Maximal almost-periodic solutions for Lagrangian equations on infinite dimensional tori

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1. Introduction

We shall briefly discuss the generalization to infinite dimensions of the existence of *maximal quasi-periodic* solutions for the following Euler-Lagrange equations on $\mathbf{T}^N \equiv (\mathbf{R}/2\pi\mathbf{Z})^N$:

$$\ddot{x}_i = V_{x_i}(x) \ , \qquad i = 1, \ldots, N \tag{1.1}$$

associated to the Lagrangian

$$L(\dot{x}, x) = \frac{1}{2}\sum_{i=1}^{N} \dot{x}_i^2 + V(x) \tag{1.2}$$

$V(x) = V(x_1, \ldots, x_N)$ being a smooth function 2π-periodic in x_i; $(\cdot)_{x_i}$ denotes partial differentiation: $V_{x_i} \equiv \frac{\partial V}{\partial x_i}$. If $\omega \in \mathbf{R}^N$ is a "Diophantine vector" and if V is small enough it follows from KAM theory (see e.g. [SZ], [CC1], [CC2] for Lagrangian KAM theory) that (1.1) admits *quasi-periodic* solutions $x(t) = \omega t + u(\omega t)$ where $u\colon \mathbf{T}^N \to \mathbf{R}^N$ is a smooth function. We recall that a vector $\omega \in \mathbf{R}^N$ is called (γ, τ)-Diophantine if

$$\Big|\sum_{i=1}^{N} \omega_i n_i\Big| \geq \frac{1}{\gamma |n|^\tau}, \qquad \forall\ n \in \mathbf{Z}^N \backslash \{0\} \tag{1.3}$$

for some positive numbers γ and τ; $|n| \equiv \sum_{i=1}^{N} |n_i|$.

A simple rescaling argument ($\omega \to \omega/\epsilon, \quad 0 < \epsilon << 1$) shows that *for any* potential (smooth enough) there always exist plenty of quasi-periodic solutions with *large* frequencies.

In this note we present some results concerning the existence of many *maximal almost-periodic* solutions in infinite dimensions. Such results are, in a suitable sense, a generalization to infinite dimensions of [CZ] and complete proofs will appear elsewhere.

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Before introducing the precise infinite dimensional set-up that generalizes the Lagrangian system (1.1)-(1.2), let us consider two examples to which our techniques will apply. The first is a so-called "finite-range system of infinitely many coupled rotators", which can be viewed as an infinite system of coupled second order differential equations:

$$\frac{d^2}{dt^2}x_i(t) = a_i \cos(x_i - x_{i-1}) - a_{i+1}\cos(x_{i+1} - x_i) \tag{1.4}$$

where $i \in \mathbf{Z}$; the a_i's are real constants with $\sup_{i\in\mathbf{Z}} |a_i| < \infty$.

Systems of this kind (and their finite dimensional approximations) are of interest in statistical mechanics (from which the above terminology has been borrowed) and have been extensively studied by many authors; see e.g. [VB], [W], [FSW], [P].

The second example involves a "*long-range* interaction" and is given by the following equations:

$$\frac{d^2}{dt^2}x_i(t) = \cos x_i \sum_{n\in\mathbf{Z}} a_n \prod_{0\neq m\in\mathbf{Z}} (1 + a_{n+m}\sin x_{i+m}) \tag{1.5}$$

where $a_n \in \mathbf{R}$ and $\sum_{n\in\mathbf{Z}} |a_n| < \infty$. Notice that *formally* (1.4) and (1.5) are the Euler-Lagrange equations associated to the *formal* Lagrangians

$$L(x,\dot{x}) = \frac{1}{2}\sum_i \dot{x}_i^2 + \sum_i a_i \sin(x_i - x_{i-1})$$

and

$$L(x,\dot{x}) = \frac{1}{2}\sum_i \dot{x}_i^2 + \sum_i \prod_j [1 + a_j \sin x_{i+j}].$$

