NORMAL FORMS FOR PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. We begin a study of normal form theorems for parabolic partial differential equations. We show that despite the presence of resonances one can construct a partial normal form for perturbations of the Ginzburg-Landau equation. The normal form transformation is expressed in terms of singular integral operators, whose behavior can be controlled in the appropriate function spaces.

1. Introduction

In hydrodynamics, as in many other phenomena described by partial differential equations, simplified equations appear naturally as descriptions of bifurcations. These simplified equations usually describe "amplitudes" or other envelope functions on time and space scales which are related to the bifurcation parameter. An example where this reduction can be proved with mathematical rigor is provided by the bifurcation near the instability threshold of the Swift-Hohenberg equation [CE1, CE2]: This equation is

$$\partial_t u \;=\; \bigl(\alpha - (1+\partial_x^2)^2\bigr)u - u^3 \;, \tag{1.1}$$

where $u = u(x,t)$, $u : \mathbf{R} \times \mathbf{R}^+ \to \mathbf{R}$. If one rescales the solutions as

$$u(x,t) \;=\; \epsilon \mathrm{Re}\; v(\epsilon x, \epsilon^2 t)e^{ix} \;, \tag{1.2}$$

with $\epsilon = \sqrt{\alpha/3}$, then the function v satisfies the equation

$$\partial_t v \;=\; (4\partial_x^2 + 3 - 4i\epsilon\partial_x^3 - \epsilon^2\partial_x^4)v - 3v|v|^2 - e^{2ix/\epsilon}v^3 \;, \tag{1.3}$$

when $\alpha > 0$. One can then show that for times of order $\mathcal{O}(1)$ in (1.3), or, equivalently, for times of order $\mathcal{O}(\epsilon^{-2})$ in (1.1), the evolution given by (1.3) is close to that of

$$\partial_t w \;=\; (4\partial_x^2 + 3 - \epsilon^2\partial_x^4)w - 3w|w|^2 \;, \tag{1.4}$$

when $w(x,0) = v(x,0)$. Thus, (1.4) is some sort of *normal form* for this type of bifurcation. We may thus ask in which sense the solutions of equations such as (1.4) are dynamically equivalent. Is the term $\epsilon^2\partial_x^4$ negligible? Do higher order nonlinear terms matter in (1.1)?

The aim of the present paper is to discuss the normal forms of some parabolic PDE's. We consider the equation

$$\partial_t v \;=\; \partial_x^2 v + \mu v - v^3 - \epsilon R(v) \;, \tag{1.5}$$

where R is a polynomial whose terms are all of degree 4 or higher. We ask whether there is a coordinate transformation (in function space), $H : v \mapsto H(v)$, depending on ϵ which *eliminates* R. Furthermore, we require H to exist *uniformly* in μ when μ is greater than or equal to zero and not too large. We shall show that the lowest order monomial of degree $n \geq 4$ in R can indeed be eliminated by a coordinate transformation, leading to a new remainder term Q, whose lowest order term is of degree $n' > n$. We would like to iterate the procedure, eliminating the lowest order term of Q, but this raises new difficulties, since Q contains terms which are not pure powers of v (as in R), but convoluted kernels. We have only preliminary results in the direction of eliminating further terms.

Before stating our result in detail, it is perhaps useful to relate it to the existing literature on normal forms. Normal forms for diffeomorphisms and flows have been studied extensively, see [PD] for a review of the literature. Our result can viewed as a first step towards an analogue of the Sternberg Linearization Theorem for PDE's of the form (1.5). In *finite* dimension, the Sternberg Linearization Theorem asserts that $\mathcal{C}^\infty$ vector fields $X : \mathbf{R}^s \to \mathbf{R}^s$ with $X(0) = 0$ can be linearized by a $\mathcal{C}^\infty$ local diffeomorphism if $DX(0)$ satisfies the "eigenvalue condition" (nonresonance condition) [N,p.38] : The eigenvalues λ_i, $i = 1, \ldots, s$ of $DX(0)$ satisfy

$$\lambda_i \neq m_1\lambda_1 + m_2\lambda_2 + \cdots + m_s\lambda_s \ , \tag{1.6}$$

for all choices of positive integers m_i satisfying $2 \leq \sum m_i$. In the infinite dimensional case, there is continuous spectrum, and more importantly, the analogue of the condition (1.6) is violated, since the continuous spectrum of the Laplacian is $\lambda_k = -k^2$, $k \in \mathbf{R}$ and it is easy to find values of k_1, k_2, k_3 for which $\lambda_{k_1} = \lambda_{k_2} + \lambda_{k_3}$.

In the case of ordinary differential equations, violation of the non-resonance condition (1.6) produces zero denominators in terms in the sum which defines the normal form transformation. (And hence, renders the transformation meaningless.) However, due to the presence of continuous spectrum, the normal form transformation is defined in terms of integrals rather than sums. Under certain conditions, we can integrate right through the resonances. This can be done because the singularity is relatively weak and the phase space volume over which we integrate is sufficiently large if the term we want to eliminate is of order 4 or higher. Note that there are weaker sorts of normal form theorems than the Sternberg theorem. For instance, the Grobman-Hartmann theorem requires only that the linear vector field be hyperbolic, at the cost of being able to conclude only that the function conjugating the vector field to the normal form is only continuous. However, even such a result (naively) fails here since for $\mu \geq 0$, zero is in the spectrum of the linearized version of (1.5).

Another fact which is often encountered in finite dimensional normal form theorems is that while it may be impossible to reduce a system to a linear normal form, there may nonetheless be a useful nonlinear normal form. The most common example of this phenomenon is a hamiltonian vector field near a hyperbolic fixed point. Relationships among the eigenvalues which are inherent in the hamiltonian nature of the problem mean that (1.6) is inevitably violated. Nonetheless one can still (often) transform the system to the Birkhoff normal form. We encounter a similar situation here. Our normal form is not linear, but contains a cubic term. Technically, the reason for this is that the integrals referred to in the previous paragraph diverge if one tries to eliminate terms of order three. On a more fundamental level, these terms cannot be eliminated because it is known that if $\mu = 0$ in (1.5), the asymptotic behavior of the solutions of the linear and non-linear equations is different, and hence they cannot be conjugate.

We conclude this introduction by reviewing some related work. An interesting paper dealing with continuous spectrum is [S], which deals with Klein-Gordon equations. In that case, one finds essentially $\lambda_k = \sqrt{k^2 + m^2}$, with $m > 0$ and such eigenvalues always satisfy the condition (1.6). Two further related papers are [Se] and [D]. In [D], the discrete Laplacian is considered in (1.5), and then μ is required to be sufficiently large as a function of the discretization parameter. In [Se], the Laplacian is not discrete but it is multiplied by a non-real coefficient which, again, prevents the appearance of resonances. This is similar to earlier work [Ni] on the nonlinear Schrödinger equation.

After a first version of this paper had been completed, the related paper of McKean and Shatah [MS] came to our attention. They also study normal forms of partial differential equations. In the result closest to ours, they show that if in (1.5) one sets $\mu = 0$, and if the cubic term is not present, then a nonlinearity $\epsilon R(u) = \epsilon u^p$ can be completely eliminated if $p \geq 4$, *i.e.* under these hypotheses one can conjugate the equation to a linear normal form. In contrast to our use of singular integrals to construct the normal form, they rely on an infinite dimensional analogue of the scattering method proof of the Sternberg theorem. In our example, this method seems unfortunately inapplicable because we do not understand sufficiently well the asymptotic behavior of the solutions of the normal form equation. Furthermore, in the case $\mu > 0$, one would need a stable manifold theorem for partial differential equations in order to apply this method, and that result is also unavailable.

