=\mathcal{O}(\epsilon)$ as $\epsilon\to 0$. $\square$

Lemma 2.7. *Let $z(t)$ be the solution of Eq.(2.2) given by Lemma 2.4. If $\xi\in\mathcal{D}(A^s)$, then $z(t)$ is continuously differentiable for all $t\ge 0$.*

Proof. First of all, we can assume without loss of generality that the Hölder exponent α of Df^s, Df^u (cf. H3) is so small that, if we set

$$
\hat\beta \;=\; \begin{cases}\beta(1+\alpha) & \text{if } \beta>0\\ \beta & \text{if } \beta\le 0\end{cases}\ ,
$$

then

$$
R_{\hat\beta}\;\equiv\;\max\left(\frac{D^s\ell^s}{\hat\beta-\lambda^s},\ \frac{D^u\ell^u}{\lambda^u-\hat\beta}\right)\;<\;1\ .
$$

Next, we differentiate Eq.(2.2) formally with respect to the time. Since $\xi\in\mathcal{D}(A^s)$, we find

$$
\begin{aligned}
Dz^s(t) &= e^{A^s t}\left(A^s\xi+f^s(z(0))\right)+\int_0^t e^{A^s(t-\tau)}Df^s(z(\tau))\cdot Dz(\tau)\,d\tau\ , \\
Dz^u(t) &= -\int_0^\infty e^{-A^u\tau}Df^u(z(t+\tau))\cdot Dz(t+\tau)\,d\tau\ ,
\end{aligned}
\tag{2.3}
$$

where $Dz^s = dz^s/dt$, $Dz^u = dz^u/dt$ and $Dz = (Dz^s, Dz^u)$. Now, we can consider Eq.(2.3) as an equation for the unknown function $Dz(t)$, given $z(t)$ the solution of Eq.(2.2). A moment's reflection shows that Eqs.(2.3) and (2.2) are integral equations of the same form: the only difference is that the Lipschitz functions f^s, f^u have been replaced by the bounded linear maps Df^s, Df^u. Thus, since $R_\beta<1$, Lemma 2.4 implies that Eq.(2.3) has a unique solution $Dz\in L_\beta$.

It remains to show that the solution $Dz(t)$ of Eq.(2.3) is the derivative of $z(t)$. In order to do that, we define for all $\epsilon>0$,

$$
\delta_\epsilon \;=\; \sup_{t\ge 0}\left(e^{-\hat\beta t}\|z(t+\epsilon)-z(t)-\epsilon Dz(t)\|\right)\ .
$$

From Lemma 2.6, we know that $\delta_\epsilon = \mathcal{O}(\epsilon)$ as $\epsilon \to 0$. We shall see that, in fact, $\delta_\epsilon = o(\epsilon)$ as $\epsilon \to 0$. This will prove that $Dz(t)$ is the right-hand derivative of $z(t)$, and since $Dz \in L_\beta$ is continuous, this will complete the proof of Lemma 2.7.

Using Eqs.(2.2), (2.3), we obtain for all $\epsilon > 0$,

$$
\begin{aligned}
z^s(t+\epsilon) \,-\, z^s(t) - \epsilon Dz^s(t) \,=&\\
&= e^{A^s t}\left\{(e^{A^s\epsilon}-1)\xi - \epsilon A^s\xi + \int_0^\epsilon \Big(e^{A^s(\epsilon-\tau)} f^s(z(\tau)) - f^s(z(0))\Big)\, d\tau\right\}\\
&+ \int_0^t e^{A^s(t-\tau)}\Big(f^s(z(\tau+\epsilon)) - f^s(z(\tau)) - Df^s(z(\tau))\cdot(z(\tau+\epsilon)-z(\tau))\Big)\, d\tau\\
&+ \int_0^t e^{A^s(t-\tau)} Df^s(z(\tau))\cdot\big(z(\tau+\epsilon) - z(\tau) - \epsilon Dz(\tau)\big)\, d\tau\;.
\end{aligned}
$$

Clearly, the quantity in brackets $\{\cdot\}$ is $o(\epsilon)$ as $\epsilon \to 0$. On the other hand, using the Mean Value Theorem, we have

$$
\begin{aligned}
&\|f^s(z(\tau+\epsilon)) - f^s(z(\tau)) - Df^s(z(\tau))\cdot(z(\tau+\epsilon)-z(\tau))\|_s \le\\
&\le \|z(\tau+\epsilon)-z(\tau)\|\cdot \sup_{w\in\Gamma}\|Df^s(w) - Df^s(z(\tau))\|\;,
\end{aligned}
$$

where $\Gamma \subset \mathcal{E}$ is the segment (straight line) joining $z(\tau)$ and $z(\tau+\epsilon)$. So, using H3 and Lemma 2.6, this term is bounded by $L^s(\gamma_\epsilon e^{\beta\tau})^{1+\alpha} \le L^s\gamma_\epsilon^{1+\alpha} e^{\hat\beta\tau}$. Finally, we also have

$$
\|Df^s(z(\tau))\cdot(z(\tau+\epsilon)-z(\tau)-\epsilon Dz(\tau))\|_s \,\le\, \ell^s\delta_\epsilon e^{\hat\beta\tau}\;.
$$

Combining these estimates, we find

$$
\|z^s(t+\epsilon) - z^s(t) - \epsilon Dz^s(t)\|_s \,\le\, e^{\hat\beta t}\left(\frac{D^s L^s}{\hat\beta-\lambda^s}\,\gamma_\epsilon^{1+\alpha} + \frac{D^s\ell^s}{\hat\beta-\lambda^s}\,\delta_\epsilon + o(\epsilon)\right)\;.
$$

Similar calculations for $z^u(t)$ yield

$$
\|z^u(t+\epsilon) - z^u(t) - \epsilon Dz^u(t)\|_u \,\le\, e^{\hat\beta t}\left(\frac{D^u L^u}{\lambda^u-\hat\beta}\,\gamma_\epsilon^{1+\alpha} + \frac{D^u\ell^u}{\lambda^u-\hat\beta}\,\delta_\epsilon\right)\;.
$$

Since (by Lemma 2.6) $\gamma_\epsilon^{1+\alpha} = o(\epsilon)$ as $\epsilon \to 0$, we conclude that $\delta_\epsilon \le R_{\hat\beta}\delta_\epsilon + o(\epsilon)$. This means that $\delta_\epsilon = o(\epsilon)$ as $\epsilon \to 0$. □

2.2. The Center-Stable Manifold

According to Lemma 2.4, all solutions of Eq.(2.2) with $\|z\|_\beta < \infty$ are contained in the graph of the map $h : \mathcal{E}^s \to \mathcal{E}^u$ defined by $h(\xi) = z^u(0)$, where $(z^s(t), z^u(t))$ is the unique solution of Eq.(2.2) for which $z^s(0) = \xi$. We are thus led to look for solutions of Eqs.(2.1) and (2.2) of the form $z(t) = (\phi_t(\xi), h(\phi_t(\xi)))$, with ϕ a semiflow in $\mathcal{E}^s$. In view of Eq.(2.2), h and ϕ must verify the integral equations

$$\begin{aligned} \phi_t(\xi) &= e^{A^s t}\xi + \int_0^t e^{A^s(t-\tau)} f^s(\phi_\tau(\xi), h(\phi_\tau(\xi)))\, d\tau , \\ h(\xi) &= -\int_0^\infty e^{-A^u \tau} f^u(\phi_\tau(\xi), h(\phi_\tau(\xi)))\, d\tau . \end{aligned} \tag{2.4}$$

In this subsection, we shall show the existence of a unique solution h, ϕ_t of Eq.(2.4) in suitable function spaces. The graph of h will be referred to as *the center-stable manifold.*