2. Set up

As "configuration space" we shall consider the Cartesian product

$$\mathbf{T}^{\mathbf{Z}^d} \equiv \bigotimes_{i\in\mathbf{Z}^d} \mathbf{T}_i \equiv \mathcal{T}, \qquad \mathbf{T}_i \equiv \mathbf{T} \equiv \mathbf{R}/2\pi\mathbf{Z}$$

endowed with the standard weak (compact) topology. Such a topology is induced by metrics: given a *weight* w (i.e. $w_i > 0$ for all i and $\sum_{i\in\mathbf{Z}^d} w_i < \infty$) we introduce on $\mathcal{T}$ the metric

$$\rho_w(x,y) \equiv \sum_{i\in\mathbf{Z}^d} w_i \ \rho(x_i, y_i), \qquad (x, y \in \mathcal{T})$$

where $\rho(x_i, y_i)$ is the standard flat metric on $\mathbf{T}$. The tangent space of $\mathcal{T}_w \equiv (\mathcal{T}, \rho_w)$ is the Banach space, $\mathcal{B}_w$, of vectors $a \in \mathbf{R}^{\mathbf{Z}^d}$ with finite norm:

$$\|a\|_w = \sum_{i\in\mathbf{Z}^d} w_i\,|a_i| < \infty.$$

Given a continuous map $f\colon \mathcal{T}_w \to \mathcal{B}_w$ we can now consider the *second order system*

$$\ddot{x}_i = f_i(x(t)) \quad , \qquad\qquad i \in \mathbf{Z}^d \tag{2.1}$$

and a *continuous map* $t \in \mathbf{R} \to x(t) \in \mathcal{T}_w$ will be called a *solution* of (2.1) if (for any $i \in \mathbf{Z}^d$) $x_i(t)$ belongs to $\mathrm{C}^2(\mathbf{R})$ and satisfy (2.1).

Equation (2.1) is an equation in $\mathcal{B}_w$ (having identified the tangent space of $\mathcal{B}_w$ with $\mathcal{B}_w$ itself). However for a solution of (2.1) it might (and will) happen that $\dot{x}(t)$ *does not* belong to $\mathcal{B}_w$ (e.g., it may happen that $\dot{x}_i(t) = \omega_i + \partial_t u_i(t)$ with ω_i growing arbitrarily fast as $|i| \to \infty$ and $|u_i| + |\partial_t u_i|$ bounded).

With standard contraction arguments one can prove the following elementary

Proposition. *Let $f\colon \mathcal{T}_w \to \mathcal{B}_w$ be a Lipschitz map (i.e. $\exists C$ s.t. $\|f(x) - f(y)\|_w \le C\rho_w(x,y)$, $\forall x, y \in \mathcal{T}_w$). Given any $x^o \in \mathcal{T}_w$ and any $y^o \in \mathbf{R}^{\mathbf{Z}^d}$, there exists a unique solution, global in time, of the Cauchy problem*

$$\begin{cases} \ddot{x}_i(t) = f_i(x(t)), & \qquad i \in \mathbf{Z}^d \\ x_i(0) = x_i^o, \qquad \dot{x}_i(0) = y_i^o. \end{cases} \tag{2.2}$$

We remark that the initial "velocity" y^o is arbitrary and is *not* required to belong to the tangent space $\mathcal{B}_w$.

Notice that the examples (1.4) and (1.5) in the introduction have a Lipschitz right hand side, *provided* one chooses suitably the weight w.

In this context the "Lagrangian structure" will be reflected by f being, in a suitable sense, a *gradient.*

As it is clear even from the "finite-range" example above we need a *generalized* notion of gradient. Such a notion will use *averages.*

The Haar measure $d\mu_i \equiv (2\pi)^{-1}dx_i$ on $\mathbf{T}_i \equiv \mathbf{T}$ induces, in a natural way, a product measure $d\mu \equiv \bigotimes_{i\in\mathbf{Z}^d} d\mu_i$ on $\mathcal{T}_w$. More precisely $d\mu$ is the unique extension on the σ-algebra generated by the cylinders,

$$\mathcal{R}_I \equiv \bigotimes_{i\in I} A_i \bigotimes_{j\in\mathbf{Z}^d\setminus I} \mathbf{T}_j, \qquad A_i \equiv \text{ open subset of } \mathbf{T}_i, \quad |I| < \infty$$