2. The Conjugation

We next describe the derivation of an equation for conjugation, in momentum space. We define the operator P by

$$(Pv)(k) \;=\; kv(k)\ .$$

We want to transform the problem (as expressed in momentum space)

$$\partial_t v \;=\; -(P^2-\mu)v - v^{*3} + U_n(v)\ ,$$

by a transformation $\mathcal{T}$ given by

$$\mathcal{T}v \;=\; v + H_n(v)\ .$$

Here, U_n is of degree $n \geq 4$ and the effect of the transformation should be to replace it with terms of degree ≥ 5. The notation v^{*3} stands for threefold convolution.

Our aim is to determine H_n. We now assume that U_n is homogeneous of degree $n \geq 4$, and we require that H_n to be homogeneous of the same degree. The transformation H_n found in this way will be adequate for any polynomial whose lowest order homogeneous term is equal to U_n. Let $w = \mathcal{T}v$. Then

$$\begin{aligned}\partial_t w \;&=\; \partial_t v + DH_n(v)\partial_t v \\ &=\; -(P^2-\mu)v - v^{*3} + U_n(v) - DH_n(v)\cdot(P^2-\mu)v - DH_n(v)v^{*3} + DH_n(v)U_n(v).\end{aligned} \tag{2.1}$$

Hence

$$\begin{aligned}\partial_t w + (P^2-\mu)w + w^{*3} \;=\;& U_n(v) + (P^2-\mu)H_n(v) - DH_n(v)(P^2-\mu)v \\ &+\; 3v^{*2} * H_n(v) - DH_n(v)\cdot v^{*3} \\ &+ DH_n(v)U_n(v) \\ &+ 3v * \big(H_n(v)\big)^{*2} \\ &+ H_n(v)^{*3}\ .\end{aligned} \tag{2.2}$$

The terms on the r.h.s. in (2.2) have been arranged by degrees and the five lines correspond to degrees n, $n+2$, $2n-1$, $2n+1$, and $3n$, respectively. We can eliminate the terms of degree n in Eq.(2.2) by setting

$$U_n(v) + (P^2-\mu)H_n(v) - DH_n(v)(P^2-\mu)v \;=\; 0\ . \tag{2.3}$$

This is our main equation, and we now rewrite it in more explicit form. Since U_n is typically v^{*n}, we will restrict our attention to operators of the form

$$U_n(v)(k) = \int \delta(k - \sum_{i=1}^{n} p_i) u(k, p_1, \ldots, p_n) \prod_{i=1}^{n} v(p_i) dp_i \ , \tag{2.4}$$

and try to construct H_n in the same form, i.e.,

$$H_n(v)(k) = \int \delta(k - \sum_{i=1}^{n} p_i) h(k, p_1, \ldots, p_n) \prod_{i=1}^{n} v(p_i) dp_i \ . \tag{2.5}$$

Despite the functional notation, the kernels u, h, etc., should be thought of as tempered distributions (of some relatively mild type) on the manifold defined by $k = \sum p_i$. In terms of these kernels, Eq.(2.3) is equivalent to

$$u(k, p_1, \ldots, p_n) = \Big(\sum_{i=1}^{n} p_i^2 - k^2 - (n-1)\mu\Big) h(k, p_1, \ldots, p_n) \ . \tag{2.6}$$

Conversely, if h is a solution of this division problem which defines an operator $v \mapsto H_n(v)$ on the space of functions v of interest to us, then the main equation (2.3) will hold. It is always possible to define, for $\varepsilon \neq 0$,

$$h_\varepsilon(k, p_1, \ldots, p_n) = \frac{1}{\sum_{j=1}^{n} p_j^2 - k^2 - (n-1)\mu - i\varepsilon} u(k, p_1, \ldots, p_n) \ , \tag{2.7}$$

as a tempered distribution on the manifold $k = \sum p_i$, as well as the function

$$H_{n,\varepsilon}(k) = \int \frac{\delta(k - \sum_{j=1}^{n} p_j)}{\sum_{j=1}^{n} p_j^2 - k^2 - (n-1)\mu - i\varepsilon} u(k, p_1, \ldots, p_n) \prod_{j=1}^{n} v(p_j) dp_j \ , \tag{2.8}$$

since this just amounts to multiplying u by a $\mathcal{C}^\infty$ function and integrating. If the limit of h_ε as $\varepsilon \downarrow 0$ exists as a tempered distribution on the manifold defined by $k = \sum p_i$, it is a solution of the division problem (2.6), and it only remains to see whether the limit $H_n = \lim_{\varepsilon \downarrow 0} H_{n,\varepsilon}$ defines a continuous operator on the right space.

There are now two problems we want to address:

i) On which space of functions can the map $v \mapsto H_n(v)$ be defined?

ii) Is the map $v \mapsto v + H_n(v)$ invertible?

3. Statement of Results

We want to study H_n for the case $u \equiv 1$. We change variables of integration in the Eq.(2.8) to $p_i = k/n + q_i$. Then,

$$\begin{aligned} H_n(v) \;&=\; \lim_{\epsilon \downarrow 0} H_{n,\epsilon}(v) \;, \\ H_{n,\epsilon}(v)(k) \;&=\; \int_0^\infty d\rho \frac{2\rho}{\rho^2 - k^2(1-1/n) - (n-1)\mu - i\epsilon} W_n(k,\rho,v) \;, \end{aligned}$$

where

$$W_n(k,\rho,v) \;=\; \int \delta(\rho^2 - \sum_{i=1}^n q_i^2)\delta(\sum_{i=1}^n q_i) \prod_{i=1}^n v(\tfrac{k}{n} + q_i) dq_i \;. \tag{3.1}$$

It is also convenient to define, for $\rho > 0$ and real k, the corresponding linear functional

$$V_n(k,\rho,\varphi) \;=\; \int \delta(\rho^2 - \sum_{i=1}^n q_i^2)\delta(\sum_{i=1}^n q_i)\, \varphi(\tfrac{k}{n} + q_1, \ldots, \tfrac{k}{n} + q_n)\, \textstyle\prod_{i=1}^n dq_i \;. \tag{3.2}$$

Thus

$$W_n(k,\rho,v) \;=\; V_n(k,\rho, v \otimes \ldots \otimes v) \;.$$

With these notations, we have

$$H_n(v)(k) \;=\; F_n(k, \sqrt{(n-1)(k^2/n + \mu) + i0}, v) \;, \tag{3.3}$$

where

$$F_n(k,\zeta,v) \;=\; \int_0^\infty \frac{2\rho}{\rho^2 - \zeta^2} W_n(k,\rho,v) d\rho \;. \tag{3.4}$$

Throughout, we consider the norms

$$\begin{aligned} |v|_\beta \;&=\; \sup_{p \in \mathbf{R}} (1+p^2)^\beta |v(p)| \;, \\ \|v\|_\beta \;&=\; \max\{|v|_\beta, |v'|_\beta\} \;. \end{aligned} \tag{3.5}$$

We call $\mathcal{V}_\beta$ and $\mathcal{V}'_\beta$ the corresponding Banach spaces of complex functions. Note that functions in $\mathcal{V}'_\beta$ need not be continuously differentiable. In particular, for $\mu = 0$, the function $k \mapsto H_n(v)(k)$ defined by (3.3) has, in general, a discontinuous derivative at $k = 0$, even when v is $\mathcal{C}^\infty$. In Appendix B we show, by an explicit calculation, that this happens even for the case of a Gaussian v. For functions of n real variables, we define analogously

$$
\begin{aligned}
|\varphi|_{n,\beta} &= \sup_{p_1,\ldots,p_n\in\mathbf{R}} |\varphi(p_1,\ldots,p_n)| \prod_{i=1}^{n}(1+p_i^2)^\beta \;, \\
\|\varphi\|_{n,\beta} &= \max_{0\le\alpha_i\le 1} \; \sup_{p_1,\ldots,p_n\in\mathbf{R}} |D^\alpha\, \varphi(p_1,\ldots,p_n)| \prod_{i=1}^{n}(1+p_i^2)^\beta \;.
\end{aligned}
\tag{3.6}
$$

We denote $\mathcal{V}_{n,\beta}$ and $\mathcal{V}'_{n,\beta}$ the corresponding Banach spaces.