We first introduce the function spaces for h and ϕ. For $\sigma \in [0,1]$, $\beta \in (\lambda^s, \lambda^u)$, we define

$$\begin{aligned} H_\sigma &= \{h : \mathcal{E}^s \to \mathcal{E}^u \mid h(0) = 0\,;\ \|h(\xi) - h(\tilde\xi)\|_u \le \sigma\|\xi - \tilde\xi\|_s,\ \ \forall \xi, \tilde\xi \in \mathcal{E}^s\} , \\ K_\beta &= \big\{\phi : \mathbf{R}_+ \times \mathcal{E}^s \to \mathcal{E}^s \mid \phi_0(\xi) = \xi,\ \ \forall \xi \in \mathcal{E}^s\,; \\ &\qquad \phi_t(0) = 0\ \ \forall t \in \mathbf{R}_+\,;\ \phi \text{ is continuous in } t\,; \\ &\qquad \|\phi_t(\xi) - \phi_t(\tilde\xi)\|_s \le D^s e^{\beta t}\|\xi - \tilde\xi\|_s,\ \ \forall t \in \mathbf{R}_+,\ \ \forall \xi, \tilde\xi \in \mathcal{E}^s\big\} . \end{aligned}$$

As is easily verified, H_σ and K_β are complete metric spaces if equipped with the distances

$$d_H(h, \tilde h) = \sup_{\xi \ne 0} \frac{\|h(\xi) - \tilde h(\xi)\|_u}{\|\xi\|_s} , \qquad d_K(\phi, \tilde\phi) = \sup_{t \ge 0} \sup_{\xi \ne 0} \left(e^{-\beta t} \frac{\|\phi_t(\xi) - \tilde\phi_t(\xi)\|_s}{\|\xi\|_s} \right) .$$

With these definitions, we have the following result (see also [3]):

Lemma 2.8. *If $\beta \in (\lambda^s, \lambda^u)$ and if*

(C1) $$\sigma \equiv \max\left(\frac{D^s \ell^s}{\beta - \lambda^s} D^s , \frac{D^u \ell^u}{\lambda^u - \beta} D^s \right) < \frac{1}{2} ,$$

then Eq.(2.4) has a unique solution h, ϕ in $H_\sigma \times K_\beta$.

Proof. To simplify the notations, we set $f_h^s(\xi) = f^s(\xi, h(\xi))$ and $f_h^u(\xi) = f^u(\xi, h(\xi))$ for all $\xi \in \mathcal{E}^s$. Now, for all $h \in H_\sigma$, $\phi \in K_\beta$, we define

$$\begin{aligned} F(h, \phi)(\xi) &= -\int_0^\infty e^{-A^u \tau} f_h^u(\phi_\tau(\xi))\, d\tau , \\ G(h, \phi)_t(\xi) &= e^{A^s t}\xi + \int_0^t e^{A^s(t-\tau)} f_h^s(\phi_\tau(\xi))\, d\tau . \end{aligned}$$

We shall show that, if the condition C1 is fulfilled, the map (F, G) is a contraction in $H_\sigma \times K_\beta$ and thus has a unique fixed point.

We first show that $F(h,\phi) \in H_\sigma$. Since $h \in H_\sigma$ and $\sigma \leq 1$, $f_h^u : \mathcal{E}^s \to \mathcal{E}^u$ is Lipschitz with the same constant ℓ^u as f^u (this follows from the definition of the norm in $\mathcal{E}$). As a consequence,

$$\|F(h,\phi)(\xi) - F(h,\phi)(\tilde{\xi})\|_u \leq \int_0^\infty \|e^{-A^u\tau}\|_u \, \|f_h^u(\phi_\tau(\xi)) - f_h^u(\phi_\tau(\tilde{\xi}))\|_u \, d\tau$$
$$\leq \int_0^\infty (D^u e^{-\lambda^u\tau})\, \ell^u D^s e^{\beta\tau} \|\xi - \tilde{\xi}\|_s \, d\tau \;=\; \frac{D^u \ell^u}{\lambda^u - \beta} D^s \|\xi - \tilde{\xi}\|_s \,.$$

Thus, the integral defining $F(h,\phi)$ converges absolutely, and (by definition of σ) we have $\|F(h,\phi)(\xi) - F(h,\phi)(\tilde{\xi})\|_u \leq \sigma \|\xi - \tilde{\xi}\|_s$ for all $\xi, \tilde{\xi} \in \mathcal{E}^s$. Since obviously $F(h,\phi)(0) = 0$, this means that $F(h,\phi) \in H_\sigma$.

We next show that $G(h,\phi) \in K_\beta$. It follows immediately from the definitions that $G(h,\phi)_0(\xi) = \xi$, $G(h,\phi)_t(0) = 0$, and that $G(h,\phi)_t(\xi)$ is continuous in t for all $\xi \in \mathcal{E}^s$. As above, $f_h^s : \mathcal{E}^s \to \mathcal{E}^s$ is Lipschitz with the same constant ℓ^s as f^s, and

$$\begin{aligned}
&\|G(h,\phi)_t(\xi) - G(h,\phi)_t(\tilde{\xi})\|_s \\
&\qquad \leq \|e^{A^s t}\|_s \|(\xi - \tilde{\xi})\| + \int_0^t \|e^{A^s(t-\tau)}\|_s \|f_h^s(\phi_\tau(\xi)) - f_h^s(\phi_\tau(\tilde{\xi}))\|_s \, d\tau \\
&\qquad \leq D^s e^{\lambda^s t} \|\xi - \tilde{\xi}\|_s + \int_0^t (D^s e^{\lambda^s(t-\tau)})\, \ell^s D^s e^{\beta\tau} \|\xi - \tilde{\xi}\|_s d\tau \\
&\qquad = D^s \|\xi - \tilde{\xi}\|_s \left(e^{\lambda^s t} + \frac{D^s \ell^s}{\beta - \lambda^s}(e^{\beta t} - e^{\lambda^s t}) \right) \leq D^s e^{\beta t} \|\xi - \tilde{\xi}\|_s \,,
\end{aligned}$$

for all $t \in \mathbf{R}_+$ and all $\xi, \tilde{\xi} \in \mathcal{E}^s$. This means that $G(h,\phi) \in K_\beta$.

It remains to show that F and G are contractions. Let $h, \tilde{h} \in H_\sigma$ and $\phi, \tilde{\phi} \in K_\beta$; for all $\xi \in \mathcal{E}^s$, we have

$$\begin{aligned}
\|f_h^u(\phi_\tau(\xi)) - f_h^u(\tilde{\phi}_\tau(\xi))\|_u &\leq \ell^u \|\phi_\tau(\xi) - \tilde{\phi}_\tau(\xi)\|_s \leq \ell^u \, d_K(\phi, \tilde{\phi}) e^{\beta\tau} \|\xi\|_s \,, \\
\|f_h^u(\tilde{\phi}_\tau(\xi)) - f_{\tilde{h}}^u(\tilde{\phi}_\tau(\xi))\|_u &\leq \ell^u \|h(\tilde{\phi}_\tau(\xi)) - \tilde{h}(\tilde{\phi}_\tau(\xi))\|_u \leq \ell^u d_H(h, \tilde{h}) D^s e^{\beta\tau} \|\xi\|_s \\
\|f_h^u(\phi_\tau(\xi)) - f_{\tilde{h}}^u(\tilde{\phi}_\tau(\xi))\|_u &\leq \ell^u D^s (d_H(h,\tilde{h}) + d_K(\phi, \tilde{\phi})) e^{\beta\tau} \|\xi\|_s \,,
\end{aligned}$$

the last line following from the preceding ones by the triangle inequality. We thus find

$$\begin{aligned}
\|F(h,\phi)(\xi) - F(\tilde{h},\tilde{\phi})(\xi)\|_u &\leq \int_0^\infty \|e^{-A^u\tau}\|_u \, \|f_h^u(\phi_\tau(\xi)) - f_{\tilde{h}}^u(\tilde{\phi}_\tau(\xi))\|_u \, d\tau \\
&\leq \frac{D^u \ell^u}{\lambda^u - \beta} D^s (d_H(h,\tilde{h}) + d_K(\phi,\tilde{\phi})) \|\xi\|_s \,.
\end{aligned}$$