($|I| \equiv$ cardinality of I), satisfying

$$\mu(\mathcal{R}_I) = \prod_{i\in I} \mu_i(A_i).$$

Fubini's theorem holds and one can use integrations as projection operators on finite dimensional function spaces. If $g\colon \mathcal{T}_w \to \mathbf{R}$ is a bounded measurable function and I is a finite subset of $\mathbf{Z}^d$, we define $g^{[I]}(x^{(I)})$ as the bounded measurable function of $x^{(I)} \in \mathbf{T}^{|I|} \equiv \bigotimes_{i\in I} \mathbf{T}_i$ obtained by integrating over $\bigotimes_{i\notin I} \mathbf{T}_i$:

$$g^{[I]} \equiv \int g \bigotimes_{i\notin I} d\mu_i \equiv \lim_{j\to\infty} \int g \bigotimes_{k=1}^{j} d\mu_{n_k}$$

where $k \to n_k$ is any one-to-one map from $\mathbf{N}$ onto $\mathbf{Z}^d \backslash I$. By Fubini's theorem the function $g^{[I]}$ does not depend on the choice of the sequence $\{n_k\}$.

Definition 2.1 *A continuous function $f\colon \mathcal{T}_w \to \mathcal{B}_w$ is a g-gradient if for any finite $I \subset \mathbf{Z}^d$ there exists a $\mathrm{C}^1(\mathbf{T}^{|I|}, \mathbf{R})$ function $V^{(I)}(x)$ so that*

$$f_i^{[I]}(x) = \partial_{x_i} V^{(I)}(x), \quad \forall\, i \in I \quad \forall\, x \in \mathbf{T}^{|I|} \quad .$$

We remark that the right hand side of (1.4), (1.5) are *g-gradients* .

3. Results

To state our main results we need some more definitions .
Definition 3.1 *A vector $\omega \in \mathbf{R}^{\mathbf{Z}^d}$ is said to be rationally independent if*

$$\sum_{i\in I} \omega_i n_i = 0 \ \ \textit{for some finite } I \subset \mathbf{Z}^d \Rightarrow n_i = 0 \quad \forall\, i \in I.$$

A rationally independent vector ω will be also called a $\mathcal{T}$-frequency vector.
Definition 3.2 *Given a $\mathcal{T}$-frequency vector ω, a continuous function $q(t)$ is called ω-almost-periodic if there exists a function $Q \in \mathrm{C}(\mathcal{T}_w, \mathbf{R})$ such that $q(t) = Q(\omega t)$. A solution $x(t)$ of (2.1) is called maximal almost-periodic (with frequencies $\omega \in \mathbf{R}^{\mathbf{Z}^d}$) if $x_i(t) - \omega_i t$ is ω- almost-periodic for all $i \in \mathbf{Z}^d$.*
Remarks:
1) The word *maximal* refers to the rational independence property of ω.

2) An ω-*almost-periodic* function q is just an *almost-periodic* function in the sense of H.Bohr with frequency modulus

$$\sigma(q) = \left\{ \sum_{i\in I} \omega_i n_i : \ I \subset \mathbf{Z}^d, \ |I| < \infty, \ n_i \in \mathbf{Z} \right\}.$$

Finally we introduce the smoothness properties we shall consider.

Definition 3.3 *A g-gradient f is called uniformly weakly real-analytic if there exists a real number $\xi > 0$ such that for any finite set $I \subset \mathbf{Z}^d$, $V^{(I)}(x)$ is real-analytic on $\mathbf{T}^{|I|}$ and can be analytically continued to the set $\{z \in \mathbf{C}^{|I|}, \ |\mathrm{Im}\, z_i| \le \xi\}$.*

The *g-gradients* of the examples in the introduction are uniformly weakly real-analytic and as parameter ξ one can take any positive number.