Theorem 3.1. *Suppose $v \in \mathcal{V}'_\beta$ for some $\beta > 1/2$. Then for all $n \ge 4$, the function $k \mapsto \big(H_n v\big)(k) = F_n(k, \sqrt{(n-1)(k^2/n+\mu)+i0}, v)$ is continuous on $\mathbf{R}$. If $\mu > 0$, this function has a Hölder continuous derivative on $\mathbf{R}$. If $\mu = 0$, its restrictions to $[0,\infty)$ and to $(-\infty,0]$ have Hölder continuous derivatives. In these two cases, there is a constant B_n such that*

$$
\|H_n(v)\|_\beta \le B_n \|v\|_\beta^n \;. \tag{3.7}
$$

Remark. If $\mu > 0$, the linear part of the r.h.s. of Eq.(1.5) is unstable, for $\mu = 0$ it is marginally stable, and, for $\mu < 0$ it is globally stable. In this last case, the function $u = 0$ is a stable fixed point in $L^2 \cap L^\infty$. Our theorem covers the unstable and marginally stable cases in a uniform setting.

The operator H_n, as defined above does not map real functions into real functions. But if v is real, the operator defined by $v \mapsto \operatorname{Re} H_n(v)$ is also a solution to our problem. The proof of Theorem 3.1 will be based on inductive bounds on the functions W_n and V_n in Section 4. These bounds will then be combined with estimates on the integral (3.4) in Section 5.

Corollary 3.2. *Suppose $v \in \mathcal{V}'_\beta$ for some $\beta > 1/2$ and $\mu \ge 0$. Then for all $n \ge 4$, the map $v \mapsto v + H_n(v)$ is bounded and invertible on a small ball in $\mathcal{V}'_\beta$, centered at the origin.*

Proof. It follows at once from Theorem 3.1 that, on a sufficiently small ball centered at the origin in $\mathcal{V}'_\beta$, the map $v \mapsto H_n(v)$ is a contraction.

Remark. In this paper, the function $\zeta \mapsto \log(\zeta)$ is always understood as being defined in $\mathbf{C} \setminus \mathbf{R}_-$ and real for $\zeta > 0$. Similarly $\sqrt{\zeta} = \exp(\log(\zeta)/2)$ is positive on $\mathbf{R}_+$.

4. Phase Space Bounds

We assume throughout $\beta > 1/2$. The bounds on W_n and V_n are summarized in

Proposition 4.1. *For every $n \geq 3$, there is a constant C_n such that, for any $v \in \mathcal{V}_\beta$,*

$$|W_n(k,\rho,v)| \leq \frac{C_n \rho^{n-3}}{(1+\rho)^{n-3}(1+k^2+\rho^2)^{\beta+1/2}} |v|_\beta^n \ . \tag{4.1}$$

If $\varphi \in \mathcal{V}_{n,\beta}$, then

$$|V_n(k,\rho,\varphi)| \leq \frac{C_n \rho^{n-3}}{(1+\rho)^{n-3}(1+k^2+\rho^2)^{\beta+1/2}} |\varphi|_\beta \ . \tag{4.2}$$

Proof of Proposition 4.1. Throughout, the letters C, C_n denote constants which can vary between equations, and which may depend on β, but not on v. ¿From Eq.(3.2), we have for $\rho > 0$, $n > 1$, by a change of variables,

$$W_n(k,\rho,v) = \rho^{n-3} \int \delta(1 - \sum_{i=1}^n z_i^2)\, \delta(\sum_{i=1}^n z_i) \prod_{i=1}^n v(\tfrac{k}{n} + \rho z_i) dz_i \ , \tag{4.3}$$

$$V_n(k,\rho,\varphi) = \rho^{n-3} \int \delta(1 - \sum_{i=1}^n z_i^2)\, \delta(\sum_{i=1}^n z_i) \varphi(\tfrac{k}{n} + \rho z_1, \ldots, \tfrac{k}{n} + \rho z_n) \prod_{i=1}^n dz_i. \tag{4.4}$$

In particular,

$$W_n(k,\rho,1) \ = \ \rho^{n-3} L_n \ ,$$

with

$$L_n \ = \ \int \delta(1 - \sum_{i=1}^n z_i^2)\, \delta(\sum_{i=1}^n z_i) \prod_{i=1}^n dz_i \ . \tag{4.5}$$

Clearly, L_n is nothing but the volume of the $n-2$ dimensional unit sphere. Thus, if v is bounded, then

$$|W_n(k,\rho,v)| \leq \rho^{n-3} L_n \|v\|_{L^\infty}^n \ . \tag{4.6}$$

This bound is valid for all k. However, if $|k| > 2n\rho$, then in the integrand of (4.3), the argument of every v has modulus at least $|k|/2n$, and therefore $v \in \mathcal{V}_\beta$ implies

$$|W_n(k,\rho,v)| \le \rho^{n-3} L_n (1+k^2/4n^2)^{-n\beta} |v|_\beta^n . \tag{4.7}$$

Note also that if $\varphi \in \mathcal{V}_{n,\beta}$, $n \ge 2$,

$$|V_n(k,\rho,\varphi)| \le W_n(k,\rho,u)|\varphi|_\beta , \quad u(p) = (1+p^2)^{-\beta} , \tag{4.8}$$

so that Eq.(4.2) follows from Eq.(4.1).

In view of the bounds (4.6) and (4.7), we can distinguish the two cases $\rho \le 1$ and $\rho > 1$.

Case 1: $\rho \le 1$**.** Combining (4.6) and (4.7) yields Eq.(4.1) for $\rho \le 1$. The proof of Eq.(4.2) is then obvious in the case $\rho \le 1$.

Case 2: $\rho > 1$**.** In this case, we proceed by induction on n, starting from $n = 2$. From Eq.(3.2), we have

$$V_2(k,\rho,\varphi) = \tfrac{\sqrt{2}}{4\rho}\varphi(\tfrac{k}{2}+\tfrac{\rho}{\sqrt{2}}, \tfrac{k}{2}-\tfrac{\rho}{\sqrt{2}}) + \tfrac{\sqrt{2}}{4\rho}\varphi(\tfrac{k}{2}-\tfrac{\rho}{\sqrt{2}}, \tfrac{k}{2}+\tfrac{\rho}{\sqrt{2}}) . \tag{4.9}$$

Hence,

$$|V_2(k,\rho,\varphi)| \le \frac{2^{2\beta-1/2}}{\rho(1+k^2+\rho^2)^\beta}|\varphi|_\beta . \tag{4.10}$$