In the same way, we have

$$\begin{aligned}
\|f_h^s(\phi_\tau(\xi)) - f_{\tilde h}^s(\tilde\phi_\tau(\xi))\|_s \;&\le\; \ell^s D^s (d_H(h,\tilde h) + d_K(\phi,\tilde\phi)) e^{\beta\tau} \|\xi\|_s \;, \\
\|G(h,\phi)_t(\xi) - G(\tilde h,\tilde\phi)_t(\xi)\|_s \;&\le\; \int_0^t \|e^{A^s(t-\tau)}\|_s \|f_h^s(\phi_\tau(\xi)) - f_{\tilde h}^s(\tilde\phi_\tau(\xi))\|_s \, d\tau \\
&\le\; \frac{D^s \ell^s}{\beta - \lambda^s} D^s (d_H(h,\tilde h) + d_K(\phi,\tilde\phi)) e^{\beta t} \|\xi\|_s \;.
\end{aligned}$$

(Note that the "inhomogeneous term" $e^{A^s t}\xi$ of Eq.(2.4) drops in this calculation.) Combining these results, we see that

$$d_H(F(h,\phi), F(\tilde h,\tilde\phi)) + d_K(G(h,\phi), G(\tilde h,\tilde\phi)) \;\le\; 2\sigma (d_H(h,\tilde h) + d_K(\phi,\tilde\phi)) \;,$$

for all $h, \tilde h \in H_\sigma$ and all $\phi, \tilde\phi \in K_\beta$. Since $\sigma < 1/2$ by C1, this means that (F,G) is a contraction in $H_\sigma \times K_\beta$. □

We next point out the relation between the solutions of Eqs.(2.4) and (2.2):

Lemma 2.9. *Let $h \in H_\sigma$, $\phi \in K_\beta$ be the solution of Eq.(2.4) given by Lemma 2.8. Then $\phi_{t_1+t_2} = \phi_{t_1} \circ \phi_{t_2}$ for all $t_1, t_2 \in \mathbf{R}_+$, and, for all $\xi \in \mathcal{E}^s$, the function $z(t) = (\phi_t(\xi), h(\phi_t(\xi)))$ is the unique solution of Eq.(2.2) in the sense of Lemma 2.4.*

Proof. Since ϕ_t is a solution of Eq.(2.4), it is not difficult to see that

$$\phi_{t_1}(\phi_{t_2}(\xi)) - \phi_{t_1+t_2}(\xi) \;=\; \int_0^{t_1} e^{A^s(t_1-\tau)} \left(f_h^s(\phi_\tau(\phi_{t_2}(\xi))) - f_h^s(\phi_{\tau+t_2}(\xi)) \right) \, d\tau \;,$$

for all $\xi \in \mathcal{E}^s$ and all $t_1, t_2 \in \mathbf{R}_+$. Let

$$K \;=\; \sup_{t_1 \ge 0} \sup_{t_2 \ge 0} \sup_{\xi \ne 0} \left(e^{-\beta(t_1+t_2)} \frac{1}{\|\xi\|_s} \|\phi_{t_1}(\phi_{t_2}(\xi)) - \phi_{t_1+t_2}(\xi)\|_s \right) \;.$$

Clearly, $K \le 2(D^s)^2 < \infty$, since $\phi \in K_\beta$. Now, it follows from the identity above that

$$\begin{aligned}
\|\phi_{t_1}(\phi_{t_2}(\xi)) - \phi_{t_1+t_2}(\xi)\|_s \;&\le\; \int_0^{t_1} D^s e^{\lambda^s(t_1-\tau)} \ell^s K e^{\beta(\tau+t_2)} \|\xi\|_s \, d\tau \\
&\le\; \frac{D^s \ell^s}{\beta - \lambda^s} K e^{\beta(t_1+t_2)} \|\xi\|_s \;\le\; \frac{K}{2} e^{\beta(t_1+t_2)} \|\xi\|_s \;.
\end{aligned}$$

Hence $K \le K/2$, so that $K = 0$; this proves the semigroup property for ϕ_t. Using this result, it is now obvious from Eq.(2.4) that $z(t) = (\phi_t(\xi), h(\phi_t(\xi)))$ verifies Eq.(2.2). Note also that C1 ensures that $R_\beta < 1$. □

We now show that h, ϕ (given by Lemma 2.8) are differentiable with respect to $\xi \in \mathcal{E}^s$, and that the derivatives $Dh, D\phi$ are Hölder continuous with exponent α. In order to do that, we follow the same strategy as in the proof of Lemma 2.7. First, differentiating Eq.(2.4) formally with respect to ξ, we obtain

$$\begin{aligned} D\phi_t(\xi) &= e^{A^s t} + \int_0^t e^{A^s(t-\tau)} Df_h^s(\phi_\tau(\xi)) \cdot (D\phi_\tau(\xi)\,,\, Dh(\phi_\tau(\xi)) \cdot D\phi_\tau(\xi))\, d\tau \ , \\ Dh(\xi) &= -\int_0^\infty e^{-A^u \tau} Df_h^u(\phi_\tau(\xi)) \cdot (D\phi_\tau(\xi)\,,\, Dh(\phi_\tau(\xi)) \cdot D\phi_\tau(\xi))\, d\tau \ , \end{aligned} \tag{2.5}$$

where $Df_h^u(\xi) \equiv Df^u(\xi, h(\xi))$ and $Df_h^s(\xi) \equiv Df^s(\xi, h(\xi))$. We then consider Eq.(2.5) as an equation for the unknown functions $Dh, D\phi$, given h, ϕ as defined by Eq.(2.4). Appropriate function spaces for $Dh, D\phi$ are:

$$\begin{aligned} \hat{H}_{\sigma,\rho} &= \{Dh : \mathcal{E}^s \to \mathcal{L}(\mathcal{E}^s, \mathcal{E}^u) \mid Dh(0) = 0 \ ; \ \|Dh(\xi)\| \le \sigma, \ \ \forall \xi \in \mathcal{E}^s \ ; \\ &\qquad \|Dh(\xi) - Dh(\tilde{\xi})\| \le \rho \|\xi - \tilde{\xi}\|_s^\alpha, \ \ \forall \xi, \tilde{\xi} \in \mathcal{E}^s\} \ , \\ \hat{K}_{\beta,\rho} &= \{D\phi : \mathbf{R}_+ \times \mathcal{E}^s \to \mathcal{L}(\mathcal{E}^s, \mathcal{E}^s) \mid D\phi_0(\xi) = \mathbf{1}, \ \forall \xi \in \mathcal{E}^s \ ; \\ &\qquad D\phi_t(0) = e^{A^s t}, \ \ \forall t \in \mathbf{R}_+ \ ; \ D\phi \text{ is strongly continuous in } t, \ \ \forall \xi \in \mathcal{E}^s \ ; \\ &\qquad \|D\phi_t(\xi)\| \le D^s e^{\beta t}, \ \ \forall \xi \in \mathcal{E}^s, \ \ \forall t \in \mathbf{R}_+ \ ; \\ &\qquad \|D\phi_t(\xi) - D\phi_t(\tilde{\xi})\| \le \rho e^{\hat{\beta} t} \|\xi - \tilde{\xi}\|_s^\alpha, \ \ \forall t \in \mathbf{R}_+, \ \ \forall \xi, \tilde{\xi} \in \mathcal{E}^s\} \ , \end{aligned}$$

for some sufficiently large $\rho > 0$. Here and in the sequel, we write

$$\hat{\beta} = \max\bigl(\beta\,,\, \beta(1+\alpha)\bigr) = \begin{cases} \beta(1+\alpha) & \text{if } \beta > 0 \\ \beta & \text{if } \beta \le 0 \end{cases} .$$

As is easily verified, $\hat{H}_{\sigma,\rho}$ and $\hat{K}_{\beta,\rho}$ are complete metric spaces if equipped with the distances

$$\begin{aligned} d_{\hat{H}}(Dh, Dh') &= \sup_{\xi \in \mathcal{E}^s} \|Dh(\xi) - Dh'(\xi)\| \ , \\ d_{\hat{K}}(D\phi, D\phi') &= \sup_{t \ge 0} \sup_{\xi \in \mathcal{E}^s} \Bigl(e^{-\beta t} \|D\phi_t(\xi) - D\phi'_t(\xi)\| \Bigr) \ . \end{aligned}$$

