DROPLET DYNAMICS FOR ASYMMETRIC ISING MODEL

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Abstract. Nucleation from a metastable state is studied for an anisotropic Ising model at very low temperatures. It turns out that the critical nucleus as well as configurations on a typical path to it differ from the Wulff shape of an equilibrium droplet.

1. Introduction

We report some new results concerning stochastic ferromagnetic Ising models in the so called metastable region. Namely, we consider stochastic (Glauber) dynamics whose stationary states are given by Gibbs measures for Ising-like systems that at infinite volume, low temperature, and zero magnetic field exhibit a phase transition with spin-flip symmetry breaking. We study, similarly to what Jordão-Neves and Schonmann did for the standard Ising model [NS, S], a single spin-flip Glauber dynamics in a large but finite volume for small (positive) magnetic fields and very low temperatures. In particular, we are interested in asymmetric models — models for which the Wulff shape (equilibrium droplet) at zero temperature is not a cube. Ising models are believed to have some relevance for a dynamical description of the crystal growth (cf. [BCF]). The reason to consider an asymmetry stems from the fact that these models turn out to be simple cases where a difference between equilibrium and dynamical droplets shows up. The present note is devoted to a study of the simplest such model, namely, an anisotropic Ising model.

We analyze the nucleation of the stable plus-phase starting from the metastable minus-phase. In particular, we are iterested in a description of the first passage from the configuration $-\underline{1}$ (all spins in Λ equal -1) to the configuration $+\underline{1}$ (all spins in Λ equal $+1$). We show that with high probability in the limit of very low temperatures this transition takes place

i) via a formation of a critical nucleus whose shape may not be Wulff and
ii) following a path that is typically given by a sequence of "non-Wulff" configurations.

In particular, for a two-dimensional anisotropic nearest neighbour Ising model with coupling constants $J_1 > J_2 > 0$ along the axes, the critical droplet is actually a square of edge $\frac{2J_2}{h}$ (h is the magnetic field), while the Wulff shape is a rectangle with edges proportional to J_1, J_2.

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Another model suitable for a detection of a difference between equilibrium and dynamical droplet is a ferromagnetic Ising model with isotropic nearest neighbour and next nearest neighbour interaction. The critical droplet in this case turns out to have the optimal Wulff shape — at zero temperature it is a (non-regular) octagon with lengths of its sides determined from the ratio of the nearest neighbour and the next nearest neighbour coupling constants. However, the growth of a droplet follows (with high probability for very low temperatures) a somehow complicated path through non-Wulff shapes: up to a certain size it is along a sequence of regular octagons then some edges remain constant whereas other grow up to the critical (Wulff) shape. A study of this model involves an additional time scale (in this respect it reminds the standard three-dimensional Ising model) and is discussed in a separate publication [KO 1]. The results for both models were summarized in [KO 2].

To cover more general cases than the standard nearest neighbour Ising model, we developped slightly different arguments and constructions than those by Jordão-Neves and Schonmann [NS, S]. We present here our new approach even though, in the particular case considered in the present paper, we could probably have worked out an extension of the somehow simpler methods of Jordão-Neves and Schonmann. In their present form, however, their methods do not apply to our situation. We believe that our alternative is of some interest not only due to its more general applicability, but also because it clarifies some other aspects of the problem.

2. Statement of the results

We consider a discrete time Metropolis dynamics for a two-dimensional nearest neighbour ferromagnetic *asymmetric* Ising model. The space of the process is $\{-1,1\}^\Lambda$ with Λ being a two-dimensional torus: the set $\{1,\dots,M\}^2$ with periodic boundary conditions. A *configuration* $\sigma \in \{-1,1\}^\Lambda$ is a function

$$\sigma\colon \Lambda \to \{-1,1\}.$$

The value $\sigma(x)$ is called the spin at the site $x \in \Lambda$. The *energy* of a configuration σ is

$$H(\sigma) = -\frac{J_1}{2} \sum_{(x,y)\in\mathcal{H}(\Lambda)} \sigma(x)\sigma(y) - \frac{J_2}{2} \sum_{(x,y)\in\mathcal{V}(\Lambda)} \sigma(x)\sigma(y) - \frac{h}{2}\sum_{x\in\Lambda}\sigma(x), \tag{1}$$

where $\mathcal{H}(\Lambda)$ is the set of horizontal nearest neighbour pairs in Λ and $\mathcal{V}(\Lambda)$ is the set of vertical nearest neighbour pairs in Λ. We suppose that

$$J_1 \geq J_2 \gg h > 0 \text{ and } M \geq \Big(\frac{2J_1}{h}\Big)^3. \tag{2}$$

Further, to avoid some "diofantine" problems, we assume that $\frac{2J_1}{h}$, $\frac{2J_2}{h}$, as well as their difference, are not integers.

The *dynamics* is prescribed by the following updating rule:

Given a configuration σ at time t, we first choose randomly a site $x \in \Lambda$ with uniform probability $\frac{1}{|\Lambda|}$. Then we flip the spin at the site x with probability

$$\exp(-\beta \Delta_x H(\sigma))^+, \tag{3}$$

where

$$\Delta_x H(\sigma)) = H(\sigma^{(x)}) - H(\sigma)$$

with

$$\sigma^{(x)}(y) = \begin{cases} \sigma(y) \text{ whenever } y \neq x, \\ -\sigma(y) \text{ for } y = x, \end{cases}$$

and $(c)^+ = \min(c, 0)$ for every $c \in \mathbb{R}$, β is the inverse temperature.

Our dynamics is *reversible* with respect to the Gibbs measure

$$\mu_\Lambda(\sigma) = \frac{\exp(-\beta H(\sigma))}{Z_\Lambda}$$

with the partition function

$$Z_\Lambda = \sum_{\sigma \in \{-1,1\}^\Lambda} \exp(-\beta H(\sigma)),$$

in the sense that the transition probabilities of the Markov chain $P(\sigma \to \sigma')$ satisfy the equalities

$$\mu_\Lambda(\sigma) P(\sigma \to \sigma') = \mu_\Lambda(\sigma') P(\sigma' \to \sigma).$$

The *space of trajectories* of the process is

$$\Omega \equiv \left(\{-1,1\}^\Lambda\right)^{\mathbb{N}}.$$

An element in Ω is denoted by ω; it is a function

$$\omega \colon \mathbb{N} \to \{-1,1\}^\Lambda.$$

We often write $\omega = \sigma_0, \sigma_1, \ldots, \sigma_t, \ldots$.

Given any set of configurations $A \subset \{-1,1\}^\Lambda$, we use τ_A to denote the *first hitting time* to A:

$$\tau_A = \inf\{t \geq 0 \colon \sigma_t \in A\}. \tag{4}$$

Sometimes we use $P_\eta(\cdot)$ to denote the probability distribution over the process starting at $t = 0$ from the configuration η. We use $-\underline{1}$, $+\underline{1}$ to denote the configurations with all spins in Λ equal to -1, $+1$, respectively.

We are interested in dynamics at very low temperatures. Namely, we will discuss the asymptotic behaviour, in the limit $\beta \to \infty$, of typical paths of the first escape from $-\underline{1}$ to $+\underline{1}$.

FIG. 1

Having in mind a low temperature dynamics, it is natural to describe configurations in terms of their Peierls contours. Namely, for every $\sigma \in \{-1, 1\}^{\Lambda}$ we consider the union $C(\sigma)$ of all closed unit squares centered at sites x with $\sigma(x) = +1$. Connected components of the boundary of $C(\sigma)$ are called *contours*. A contour γ is thus a polygon connecting vertices of dual lattice $\mathbb{Z}^2 + (\frac{1}{2}, \frac{1}{2})$ such that any vertex is contained in an even number (0, 2, or 4) of unit segments belonging to γ. Often we shall identify a configuration σ with the corresponding set $C(\sigma)$.