To simplify the notation, we formulate our further calculations for functions φ of the form $\varphi = v_1 \otimes \ldots \otimes v_n$. The extension to general φ is immediate. Starting from (3.2) for $n+1$ instead of n, and writing $q_j = y_j - q/n$ for $j \le n$, and $q_{n+1} = q$, we find:

$$\begin{aligned} &V_{n+1}(k,\rho,v_1 \otimes \ldots \otimes v_{n+1}) \\ &= \int_{-\sqrt{\frac{n}{n+1}}\rho}^{\sqrt{\frac{n}{n+1}}\rho} dq\, V_n\left(\tfrac{nk}{n+1} - q, \sqrt{\rho^2 - \tfrac{n+1}{n}q^2},\ v_1 \otimes \ldots \otimes v_n\right) v_{n+1}\left(\tfrac{k}{n+1}+q\right) , \end{aligned} \tag{4.11}$$

and, by a further change of variables,

$$\begin{aligned} &V_{n+1}(k,\rho,v_1 \otimes \ldots \otimes v_{n+1}) \\ &= \rho\sqrt{\tfrac{n}{n+1}} \int_{-1}^{1} dt\, V_n\left(\tfrac{nk}{n+1} - \sqrt{\tfrac{n}{n+1}}\rho t, \rho\sqrt{1-t^2}, v_1 \otimes \ldots \otimes v_n\right) v_{n+1}\left(\tfrac{k}{n+1} + \sqrt{\tfrac{n}{n+1}}\rho t\right) . \end{aligned} \tag{4.12}$$

Before using Eqs.(4.11) and (4.12) to obtain iterative bounds on W_n, we note a simplifying effect of the symmetry of Eq.(3.1). The integral of Eq.(4.3) is also equal to n times the integral over the region where $z_j \le z_n$. In this region, because of the constraint $\sum_{i=0}^{n} z_i = 0$, z_n must be strictly positive. Suppose $z_n \le x$ and that there are p of the z_j taking values > 0: these add up to at most px. The other variables add up to the opposite amount, so that the sum of their squares is at most $(px)^2$. Hence we have $1 = \sum z_i^2 \le (p+1)px^2 \le n(n-1)x^2$. Thus, in this region, $z_n^2 \ge \frac{1}{n(n-1)}$. Therefore,

$$\begin{aligned}&|W_{n+1}(k,\rho,v)| \\ &\le \rho^{n-2}(n+1)\int_{z_{n+1}\ge 1/\sqrt{n(n+1)}} \delta(1-\sum_{i=1}^{n+1} z_i^2)\delta(\sum_{i=1}^{n+1} z_i)\prod_{i=1}^{n+1}|v|(\tfrac{k}{n+1}+\rho z_i)dz_i \ .\end{aligned} \tag{4.13}$$

Going through the same changes of variables as before, we obtain:

$$\begin{aligned}&|W_{n+1}(k,\rho,v)| \\ &\le \rho\sqrt{n(n+1)}\int_{1/n}^{1} dt\, W_n\left(\tfrac{nk}{n+1}-\sqrt{\tfrac{n}{n+1}}\rho t, \rho\sqrt{1-t^2}, |v|\right)|v|\left(\tfrac{k}{n+1}+\sqrt{\tfrac{n}{n+1}}\rho t\right) \ .\end{aligned} \tag{4.14}$$

In view of (4.8), it suffices to prove, for $\rho > 1$,

$$W_n(k,\rho,u) \le \frac{C_n}{(1+k^2+\rho^2)^{\beta+1/2}} \ , \tag{4.15}$$

where $u(p) = (1+p^2)^{-\beta}$. The corresponding result for V_n is then obvious. Note also that $W_n(k,\rho,v) = W_n(-k,\rho,\widetilde{v})$, with $\widetilde{v}(p) = v(-p)$. It suffices therefore to consider only $k \ge 0$.

We perform now an inductive step from $n \ge 3$ to $n+1$. The step from $n = 2$ to $n = 3$ will be dealt with later. Suppose that (4.1) holds for some $n \ge 3$. To prove that it holds for $n+1$, we use Eq.(4.14), substituting the postulated bound for $|W_n|$, and set $k = (n+1)a$, $\rho = b\sqrt{(n+1)/n}$. Some elementary estimates on the resulting integral, and variants of these estimates we will need later are stated in the following lemma.

Lemma 4.2. *Let $\beta > \frac{1}{2}$ and $\gamma \ge \beta + 1/2$. For any $n \ge 2$ there is a constant M_n such that for all $a > 0$, $b > 0$,*

$$I_n(a,b) = \int_{1/n}^{1} dt \frac{b}{[1+(na-bt)^2+b^2(1-t^2)]^\gamma[1+(a+bt)^2]^\beta}$$

$$\leq \frac{M_n}{(1+a^2+b^2)^{\beta+1/2}} \ , \tag{4.16}$$

$$J_n(a,b) = \int_{1/n}^{1} dt \, \frac{1}{\sqrt{1-t^2}[1+(na-bt)^2+b^2(1-t^2)]^\beta[1+(a+bt)^2]^\beta}$$

$$\leq \frac{M_n}{(1+a^2+b^2)^{\beta+1/2}} \ , \tag{4.17}$$

$$K_n(a,b) = \int_{-1}^{1} dt \, \frac{b}{\sqrt{1-t^2}[1+(na-bt)^2+b^2(1-t^2)]^\gamma[1+(a+bt)^2]^\beta}$$

$$\leq \frac{M_n}{(1+a^2+b^2)^{\beta}} \ . \tag{4.18}$$

Proof. The proofs are given in Appendix A.

It is now clear that Eq.(4.16) proves the induction step $n \geq 3 \to n+1$. We next consider the step from $n = 2$ to $n = 3$. Using the bounds Eq.(4.10) and (4.14), we get the desired estimate from Eq.(4.17).

Bounds on First Derivatives

In addition to the bounds on W_n and V_n of Proposition 4.1, we shall also need bounds on their first and second derivatives. W_n and V_n. Since

$$\partial_k W_n(k,\rho,v) = V_n(k,\rho,v\otimes\ldots\otimes v\otimes v') \ , \tag{4.19}$$

we have immediately from Proposition 4.1 and Eq.(4.10):

Corollary 4.3. *For all $n \geq 3$, there is a constant C_n such that, whenever v and v' are both in $\mathcal{V}_\beta$,*

$$|\partial_k W_n(k,\rho,v)| \leq \frac{C_n \rho^{n-3}}{(1+\rho)^{n-3}(1+k^2+\rho^2)^{\beta+1/2}} |v|_\beta^{n-1}|v'|_\beta \ . \tag{4.20}$$

For $n = 2$,

$$|\partial_k W_2(k,\rho,v)| \leq \frac{C_n}{\rho(1+k^2+\rho^2)^{\beta}} |v|_\beta|v'|_\beta \ . \tag{4.21}$$

We consider next the ρ-derivative. For all $n \geq 3$, we get from (4.3),

$$\begin{aligned}\partial_\rho W_n(k,\rho,v) &= \tfrac{n-3}{\rho} W_n(k,\rho,v)\\ &+ n\rho^{n-3}\int \delta(1-\sum_{i=1}^n z_i^2)\delta(\sum_{i=1}^n z_i)\prod_{i=1}^{n-1} v(\tfrac{k}{n}+\rho z_i)dz_i \cdot z_n \cdot v'(\tfrac{k}{n}+\rho z_n)dz_n \ .\end{aligned} \tag{4.22}$$

The second term is bounded in modulus by $nV_n(|v| \otimes \ldots \otimes |v| \otimes |v'|)$. Therefore, using Eq.(4.2), we find:

Proposition 4.4. *For all $n \geq 4$ there is a constant K'_n such that, whenever v and v' are both in $\mathcal{V}_\beta$,*

$$|\partial_\rho W_n(k,\rho,v)| \leq \frac{K'_n \rho^{n-4}}{(1+\rho)^{n-4}(1+k^2+\rho^2)^{\beta+1/2}} |v|_\beta^{n-1} |v'|_\beta \ . \tag{4.23}$$

For $n = 3$,

$$|\partial_\rho W_3(k,\rho,v)| \leq \frac{3C_3}{(1+k^2+\rho^2)^{\beta+1/2}} |v|_\beta^2 |v'|_\beta \ . \tag{4.24}$$

Bounds on Second Derivatives

We shall assume throughout that $v,\ v' \in \mathcal{V}_\beta$. We start by recasting Eq.(4.22) in the following "inductive" form, valid for $n \geq 2$:

$$\begin{aligned} \partial_\rho W_{n+1}(k,\rho,v) \ &= \ \tfrac{n-2}{\rho} W_{n+1}(k,\rho,v) \\ &+ (n+1) \int_{-\sqrt{\frac{n}{n+1}}\rho}^{\sqrt{\frac{n}{n+1}}\rho} dq \ W_n \left(\tfrac{nk}{n+1} - q, \sqrt{\rho^2 - \tfrac{n+1}{n} q^2},\ v \right) \frac{q}{\rho}\ v' \left(\tfrac{k}{n+1} + q \right) \ . \end{aligned} \tag{4.25}$$

– **Bound on** $\partial_\rho^2 W_n$

We now take $n \geq 3$ and differentiate in ρ. This yields several terms which we call $T_1 + \ldots + T_6$.