Theorem 1. *Let f be a uniformly weakly real analytic g-gradient. Then there exist uncountably many maximal almost-periodic solutions of (2.1).*

This Theorem is an immediate corollary of the following more detailed statement

Theorem 2. *Let f as in* **Theorem 1** *(recall Definition 2.1) and assume that for some finite $I \subset \mathbf{Z}^d$ the equation*

$$\ddot{x}^{(I)} = \partial_{x^{(I)}} V^{(I)}(x^{(I)}), \qquad (x^{(I)} \in \mathbf{T}^{|I|}) \tag{3.1}$$

admits a non-degenerate real-analytic quasi-periodic solution with a (γ, τ)-Diophantine frequency $\omega^{(I)} \in \mathbf{R}^{|I|}$ (i.e. $x^{(I)}(t) = \omega^{(I)} t + u^{(I)}(\omega^{(I)} t)$ for a suitable real-analytic function $u^{(I)} : \mathbf{T}^{|I|} \to \mathbf{R}^{|I|}$ such that $\det(\mathrm{Id} + \partial u^{(I)}) \neq 0$). Then there exist uncountably many ω-almost-periodic solutions with $\omega_i = \omega_i^{(I)}, \forall i \in I$. All such frequencies ω are "γ-Diophantine" in the sense that for all $J \supset I$ with $|J \backslash I| = m$ it is:

$$\Big| \sum_{j\in J} \omega_j n_j \Big| \ge \frac{1}{\gamma (\sum_{j\in J} |n_j|)^{\tau+m}}, \qquad \forall \ (\{n_j\}_{j\in J}) \in \mathbf{Z}^{|J|} \backslash \{0\}. \tag{3.2}$$

Moreover for a suitable (small) constant $\epsilon > 0$ one has

$$\|x_i - x_i^{(I)}\|_{\mathrm{C}^2} \le \epsilon \quad (i \in I) \qquad , \quad \|x_i - \omega_i t\|_{\mathrm{C}^2} \le \epsilon \quad (i \notin I)$$

where $\| \cdot \|_{\mathrm{C}^2}$ is the standard C^2 norm.

The frequencies in the above theorems will be such that $|\omega_j|$ grows rapidly as $|j| \to \infty$. In fact rough estimates based on the technique used to prove the above results, indicate

the behaviour $|\omega_j| \sim (|j|!)^c$. Of course better estimates can be obtained by softening the numerical properties (3.2) of the frequencies.
Establishing the existence of maximal almost-periodic solutions with slowly growing frequencies is an interesting (and difficult) problem especially in view of more "realistic" models such as perturbed integrable P.D.E.'s (where, typically, the integrable frequencies grow as $|j|^c$ compare [K]).

The reason for the abundance of maximal almost-periodic solutions with $|\omega_j|$ growing rapidly may be justified as follows. Imagine that a subsystem of our model admits a quasi-periodic solution (see the assumption of Theorem 2), and imagine to "turn on one mode at time". Then, intuitively speaking, if the "turned–on mode" rotates fast enough, its effect will be mostly "averaged out" and it will weakly interact with the pre-existing motion.

In the rest of this note we present the main ideas beyond the proof of the above theorems.

4. Sketch of proofs

Theorems 1 is an immediate corollary of Theorem 2: For example one could take $I = \{i_o\}$ (with any $i_o \in \mathbf{Z}^d$) in Theorem 2: in this case (3.1) becomes trivial and it always admits non-degenerate quasi-periodic (in this case simply periodic) solutions.
The proof of Theorem 2 is based on a recursive argument.

Fix once for all a one-to-one map, j_n, from $\mathbf{N}$ onto $\mathbf{Z}^d \backslash I$ and let $I_o \equiv I$, $I_1 \equiv I_o \cup \{j_1\}$, $I_{n+1} \equiv I_n \cup \{j_{n+1}\}$, (so that $|I_n| = |I| + n$ and $I_o \subset I_1 \subset \ldots I_n \uparrow \mathbf{Z}^d$).