With a certain dose of imagination, one could view an evolution of a configuration σ with energy (1) as a movement of a point in a complicated energy landscape (in "phase space") — like that shown on Fig. 1, simplifying, however, the multidimensional space of configurations to a two-dimensional space — with a natural tendency to follow a downhill path and an occasional, random and rather unprobable, uphill move. An important role is played by local minima of this landscape. Namely, let us introduce $\mathcal{R}(L_1, L_2)$ as the set of all configurations (up to a a translation) with all spins -1 except for those in a rectangle $R(L_1, L_2)$ with corners on the dual lattice and horizontal (vertical) sides of length L_1 (L_2). It is easy to verify that, for any (L_1, L_2) with $\min(L_1, L_2) > 1$, these configurations correspond to such local minima. It turns out (see [NS]) that small rectangles, namely those with small values of $\min(L_1, L_2)$, tend to shrink, while large rectangles tend to grow. The dynamical mechanism responsible for this behaviour has been clarified in [NS]: it is based on a competition between the creation of a unit square protuberance attached to an edge of the rectangle (and consequently a growth of a side of the rectangle by one) and an erosion of an edge. When deciding which tendency wins, one has to realize that the typical time for a creation of a protuberance on a vertical or horizontal edge is of the order $\exp\{2\beta\frac{J_1-h}{h}\}$, $\exp\{2\beta\frac{J_2-h}{h}\}$, respectively, while the typical time for eroding an edge of the length l is $\sim \exp\{\beta h(l-1)\}$ (see [NS] for more details).

Notice that the anisotropy reveals itself, in addition to different growth rates in different directions, also in an anisotropy of "interactions" between separate droplets. Namely, two droplets approaching each other in the horizontal direction will coalesce, while if they approach in the vertical direction they have to overcome an energetical barrier and the time needed to make their coalescence probable is of the order $\exp\{2\beta\frac{J_1-J_2}{h}\}$. If a droplet is too small, its existence is too ephemere to participate in a coalescence. This will be an important factor when discussing the detailed definition of a set of configurations attracted to the configuration $-\underline{1}$.

The configurations in $\mathcal{R}(L_1, L_2)$ are characterized by the point $\boldsymbol{L} \equiv (L_1, L_2)$ in $(\mathbb{Z}_+)^2$. The origin $\mathbf{0} \in (\mathbb{Z}_+)^2$ represents the configuration $-\underline{1}$. Points with L_1 or $L_2 = M$ represent rectangles winding around the torus. We use $\mathcal{R}$ to denote the set of all rectangular configurations,

$$\mathcal{R} = \bigcup_{L_1, L_2} \mathcal{R}(L_1, L_2).$$

In $(\mathbb{Z}_+)^2$ we introduce the distance

$$d(\boldsymbol{x}, \boldsymbol{y}) = \max(|x_1 - x_2|, |y_1 - y_2|)$$

for $\boldsymbol{x} \equiv (x_1, x_2), \boldsymbol{y} \equiv (y_1, y_2) \in (\mathbb{Z}_+)^2$. A *saddle point* between two neighbouring local minima, say $(L_1, L_2 - 1)$ and (L_1, L_2), is any configuration $\bar{\sigma}$ such that

$$H(\bar{\sigma}) = \min_{\omega:\ R(L_1, L_2-1) \to R(L_1, L_2)} \max_{\sigma \in \omega} H.$$

It is easy to see that it is represented by the set $C(\sigma)$ consisting of a rectangle $R(L_1, L_2 - 1)$ with a unit square attached to one of its sides of length L_1. We will use $\mathcal{P}(L_1, L_2 - 1; L_1, L_2)$ to denote the set of all such configurations.

A *global saddle point* is any configuration $\bar{\sigma}$ such that

$$H(\bar{\sigma}) = \min_{\omega:\ -\underline{1} \to +\underline{1}} \max_{\sigma \in \omega} H(\sigma). \tag{5}$$

It turns out (see Remark at the end of Section 3.1 below) that the set of all global saddle points is the set $\mathcal{P}$ of all configurations $\bar{\sigma}$ giving rise to a unique contour γ consisting of a rectangle with sides L_2^*, $L_2^* - 1$, and a unit square attached to one of its longer sides (see Fig. 2). Here

$$L_2^* = \left[\frac{2J_2}{h}\right] + 1, \tag{6a}$$

where $[\cdot]$ denotes the integer part. We also introduce

$$L_1^* = \left[\frac{2J_1}{h}\right] + 1. \tag{6b}$$

For any $\bar{\sigma} \in \mathcal{P}$ one has

$$H(\bar{\sigma}) - H(-\underline{1}) \equiv \Gamma(h) = (2J_1 + 2J_2)L_2^* - h\big[(L_2^*)^2 - L_2^* + 1\big]$$

for the "height" of the global saddle point.

FIG. 2

Using this fact we shall prove that the first excursion from $-\underline{1}$ to $+\underline{1}$ typically passes through a configuration from $\mathcal{P}$ and the time needed for this to happen is of the order $\exp(\beta\Gamma)$.

Theorem 1.

$$\lim_{\beta\to\infty} P_{-\underline{1}}(\tau_{\mathcal{P}} < \tau_{+\underline{1}}) = 1. \tag{7}$$

Theorem 2.

$$\lim_{\beta\to\infty} P_{-\underline{1}}(\exp[\beta(\Gamma - \varepsilon)] < \tau_{+\underline{1}} < \exp[\beta(\Gamma + \varepsilon)]) = 1 \tag{8}$$

for every $\varepsilon > 0$.

Notice, as already mentioned in Introduction, that a droplet of least overall surface tension covering a fixed area is given by the Wulff construction [W, H, RW, BN, DKS] and, at very low temperatures, is close to a rectangle proportional to $R(L_1^*, L_2^*)$. In spite of this, escaping trajectories are passing through $\mathcal{P}$ — a configuration close to critical nucleus $R(L_2^*, L_2^*)$.

Moreover, we shall see that a typical first excursion follows a rather well specified path that visits certain growing rectangular, almost square, configurations in well defined moments.

To state this result, we first introduce *a standard tube* (of rectangles) as a subset $\mathcal{T}$ of $(\mathbb{Z}_+)^2$ consisting of points corresponding either to "almost squares" or "large rectangles" (with either $x_2 = L_2^*$ or $x_1 = M$):

$$\mathcal{T} = \{\boldsymbol{x} \in (\mathbb{Z}_+)^2 \colon d(\boldsymbol{x}, \mathcal{L}_1) \le 1\} \cup \mathcal{L}_2 \cup \mathcal{L}_3. \tag{9}$$

Here

$$\begin{aligned}
\mathcal{L}_1 &= \{(x_1, x_2) \in (\mathbb{Z}_+)^2 \colon 1 \le x_1 = x_2 \le L_2^* - 1\} \\
\mathcal{L}_2 &= \{(x_1, x_2) \in (\mathbb{Z}_+)^2 \colon x_2 = L_2^*, L_2^* \le x_1 \le M\} \qquad (10) \\
\mathcal{L}_3 &= \{(x_1, x_2) \in (\mathbb{Z}_+)^2 \colon x_1 = M, L_2^* \le x_2 \le M\}.
\end{aligned}$$

We call *a standard sequence of rectangles* any sequence $\boldsymbol{x}^{(1)}, \dots, \boldsymbol{x}^{(2M-1)}$, $\boldsymbol{x}^{(i)} \in (\mathbb{Z}_+)^2$ such that

(1) $\{\boldsymbol{x}^{(i)}\}_{i=1,\dots,2M-1} \in \mathcal{T}$,

(2) $\boldsymbol{x}^{(1)} = (1,1)$ and the sequence $\{\boldsymbol{x}^{(i)}\}_{i=1,\dots,2M-1}$ is monotonous and consists of nearest neighbours in the sense

$$\boldsymbol{x}^{(i+1)} \equiv (x_1^{(i+1)}, x_2^{(i+1)}) = (x_1^{(i)}, x_2^{(i)}) + \boldsymbol{e},$$

where $\boldsymbol{e}$ is either $\boldsymbol{e}_1 = (1,0)$ or $\boldsymbol{e}_2 = (0,1)$.

FIG. 3. A standard sequence of rectangles.