1) We consider the first term in (4.25) reexpressed with the help of (4.3) for $n+1$:

$$\frac{n-2}{\rho} W_{n+1}(k,\rho,v) \;=\; (n-2)\rho^{n-3} \int \delta(1-\sum_{i=1}^{n+1} z_i^2)\, \delta(\sum_{i=1}^{n+1} z_i) \prod_{i=1}^{n+1} v(\tfrac{k}{n+1}+\rho z_i) dz_i \;.$$

Differentiating with respect to ρ, this gives two terms, T_1 and T_2:

$$T_1 \;=\; (n-2)(n-3)\rho^{-2} W_{n+1}(k,\rho,v) \;, \tag{4.26}$$

$$\begin{aligned} T_2 \;=\; & (n-2)(n+1)\rho^{n-3} \int \delta(1-\sum_{i=1}^{n+1} z_i^2)\delta(\sum_{i=1}^{n+1} z_i) \\ & \times \prod_{i=1}^{n} v(\tfrac{k}{n+1}+\rho z_i) dz_i \cdot z_{n+1} \cdot v'(\tfrac{k}{n+1}+\rho z_{n+1}) dz_{n+1} \;. \end{aligned} \tag{4.27}$$

The first term is bounded using Eq.(4.1), and yields

$$|T_1| \;\leq\; \frac{(n-2)(n-3) C_{n+1} \rho^{n+1-5}}{(1+\rho)^{n+1-3}(1+k^2+\rho^2)^{\beta+1/2}} \,|v|_\beta^{n+1} \;.$$

Since $|z_{n+1}| \leq 1$, the term T_2 can be bounded using Eq.(4.2), yielding

$$|T_2| \;\leq\; \frac{(n-2)(n+1) C_{n+1} \rho^{n+1-4}}{(1+\rho)^{n+1-3}(1+k^2+\rho^2)^{\beta+1/2}} \,|v|_\beta^{n}\,|v'|_\beta \;.$$

Since $T_1 = 0$ for $n=3$, we find

$$|T_1| + |T_2| \;\leq\; \frac{C}{(1+k^2+\rho^2)^{\beta+1/2}} \,\|v\|_\beta^{n+1} \;.$$

2) We consider now the integral in (4.25) and differentiate with respect to ρ. This gives two boundary terms, T_3, T_4, and two terms T_5, T_6 from differentiating the factor ρ^{-1} and the square root in W_n, respectively. The boundary terms are

$$\begin{aligned} T_3 \;=\; & \;\; nW_n\left(\tfrac{nk}{n+1} - \sqrt{\tfrac{n}{n+1}}\rho,\; 0, v\right)\; v'\left(\tfrac{k}{n+1} + \sqrt{\tfrac{n}{n+1}}\rho\right) \;, \\ T_4 \;=\; & -nW_n\left(\tfrac{nk}{n+1} + \sqrt{\tfrac{n}{n+1}}\rho,\; 0, v\right)\; v'\left(\tfrac{k}{n+1} - \sqrt{\tfrac{n}{n+1}}\rho\right) \;. \end{aligned} \tag{4.28}$$

Using again Eq.(4.1) we see that they vanish for $n > 3$, and when $n = 3$, the first of these two terms decreases slower at infinity, and their total contribution is bounded by

$$|T_3| + |T_4| \le \frac{C}{(1+k^2+\rho^2)^\beta} \, \|v\|_\beta^{n+1} \ .$$

The term T_5 is bounded by $(1/\rho)(n+1)V_{n+1}(k,\rho,|v\otimes\ldots\otimes v\otimes v'|)$, which by Eq.(4.1) leads to

$$|T_5| \le \frac{C}{(1+k^2+\rho^2)^{\beta+1/2}} \, \|v\|_\beta^{n+1} \ .$$

The term T_6 is equal to

$$\begin{aligned} T_6 =& (n+1) \int_{-\sqrt{\frac{n}{n+1}}\rho}^{\sqrt{\frac{n}{n+1}}\rho} dq \ \partial_\rho W_n \left(\tfrac{nk}{n+1} - q, \ \sqrt{\rho^2 - \tfrac{n+1}{n} q^2}, \ v \right) \\ & \times \frac{q}{\sqrt{\rho^2 - \frac{n+1}{n} q^2}} \, v' \left(\tfrac{k}{n+1} + q \right) \ . \end{aligned} \tag{4.29}$$

After changing variables $q = \rho t \sqrt{\frac{n}{(n+1)}}$, and using $|t| \le 1$, this term can be estimated by Eq.(4.18) and yields

$$|T_6| \le \frac{C}{(1+k^2+\rho^2)^\beta} \, \|v\|_\beta^{n+1} \ .$$

Combining the bounds, and replacing $n+1$ by n, we get the

Proposition 4.5. *For every $n \ge 4$, there is a constant C_n such that*

$$|\partial_\rho^2 W_n(k,\rho,v)| \le \frac{C_n}{(1+k^2+\rho^2)^\beta} \, \|v\|_\beta^n \ . \tag{4.30}$$

– **Bound on $\partial_k \partial_\rho W_n$**

In analogy with Proposition 4.5, we have

Proposition 4.6. *For every $n \geq 4$, there is a constant C_n such that*

$$|\partial_k \partial_\rho W_n(k,\rho,v)| \leq \frac{C_n}{(1+k^2+\rho^2)^\beta} \, \|v\|_\beta^n \,. \tag{4.31}$$

Proof. We go back to Eq.(4.25) for $n \geq 3$, and differentiate in k. This gives a sum of three terms

$$\begin{aligned}
&\partial_k \partial_\rho W_{n+1}(k,\rho,v) \\
&\quad = \tfrac{n-2}{\rho} \partial_k W_{n+1}(k,\rho,v) \\
&\quad + n \int_{-\sqrt{\frac{n}{n+1}}\rho}^{\sqrt{\frac{n}{n+1}}\rho} dq\; \partial_k W_n \left(\tfrac{nk}{n+1} - q,\, \sqrt{\rho^2 - \tfrac{n+1}{n} q^2},\; v \right) \frac{q}{\rho}\; v' \left(\tfrac{k}{n+1} + q \right) \\
&\quad + \int_{-\sqrt{\frac{n}{n+1}}\rho}^{\sqrt{\frac{n}{n+1}}\rho} dq \quad W_n \left(\tfrac{nk}{n+1} - q,\, \sqrt{\rho^2 - \tfrac{n+1}{n} q^2},\; v \right) \frac{q}{\rho}\; v'' \left(\tfrac{k}{n+1} + q \right) \\
&\quad = S_1 + S_2 + S_3 \,.
\end{aligned}$$