The **main step** consists in showing how, assuming to have a non-degenerate quasi-periodic solution of

$$\ddot{x} = \partial_x V^{(I_n)}(x), \qquad (x \in \mathbf{T}^{|I_n|}) \tag{4.1$_n$}$$

with frequencies $\omega \equiv \omega^{(n)} \equiv (\{\omega_j\}_{j \in I_n})$, one can construct, via a "KAM theorem", a quasi-periodic non-degenerate solution of of $(4.1)_{n+1}$ with frequencies $\omega' \equiv \omega^{(n+1)} \equiv (\omega, \alpha) \equiv (\{\omega_j\}_{j \in I_{n+1}})$ provided $\alpha \equiv \omega_{j_{n+1}}$ is large enough and suitably chosen.

Once this step is carried out and *made quantitative* a solution $x(t)$ of (2.1) (with f as in Theorem 2) will be easily obtained by taking, in a suitable (but natural) sense the limit

$$x_j(t) \equiv \lim_{n \to \infty} x_j^{(n)}(t) \tag{4.2}$$

where $x_j^{(n)}(t) \equiv \omega_j^{(n)} t + u_j^{(n)}(\omega^{(n)} t)$ is the quasi-periodic solution of $(4.1)_n$.

The abundance of (rapidly oscillating) almost-periodic solutions is related to the choice of $\omega_{j_{n+1}}$ *given* $\omega^{(n)}$: $\omega_{j_{n+1}}$ can be *arbitrarily chosen* in a Cantor set $\Omega^{(n+1)} \subset$

$[\overline{\alpha}, \infty)$ (for a suitable $\overline{\alpha} = \overline{\alpha}(\omega^{(n)})$ large enough) having the property that

$$\lim_{R\to\infty} \ell(\Omega^{(n+1)} \cap [R, R+1]) = 1 \tag{4.3}$$

where ℓ denotes Lebesgue measure. We shall not work out here the straightforward details necessary to give meaning to (4.2), while we shall concentrate on the main step.

As already mentioned the main technical tool comes from KAM theory: we shall use a minor modification (see [Pe]) of the "configurational KAM theorem" given in [CC1] (see also [SZ] and [CC2]).

Consider (1.1) with V real-analytic on $\mathbf{T}^N$ admitting a holomorphic extension to

$$\Delta^N_{\xi_o} \equiv \{x \in \mathbf{C}^N \ : |\mathrm{Im} x_j| \le \xi_o \ j = 1, \ldots, N\}.$$

It is easy to see that having a non-degenerate quasi-periodic solution $x(t) = \omega t + u(\omega t)$ (with $u\colon \mathbf{T}^N \to \mathbf{R}^N$ and $\omega \in \mathbf{R}^N$ rationally independent) is *equivalent* to require that u satisfies

$$D^2 u(\theta) = V_x(\theta + u(\theta)) \tag{4.4}$$

where $Du \equiv \omega \cdot \partial_\theta u \equiv \sum_{k=1}^N \omega_k \partial_{\theta_k} u$; "non-degenerate" means that $\det(\mathrm{Id} + u_\theta) \ne 0$.

Theorem K (cfr. [CC1]). *Let ω verify (1.3) and let $v\colon \mathbf{T}^N \to \mathbf{R}^N$ be a holomorphic function on $\Delta^N_{\xi_*}$, with $\xi_* < \xi_o$, such that:*

i) $\{\theta + v(\theta) : \theta \in \Delta^N_{\xi_*}\} \subset \Delta^N_{\xi_o}$, *ii)* $\sup_{\Delta^N_{\xi_*}} \|(\mathrm{Id} + v_\theta)^{-1}\| \equiv \widetilde{M} < \infty$.