Now, let $\bar{\tau}_{-\underline{1}}$ be the last instant in which $\sigma_t = -\underline{1}$ before $\tau_{+\underline{1}}$:

$$\bar{\tau}_{-\underline{1}} = \max\{t < \tau_{+\underline{1}} \colon \sigma_t = -\underline{1}\}. \qquad (11)$$

Let $\tau_0, \tau_1, \dots, \tau_n, \dots$ be random times after $\bar{\tau}_{-\underline{1}}$ in which σ_t visits the set $\mathcal{R}$ of rectangular configurations (after a change):

$$\begin{aligned}
\tau_0 &= \bar{\tau}_{-\underline{1}} \\
\tau_{n+1} &= \min\{t > \tau_n \colon \sigma_t \in \mathcal{R} \setminus \{\sigma_{\tau_n}\}\}, n = 0, 1, 2, \dots
\end{aligned} \qquad (12)$$

We say that σ_t is an *ε-standard path* if

(1) $\sigma_{\tau_0} = -\underline{1}$, $\{\sigma_{\tau_n}\}_{n=0,1,\ldots}$ is a standard sequence of rectangles,

(2) random times τ_n satisfy the following conditions:

(a) $\tau_1 < e^{\varepsilon\beta}, \quad \tau_2 - \tau_1 < e^{\varepsilon\beta}$,

(b) $\exp\{\beta(h(L-2)-\varepsilon) \leq \tau_n - \tau_{n-1} \leq \exp\{\beta(h(L-2)+\varepsilon)\}$, (13)
whenever $\sigma_{\tau_n} \in \mathcal{R}(L,L)$ for $2 \leq L \leq L_2^*$,
or $\sigma_{\tau_n} \in \mathcal{R}(L-1,L)$ for $3 \leq L \leq L_2^*$,

(c) $\exp\{\beta(2J_2 - h - \varepsilon) \leq \tau_n - \tau_{n-1} \leq \exp\{\beta(2J_2 - h + \varepsilon)\}$, (14)
whenever $\sigma_{\tau_n} \in \mathcal{R}(L, L_2^*)$ for $L_2^* + 1 \leq L \leq M$, and

(d) $\exp\{\beta(2J_1 - h - \varepsilon) \leq \tau_n - \tau_{n-1} \leq \exp\{\beta(2J_1 - h + \varepsilon)\}$, (15)
whenever $\sigma_{\tau_n} \in \mathcal{R}(M, L)$ for $L_2^* + 1 \leq L \leq M$.

We use $\mathcal{S}_\varepsilon$ to denote the set of all ε-standard paths.

Our results about the asymptotics of the first excursion from $-\underline{1}$ to $+\underline{1}$ are then summarized in the following theorem.

Theorem 3.

$$\lim_{\beta\to\infty} P_{-\underline{1}}(\{\sigma_t\}_{t=1,2,\ldots} \in \mathcal{S}_\varepsilon) = 1. \tag{16}$$

Before indicating the proof of our crucial result, namely Theorem 2, let us state several lemmas. First, we use the reversibility to prove that the time necessary to reach a particular configuration is the exponential of the concerned energy difference multiplied by inverse temperature.

Lemma 1. *For every $\varepsilon > 0$ and any configuarations η, σ, such that $H(\eta) > H(\sigma)$, one has*

$$\lim_{\beta\to\infty} P_\sigma(\tau_\eta < \exp\{\beta(H(\eta) - H(\sigma) - \varepsilon)\}) = 0. \tag{17}$$

Proof. Given $T \in \mathbb{N}$, one has

$$P_\sigma(\tau_\eta < T) = \sum_{s=1}^{T-1} \sum_{\bar\sigma_1,\ldots,\bar\sigma_{s-1}\neq\eta} P(\sigma_0 = \sigma, \sigma_1 = \bar\sigma_1, \ldots, \sigma_{s-1} = \bar\sigma_{s-1}, \sigma_s = \eta) =$$

$$= \exp\{-\beta(H(\eta) - H(\sigma))\} \sum_{s=1}^{T-1} \sum_{\bar\sigma_1,\ldots,\bar\sigma_{s-1}\neq\eta} P(\sigma_0 = \eta, \sigma_1 = \bar\sigma_{s-1}, \ldots, \sigma_{s-1} = \bar\sigma_1, \sigma_s = \sigma) \leq$$

$$\leq T \exp\{-\beta(H(\eta) - H(\sigma))\}. \tag{18}$$

To conclude the proof we choose

$$T = \left[\exp\{\beta(H(\eta) - H(\sigma) - \delta)\}\right]$$

with $\delta = \min(\varepsilon, \frac{H(\eta)-H(\sigma)}{2})$ and take β sufficiently large. $\square$

For any $(L_1, L_2) \in \mathbb{Z}_+^2$, let us denote $l = \min(L_1, L_2)$ and $L = \max(L_1, L_2)$. The following three lemmas are direct consequences of Theorem 1 from [NS] or of the arguments used in its proof.

The first lemma claims that the size of a critical droplet is L_2^* and indicates what barrier one has to pass when starting from a local minimum.

Lemma 2 [NS] *Using P_{L_1,L_2} to denote P_σ with $\sigma \in \mathcal{R}(L_1, L_2)$, we have*

$$\lim_{\beta\to\infty} P_{L_1,L_2}(\tau_{-\underline{1}} < \tau_{+\underline{1}}) = 1 \tag{19}$$

and

$$\lim_{\beta\to\infty} P_{L_1,L_2}(\tau_{-\underline{1}} > \exp\{\beta(h(l-1)+\varepsilon)\}) = 0, \tag{20}$$

whenever L_1 and L_2 are such that $l = \min(L_1, L_2) < L_2^*$ and $L = \max(L_1, L_2) < M$. Further,

$$\lim_{\beta\to\infty} P_{L_1,L_2}(\tau_{+\underline{1}} < \tau_{-\underline{1}}) = 1 \tag{21}$$

and

$$\lim_{\beta\to\infty} P_{L_1,L_2}(\tau_{+\underline{1}} > \exp\{\beta(2J_2 - h + \varepsilon)\}) = 0, \tag{22}$$

whenever L_1 and L_2 are such that $\min(L_1, L_2) \geq L_2^*$.

The following lemma says that subcritical shrinking is isotropical. Namely, starting from a subcritical rectangular configuration, it is very probable that we will first cut a shorter edge in the time given by the height of the corresponding barrier.

Lemma 3 [NS] *Starting from $\sigma_0 \in \mathcal{R}(L_1, L_2)$, let*

$$\tilde{\tau}_{\mathcal{R}} = \inf\{t > 0 \colon \sigma_t \in \mathcal{R} \setminus \{\sigma_0\}\}. \tag{23}$$

If $l = \min(L_1, L_2) < L_2^*$ and $L = \max(L_1, L_2) < M$, then

i) $\lim_{\beta\to\infty} P_{L_1,L_2}(\sigma_{\tilde{\tau}_{\mathcal{R}}} \in \mathcal{R}(L_1', L_2')) = 1$, *whenever L_1' and L_2' are such that* *if $L_1 \neq L_2$, $\max(L_1', L_2') = L - 1$, and $\min(L_1', L_2') = l$,*
or
if $L_1 = L_2 = L$, $\max(L_1', L_2') = L$, and $\min(L_1', L_2') = L - 1$,

ii) $\lim_{\beta\to\infty} P_{L_1,L_2}(\exp\{\beta(h(l-1)-\varepsilon)\} < \tilde{\tau}_{\mathcal{R}} < \exp\{\beta(h(l-1)+\varepsilon)\}) = 1$ *for any $\varepsilon > 0$.*

Finally, the third lemma states that a supercritical droplet first grows in the "easier" direction, and only after L_1 hits M, the side L_2 starts to increase.

Lemma 4 [NS] *Whenever $l \geq L_2^*$, $L_1 < M$, and $\varepsilon > 0$, one has*

$$\lim_{\beta\to\infty} P_{L_1,L_2}(\sigma_{\tilde{\tau}_{\mathcal{R}}} \in \mathcal{R}(L_1 + 1, L_2)) = 1 \tag{24}$$

and

$$\lim_{\beta\to\infty} P_{L_1,L_2}(\exp\{\beta(2J_2 - h - \varepsilon)\} < \tilde{\tau}_{\mathcal{R}} < \exp\{\beta(2J_2 - h + \varepsilon)\}) = 1. \tag{25}$$

For $L_1 = M$, $L_2 \geq 2$, and any $\varepsilon > 0$ one has

$$\lim_{\beta\to\infty} P_{L_1,L_2}(\sigma_{\tilde{\tau}_{\mathcal{R}}} \in \mathcal{R}(M, L_2+1)) = 1 \tag{26}$$

and

$$\lim_{\beta\to\infty} P_{L_1,L_2}(\exp\{\beta(2J_1 - h - \varepsilon)\} < \tilde{\tau}_{\mathcal{R}} < \exp\{\beta(2J_1 - h + \varepsilon)\}) = 1. \tag{27}$$

Remark. It is possible to prove a stronger version of Theorem 3 giving rise to a more accurate description of the characteristics of typical paths of the first excursion from $-\underline{1}$ to $+\underline{1}$. In particular one can prove that, with probability going to 1 as $\beta \to \infty$, during the transition from $-\underline{1}$ to a protocritical configuration (corresponding to the part $\mathcal{L}_1$ of the standard tube), the pluses form a connected cluster C without holes and with a monotone boundary ∂C. Here, "monotone" means that ∂C intersects the four edges of its rectangular envelope $R(C)$ in four intervals, and its length equals that of the perimeter $R(C)$. All these properties follow from stronger versions of Lemmas whose proof can again be found in [NS].