The terms S_1 and S_2 are readily bounded by

$$|S_1| + |S_2| \leq \frac{C}{(1+k^2+\rho^2)^\beta} \|v\|_\beta^{n+1} \,,$$

using Corollary 4.3 for S_1, then (4.19), (4.11), and (4.2) for S_2. We now deal with S_3. Here, the v'' must be interpreted in the weak sense, i.e., it has to be integrated by parts with respect to the variable q. (Alternatively, the same expression can be obtained without ever mentioning v'', e.g., by changing to the integration variable $q + k/(n+1)$, differentiating in k, then changing back to q.) This yields:

$$S_3 = S_{31} + \ldots + S_{35} \,. \tag{4.32}$$

The term S_{31} arises, when integrating by parts, from differentiating the first argument of the W_n factor. It is equal to S_2/n. The boundary terms give S_{32} and S_{33}:

$$\begin{aligned}
S_{32} &= \sqrt{\tfrac{n}{n+1}} W_n \left(\tfrac{nk}{n+1} - \sqrt{\tfrac{n}{n+1}}\rho, 0, v \right)\; v' \left(\tfrac{k}{n+1} + \sqrt{\tfrac{n}{n+1}}\rho \right) \,, \\
S_{33} &= \sqrt{\tfrac{n}{n+1}} W_n \left(\tfrac{nk}{n+1} + \sqrt{\tfrac{n}{n+1}}\rho, 0, v \right)\; v' \left(\tfrac{k}{n+1} - \sqrt{\tfrac{n}{n+1}}\rho \right) \,.
\end{aligned}$$

These terms are handled like T_3 and T_4. They vanish for $n > 3$, and for $n = 3$ they are bounded by

$$|S_{32}| + |S_{33}| \le C(1 + k^2 + \rho^2)^{-\beta} \|v\|_\beta^{n+1} .$$

Upon integrating by parts, there is a term coming from $\partial_q q/\rho$ which is equal to

$$S_{34} = (1/\rho) V_{n+1}(k, \rho, v \otimes \ldots \otimes v \otimes v') .$$

This term is bounded by $|S_{34}| \le C(1 + k^2 + \rho^2)^{-\beta-1/2} \|v\|_\beta^{n+1}$. Finally there is a term where the derivative acts on the second argument of W_n:

$$\begin{aligned} S_{35} \;=\; & \tfrac{n+1}{n} \int_{-\sqrt{\frac{n}{n+1}}\rho}^{\sqrt{\frac{n}{n+1}}\rho} dq \; \partial_\rho W_n \left(\tfrac{nk}{n+1} - q,\; \sqrt{\rho^2 - \tfrac{n+1}{n} q^2},\; v \right) \\ & \times \frac{q}{\rho} \frac{q}{\sqrt{\rho^2 - \frac{n+1}{n} q^2}} \, v' \left(\tfrac{k}{n+1} + q \right) dq \, . \end{aligned}$$

It is estimated exactly in the same way as the term T_6 in Eq.(4.29), and is bounded by $C(1 + k^2 + \rho^2)^{-\beta} \|v\|_\beta^{n+1}$. The proof of Proposition 4.6 is complete.

– **Bound on $\partial_k^2 W_n$**

Finally, we bound the second derivative with respect to k:

Proposition 4.7. *For any $n \ge 4$, there is a constant C_n such that*

$$|\partial_k^2 W_n(k, \rho, v)| \le \frac{C_n}{(1 + k^2 + \rho^2)^\beta} \|v\|_\beta^n . \tag{4.33}$$

Proof. For $n \ge 3$, a formula for $\partial_k^2 W_{n+1}$ is obtained in the same way as in the case of $\partial_k \partial_\rho W_{n+1}$ (using formal integration by parts). This gives

$$\partial_k^2 W_{n+1}(k, \rho, v) = R_1 + \ldots + R_4 , \tag{4.34}$$

with

$$R_1 = V_{n+1}(k, \rho, v \otimes \ldots \otimes v \otimes v' \otimes v') , \tag{4.35}$$

$$
\begin{aligned}
R_2 = & \frac{1}{n} \int_{-\sqrt{\frac{n}{n+1}\rho}}^{\sqrt{\frac{n}{n+1}\rho}} \partial_\rho W_n \left(\tfrac{nk}{n+1} - q, \ \sqrt{\rho^2 - \tfrac{n+1}{n} q^2}, \ v \right) \\
& \times \frac{q}{\sqrt{\rho^2 - \frac{n+1}{n} q^2}} \ v' \left(\tfrac{k}{n+1} + q \right) dq
\end{aligned}
\tag{4.36}
$$

$$
\begin{aligned}
R_3 &= \tfrac{1}{n+1} W_n \left(\tfrac{nk}{n+1} - \sqrt{\tfrac{n}{n+1}}\rho, 0, v \right) \ v' \left(\tfrac{k}{n+1} + \sqrt{\tfrac{n}{n+1}}\rho \right), \\
R_4 &= \tfrac{-1}{n+1} W_n \left(\tfrac{nk}{n+1} + \sqrt{\tfrac{n}{n+1}}\rho, 0, v \right) \ v' \left(\tfrac{k}{n+1} - \sqrt{\tfrac{n}{n+1}}\rho \right) .
\end{aligned}
\tag{4.37}
$$

The details are now essentially the same as in the case of $\partial_k \partial_\rho$ and are left to the reader.

5. Properties of F_n

In this section, v is fixed and such that $|v|_\beta \le 1$ and $|v'|_\beta \le 1$. Recall that F_n has been defined as:

$$
\begin{aligned}
F_n(k, \zeta, v) &= \int_0^\infty \frac{2\rho}{\rho^2 - \zeta^2} W_n(k, \rho, v) d\rho \\
&= \int_0^\infty \left(\frac{1}{\rho - \zeta} + \frac{1}{\rho + \zeta} \right) \ W_n(k, \rho, v) d\rho \ .
\end{aligned}
\tag{5.1}
$$

We take $n \ge 4$, so that the bounds established in the previous section give:

$$
|W_n(k, \rho, v)| + |\partial_k W_n(k, \rho, v)| \le \frac{C\rho}{(1+\rho)(1+k^2+\rho^2)^{\beta+1/2}} \ , \tag{5.2}
$$

$$
|\partial_\rho W_n(k, \rho, v)| \le \frac{C}{(1+k^2+\rho^2)^{\beta+1/2}} \ , \tag{5.3}
$$

$$
|\partial^m W_n(k, \rho, v)| \le \frac{C}{(1+k^2+\rho^2)^{\beta}} \ \text{for } |m| = 2 \ . \tag{5.4}
$$

Because of these bounds, $\zeta \mapsto F_n(k, \zeta, v)$, as defined by Eq.(5.1), is even and holomorphic in ζ for $\mathrm{Im}\, \zeta \ne 0$, and has (not necessarily even) Hölder continuous boundary values on $\mathbf{R}$ from the upper and lower half-planes. We note the following identities:

$$
\partial_k F_n(k, \zeta, v) = \int_0^\infty \frac{2\rho d\rho}{\rho^2 - \zeta^2} \partial_k W_n(k, \rho, v) \ , \tag{5.5}
$$

$$
\partial_\zeta F_n(k, \zeta, v) = \int_0^\infty \frac{2\zeta d\rho}{\rho^2 - \zeta^2} \partial_\rho W_n(k, \rho, v) \ . \tag{5.6}
$$

We now propose to find bounds on $|F_n(k,\zeta,v)|$, $|\partial_k F_n(k,\zeta,v)|$, and $|\partial_\zeta F_n(k,\zeta,v)|$ when $\zeta = \xi + i\eta$, for real ξ and small, positive η. It suffices to deal with the case $k \ge 0$. We give all details for the case of $\partial_\zeta F_n(k,\zeta,v)$ and give indications on the changes needed for the two other cases.