Finally let $0 < \xi' < \xi_*$. Let be a constant $K = K(\|V\|_{C^3, \Delta^N_{\xi_o}}, \widetilde{M}, \gamma, \tau, \xi_* - \xi') > 1$ such that if

$$K \ \|D^2 v - V_x(\theta + v)\|_{\Delta^N_{\xi_*}} \equiv KE \le 1 \tag{4.5}$$

then there exists a solution $u\colon \mathbf{T}^N \to \mathbf{R}^N$ of (4.4) with a holomorphic extension to $\Delta^N_{\xi'}$; such a solution is (locally) unique if we require that $\int u d\theta = \int v d\theta$. The constant K can be taken to be

$$K \equiv \lambda \gamma^2 \frac{\overline{M}^{10} \widetilde{M}^8 (N!)^4 \, 2^{34N+12}}{(\xi_* - \xi')^{8N+1}} \tag{4.6}$$

with

$$\lambda \equiv \max\Big\{1 \ , \gamma^2 \, \|V_{xxx}\|_{\Delta^N_{\xi_o}}\Big\}, \quad M \equiv \sup_{\Delta^N_{\xi_*}} \|\mathrm{Id} + v_\theta\|. \tag{4.7}$$

The solution u is close to v in the following sense:

$$\begin{aligned} &\max\Big\{\|u \ - \ v\|_{\Delta^N_{\xi'}} \, , \ \|u_\theta \ - \ v_\theta\|_{\Delta^N_{\xi'}}\Big\} \le KE \\ &\max\Big\{\gamma \, \|Du \ - \ Dv\|_{\Delta^N_{\xi'}} \, , \ \gamma^2 \, \|D^2 u \ - \ D^2 v\|_{\Delta^N_{\xi'}}\Big\} \le KE\Lambda \end{aligned} \tag{4.8}$$

where $\quad \Lambda = \max\Big\{1,\ \|V_{xx}\|_{\Delta^N_{\xi_o}}/\|V_{xxx}\|_{\Delta^N_{\xi_o}}\Big\}.$

Remarks:

1) In other words: near any (non-degenerate) *approximate* solution of (4.4) there is a true (non-degenerate) solution, provided the approximate solution solves equation (4.4) up to a small *enough* error.

2) The (minor) changes with respect to the version in [CC1] are the estimates in (4.6), (4.8) and the introduction of the parameter ξ' (the domain of holomorphy of u) which, here, is left free.

Assume now to have a non-degenerate solution of $(4.1)_n$, $x^{(n)}(t) = \theta' + \omega^{(n)}t + u'(\theta' + \omega^{(n)}t)$, $\theta' \in \mathbf{T}^{|I|+n}$, $u'\colon \mathbf{T}^{|I|+n} \to \mathbf{R}^{|I|+n}$. Let $N \equiv |I| + n + 1$ and let ξ_o be the parameter measuring the analyticity of the *g-gradient* f (see Definition 3.3: $\xi_o \equiv \xi$). The function $u'\colon \mathbf{T}^{|I|+n} \to \mathbf{R}^{|I|+n}$ is assumed to be such that $|\mathrm{Im}(\theta'_i + u'_i(\theta'))| \le \xi_o$ for any $i \in I_n$, for any $\theta' \in \Delta^{N-1}_\xi$, for a suitable ξ; the non-degeneracy of $x^{(n)}(t)$ means that $\|(\mathrm{Id} + u'_{\theta'})^{-1}\|_{\Delta^{N-1}_\xi} < \infty$. To attack the problem with one more degree-of-freedom on $\mathbf{T}^N$ we shall look directly at the P.D.E. (4.4) for a new quasi-periodic function $x^{(n+1)}(t) = \omega^{(n+1)}t + u(\omega^{(n+1)}t)$, u *and* $\alpha \equiv \omega_{j_{n+1}}$ being the *unknowns*: of course V in (4.4) is now $V^{(I_{n+1})}$ (recall that $V^{(I_n)} = \int_0^{2\pi} V^{(I_{n+1})} d\theta_{j_{n+1}}/2\pi$ up t o a constant playing no role).

The new frequency $\alpha = \omega_{j_{n+1}}$ is taken to be in the (Cantor) set

$$\Omega_{\alpha_o} \equiv \Big\{\alpha \in \mathbf{R} : |\omega^{(n)} \cdot n + \alpha m| \ge \frac{\alpha_o}{|n|^N},\ \alpha \ge \alpha_o, \quad \forall\, n \in \mathbf{Z}^{N-1}, \quad \forall\, m \in \mathbf{Z}\backslash\{0\}\Big\}$$

where α_o is (at the moment) an arbitrary parameter larger than $1/\gamma$ (γ, τ are the Diophantine constants associated to $\omega^{(o)}$: cfr. hypotheses of Theorem 2). It is not difficult to see that $\ell([\alpha_o, \infty)\backslash\Omega_{\alpha_o}) \le c|\omega^{(n)}|$ with a suitable constant c depending only on N, τ. Thus the set Ω_{α_o} verifies (4.3).