3. Proof of Theorems

The most crucial is the proof Theorem 2. It will consist of two steps.

First, we define a set $\mathcal{A} \subset \{-1, 1\}^{\Lambda}$ satisfying the following three properties:

(1) For every $\sigma \in \mathcal{A}$ and any $\varepsilon > 0$ one has

$$\lim_{\beta\to\infty} P_{\sigma}(\tau_{-\underline{1}} < \tau_{+\underline{1}}) = 1 \tag{28}$$

and

$$\lim_{\beta\to\infty} P_{\sigma}(\tau_{-\underline{1}} < \exp\{\beta(h(L_2^* - 2) + \varepsilon)\}) = 1. \tag{29}$$

(2) Any path $\{\sigma_t\}_{t\in\mathbb{N}}$ such that $\sigma_0 = -\underline{1}$ and $\sigma_t = +\underline{1}$ for some t has to pass through the "boundary" $\partial\mathcal{A}$ of the set $\mathcal{A}$ defined by

$$\partial\mathcal{A} = \{\eta = \sigma^{(x)} \text{ for some } x;\ \sigma \in \mathcal{A}, \eta \notin \mathcal{A}\}. \tag{30}$$

Namely, there exists $s < t$ such that $\sigma_s \in \partial\mathcal{A}$.

(3) The minimal energy in $\partial\mathcal{A}$ is attained for "protocritical" (global saddle) configurations $\sigma \in \mathcal{P}$; namely,

$$\begin{aligned} &\min_{\sigma\in\partial\mathcal{A}} (H(\sigma) - H(-\underline{1})) = H(\mathcal{P}) - H(-\underline{1}) = \Gamma \\ &\min_{\sigma\in\partial\mathcal{A}\setminus\mathcal{P}} (H(\sigma) - H(\mathcal{P})) \geq h > 0. \end{aligned} \tag{31}$$

The second step will be to prove, for any $\varepsilon > 0$, that before the time given by the upper bound from (8), one certainly reaches the boundary of $\mathcal{A}$; namely,

$$\lim_{\beta\to\infty} P_{-\underline{1}}(\tau_{\partial\mathcal{A}} \geq T(\varepsilon)) = 0 \tag{32}$$

with

$$T(\varepsilon) = \exp\{\beta(\Gamma + \varepsilon)\}. \tag{33}$$

Once the set $\mathcal{A}$ satisfying the conditions (1)–(3) is constructed and the equality (32) is assured, the proof can be easily completed.

Indeed, starting from $\sigma \in \mathcal{P}$, the probability of flipping a spin -1 adjacent to the unit square proturberance in such a way that a stable "proturberance of length 2" is created, is not smaller than $1/|\Lambda|$ (see [NS] for more details). Then, for any $\varepsilon > 0$, it follows from Lemmas 2 and 4 that the probability to reach $+\underline{1}$ before reaching $-\underline{1}$, and to reach it in a time needed to create a minimal proturberance, can be bounded from below:

$$\begin{aligned} &P_{\mathcal{P}}(\tau_{+\underline{1}} < \tau_{-\underline{1}}) \geq \frac{1}{|\Lambda|} \\ &\lim_{\beta\to\infty} P_{\mathcal{P}}(\tau_{+\underline{1}} < \exp\{2J_2 - h + \varepsilon\} \mid \tau_{+\underline{1}} < \tau_{-\underline{1}}) = 1. \end{aligned} \tag{34}$$

On the other hand, Lemma 1 and the property (3) of $\mathcal{A}$ imply that one needs much longer time than $T(\varepsilon)$ to reach $\partial\mathcal{A} \setminus \mathcal{P}$,

$$\lim_{\beta\to\infty} P_{-\underline{1}}(\tau_{\partial\mathcal{A}\setminus\mathcal{P}} < \exp\{\beta(\Gamma + h - \varepsilon)\}) = 0. \tag{35}$$

Clearly,

$$P_{-\underline{1}}(\tau_{\partial\mathcal{A}\setminus\mathcal{P}} < \tau_{\partial\mathcal{A}}) \leq P_{-\underline{1}}(\tau_{\partial\mathcal{A}\setminus\mathcal{P}} < T(\varepsilon)) + P_{-\underline{1}}(\tau_{\partial\mathcal{A}} \geq T(\varepsilon)). \tag{36}$$

Taking $\varepsilon < \frac{h}{2}$, the equation (35) implies that the first term on the right hand side of (36) vanishes. Thus the relations (34), (36), and the strong Markov property allow to reduce the proof of the equality

$$\lim_{\beta\to\infty} P_{-\underline{1}}(\tau_{+\underline{1}} > T(\varepsilon)) = 0 \tag{37}$$

to the proof of (32).

Finally, from Lemma 1 and the properties (2) and (3) of $\mathcal{A}$ it follows directly that one cannot reach $+\underline{1}$ in a too short time,

$$\lim_{\beta\to\infty} P_{-\underline{1}}(\tau_{+\underline{1}} < \exp\{\beta(\Gamma - \varepsilon)\}) = 0. \tag{38}$$

Thus, to complete the proof, it remains to construct the set $\mathcal{A}$ and to prove the equality (32).

3.1. The construction of $\mathcal{A}$

First we introduce the notion of acceptable configurations $\sigma \in \{-1,1\}^\Lambda$. Any configuration σ can be identified with the collection $\{C_1, \ldots, C_k\}$ of its maximal connected components of plus spins (considering the union of all closed unit squares centered at the sites occupied by plus spin). To any such component C we assign its *rectangular envelope* defined as the minimal closed rectangle $R(C)$ (with edges parallel to the coordinate axes and vertices on the dual lattice) containing C. As before, we consider a strip winding around the torus to be a rectangle with a side of length M. If none of the rectangles $R(C_1), \ldots, R(C_k)$ is winding around the torus, we call the corresponding configuration *acceptable*.

For any acceptable configuration σ, there always exists a unique component of minuses winding around the torus. The contours touching it are *outer contours*. Given any outer contour γ, we use $C(\gamma)$ to denote the region enclosed in it and $R(\gamma)$ to denote the rectangular envelope $R(C(\gamma))$. Notice that every edge of $R(\gamma)$ contains at least one unit segment belonging to γ.

Now, for any acceptable σ, we shall construct a new configuration

$$\hat{\sigma} = S\sigma$$

by "filling up" and "gluing" together some its rectangular envelopes. To this end we first introduce the notion of interacting rectangles and chains of them. Two rectangles $R = R(L_1, L_2)$ and $R' = R(L'_1, L'_2)$ are said to be *interacting* if one of the following three possibilities occurs:

i) the rectangles R and R' intersect, or

ii) there exists a unit square centered at some lattice site such that one its vertical edge is contained in R and the other in R', or

iii) there exists a unit square centered at some lattice site such that one its horizontal edge is contained in R and the other in R' and, in the same time, $\min(L_1, L_2, L'_1, L'_2) \geq l^*$, where

$$l^* = \left[\frac{2(J_1 - J_2)}{h}\right] + 1.$$

(Neither R nor R' is *ephemere*.)

A set of rectangles $R_1, \ldots, R_m$ is said to form a *chain* $\mathcal{C}$ if every pair (R_i, R_j) of them can be linked by a sequence $\{R_{i_1}, \ldots, R_{i_n}\}$ of pairwise interacting rectangles from $\mathcal{C}$; $R_{i_1} = R_i$, $R_{i_n} = R_j$, and R_{i_l} and $R_{i_{l+1}}$ are interacting for all $l = 1, \ldots, n-1$.