With these definitions, we have the following result:

Lemma 2.10. *Let $h \in H_\sigma$, $\phi \in K_\beta$ be the solution of Eq.(2.4) given by Lemma 2.8. Assume furthermore that $\hat{\beta} \in (\lambda^s, \lambda^u)$ and that*

(C2) $$\max\left(\frac{D^s \ell^s}{\hat{\beta} - \lambda^s} (D^s)^{1+\alpha}\,,\, \frac{D^u \ell^u}{\lambda^u - \hat{\beta}} (D^s)^{1+\alpha} \right) < \frac{1}{2} \, .$$

Then Eq.(2.5) has a unique solution $Dh, D\phi$ *in* $\hat{H}_{\sigma,\rho} \times \hat{K}_{\beta,\rho}$, *if* ρ *is sufficiently large.*

Proof. For all $Dh \in \hat{H}_{\sigma,\rho}$, $D\phi \in \hat{K}_{\beta,\rho}$, we define

$$\hat{F}(Dh, D\phi)(\xi) = -\int_0^\infty e^{-A^u\tau} Df_h^u(\phi_\tau(\xi)) \cdot \big(D\phi_\tau(\xi)\,,\ Dh(\phi_\tau(\xi)) \cdot D\phi_\tau(\xi)\big)\, d\tau\ ,$$

$$\hat{G}(Dh, D\phi)_t(\xi) = e^{A^s t} + \int_0^t e^{A^s(t-\tau)} Df_h^s(\phi_\tau(\xi)) \cdot \big(D\phi_\tau(\xi)\,,\ Dh(\phi_\tau(\xi)) \cdot D\phi_\tau(\xi)\big)\, d\tau\ ,$$

where the integrals in the right-hand side are only *strongly* convergent in $\mathcal{L}(\mathcal{E}^s, \mathcal{E}^u)$, $\mathcal{L}(\mathcal{E}^s, \mathcal{E}^s)$ respectively. We shall show that, provided that ρ is sufficiently large, the map $(\hat{F}, \hat{G})$ is a contraction in $\hat{H}_{\sigma,\rho} \times \hat{K}_{\beta,\rho}$, and thus has a unique fixed point.

We first show that $\hat{F}(Dh, D\phi) \in \hat{H}_{\sigma,\rho}$, $\hat{G}(Dh, D\phi) \in \hat{K}_{\beta,\rho}$. Clearly, $\hat{G}(Dh, D\phi)_0 = e^{A^s t}$, $\hat{G}(Dh, D\phi)_t(\xi)$ is strongly continuous in t, and since $Df^u(0) = 0$, $Df^s(0) = 0$, we see that $\hat{F}(Dh, D\phi)(0) = 0$, $\hat{G}(Dh, D\phi)_t(0) = \mathbf{1}$. Furthermore, for all $\xi \in \mathcal{E}^s$ and all $t \in \mathbf{R}_+$, we have $\|Df_h^u(\xi)\| \le \ell^u$, $\|Df_h^s(\xi)\| \le \ell^s$, $\|D\phi_t(\xi)\| \le D^s e^{\beta t}$, $\|Dh(\xi)\| \le \sigma$, and thus

$$\begin{aligned}
\|\hat{F}(Dh, D\phi)(\xi)\| &\le \int_0^\infty D^u e^{-\lambda^u\tau} \ell^u D^s e^{\beta\tau}\, d\tau \le \frac{D^u \ell^u}{\lambda^u - \beta} D^s \le \sigma\ ,\\
\|\hat{G}(Dh, D\phi)_t(\xi)\| &\le D^s e^{\lambda^s t} + \int_0^t D^s e^{\lambda^s(t-\tau)} \ell^s D^s e^{\beta\tau}\, d\tau\\
&\le D^s \left(e^{\lambda^s t} + \frac{D^s \ell^s}{\beta - \lambda^s} (e^{\beta t} - e^{\lambda^s t}) \right) \le D^s e^{\beta t}\ .
\end{aligned}$$

Finally, for all $\xi, \tilde{\xi} \in \mathcal{E}^s$ and all $\tau \in \mathbf{R}_+$, we have

$$\begin{aligned}
&\|D\phi_\tau(\xi) - D\phi_\tau(\tilde{\xi})\| \le \rho e^{\hat{\beta}\tau} \|\xi - \tilde{\xi}\|_s^\alpha\ ,\\
&\|Dh(\phi_\tau(\xi)) - Dh(\phi_\tau(\tilde{\xi}))\| \le \rho \left(D^s e^{\beta\tau} \|\xi - \tilde{\xi}\|_s \right)^\alpha\ ,\\
&\|Dh(\phi_\tau(\xi)) \cdot D\phi_\tau(\xi) - Dh(\phi_\tau(\tilde{\xi})) \cdot D\phi_\tau(\tilde{\xi})\| \le 2(D^s)^{1+\alpha} \rho e^{\hat{\beta}\tau} \|\xi - \tilde{\xi}\|_s^\alpha\ ,\\
&\|Df_h^{u,s}(\phi_\tau(\xi)) - Df_h^{u,s}(\phi_\tau(\tilde{\xi}))\| \le L^{u,s} \left(D^s e^{\beta\tau} \|\xi - \tilde{\xi}\|_s \right)^\alpha\ .
\end{aligned}$$

Therefore, we find

$$\begin{aligned}
&\|\hat{F}(Dh, D\phi)(\xi) - \hat{F}(Dh, D\phi)(\tilde{\xi})\|\\
&\qquad \le \left\{ \frac{D^u L^u}{\lambda^u - \hat{\beta}} (D^s)^{1+\alpha} + 2 \frac{D^u \ell^u}{\lambda^u - \hat{\beta}} (D^s)^{1+\alpha} \rho \right\} \|\xi - \tilde{\xi}\|_s^\alpha\ ,\\
&\|\hat{G}(Dh, D\phi)_t(\xi) - \hat{G}(Dh, D\phi)_t(\tilde{\xi})\|\\
&\qquad \le \left\{ \frac{D^s L^s}{\hat{\beta} - \lambda^s} (D^s)^{1+\alpha} + 2 \frac{D^s \ell^s}{\hat{\beta} - \lambda^s} (D^s)^{1+\alpha} \rho \right\} e^{\hat{\beta}\tau} \|\xi - \tilde{\xi}\|_s^\alpha\ .
\end{aligned}$$

In view of C2, the quantities in brackets $\{\cdot\}$ are smaller than ρ, if ρ is sufficiently large. This means that $\hat{F}(Dh, D\phi) \in \hat{H}_{\sigma,\rho}$, $\hat{G}(Dh, D\phi) \in \hat{K}_{\beta,\rho}$.