The idea now is to try to construct out of $u'(\theta')$ an approximate solution $v\colon \mathbf{T}^N \to \mathbf{R}^N$ for (4.4) with $V = V^{(I_{n+1})}$ and to use Theorem K to show the existence of a solution, u, close to u'. The job is done by setting ($\theta \equiv (\theta', \theta_{j_{n+1}}) \in \mathbf{T}^N$):

$$\begin{aligned} v(\theta) \equiv & D^{-2}_{\omega^{(n+1)}} V_x^{(I_{n+1})}(\theta' + u'(\theta'), \theta_{j_{n+1}}) \\ = & - \sum_{\substack{n \in \mathbf{Z}^{N-1}, m \in \mathbf{Z} \\ (n,m) \ne (0,0)}} e^{(in\cdot\theta' + m\theta_{j_{n+1}})} \frac{C_{n,m}}{(\omega^{(n)} \cdot n + \alpha m)^2} \end{aligned} \tag{4.9}$$

where $x \in \mathbf{T}^N$ ($x = \{x_i\}_{i \in I_{n+1}}$), $C_{n,m}$ denotes the Fourier coefficient of the (vector-valued) function $C(\theta) = V_x^{(I_{n+1})}(\theta' + u'(\theta'), \theta_{j_{n+1}})$, $D_{\omega^{(n+1)}} \equiv (\omega^{(n)}, \alpha) \cdot \partial_\theta \equiv \omega^{(n)} \cdot$

$\partial_{\theta'} + \alpha\partial_{\theta_{j_{n+1}}}$ and if $g(\theta)$ is a function with zero average we denote by $D^{-2}g$ the *unique* solution of $D^2 f = g$ with vanishing mean value.
Notice that in (4.9) it is implicitly stated that

$$\int_{\mathbf{T}^N} V_x^{(I_{n+1})}(\theta' + u'(\theta'), \theta_{j_{n+1}}) \frac{d\theta}{(2\pi)^N} = 0, \tag{4.10}$$

a fact that is deduced by recalling that u' solves equation (4.4) with $V = V^{(I_n)}$ and that

$$\int_0^{2\pi} V_x^{(I_{n+1})}(\theta' + u'(\theta'), \theta_{j_{n+1}}) \frac{d\theta_{j_{n+1}}}{(2\pi)} = V_{x'}^{(I_n)}(\theta' + u'(\theta')).$$

This fact justify the application of the operator D^{-2}.

Now, notice that since u' solves (4.4) with $V = V^{(I_n)}$, it is $u'(\theta') = D^{-2}_{\omega^{(n)}}[V_{x'}^{(I_n)}(\theta' + u')]$ thus

$$v = -\sum_{m \neq 0} e^{(\imath n \cdot \theta' + m\theta_{j_{n+1}})} \frac{C_{n,m}}{(\omega^{(n)} \cdot n + \alpha m)^2} + (u', 0) \equiv \tilde{v} + (u', 0).$$

But recalling the definition of Ω_{α_o} it is easy to see that the norm of $\tilde{v}$ is $O(1/(\alpha_o)^2)$ and it can be made arbitrarily small by choosing the free parameter α_o large enough. How large one has to choose α_o is dictated by the conditions (4.5) in Theorem K. In fact it is not difficult to show that also $\|D^2 v - V^{(I_{n+1})}(\theta + v)\|$ is of size $O(1/(\alpha_o^2))$; thus if one takes $\alpha_o = O(K^{1/2})$ one can apply Theorem K obtaining in such a way, a quasi-periodic solution $x^{(n+1)}(t)$. This concludes our discussion of the main step.
Complete detailed proofs will appear elsewhere.

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