Given a collection of chains $\mathcal{C}_1, \ldots, \mathcal{C}_n$ we start the following iterative procedure:

(1) The chains $\mathcal{C}_j^{(1)}$ of the "first generation" are identical to $\mathcal{C}_j$, $j = 1, \ldots, n$.

(2) Having defined $\mathcal{C}_j^{(r)}$, we construct rectangular envelopes $R_j^{(r)}$ of the sets

$$\bigcup_{R \in \mathcal{C}_j^{(r)}} R$$

and the maximal chains $\mathcal{C}_j^{(r+1)}$ of them.

The procedure ends once we reach a set of chains, each consisting of a single rectangle. Notice that every pair from the resulting set of noninteracting rectangles $\bar{R}_1, \dots, \bar{R}_s$ is such that either

- their distance is at least $\sqrt{2}$, or
- (if their distance is 1) they are either "almost touching by corners" (see Fig. 4A) or they are placed at distance 1 in vertical direction and at least one of the two, say $R(L_1, L_2)$, is ephemere, $\min(L_1, L_2) < l^*$ (see Fig. 4B).

FIG. 4A

FIG. 4B

Starting now from any acceptable configuration σ, we apply the above construction on chains of rectangular envelopes of its outer contours and define $\hat{\sigma}$ as the configuration obtained by placing the spin $+1$ at all sites inside the resulting rectangles $\bar{R}_1, \dots, \bar{R}_s$ (filling up the rectangles).

It is easy to verify that

$$H(\sigma) \geq H(\hat{\sigma}). \tag{39}$$

Indeed, notice that whenever a configuration ξ has contours γ', γ'' with interacting rectangular envelopes $R' = R(\gamma')$, $R'' = R(\gamma'')$, we will decrease the energy by filling the rectangular envelope of the union of R' and R''. This is evident in the case i) of the definition of interacting rectangles (the number of horizontal and vertical bonds, separately, is nonincreasing, the volume occupied by pluses is increasing) and in the case ii) (flipping the minus spin in the centre of the unit square touching R' and R'' the energy decreases since $J_1 \geq J_2$). In the case iii) we observe that when filling the rectangular envelope of $R' \cup R''$ with pluses, one gains at least hl^* that suffices, according to the definition of l^*, to compensate the loss of no more than $2(J_1 - J_2)$. Using this observation in an iterative manner, we can construct a sequence of configurations of decreasing energy starting with σ and ending with $\hat{\sigma}$.

Now we are ready to define the set $\mathcal{A}$. Namely, we introduce $\mathcal{A}$ as the set of all configurations σ such that every resulting rectangle $\bar{R}(L_1, L_2)$ from the configuration $\hat{\sigma}$ is subcritical, $l = \min(L_1, L_2) < L_2^*$ and $L = \max(L_1, L_2) < M$.

The property 1) of the set $\mathcal{A}$ is a direct consequence of the definition of $\mathcal{A}$ and of Lemma 2. Since the property 2) is obvious, again directly from the definition, our next task is to analyse the boundary $\partial\mathcal{A}$ and to prove the property 3). Let $\eta \in \partial\mathcal{A}$. Then there exists $\sigma \in \mathcal{A}$ and x such that $\eta = \sigma^{(x)} \notin \mathcal{A}$. It is clear that the mapping $\xi \to \hat{\xi}$ is monotonous. Namely, if $\xi_1 \prec \xi_2$ (i.e., by definition, $\{x \mid \xi_1(x) = +1\} \subset \{x \mid \xi_2(x) = +1\}$), then also $\hat{\xi}_1 \prec \hat{\xi}_2$. As a consequence $\sigma(x)$ is necessarily -1; otherwise $\sigma \in \mathcal{A}$ would imply also $\eta \in \mathcal{A}$. Moreover, the site x lies outside of all rectangles $\bar{R}_1, \ldots, \bar{R}_s$ corresponding to $\hat{\sigma}$. Among the rectangles corresponding to $\hat{\eta}$ there exists a rectangle $\bar{R}(L_1, L_2)$ with the following properties:

i) $\bar{R}$ is supercritical, $\min(L_1, L_2) \geq L_2^*$ or $2 \leq \min(L_1, L_2) < L_2^*$ and $L = \max(L_1, L_2) = M$,

ii) it contains the site x and several rectangles $\bar{R}_i$, say, $\bar{R}_1, \ldots, \bar{R}_k$, corresponding to $\hat{\sigma}$; the remaining rectangles $\bar{R}_{k+1}, \ldots, \bar{R}_s$ are also rectangles of $\hat{\eta}$.

Our aim now is to prove that

$$H(\eta) - H(-\underline{1}) \geq (2J_1 + 2J_2)L_2^* - h\big[(L_2^*)^2 - L_2^* + 1\big]. \tag{40}$$

If $\bar{R}$ is winding around the torus, the inequality (40) is clearly satisfied (the overall length of the boundaries of rectangles $\bar{R}_1, \ldots, \bar{R}_k$ is necessarily at least M.) Let us thus suppose that $\bar{R}$ is not wrapped around the torus. Consider first the configuration $\tilde{\eta}$, $H(\eta) \geq H(\tilde{\eta})$, whose set of pluses consists of the site x and the rectangles $\bar{R}_1, \ldots, \bar{R}_k$ (energy decreases when skipping the subcritical rectangles $\bar{R}_{k+1}, \ldots, \bar{R}_s$). Further, consider the set $C^{(0)}$ consisting of the union of the unit square $q(x)$ centered at the site x and those rectangles among $\bar{R}_1, \ldots, \bar{R}_k$ that intersect $q(x)$ along its edge. Let us take now the rectangular envelope $\bar{R}^{(1)}$ of $C^{(0)}$ and distinguish two cases, either the rectangle $\bar{R}^{(1)}$ is supercritical or not. (Notice that both, $C^{(0)}$ and $\bar{R}^{(1)}$ may actually coincide with $q(x)$.)

If $\bar{R}^{(1)}$ is supercritical, we decrease the energy of $\tilde{\eta}$ further by erasing all rectangles among $\bar{R}_1, \ldots, \bar{R}_k$ that were not contributing to the set $C^{(0)}$ and consider the configuration yielded by the set $C^{(0)}$. Since $\bar{R}^{(1)}$ is supercritical, there exists an $L_2^* \times L_2^*$ square Q^* contained in $\bar{R}^{(1)}$ and containing the site x. Given the fact that the rectangles $\bar{R}_1, \ldots, \bar{R}_k$ are mutually noninteracting, there are at most two among these rectangles that contribute to $C^{(0)}$. Moreover, if they are two, they intersect the unit square $q(x)$ along its opposite edges. Hence, always either the row or the column passing through x intersect the set $C^{(0)}$ only in $q(x)$. We decrease the energy by shrinking furher the configuration corresponding to $C^{(0)}$ to its intersection with Q^* (taking into account the fact that the concerned rectangles are subcritical), obtaining thus the configuration $\bar{\eta}$ whose boundary necessarily contains $2L_2^*$ horizontal and $2L_2^*$ vertical bonds (as Q^* does), while its pluses are covering the area that is at least by $L_2^* - 1$ sites of the mentioned row or column smaller then the area of Q^*. This yield the bound (40)

Next, consider the case when $\bar{R}^{(1)}$ is subcritical. Take $\bar{R}^{(1)}$ and all rectangles among $\bar{R}_1, \ldots, \bar{R}_k$ that were not used for $C^{(0)}$ and construct from them the set of chains $\mathcal{C}_j^{(1)}$ of the first generation. A sequence $\mathcal{C}_j^{(r)}, r = 1, \ldots, m$ of chains of following generations is obtained from it by iteration. Since the rectangles $\bar{R}_1, \ldots, \bar{R}_k$ are mutually noninteracting, for every generation r we get a chain, say $\mathcal{C}_1^{(r)}$, consisting of a rectangle $\bar{R}^{(r)}$ containing the site x and certain subset of $\bar{R}_1, \ldots, \bar{R}_k$. The remaining chains $\mathcal{C}_j^{(r)}, j = 2, \ldots$ contain each just one rectangle from those among $\bar{R}_1, \ldots, \bar{R}_k$ that have not appeared in $\mathcal{C}_1^{(p)}$, $p \leq r$, in the preceding steps. Clearly, there is only one chain in the last generation, $\mathcal{C}_1^{(m)} = \{\bar{R}^{(m)}\} \equiv \{\bar{R}\}$.