We rewrite Eq.(5.6) as

$$\partial_\zeta F_n(k,\zeta,v) \;=\; \psi(\zeta)\partial_\rho W_n(k,0,v) + R_+(k,\zeta) + R_-(k,\zeta)\;, \tag{5.7}$$

$$\psi(\zeta) \;=\; \int_{-\infty}^{\infty} \frac{d\rho}{(\rho-\zeta)(1+\rho^2)^2} \;=\; -\frac{\pi(\zeta+2i)}{2(\zeta+i)^2}\;, \tag{5.8}$$

$$R_\pm(k,\zeta) \;=\; \pm\int_{-\infty}^{\infty} \frac{\varphi(k,\rho)d\rho}{\rho \mp \zeta}\;, \tag{5.9}$$

$$\varphi(k,\rho) \;=\; \theta(\rho)\left(\partial_\rho W_n(k,\rho,v) - \partial_\rho W_n(k,0,v)(1+\rho^2)^{-2}\right)\;. \tag{5.10}$$

(Here we make the inessential assumption that $\beta \le 3/2$. Otherwise the definition of ψ and φ would be changed by replacing $(1+\rho^2)^{-2}$ with $(1+\rho^2)^{-p}$ with $p \ge \beta + 1/2$.) Each of the functions $\zeta \mapsto R_\pm(k,\zeta)$ is a Cauchy transform of the Lipschitz function $\rho \mapsto \varphi(k,\rho)$, and its boundary value as $\mathrm{Im}\,\zeta \downarrow 0$ is therefore Hölder continuous (of any exponent < 1) in the variable ζ.

We now fix $\zeta = \xi + i\eta$, $\eta > 0$ and $0 < h < 1/2$. We have by Eq.(5.3):

$$|\varphi(k,\rho)| \le \frac{C}{(1+k^2+\rho^2)^{\beta+1/2}}\;. \tag{5.11}$$

For all $\alpha \in [0,1)$, combining Eq.(5.3) and (5.4), we have:

$$|\varphi(k+h,\rho) - \varphi(k,\rho)| \le \frac{Ch^{1-\alpha}}{(1+k^2+\rho^2)^{\beta+\alpha/2}}\;. \tag{5.12}$$

Finally, if $|\rho - \xi| < 1/2$, we have for all $\theta \in [0,1]$, by Eq.(5.3) and (5.4):

$$|\varphi(k,\rho) - \varphi(k,\xi)| \le \frac{C|\rho-\xi|^{1-\theta}}{(1+k^2+\rho^2)^{\beta+\theta/2}}\;, \tag{5.13}$$

and, by interpolation,

$$|\varphi(k+h,\rho) - \varphi(k,\rho) - \varphi(k+h,\xi) + \varphi(k,\xi)| \;\le\; \frac{Ch^{(1-\alpha)(1-\theta)}|\rho-\xi|^{(1-\alpha)\theta}}{(1+k^2+\rho^2)^{\beta+\alpha/2}}\;. \tag{5.14}$$

Analogous estimates can be obtained for the case of the function $F_n(k,\zeta,v)$, with φ replaced by $\theta(\rho)W_n(k,\rho,v)$ and for $\partial_k F_n(k,\zeta,v)$, with φ replaced by $\theta(\rho)\partial_k W_n(k,\rho,v)$.

Bounds on $|R_\pm(k,\zeta)|$

We choose $a \in (0, \frac{1}{2})$ and split the integral (5.9) representing $R_+(k,\zeta)$

$$\int_{-\infty}^{\infty} \frac{\varphi(k,\rho)d\rho}{\rho-\zeta}$$

into three contributions, denoted K_1, K_2, and K_3, corresponding to the ranges of integration $(-\infty, \xi-a]$, $[\xi-a, \xi+a]$, and $[\xi+a, \infty)$, respectively. Then, using the bound (5.11),

$$|K_1| \le \frac{C}{(1+k^2)^{\beta-1/2}} \int_{-\infty}^{\xi-a} \frac{d\rho}{(\xi-\rho)(1+k^2+\rho^2)} \ .$$

The last integral is equal to

$$\frac{1}{2\pi i} \int_{\mathcal{C}} d\rho \, \frac{\log(\rho-\xi+a)}{(\xi-\rho)(1+k^2+\rho^2)} \ ,$$

where the contour $\mathcal{C}$ follows the boundary of $\mathbf{C} \setminus (\xi - a + \mathbf{R}_-)$ in the positive direction. It can thus be computed by residues, yielding

$$\frac{\log a}{(1+k^2+\xi^2)} + 2\mathrm{Re}\, \frac{\log(i\sqrt{1+k^2}-\xi+a)}{(\xi-i\sqrt{1+k^2})(2i\sqrt{1+k^2})}$$

The term K_3 is treated in a similar manner. Therefore:

$$|K_1| + |K_3| \le \frac{C(1+|\log a| + \log(1+|k|) + \log(1+|\xi|)\,)}{(1+k^2)^\beta (1+k^2+\xi^2)^{1/2}} \ . \tag{5.15}$$

We next split K_2 as $K_{21} + K_{22}$:

$$K_{21} \;=\; \int_{\xi-a}^{\xi+a} \frac{\varphi(k,\rho)-\varphi(k,\xi)}{\rho-\xi} \ .$$

This is bounded by

$$\begin{aligned} |K_{21}| &\le \frac{C}{(1+k^2+\xi^2)^{\beta+\theta/2}} \int_{\xi-a}^{\xi+a} |\rho-\xi|^{-\theta} d\rho \\ &\le \frac{2Ca^{1-\theta}}{(1-\theta)(1+k^2+\xi^2)^{\beta+\theta/2}} \ . \end{aligned} \tag{5.16}$$

The term K_{22} is equal to

$$K_{22} \;=\; \varphi(k,\xi) \int_{\xi-a}^{\xi+a} \frac{d\rho}{\rho-\zeta} \;=\; \varphi(k,\xi) \log \frac{a-i\eta}{-a-i\eta} \ ,$$

so that

$$|K_{22}| \leq \frac{C}{(1+k^2+\xi^2)^{\beta+1/2}} \ . \tag{5.17}$$

The quantity $|R_-(k,\zeta)|$ is estimated in the same way. Combining these estimates, we get

$$|R_\pm(k,\xi+i\eta)| \leq \frac{C_\varepsilon}{(1+k^2)^\beta(1+k^2+\xi^2)^{1/2-\varepsilon}} \ , \tag{5.18}$$

for all sufficiently small $\varepsilon > 0$.