We now show the contraction property. Let $Dh, Dh' \in \hat{H}_{\sigma,\rho}$, $D\phi, D\phi' \in \hat{K}_{\beta,\rho}$. For all $\xi \in \mathcal{E}^s$ and all $\tau \in \mathbf{R}_+$, we have

$$\begin{aligned}
\|D\phi_\tau(\xi) - D\phi'_\tau(\xi)\| &\leq d_{\hat{K}}(D\phi, D\phi')e^{\beta\tau} , \\
\|Dh(\phi_\tau(\xi)) - Dh'(\phi_\tau(\xi))\| &\leq d_{\hat{H}}(Dh, Dh') , \\
\|Dh(\phi_\tau(\xi)) \cdot D\phi_\tau(\xi) - Dh'(\phi_\tau(\xi)) \cdot D\phi'_\tau(\xi)\| &\leq D^s \Delta e^{\beta\tau} ,
\end{aligned}$$

where $\Delta = d_{\hat{H}}(Dh, Dh') + d_{\hat{K}}(D\phi, D\phi')$. Therefore, we easily find

$$\begin{aligned}
\|\hat{F}(Dh, D\phi)(\xi) - \hat{F}(Dh', D\phi')(\xi)\| &\leq \frac{D^u \ell^u}{\lambda^u - \beta} D^s \Delta \leq \sigma\Delta , \\
\|\hat{G}(Dh, D\phi)_t(\xi) - \hat{G}(Dh', D\phi')_t(\xi)\| &\leq \frac{D^s \ell^s}{\beta - \lambda^s} D^s \Delta e^{\beta t} \leq \sigma\Delta e^{\beta t} ,
\end{aligned}$$

so that

$$\begin{aligned}
&d_{\hat{H}}(\hat{F}(Dh, D\phi), \hat{F}(Dh', D\phi')) + d_{\hat{K}}(\hat{G}(Dh, D\phi), \hat{G}(Dh', D\phi')) \\
&\leq 2\sigma(d_{\hat{H}}(Dh, Dh') + d_{\hat{K}}(D\phi, D\phi')) ,
\end{aligned}$$

for all $Dh, Dh' \in \hat{H}_{\sigma,\rho}$ and all $D\phi, D\phi' \in \hat{K}_{\beta,\rho}$. Since $2\sigma < 1$, this means that $(\hat{F}, \hat{G})$ is a contraction in $\hat{H}_{\sigma,\rho} \times \hat{K}_{\beta,\rho}$. □

It remains to verify that $Dh, D\phi$ are the derivatives of h, ϕ:

Lemma 2.11. *Under the assumptions of Lemma 2.8 and Lemma 2.10, the solution (h, ϕ) of Eq.(2.4) in $H_\sigma \times K_\beta$ is differentiable with respect to $\xi \in \mathcal{E}^s$, and its derivative $(Dh, D\phi)$ is the solution of Eq.(2.5) in $\hat{H}_{\sigma,\rho} \times \hat{K}_{\beta,\rho}$.*

Proof. Let $h \in H_\sigma$, $\phi \in K_\beta$ be the solution of Eq.(2.4), and $Dh \in \hat{H}_{\sigma,\rho}$, $D\phi \in \hat{K}_{\beta,\rho}$ be the solution of Eq.(2.5). We have to show that Dh is the derivative of h and $D\phi$ the derivative of ϕ. For all $\epsilon > 0$, we define $\eta(\epsilon) = \max(\eta_1(\epsilon), \eta_2(\epsilon))$, where

$$\begin{aligned}
\eta_1(\epsilon) &= \sup_{\xi\in\mathcal{E}^s} \sup_{\|k\|=\epsilon} \|h(\xi + k) - h(\xi) - Dh(\xi) \cdot k\|_u , \\
\eta_2(\epsilon) &= \sup_{\xi\in\mathcal{E}^s} \sup_{t\geq 0} \sup_{\|k\|=\epsilon} \left(\|\phi_t(\xi + k) - \phi_t(\xi) - D\phi_t(\xi) \cdot k\|_s e^{-\hat{\beta}t}\right) .
\end{aligned}$$

By construction, $\eta(\epsilon) \leq 2D^s\epsilon$ for all $\epsilon > 0$. We shall show that, in fact, $\eta(\epsilon) = \mathcal{O}(\epsilon^{1+\alpha})$, and this will prove Lemma 2.11.

Let $k \in \mathcal{E}^s$, $\|k\| = \epsilon$. In view of Eqs.(2.4) and (2.5), we have the identity

$$
\begin{aligned}
h(\xi+k) - h(\xi) - Dh(\xi)\cdot k \;=& \\
& - \int_0^\infty e^{-A^u\tau} \Big\{ f_h^u(\phi_\tau(\xi+k)) - f_h^u(\phi_\tau(\xi)) - Df_h^u(\phi_\tau(\xi)) \cdot \\
& \qquad \cdot \Big(\phi_\tau(\xi+k) - \phi_\tau(\xi)\,,\, h(\phi_\tau(\xi+k)) - h(\phi_\tau(\xi)) \Big) \Big\}\, d\tau \\
& - \int_0^\infty e^{-A^u\tau} Df_h^u(\phi_\tau(\xi)) \cdot \Big(\phi_\tau(\xi+k) - \phi_\tau(\xi) - D\phi_\tau(\xi)\cdot k\,, \\
& \qquad h(\phi_\tau(\xi+k)) - h(\phi_\tau(\xi)) - Dh(\phi_\tau(\xi))\cdot D\phi_\tau(\xi)\cdot k \Big)\, d\tau\,.
\end{aligned}
$$

We have to estimate the various terms in the right-hand side. In the first integral, we use the Mean Value Theorem to bound the expression in brackets $\{\cdot\}$ by

$$
\|\phi_\tau(\xi+k) - \phi_\tau(\xi)\|_s \cdot \sup_{z\in\Gamma} \|Df^u(z) - Df_h^u(\phi_\tau(\xi))\|\,,
$$

where $\Gamma \subset \mathcal{E}$ is the segment joining $(\phi_\tau(\xi), h(\phi_\tau(\xi)))$ and $(\phi_\tau(\xi+k), h(\phi_\tau(\xi+k)))$; we thus obtain the bound $L^u\big(D^s e^{\beta\tau}\epsilon\big)^{1+\alpha}$. For the second integral, we note that

$$
\|\phi_\tau(\xi+k) - \phi_\tau(\xi) - D\phi_\tau(\xi)\cdot k\|_s \le \eta(\epsilon) e^{\hat\beta\tau}\,,
$$

and we rewrite the last line as

$$
\begin{aligned}
& h(\phi_\tau(\xi+k)) - h(\phi_\tau(\xi)) - Dh(\phi_\tau(\xi))\cdot\big(\phi_\tau(\xi+k) - \phi_\tau(\xi)\big) \\
& + Dh(\phi_\tau(\xi))\cdot\big(\phi_\tau(\xi+k) - \phi_\tau(\xi) - D\phi_\tau(\xi)\cdot k\big)\,.
\end{aligned}
$$

Using again the Mean Value Theorem, the first line of this expression can be bounded by

$$
\|\phi_\tau(\xi+k) - \phi_\tau(\xi)\|_s \cdot \sup_{w\in\Gamma'} \|Dh(w) - Dh(\phi_\tau(\xi))\|\,,
$$

where $\Gamma' \subset \mathcal{E}^s$ is now the segment joining $\phi_\tau(\xi)$ and $\phi_\tau(\xi+k)$; this yields the bound $\rho\big(D^s e^{\beta\tau}\epsilon\big)^{1+\alpha}$. The second line is simply bounded by $\sigma\eta(\epsilon)e^{\hat\beta\tau}$.

Combining all these estimates, we obtain for all $\xi \in \mathcal{E}^s$ and all $k \in \mathcal{E}^s$ with $\|k\| = \epsilon$,

$$
\|h(\xi+k) - h(\xi) - Dh(\xi)\cdot k\|_u \le \left\{ \frac{D^u(L^u + \ell^u\rho)}{\lambda^u - \hat\beta}(D^s\epsilon)^{1+\alpha} + \eta(\epsilon)\frac{D^u\ell^u}{\lambda^u - \hat\beta} \right\}\,.
$$

By similar calculations, we also find for all $t \in \mathbf{R}_+$,

$$
\|\phi_t(\xi+k) - \phi_t(\xi) - D\phi_t(\xi)\cdot k\|_s \le e^{\hat\beta t} \left\{ \frac{D^s(L^s + \ell^s\rho)}{\hat\beta - \lambda^s}(D^s\epsilon)^{1+\alpha} + \eta(\epsilon)\frac{D^s\ell^s}{\hat\beta - \lambda^s} \right\}\,.
$$

Using C2 and the definition of $\eta(\epsilon)$, we conclude that $\eta(\epsilon) = \mathcal{O}(\epsilon^{1+\alpha})$. $\square$

Using all these results, it is now easy to complete the proof of the global center-stable manifold theorem.