Let us consider now the last rectangle $\bar{R}^{(p)}$ among $\bar{R}^{(r)}$, $r = 1, \dots, m$ that is subcritical and take the chain $\mathcal{C}_1^{(p)}$ with the rectangles in $\mathcal{C}_1^{(p)} \setminus \bar{R}^{(p)}$ ordered in a particular way, say in lexicographic order of their left upper corner. Let us unite them, one by one in the given order, with the rectangle $\bar{R}^{(p)}$ until the circumscribed rectangle is supercritical. Cutting off the remaining rectangles from the chain $\mathcal{C}_1^{(p)}$ we get the chain $\tilde{\mathcal{C}}_1^{(p)} \subset \mathcal{C}_1^{(p)}$. Let us use $\tilde{R}'$ to denote the last rectangle that was attached to form the chain $\tilde{\mathcal{C}}_1^{(p)}$ and $\tilde{R}$ the circumscribed rectangle to the union of rectangles from $\tilde{\mathcal{C}}_1^{(p)} \setminus \{\tilde{R}'\}$. Clearly, $\tilde{R}$ and $\tilde{R}'$ are subcritical interacting rectangles with a supercritical envelope R^* of their union. Hence, there exists an $L_2^* \times L_2^*$ square Q^* contained in R^* such that it intersects both rectangles $\tilde{R}$ and $\tilde{R}'$ in nondegenerated rectangles R and R', $R = \tilde{R} \cap Q^*$ and $R' = \tilde{R}' \cap Q^*$, and contains the intersection $\tilde{R} \cap \tilde{R}'$ (if it is nonempty). One has

$$H(\tilde{\eta}) \geq H(R) + H(R'),$$

where $H(R)$ and $H(R')$ denotes the energy of the configuration with pluses in R and R', respectively. To see this, it is enough to realize that the energy of $\tilde{\eta}$ is certainly higher than the energy of all among original rectangles $\bar{R}^{(1)}, \bar{R}_1, \dots, \bar{R}_k$ that were subsequently used in the construction of $\tilde{\mathcal{C}}_1^{(p)} \setminus \{\tilde{R}'\}$ plus the energy associated to the rectangle $\tilde{R}'$ (notice that $\tilde{R}'$ is not intersecting the remaining rectangles from the original set $\bar{R}_1, \dots, \bar{R}_k$ used to construct $\tilde{\mathcal{C}}_1^{(p)}$ and it can only touch by its corners the set $C^{(0)}$). The first term can be subsequently bounded from below by $H(\tilde{R})$ and we get the above inequality observing that $\tilde{R}$ and $\tilde{R}'$ are subcritical and thus one gains energy by reducing them to R and R'.

Now, if we replace the rectangles R and R' by their circumscribed rectangle Q^*, the sum on the right hand side above decreases by at least $(L_2^* - 1)h$ leading thus to the inequality (40).

Indeed, if the rectangles R and R' intersect in more than one point, there is a surplus of at least two bonds in the sum of their boundaries yielding at least $2J_2 > (L_2^* - 1)h$. If R and R' just touch in the corner, the boundary has the same number of horizontal and vertical bonds as in Q^* and there is at least $2(L_2^* - 1)$ minus sites inside of Q^*. If R and R' are interacting according to the case ii) from the definition of interacting rectangles, the surplus of at least two vertical edges compensates for the lack of two horizontal edges, while there is at least L_2^* minuses inside Q^*.

Finally, consider the case iii). Notice first that since $\tilde{R}$ and $\tilde{R}'$ are interacting and subcritical, the appearance of the case iii) necessarily means that $L_2^* > l^*$, namely, the coupling constants satisfy the inequality $J_1 < 2J_2$. The rectangles R and R' are separated by a row of minuses and there must exist a unit square q (in the concerned row) whose opposite horizontal edges intersect R and R'. The column passing through this square intersects the boundary of $R \cup R'$ in at least four horizontal bonds — two of them are the edges of q. Suppose first that this is the only such column (and q is the only unit square with the property stated above), the rows below and above the considered separating row contain together at least $L_2^* - 1$ minuses (in addition to L_2^* minuses in the concerned row) in Q^* (see Fig. 5). Then we see that, in the configuration with pluses at all sites inside Q^*, the at least four horizontal edges in the considered column are replaced by only two, with two new vertical edges added in the considered row. The possible increase in energy associated with the replacement of two "weak" horizontal edges by two "strong" vertical edges is at most $2(J_1 - J_2)$ and is compensated filling up L_2^* minuses of the concerned row (recall that in the present case $J_1 < 2J_2$ and thus $2(J_1 - J_2) < L_2^* h$ by definition of L_2^*). The additional $L_2^* - 1$ bonds found above and below the considered row of minuses will then contribute to the second term on the right hand side of (40) leading to its verification in the considered case. If there exist at least two unit squares q_1 and q_2 with the property formulated above, one has at least four surplus horizontal edges to compensate for the only two additional vertical edges (again, use $J_1 < 2J_2$). The L_2^* minuses of the concerned row are then used for the additional term $(L_2^* - 1)h$ when comparing the right hand side of (40) with the energy of Q^*.

FIG. 5

Thus, we are left with the task of finding configurations for which the energy equals the right hand side in (40). Thus, let us suppose that η satisfies the equality in (40). Then, necessarily, $\bar{R}^{(1)}$ in the construction above is supercritical. If it were not and other steps of the construction and filling of final chains would follow, we would run into contradiction since in every of those steps the energy strictly decreeses with, as the proof shows, the inequality (40) maintained. This excludes a possibility to have an equality for the starting configuration η. Similar reasoning also shows that $\eta = \bar{\eta}$. Moreover, if two rectangles from among $\bar{R}_1, \dots, \bar{R}_k$ were contained in $C^{(0)}$, the bound (40) would be valid with the strict inequality. Hence, $C^{(0)}$ consists of $q(x)$ attached to a single rectangle and to get the equality in (40) it must be $(L_2^* - 1) \times L_2^*$ rectangle (see Fig. 2). Thus, the only possibility to achieve an equality in (40) is to take $\eta \in \mathcal{P}$ and we can conclude that

$$\min_{\sigma \in \partial \mathcal{A}} H(\sigma) = H(\mathcal{P})$$

and

$$\min_{\sigma \in \partial \mathcal{A} \setminus \mathcal{P}} H(\sigma) = H(\mathcal{P}) + h,$$

namely, the property 3) of $\mathcal{A}$.

Remark. It is easy to convince oneself that, for every σ in $\mathcal{P}$, there exists a path $\bar{\omega}$, starting in σ and ending in $+\underline{1}$, such that it first passes through $\mathcal{R}(L_2^*, L_2^*)$, then it visits the whole sequence of rectangles $R(L_1, L_2)$ in the sets $\mathcal{L}_2$ and $\mathcal{L}_3$ of the standard tube (see the equation (10)) as well as the saddle points $P(L_1, L_2^*; L_1 + 1, L_2^*)$ and $P(M, L_2; M, L_2 + 1)$ between them, and in the same time

$$\max_{\sigma \in \bar{\omega}} H(\sigma) = H(\mathcal{P}).$$

In other words, along this path the energy is almost monotonously decreasing with some uphill jump over barriers that never overcome the initial height.

It follows from the properties (2) and (3) of the set $\mathcal{A}$ and from the existence of the above mentioned path $\bar{\omega}$ that

$$H(\mathcal{P}) = \min_{\omega :\ -\underline{1} \to +\underline{1}} \max_{\sigma \in \omega} H(\sigma),$$

so that, indeed, any σ in $\mathcal{P}$ is a global saddle point. On the other hand, again by the property (3) of $\mathcal{A}$, it follows that any such a global saddle point is in $\mathcal{P}$.

3.2. Proof of the equality (32)

The strategy of the proof is as follows: we introduce an event $\mathcal{E}_\sigma$ consisting of a set of trajectories starting from $\sigma \in \mathcal{A}$, taking place over an interval of time T_1, and such that

(1) if $\mathcal{E}_\sigma$ takes place, one necessarily reaches the set $\partial \mathcal{A}$ (in a particular manner) before the time T_1.