Hölder continuity of $k \mapsto R_+(k,\zeta)$

We apply the same treatment as above to

$$R_+(k+h,\zeta) - R_+(k,\zeta) \ = \ \int_{-\infty}^{\infty} \frac{[\varphi(k+h,\rho)-\varphi(k,\rho)]d\rho}{\rho-\zeta} \ ,$$

using now the bounds (5.12) and (5.14) instead of (5.11) and (5.13). We have to choose here $\alpha \geq 2-2\beta$. All the estimates are completely analogous to those for $R_+(k,\zeta)$ and we find that for every $\alpha \in (2-2\beta,1)$, $\kappa \in (0,1-\alpha)$ and sufficiently small $\varepsilon > 0$, there is a constant C such that

$$|R_+(k+h,\zeta) - R_+(k,\zeta)| \leq \frac{Ch^\kappa}{(1+k^2)^{\beta-(1-\alpha)/2}(1+k^2+\xi^2)^{1/2-\varepsilon}} \ . \tag{5.19}$$

The same holds for R_-. It is clear that the same kind of estimates work for $R_\pm(k,\zeta+h) - R_\pm(k,\zeta)$, with the same result. Therefore,

Theorem 5.1. *Let $n \geq 4$. The function $(k,\zeta) \mapsto F_n(k,\zeta,v)$ extends to a $\mathcal{C}^1$ function in $\mathbf{R}\times\{\zeta \,:\, \mathrm{Im}\,\zeta \geq 0\}$. For each sufficiently small $\varepsilon > 0$, there is a constant $C_n(\varepsilon)$ such that for all real k and ξ, and every bi-index m with $|m| \leq 1$,*

$$|D^m\, F_n(k,\xi+i0,v)| \ \leq \frac{C_n(\varepsilon)}{(1+k^2)^\beta(1+k^2+\xi^2)^{1/2-\varepsilon}} \ . \tag{5.20}$$

Moreover, if $0 < \kappa < 2\beta-1$, there is a $C_n'(\varepsilon,\kappa)$ such that for $|h_1|, |h_2| \leq 1/2$,

$$\begin{aligned} &|D^m\, F_n(k+h_1,\xi+h_2+i0,v) - D^m\, F_n(k,\xi+i0,v)| \\ &\quad \leq \frac{C_n'(\varepsilon,\kappa)(|h_1|^\kappa+|h_2|^\kappa)}{(1+k^2)^{\beta-\kappa/2}(1+k^2+\xi^2)^{1/2-\varepsilon}} \ . \end{aligned} \tag{5.21}$$

To prove Theorem 3.1, it suffices to apply Theorem 5.1 with $\zeta = \sqrt{(n-1)(k^2/n+\mu)+i0}$. If $\mu = 0$, then this means $\zeta = |k|\sqrt{1-1/n}+i0$, while for $\mu > 0$, the square root depends differentiably on k. If $\mu < 0$, the square root is not differentiable at $k = \pm\sqrt{n|\mu|}$.

Appendix A

Here, we give the proof of Lemma 4.2. We begin by proving Eq.(4.16). We distinguish two cases:

1) $na \geq 2b$: in that case

$$I_n(a,b) \leq \frac{b}{[1+(na/2)^2]^\gamma [1+a^2]^\beta} \quad \leq \quad \frac{C}{(1+a^2+b^2)^{\beta+1/2}} \ .$$

2) $na < 2b$: then

$$\begin{aligned} I_n(a,b) &\leq \frac{1}{(1+a^2+b^2/n^2)^\beta} \int_0^1 \frac{bdt}{[1+b^2(1-t)]^\gamma} \\ &= \frac{1}{(1+a^2+b^2/n^2)^\beta} \int_0^1 \frac{bdy}{[1+b^2 y]^\gamma} \ . \end{aligned}$$

The last integral is bounded by $1/b(\gamma-1)$ and also bounded by 1 if $b \leq 1$, hence in the present case,

$$I_n(a,b) \leq \frac{2\gamma}{(\gamma-1)(1+a^2+b^2/n^2)^\beta(1+b)} \leq \frac{C}{(1+a^2+b^2)^{\beta+1/2}} \ .$$

This proves Eq.(4.16).

We next consider Eq.(4.17). We again distinguish two cases:

1) $na \geq 2b$: then

$$J_n(a,b) \leq \frac{1}{[1+(na/2)^2]^\beta [1+a^2]^\beta} \int_0^1 \frac{1}{\sqrt{1-t^2}} \leq \frac{C}{(1+a^2+b^2)^{2\beta}} \ .$$

2) $na < 2b$: then

$$\begin{aligned} J_n(a,b) \leq& \frac{1}{(1+a^2+b^2/n^2)^\beta} \int_{1/n}^1 \frac{ntdt}{[1+b^2(1-t^2)]^\beta \sqrt{1-t^2}} \\ \leq& \frac{n}{(1+a^2+b^2/n^2)^\beta} \int_0^1 \frac{dy}{(1+b^2y^2)^\beta} \ . \end{aligned}$$

The last integral is bounded by 1 and also by

$$\frac{1}{b}\int_0^1 \frac{2^\beta bdy}{(1+by)^{2\beta}} \quad \leq \quad \frac{2^\beta}{b(2\beta-1)} \ ,$$

and the assertion follows.

Finally, we prove Eq.(4.18). Note that if $b \le 1$, the integral is bounded. If $b \le a/2$ the integral is bounded by $Cb/(1+a^2)^{\beta+\gamma}$. Therefore we can restrict our attention to the case when $b \ge 1$ and $a < 2b$. The part of the integral from $t = 1/n$ to $t = 1$ can be estimated by (4.17); that part is bounded by

$$\frac{Cb}{(1+a^2+b^2)^{\beta+1/2}} \le \frac{C}{(1+a^2+b^2)^{\beta}} .$$

We next consider the range of integration $0 \le t \le 1/n$. This contribution is bounded by

$$\int_0^{1/n} dt \, \frac{Cb}{(1+b^2)^{\gamma}[1+(a+bt)^2]^{\beta}} \le \frac{C}{(1+b^2)^{\gamma}} \int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^{\beta}} \le \frac{C}{(1+a^2+b^2)^{\gamma}} .$$

Finally, the contribution from $-1 \le t \le 0$ can be bounded by

$$\frac{Cb}{(1+a^2+b^2)^{\gamma}} \int_0^1 \frac{dt}{\sqrt{1-t^2}} \le \frac{C}{(1+a^2+b^2)^{\beta}} .$$

The proof is complete.

Appendix B

In this section, we briefly discuss the Gaussian example, i.e., the case when

$$v(p) \;=\; g(p) \equiv \exp(-Ap^2) \;, \tag{7.1}$$

with some fixed $A > 0$. Then, for any integer $n \ge 2$,

$$W_n(k,\rho,g) \;=\; e^{-A(\rho^2+k^2/n)} W_n(k,\rho,1) \;=\; L_n \, e^{-A(\rho^2+k^2/n)} \, \theta(\rho) \, \rho^{n-3} \;. \tag{7.2}$$

We now consider, for any integer $n \ge 3$,

$$F_n(k,\zeta,g) \;=\; L_n \, e^{-Ak^2/n} \int_0^{\infty} d\rho \, \frac{2\rho^{n-2} \, e^{-A\rho^2}}{\rho^2-\zeta^2} \;. \tag{7.3}$$

If n is odd and $n \ge 3$, we get

$$F_n(k,\zeta,g) \;=\; -L_n e^{-Ak^2/n} \left(\log(-\zeta^2) e^{-A\zeta^2} \zeta^{n-3} + E_n(\zeta^2)\right) \;. \tag{7.4}$$

If n is even and ≥ 4, we find:

$$F_n(k, \zeta, g) = L_n e^{-Ak^2/n} \left(i\pi \mathrm{sign}(\mathrm{Im}\ \zeta) e^{-A\zeta^2} \zeta^{n-3} + E_n(\zeta^2) \right) . \tag{7.5}$$

Here E_n is an entire function such that $E_n(z) = E_n(z^*)^*$.

If we choose $n = 4$ and $\zeta = \sqrt{(n-1)(k^2/n + \mu) + i0}$, then there are three cases:

a) $\mu > 0$: Then ζ is a holomorphic function of k in a neighborhood of $\mathbf{R}$. Therefore, $k \mapsto F_n(k, \zeta, g)$ is $\mathcal{C}^1$ on $\mathbf{R}$ with a Hölder continuous derivative.

b) $\mu = 0$: Then $\zeta = \sqrt{1 - 1/n}|k| + i0$. Therefore, $k \mapsto F_n(k, \zeta, g)$ has a Hölder continuous derivative on $\mathbf{R} \setminus 0$.

c) $\mu < 0$: Then the derivative of ζ is infinite at $k = \pm\sqrt{-n\mu}$.

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