Proof of Theorem 2.1. Let $h \in H_\sigma$, $\phi \in K_\beta$ be the solution of Eq.(2.4) given by Lemma 2.8. In view of Lemma 2.9, Lemma 2.11, only the last three assertions of Theorem 2.1 remain to be proved. If $\xi \in \mathcal{D}(A^s)$, then $z(t) = (\phi_t(\xi), h(\phi_t(\xi)))$ is a solution of Eq.(2.2) by Lemma 2.9, hence a solution of Eq.(2.1) by Proposition 2.3. In particular, $t \to \phi_t(\xi)$ is C^1, $\phi_t(\xi) \in \mathcal{D}(A^s)$ for all t and $h(\xi) = h(\phi_0(\xi)) \in \mathcal{D}(A^u)$. This proves iii) and iv). On the other hand, if $z(t)$ is any solution of Eq.(2.1) such that $\|z\|_\beta < \infty$, we know from Lemma 2.2 that $z(t)$ verifies Eq.(2.2) with $\xi = z^s(0)$. By uniqueness of the solutions of this equation, we must have $z(t) = (\phi_t(\xi), h(\phi_t(\xi)))$. This proves v). $\square$

2.3. Continuous Dependence on Parameters

The approach we used in proving Theorem 2.1 makes it easy to obtain additional information about the center-stable manifold. As an example, we shall state here a continuity result which is useful in applications. Suppose that we are given two systems like Eq.(2.1), defined by two collections $(A_1^s, A_1^u, f_1^s, f_1^u)$, $(A_2^s, A_2^u, f_2^s, f_2^u)$ verifying the hypotheses H1, H2 with the same constants D^s, D^u and ℓ^s, ℓ^u. Assume also that the condition C1 of Lemma 2.8 is fulfilled for some $\beta \in (\lambda^s, \lambda^u)$, and denote by (h_1, ϕ_1), (h_2, ϕ_2) the solutions of Eq.(2.4) corresponding to the systems 1, 2 respectively. The following result says that h_1, h_2 and ϕ_1, ϕ_2 are close to each other if A_1, A_2 and f_1, f_2 are:

Proposition 2.12. *Under the assumptions above, if*

$$\sup_{t\geq 0} \|e^{A_1^s t} - e^{A_2^s t}\|_s \, e^{-\lambda^s t} \leq D^s \epsilon \,, \qquad \sup_{t\geq 0} \|e^{-A_1^u t} - e^{-A_2^u t}\|_u \, e^{\lambda^u t} \leq D^u \epsilon \,,$$

$$\sup_{z\neq 0} \frac{1}{\|z\|} \|f_1^s(z) - f_2^s(z)\|_s \leq \ell^s \epsilon \,, \qquad \sup_{z\neq 0} \frac{1}{\|z\|} \|f_1^u(z) - f_2^u(z)\|_u \leq \ell^u \epsilon \,,$$

for some $\epsilon > 0$, *then*

$$d_H(h_1, h_2) \leq \frac{D^s \epsilon}{1 - 2\sigma} \,, \qquad d_K(\phi_1, \phi_2) \leq \frac{D^s \epsilon}{1 - 2\sigma} \,.$$

Proof. Let $\Delta = \max(d_H(h_1, h_2), d_K(\phi_1, \phi_2))$. In view of Eq.(2.4), we have

$$h_1(\xi) = -\int_0^\infty e^{-A_1^u \tau} f_1^u(\phi_{1,\tau}(\xi), h_1(\phi_{1,\tau}(\xi)))\, d\tau \,,$$

$$h_2(\xi) = -\int_0^\infty e^{-A_2^u \tau} f_2^u(\phi_{2,\tau}(\xi), h_2(\phi_{2,\tau}(\xi)))\, d\tau \,,$$

for all $\xi \in \mathcal{E}^s$. As a consequence,

$$
\begin{aligned}
&\|h_1(\xi) - h_2(\xi)\|_u \le \\
&\qquad \le \int_0^\infty \|e^{-A_1^u\tau} - e^{-A_2^u\tau}\|_u \, \|f_1^u(\phi_{1,\tau}(\xi), h_1(\phi_{1,\tau}(\xi)))\|_u \, d\tau \\
&\qquad + \int_0^\infty \|e^{-A_2^u\tau}\|_u \, \|f_1^u(\phi_{1,\tau}(\xi), h_1(\phi_{1,\tau}(\xi))) - f_2^u(\phi_{1,\tau}(\xi), h_1(\phi_{1,\tau}(\xi)))\|_u \, d\tau \\
&\qquad + \int_0^\infty \|e^{-A_2^u\tau}\|_u \, \|f_2^u(\phi_{1,\tau}(\xi), h_1(\phi_{1,\tau}(\xi))) - f_2^u(\phi_{2,\tau}(\xi), h_2(\phi_{2,\tau}(\xi)))\|_u \, d\tau \, .
\end{aligned}
$$

Proceeding as in the proof of Lemma 2.8, it is easy to bound each of the first two terms in the right-hand side by $\epsilon\sigma\|\xi\|_s$. Moreover, since

$$
\|(\phi_{1,\tau}(\xi), h_1(\phi_{1,\tau}(\xi))) - (\phi_{2,\tau}(\xi), h_2(\phi_{2,\tau}(\xi)))\| \le 2D^s \Delta e^{\beta\tau} \|\xi\|_s \, ,
$$

the third term is bounded by $2\sigma\Delta\|\xi\|_s$. We thus find

$$
d_H(h_1, h_2) \le 2\sigma\epsilon + 2\sigma\Delta \, .
$$

Similar calculations for ϕ_1, ϕ_2 yield

$$
d_K(\phi_1, \phi_2) \le D^s\epsilon + 2\sigma\Delta \, .
$$

Combining these results and recalling that $2\sigma < 1 \le D^s$, we see that $\Delta \le D^s\epsilon + 2\sigma\Delta$, or $\Delta \le D^s\epsilon/(1 - 2\sigma)$. $\square$

3. Proof of the Local Center-Stable Manifold Theorem

Using the results of Section 2, we now prove the local center-stable manifold theorem (Theorem 1.1) in the case $k \in (1, 2]$. It turns out to be convenient to deal with the cases $\lambda^s < 0$, $\lambda^s \ge 0$ separately.

3.1. The Stable Case $\lambda^s < 0$

We first state a variant of Theorem 2.1 which is adapted to our present purposes.

Corollary 3.1. *Assume that there exists an $r > 0$ such that the hypotheses H2, H3 of Theorem 2.1 are verified for $z, \tilde{z}$ restricted to the ball $B_r \equiv B_r^s \times B_r^u \subset \mathcal{E}$. Assume also that H1 holds with $D^s = 1$, $\lambda^s < 0$, and that the conditions C1, C2 are fulfilled for some $\beta \in (\lambda^s, \lambda^u)$, $\beta \le 0$. Then there exist a (unique) map $h : B_r^s \to B_r^u$ and a (unique) semiflow $\phi : \mathbf{R}_+ \times B_r^s \to B_r^s$ with the same properties (i), (ii), (iii) as in Theorem 2.1, and*

iv) For all $\xi \in \mathcal{D}(A^s) \cap B_r^s$, $\phi_t(\xi)$ is C^1 in t and $z(t) = (\phi_t(\xi), h(\phi_t(\xi)))$ is a solution of Eq.(2.1) such that $z(t) \in B_r$ for all $t \in \mathbf{R}_+$ and $\|z\|_\beta < \infty$.
v) If $z(t)$ is any solution of Eq.(2.1) such that $z(t) \in B_r$ for all $t \in \mathbf{R}_+$ and $\|z\|_\beta < \infty$, then $z(t) = (\phi_t(\xi), h(\phi_t(\xi)))$ for some $\xi \in B_r^s$.