Moreover, we suppose that there is a lower bound on the probability of this event such that, attempting many times the event $\mathcal{E}_\sigma$, we can conclude with the help of strong Markov property that after a time $T_2 > T_1$ it happens with a high probability. Namely, we are assuming that

(2) one has a uniform lower bound for the probability $\inf_\sigma P(\mathcal{E}_\sigma) \geq \alpha(T_1)$ such that

$$\lim_{\beta \to \infty} (1 - \alpha(T_1))^{\frac{T_2}{T_1}} = 0, \tag{41}$$

Hence, if we succeed in choosing times T_1, T_2 and the event $\mathcal{E}_\sigma$ so that the conditions (1) and (2) are satisfied, we will be able to conclude that, with probability approaching 1 as $\beta \to \infty$, one has to reach $\partial \mathcal{A}$ before T_2.

Next we pass to the construction of the event $\mathcal{E}_\sigma$. It can be quite special, once a correct lower bound on its probability is satisfied.

The first portion of $\mathcal{E}_\sigma$ is an essentially downhill path from σ to $-\underline{1}$. Namely, for every $\sigma \in \mathcal{A}$, $t_1 \in \mathbb{N}$, we define

$$\mathcal{E}^{(1)}_{\sigma,t_1} = \{\omega \in \Omega\colon \sigma_0 = \sigma, \tau_{-\underline{1}} = t_1\}. \tag{42}$$

Next portion of the event means simply that one is staying in the configuration $-\underline{1}$; for every $t_2 > t_1$, $t_2 \in \mathbb{N}$, we set

$$\mathcal{E}^{(2)}_{t_1,t_2} = \{\omega \in \Omega\colon \sigma_t = -\underline{1}, t_1 \le t \le t_2\}. \tag{43}$$

Now comes a very particular growth to $\mathcal{P}$ starting from $-\underline{1}$. Namely, our aim is to consider a set of paths passing through a standard sequence of rectangles, reaching the rectangles in random times of particular orders. The orders of random times are chosen so that, roughly speaking, at every basin of attraction of a particular rectangle one is allowed to stay for a time proportional to the exponent of the product of inverse temperature and the height of the energetical barrier that prevents an erosion and after that one reaches in a shortest possible time the local saddle point toward an enlarged rectangle. This saddle point is higher than the saddle toward the eroded rectangle and the exponent of the difference of the energies of these two barriers will be the main ingredient for the lower bound on the probability of the event $\mathcal{E}_\sigma$. To be more precise, simplifying the notation and writing L^* for L^*_2 and Q_{L_1,L_2} for the rectangle with horizontal edge L_1 and vertical edge L_2 and with the upper left corner in the point $(-\frac{1}{2}, +\frac{1}{2})$ (the origin of $\mathbb{Z}^2$ is the first site x in Q; the edges of Q lie on the dual lattice), for every $t_2 \in \mathbb{N}$ we set

$$\mathcal{E}^{(3)}_{t_2} = \{\omega \in \Omega\colon \sigma_{t_2} = -\underline{1}, \sigma_{t_2+1} = Q_{1,1}, \sigma_{t_2+2} = Q_{1,2}, \sigma_{t_2+4} = Q_{2,2}\}. \tag{44}$$

This is the portion of the path starting with $-\underline{1}$ and growing to $Q_{2,2}$. Further, for every $T_0 < t_{2,2} < t_{2,3} < t_{3,3} < \cdots < t_{L^*-1,L^*}$ ($t_{L_1,L_2} \in \mathbb{N}$) we set

$$\mathcal{E}^{(4)}_{t_2,t_{2,2},t_{2,3},\dots,t_{L^*-1,L^*}} = \mathcal{E}_{2,2} \cap \mathcal{E}_{2,3} \cap \cdots \cap \mathcal{E}_{L,L} \cap \mathcal{E}_{L,L+1} \cap \cdots \cap \mathcal{E}_{L^*-1,L^*-1} \cap \mathcal{E}_{L^*-1,L^*}, \tag{45}$$

where, for every $2 \le L \le L^* - 1$,

$$\mathcal{E}_{L,L} = \{\omega \in \Omega\colon \sigma_{T_{L,L}} = Q_{L,L}, \sigma_t \in \mathcal{B}(Q_{L,L}) \text{ for every } t \in [T_{L,L}, T_{L,L} + t_{L,L} - T_0], \sigma_{T_{L,L}+t_{L,L}} = Q_{L,L+1}\}, \tag{45$'$}$$

$$\mathcal{E}_{L,L+1} = \{\omega \in \Omega\colon \sigma_{T_{L,L+1}} = Q_{L,L+1}, \sigma_t \in \mathcal{B}(Q_{L,L+1}) \text{ for every } t \in [T_{L,L+1}, T_{L,L+1} + t_{L,L+1} - T_0], \sigma_{T_{L,L+1}+t_{L,L+1}} = Q_{L+1,L+1}\}, \tag{45$''$}$$

and

$$\mathcal{E}_{L^*-1,L^*} = \{\omega \in \Omega\colon \sigma_{T_{L^*-1,L^*}} = Q_{L^*-1,L^*}, \sigma_t \in \mathcal{B}(Q_{L^*-1,L^*}) \text{ for every } t \in [T_{L^*-1,L^*}, T_{L^*-1,L^*} + t_{L^*-1,L^*} - T_0], \sigma_{T_{L^*-1,L^*}+t_{L^*-1,L^*}} = S_{L^*}\}. \tag{45$'''$}$$

Here

$$T_{2,2} = t_2 + 4,$$

$$T_{L,L+1} = t_2 + 4 + t_{2,2} + t_{2,3} + \cdots + t_{L,L}$$

for every $2 \le L \le L^* - 1$, and

$$T_{L,L} = T_{L-1,L} + t_{L-1,L}$$

for every $3 \le L \le L^* - 1$. The set S_L is for every $L \le L^*$ obtained by adding a unit square to the vertical right hand edge of $Q(L-1,L)$, $S_L \in \mathcal{P}(L, L-1; L, L)$. The time T_0 is chosen so that $\frac{T_0}{2}$ is an upper bound on the time needed to monotonously decrease the energy from any configuration σ to any other (say η) through a path of "nearest neighbour configurations",

$$T_0 = \Big[\frac{2}{h}\big(\max_{\sigma\in\{-1,1\}^\Lambda} H(\sigma) - \min_{\sigma'\in\{-1,1\}^\Lambda} H(\sigma')\big)\Big] + 1. \tag{46}$$

Of course, $S_{L^*} \in \mathcal{P}$.

Further we define

$$\mathcal{E}^{(4)}_{t_2} = \bigcup_{n_{2,2}=1}^{\bar n_{2,2}} \bigcup_{n_{2,3}=1}^{\bar n_{2,3}} \cdots \bigcup_{n_{L^*-1,L^*}=1}^{\bar n_{L^*-1,L^*}} \tilde{\mathcal{E}}_{n_{2,2},\dots,n_{L^*-1,L^*}}, \tag{47}$$

where

$$\tilde{\mathcal{E}}_{t_2,n_{2,2},\dots,n_{L^*-1,L^*}} = \mathcal{E}^{(4)}_{t_2,t_{2,2}=n_{2,2}T_0,\dots,t_{L^*-1,L^*}=n_{L^*-1,L^*}T_0}, \tag{47$'$}$$

and

$$\bar n_{L-1,L-1} = \bar n_{L-1,L} = \big[\exp\{\beta[h(L-2)+\delta]\}\big] \tag{47$''$}$$

for every $L = 1, \dots, L^*$. Choosing now the times

$$\bar t_1 = \bar t_2 = \big[\exp\{\beta[h(L^*-2)+\delta]\}\big], \tag{48}$$

we define

$$\mathcal{E}_\sigma = \bigcup_{t_1=1}^{\bar t_1} \mathcal{E}^{(1)}_{\sigma,t_1} \bigcap \Big(\bigcup_{t_2=t_1+1}^{t_1+\bar t_2} [\mathcal{E}^{(2)}_{t_1,t_2} \cap \mathcal{E}^{(3)}_{t_2} \cap \mathcal{E}^{(4)}_{t_2}]\Big). \tag{49}$$

The constant δ will be fixed later when also the reason for this particular choice of the constants $\bar n$ will be apparent..

We use $\mathcal{B}(Q)$ to denote the set of all connected clusters C of pluses whose rectangular envelope is Q and such that ∂C contains at least a segment of length not shorter than 2 in any edge of Q. The set $\mathcal{B}(Q)$ is a subset of the basin of attraction of Q in the sense that any sequence of spin flips decreasing the energy and leading to some rectangle R necessarily is such that $R \equiv Q$.