Sketch of the proof. The idea is to repeat the whole proof of Theorem 2.1 while restricting the various definitions and equations to the ball $B_r \subset \mathcal{E}$, where the non-linear terms satisfy H2, H3. For example, it is easy to verify that all results of Sect. 2.1 remain valid when restricted to the family of curves $z(t)$ which stay in B_r for all $t \in \mathbf{R}_+$. The crucial point is to check that this property is left invariant by the right-hand side of Eq.(2.2); but this follows from Lemma 2.4, since $D^s = 1$ and $\beta \le 0$. Similarly, in Sect. 2.2, we consider the function spaces $H_\sigma(r), K_\beta(r)$ obtained by replacing everywhere $\mathcal{E}^s$ by B_r^s in the definitions of H_σ, K_β. Using Lemma 2.8, it is easy to see that the right-hand side of Eq.(2.4) is a contraction in $H_\sigma(r) \times K_\beta(r)$, and thus has a unique fixed point h, ϕ. The differentiability of h, ϕ with respect to $\xi \in B_r^s$ is shown in the same way, by regarding Eq.(2.5) as a fixed point problem in $\hat{H}_{\sigma,\rho}(r) \times \hat{K}_{\beta,\rho}(r)$. Combining these results as in Theorem 2.1, we conclude the proof of Corollary 3.1. □

We are now able to prove Theorem 1.1 in the case $\lambda^s < 0$. In view of the Assumption A2 (with $k = 1 + \alpha$, $\alpha \in (0, 1]$), there is an $r_0 > 0$ such that the derivatives Df^s, Df^u are (globally) α-Hölder in the ball $B_{r_0} \subset \mathcal{E}$, for some Hölder constants L^s, L^u. For all $r \le r_0$, we define

$$\ell^s(r) = \sup_{z \in B_r} \|Df^s(z)\| , \qquad \ell^u(r) = \sup_{z \in B_r} \|Df^u(z)\| .$$

Clearly, $\ell^s(r), \ell^u(r) \sim r^\alpha$ as $r \to 0$. So, given $\beta \in (\lambda^s, \lambda^u)$ such that $\beta \le 0$, we can choose $r > 0$ so small that

$$\sigma \equiv \max\left(\frac{\ell^s(r)}{\beta - \lambda^s}, \frac{\ell^u(r)}{\lambda^u - \beta}\right) < \frac{1}{2} .$$

Since $D^s = D^u = 1$ by choice of the norms in $\mathcal{E}^s, \mathcal{E}^u$, this implies that the conditions C1, C2 are fulfilled with $\ell^s = \ell^s(r)$ and $\ell^u = \ell^u(r)$. As a consequence, restricting the system (2.1) to the ball $B_r \subset \mathcal{E}$, we can apply Corollary 3.1 and we obtain a map $h \in H_\sigma(r)$ and a semiflow

$\phi \in K_\beta(r)$ with the desired properties. In particular, h is C^k, $h(0) = 0$, $Dh(0) = 0$, and the assertions (a), (b) in Theorem 1.1 follow from (iv), (v) in Corollary 3.1. □

3.2. The Center-Stable Case $\lambda^s \geq 0$

Again, we begin with a variant of Theorem 2.1:

Corollary 3.2. *Assume that there exists an $r > 0$ such that the hypotheses H2, H3 of Theorem 2.1 are verified for $z, \tilde{z}$ restricted to the cylinder $\mathcal{E}^s \times B_r^u \subset \mathcal{E}$, and such that $f^s(z) = 0$, $f^u(z) = 0$ for all $z \notin B_r^s \times \mathcal{E}^u$. Assume also that H1 holds and that the conditions C1, C2 are fulfilled for some $\beta \in (\lambda^s, \lambda^u)$. Then there exist a map $h : \mathcal{E}^s \to \mathcal{E}^u$ and a semiflow $\phi : \mathbf{R}_+ \times \mathcal{E}^s \to \mathcal{E}^s$ with the same properties (i), (ii), (iii) as in Theorem 2.1, and*

- *iv) For all $\xi \in \mathcal{D}(A^s)$, $\phi_t(\xi)$ is C^1 in t and $z(t) = (\phi_t(\xi), h(\phi_t(\xi)))$ is a solution of Eq.(2.1) satisfying $\|z\|_\beta < \infty$.*
- *v) If $z(t)$ is any solution of Eq.(2.1) such that $z(t) \in B_r \equiv B_r^s \times B_r^u$ for all $t \in \mathbf{R}_+$, then $z(t) = (\phi_t(\xi), h(\phi_t(\xi)))$ for some $\xi \in B_r^s$.*

Sketch of the proof. Adapting the results of Section 2 to the present case needs some care, because now we have no control on the non-linear terms f^s, f^u in the region $\mathcal{R} = B_r^s \times (\mathcal{E}^u \backslash B_r^u)$. In Sect. 2.2, this causes no trouble, because f^s, f^u and their derivatives Df^s, Df^u only appear in expressions like $f_h^{s,u}(\xi) \equiv f^{s,u}(\xi, h(\xi))$, $Df_h^{s,u}(\xi) \equiv Df^{s,u}(\xi, h(\xi))$, where $h \in H_\sigma$ for some $\sigma \leq 1$. But f_h^s, f_h^u are Lipschitz on $\mathcal{E}^s$, and Df_h^s, Df_h^u are α-Hölder, since the graph of h does not intersect the "forbidden" region $\mathcal{R}$. So, all the machinery of Sect. 2.2 works, and we obtain a map $h \in H_\sigma$ and a semiflow $\phi \in K_\beta$ such that $Dh \in \hat{H}_{\sigma,\rho}$, $D\phi \in \hat{K}_{\beta,\rho}$ and Eq.(2.4) is verified.

In Sect. 2.1 however, not all results remain true, since in general a curve $z(t)$ with $\|z\|_\beta < \infty$ intersects the region $\mathcal{R}$. Nevertheless, if $\xi \in \mathcal{D}(A^s)$ and if $z(t) = (\phi_t(\xi), h(\phi_t(\xi)))$, then $z(t) \notin \mathcal{R}$ for all $t \in \mathbf{R}_+$ and $z(t)$ is a solution of Eq.(2.2) by Lemma 2.9. So, using Lemma 2.5, Lemma 2.6, Lemma 2.7, we see that $z(t)$ is C^1 in t and verifies Eq.(2.1). This proves (iv). On the other hand, if $z(t)$ is any solution of Eq.(2.1) such that $z(t) \notin \mathcal{R}$ for all $t \in \mathbf{R}_+$ (for example, if $z(t) \in B_r$ for all $t \in \mathbf{R}_+$), then by Lemma 2.2 $z(t)$ is a solution of Eq.(2.2) with $\xi = z^s(0)$. As we have seen, $\hat{z}(t) = (\phi_t(\xi), h(\phi_t(\xi)))$ is also a solution of Eq.(2.2) such that $\hat{z}(t) \notin \mathcal{R}$ for all $t \in \mathbf{R}_+$, and using the uniqueness part of Lemma 2.4 (contraction property of the map T), we conclude that $z(t) \equiv \hat{z}(t)$. This proves (v). □

We are now able to prove Theorem 1.1 in the case $\lambda^s \geq 0$. First of all, since (by A3) $\lambda^u > k\lambda^s$ with $k = 1 + \alpha$, we can choose $\beta \in (\lambda^s, \lambda^u)$ such that $\hat{\beta} \equiv \beta(1 + \alpha) \in (\lambda^s, \lambda^u)$. Since $\mathcal{E}^s$ has the $C^{1+\alpha}$ extension property, there exist a radius $R > 1$ and a function χ :

$\mathcal{E}^s \to [0,1]$ such that $\chi(\xi) = 1$ if $\xi \in B_1^s$, $\chi(\xi) = 0$ if $\xi \notin B_R^s$, χ is Lipschitz with constant $\ell_\chi < \infty$ and $D\chi$ is α-Hölder with constant $L_\chi < \infty$. If $\ell^s(r), \ell^u(r)$ are as in Sect. 3.1, and if $\hat{\ell}^s(r) = (1 + R\ell_\chi)\ell^s(Rr)$, $\hat{\ell}^u(r) = (1 + R\ell_\chi)\ell^u(Rr)$, we can assume that

$$\max\left(\frac{\hat{\ell}^s(r)}{\beta - \lambda^s}, \frac{\hat{\ell}^u(r)}{\lambda^u - \hat{\beta}}\right) < \frac{1}{2},$$

by taking r sufficiently small. We next define the "localized functions" $g^s : \mathcal{E} \to \mathcal{E}^s$, $g^u : \mathcal{E} \to \mathcal{E}^u$ by

$$g^s(z^s, z^u) = f^s(z^s, z^u)\chi(z^s/r), \qquad g^u(z^s, z^u) = f^u(z^s, z^u)\chi(z^s/r).$$

Clearly, g^s, g^u are $C^{1+\alpha}$, coincide with f^s, f^u in the ball $B_r \subset \mathcal{E}$, and vanish for all $z \notin B_{Rr}^s \times \mathcal{E}^u$. Moreover, it is straightforward to verify that, in the cylinder $\mathcal{E}^s \times B_{Rr}^u \subset \mathcal{E}$, g^s, g^u are Lipschitz with constants $\hat{\ell}^s(r), \hat{\ell}^u(r)$, and Dg^s, Dg^u are α-Hölder for some constants $\hat{L}^s, \hat{L}^u$.

So, we can apply Corollary 3.2 with r replaced by Rr, $f^{s,u}$ replaced by $g^{s,u}$ in Eq.(2.1), $D^{s,u}$ replaced by 1 in H1, $\ell^{s,u}$ replaced by $\hat{\ell}^{s,u}(r)$ in H2, and $L^{s,u}$ replaced by $\hat{L}^{s,u}$ in H3. The conditions C1, C2 are fulfilled by the choice of r. We thus obtain a map $h \in H_\sigma$ and a semiflow $\phi \in K_\beta$ corresponding to the system (2.1) with f^s, f^u replaced by g^s, g^u; in particular, h is C^k, $h(0) = 0$ and $Dh(0) = 0$. Restricting h to the ball $B_r \subset \mathcal{E}$ and recalling that f^s, f^u and g^s, g^u coincide in this domain, we see that the assertions (a), (b) in Theorem 1.1 follow from (iv), (v) in Corollary 3.2. $\square$

4. The Center Manifold Theorem

In conclusion, we state (without an explicit proof) the center manifold theorem in the same setting as Theorem 1.1. Instead of A1, A3, we assume:

A1') The Banach space $\mathcal{E}$ is the direct sum of three closed, A-invariant subspaces $\mathcal{E}^s, \mathcal{E}^c$, $\mathcal{E}^u$. The corresponding restrictions $A^s = A|_{\mathcal{E}^s}$, $A^c = A|_{\mathcal{E}^c}$, $A^u = A|_{\mathcal{E}^u}$ define two strongly continuous semigroups $e^{A^s t}$, $e^{-A^u t}$ $(t \geq 0)$ and a strongly continuous group $e^{A^c t}$ $(t \in \mathbf{R})$. Furthermore, there are real numbers $\lambda^s < -\lambda^c \leq 0 \leq \lambda^c < \lambda^u$ such that

$$\sup_{t\geq 0} \|e^{A^s t}\| e^{-\lambda^s t} < \infty, \quad \sup_{t\in\mathbf{R}} \|e^{A^c t}\| e^{-\lambda^c |t|} < \infty, \quad \sup_{t\geq 0} \|e^{-A^u t}\| e^{\lambda^u t} < \infty.$$

A3') $\lambda^u > k\lambda^c$, $\lambda^s < -k\lambda^c$, and $\mathcal{E}^c$ has the C^k extension property.

The spectrum of A is thus split into three pieces: a stable part (contained in the half-plane $\mathrm{Re}\,(w) \leq \lambda^s$), an unstable part (contained in the half-plane $\mathrm{Re}\,(w) \geq \lambda^u$), and a central part (contained in the band $|\mathrm{Re}\,(w)| \leq \lambda^c$). As above, we write $z = (z^s, z^c, z^u)$ the points of $\mathcal{E}$, B_r^s, B_r^c, B_r^u the balls of radius r around the origin and $\mathcal{D}(A^s), \mathcal{D}(A^c), \mathcal{D}(A^u)$ the domains of the operators A^s, A^c, A^u. Using these notations, we have:

Theorem 4.1. (Local center manifold theorem)
Assume that the conditions A1', A2, A3' above are fulfilled. Then, for sufficiently small $r > 0$, *there is a* C^k *map* $h : B_r^c \to B_r^s \times B_r^u$ *with* $h(0) = 0$, $Dh(0) = 0$, *whose graph* $\mathcal{V} \subset B_r$ (the local center manifold) *has the following properties:*

a) *(Invariance) For all* $z_0 \in \mathcal{V}$ *such that* $z_0^c \in \mathcal{D}(A^c)$, *there exists a* C^1 *curve* $z : \mathbf{R} \to \mathcal{E}$ *with* $z(0) = z_0$ *such that, as long as* $z(t) \in B_r$, $z(t) \in \mathcal{V}$ *and Eq.(1.1) holds. If moreover* $z(t) \in B_r$ *for all* $t \in \mathbf{R}$, *then* $z(t)$ *is the unique solution of Eq.(1.1) with these properties.*
b) *(Uniqueness) If* $z(t)$ *is any solution of Eq.(1.1) such that* $z(t) \in B_r$ *for all* $t \in \mathbf{R}$, *then* $z(t) \in \mathcal{V}$ *for all* $t \in \mathbf{R}$.

In the case where the whole space $\mathcal{E}$ has the C^k extension property, an economic way to prove this theorem is to use Theorem 1.1 twice. First, one shows the existence of a local (center-stable) manifold $\mathcal{V}^{cs}$ tangent to the invariant subspace $\mathcal{E}^c \oplus \mathcal{E}^s$. Then, reversing the sign of the time t in Eq.(1.1), one applies Theorem 1.1 again to obtain a local manifold $\mathcal{V}^{cu}$ tangent to the invariant subspace $\mathcal{E}^c \oplus \mathcal{E}^u$. The local center manifold $\mathcal{V}$ is simply given by $\mathcal{V}^{cs} \cap \mathcal{V}^{cu}$.

In the general case, one has to repeat the whole proof of Theorem 1.1, with suitable modifications. This is a straightforward (but somewhat lengthy) task, and we shall not go into details. Essentially, in Sect. 2.1, one has to deal with curves $z(t)$ defined for all $t \in \mathbf{R}$ and verifying, instead of Eq.(2.2),

$$\begin{aligned} z^s(t) &= \int_{-\infty}^{t} e^{A^s(t-\tau)} f^s(z(\tau))\, d\tau \;, \\ z^c(t) &= e^{A^c t}\xi + \int_0^t e^{A^c(t-\tau)} f^c(z(\tau))\, d\tau \;, \\ z^u(t) &= -\int_t^{\infty} e^{-A^u(\tau-t)} f^u(z(\tau))\, d\tau \;, \end{aligned} \tag{2.2'}$$

with $\xi = z^c(0) \in \mathcal{E}^c$. Similarly, in Sect. 2.2, one has to show the existence of a map $h : \mathcal{E}^c \to \mathcal{E}^s \oplus \mathcal{E}^u$ and a flow $\phi_t : \mathbf{R} \times \mathcal{E}^c \to \mathcal{E}^c$ satisfying, instead of Eq.(2.4),

$$\begin{aligned} \phi_t(\xi) &= e^{A^c t}\xi + \int_0^t e^{A^c(t-\tau)} f^c(h^s(\phi_\tau(\xi)), \phi_\tau(\xi), h^u(\phi_\tau(\xi)))\, d\tau \;, \\ h^s(\xi) &= \int_{-\infty}^{0} e^{-A^s\tau} f^s(h^s(\phi_\tau(\xi)), \phi_\tau(\xi), h^u(\phi_\tau(\xi)))\, d\tau \;, \\ h^u(\xi) &= -\int_0^{\infty} e^{-A^u\tau} f^u(h^s(\phi_\tau(\xi)), \phi_\tau(\xi), h^u(\phi_\tau(\xi)))\, d\tau \;. \end{aligned} \tag{2.4'}$$

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