The crucial point in the lower bound on $P(\mathcal{E}_\sigma)$ will be the following inequality:

$$P(\{\omega\colon \sigma_0 = Q_{L-1,L}; \sigma_s \in \mathcal{B}(Q_{L-1,L}) \text{ for every } s \le t\}) \ge \big(1 - e^{\varepsilon\beta} e^{-h(L-2)\beta}\big)^t \tag{50}$$

and, similarly,

$$P(\{\omega\colon \sigma_0 = Q_{L,L}; \sigma_s \in \mathcal{B}(Q_{L,L}) \text{ for every } s \le t\}) \ge \left(1 - e^{\varepsilon\beta} e^{-h(L-1)\beta}\right)^t. \tag{51}$$

To get the estimate (50) ((51) is completely analogous), we introduce, following Freidlin and Wentzell[FW], an auxilliary Markov chain whose space of states is

$$\mathcal{X} = Q \cup \partial\mathcal{B}$$

(to simplify the notation we write Q for $Q_{L-1,L}$ and $\mathcal{B}$ for $\mathcal{B}(Q_{L-1,L})$). The boundary $\partial\mathcal{B}$ of $\mathcal{B}$ is given by

$$\partial\mathcal{B} = \{\eta = \sigma^{(x)} \text{ for some } x\colon \eta \notin \mathcal{B}, \sigma \in \mathcal{B}\}. \tag{52}$$

We introduce a sequence of times

$$v_0 < u_0 \le v_1 < u_1 \le v_2 < \dots$$

with $v_0 = 0$, $u_i, v_i \in \mathbb{N}$,

$$\begin{aligned} u_n &= \inf\{t > v_n\colon \sigma_t \ne \sigma_{t-1}\} \\ v_n &= \inf\{t \ge u_n\colon \sigma_t \in \partial\mathcal{B} \cap Q\}. \end{aligned} \tag{53}$$

We set

$$\xi_n = \sigma_{v_n}, \xi_0 = \sigma_0 = Q, \tag{54}$$

$$\nu = \inf\{n\colon \xi_n \in \partial\mathcal{B}\}. \tag{55}$$

For every $s \in \mathbb{N}$ one has

$$P_Q(\tau_{\partial\mathcal{B}} > s) \ge P_Q(\nu > s) = P(Q \to Q)^s = \left[1 - P(Q \to \partial\mathcal{B})\right]^s, \tag{56}$$

where

$$P(Q \to Q) = P(\xi_1 = Q \mid \xi_0 = Q) \tag{57}$$

and

$$P(Q \to \partial\mathcal{B}) = \sum_{\rho\in\partial\mathcal{B}} P(\xi_1 = \rho \mid \xi_0 = Q). \tag{58}$$

For every $\varepsilon > 0$ we have

$$\begin{aligned} P(Q \to \partial\mathcal{B}) \le \sum_{s=1}^{[e^{\varepsilon\beta}]} \sum_{\bar\sigma_1,\dots,\bar\sigma_{s-1}} & P(\sigma_0 = Q, \sigma_1 = \bar\sigma_1, \dots, \sigma_{s-1} = \bar\sigma_{s-1}, \sigma_s \in \partial\mathcal{B}) + \\ & + P_Q(\sigma_t \notin \mathcal{M} \text{ for every } t \in [1, [e^{\varepsilon\beta}]]), \end{aligned} \tag{59}$$

where

$$\mathcal{M} = \{\sigma \in \{-1,1\}^\Lambda\colon \sigma \text{ is a local minimum for } H\}.$$

Of course $\mathcal{M} \supset \mathcal{R}$.

We have

$$P_Q(\sigma_t \notin \mathcal{M} \text{ for every } t \in [1, [e^{\varepsilon\beta}]]) \leq (\frac{1}{2})^{[e^{\varepsilon\beta}]} \tag{60}$$

for β sufficiently large. Indeed, one can see immediately that

$$\inf_{\sigma \in \{-1,1\}^\Lambda} P_\sigma(\tau_\mathcal{M} < T_0) > (\frac{1}{|\Lambda|})^{T_0}. \tag{61}$$

Hence, by strong Markov property, we get

$$\inf_{\sigma \in \{-1,1\}^\Lambda} P_\sigma(\tau_\mathcal{M} < [e^{\varepsilon\beta}]) > \frac{1}{2} \tag{62}$$

for all $\varepsilon > 0$ and all β sufficiently large, and thus, again by strong Markov property, the bound (60) is implied.

To estimate the first sum on the right hand side of the inequality (59), we first observe that if $\eta \in \partial\mathcal{B}$, then either

(1) the recangular envelope of η is $Q' \supset Q$ with $\eta \cap (Q' \setminus Q) = \{x\}$ if $\eta = \sigma^{(x)}$, $\sigma \in \mathcal{B}$, and x is adjacent, from exterior, to Q, or

(2) η is contained in Q, but it is not connected, or, finally,

(3) η is a connected cluster whose rectangular envelope is Q, but at least on one edge it intersects η on a single unit square.

We claim that, given $\bar{Q} = Q_{L_1,L_2}$ with $L_1 \leq L_2 < L_2^*$, then

$$\min_{\sigma \in \partial\mathcal{B}(\bar{Q})} H(\sigma) = (2J_1L_2 + 2J_2L_1) - hL_1L_2 + h(L_1 - 1) = H(Q_{L_1,L_2}) + h(L_1 - 1) = H(S_{L_1,L_2}), \tag{63}$$

where S_{L_1,L_2} is obtained from Q_{L_1,L_2-1} by adding to it a unit square adjacent to its (short) vertical edge. Indeed, it is easy to see that for η like in case 1) it is

$$H(\eta) - H(Q) \geq 2J_2 - h,$$

whereas in case 2) one has

$$H(\eta) - H(Q) > h(L_1 + L_2 - 1).$$

Hence, since $L_2 < L_2^*$, we are left with the case 3) that is easily treated and leads to (63). By exploiting the reversibility like in Lemma 1, and using the inequalities (59) and (60), it is easy to see that

$$P(Q \to \partial\mathcal{B}) \leq e^{\varepsilon\beta} \exp\{-h\beta(L-2)\} \tag{64}$$

for all $\varepsilon > 0$ and all β sufficiently large. ¿From the inequalities (56) and (64) we get the estimate (50). The bound (51) follows in a similar way.

¿From the estimates (50) and (51) it is easy to deduce that for every $t_{L-1,L}, t_{L,L} \in \mathbb{N}$, all $\varepsilon > 0$, and all β sufficiently large one has

$$P(\mathcal{E}_{L-1,L}) \geq \big(1 - e^{\varepsilon\beta}\exp\{-h\beta(L-2)\}\big)^{t_{L-1,L}} \frac{1}{|\Lambda|^{T_0}} \exp\{-(2J_2 - h)\beta\} \tag{65}$$

and

$$P(\mathcal{E}_{L,L}) \geq \big(1 - e^{\varepsilon\beta}\exp\{-h\beta(L-1)\}\big)^{t_{L,L}} \frac{1}{|\Lambda|^{T_0}} \exp\{-(2J_1 - h)\beta\}. \tag{66}$$

To get the bound (65) we consider, for every $\sigma \in \mathcal{B}_{L-1,L}$, the following event

$$\begin{aligned} \bar{\mathcal{E}}_{L-1,L}(\sigma) = \{\sigma = \sigma_0, \tau_{Q_{L-1,L}} \leq T_0 - L, \sigma_{\tau_{Q_{L-1,L}}+1} = S_L, \tau_{Q_{L,L}} = \tau_{Q_{L-1,L}} + L,& \\ \sigma_t = Q_{L,L} \text{ for every } t \in [\tau_{Q_{L,L}}, \tau_{Q_{L,L}} + T_0 - L - \tau_{Q_{L-1,L}}]\}.& \end{aligned} \tag{67}$$

To put this definition into words: every path in $\bar{\mathcal{E}}_{L-1,L}(\sigma)$ starts from $\sigma \in \mathcal{B}(Q_{L-1,L})$. In a time shorter than $T_0 - L$ it reaches $Q_{L-1,L}$. For every $\sigma \in \mathcal{B}(Q_{L-1,L})$ there exists such a path along which the energy is decreasing. Then a unit square protuberance is attached to the vertical right edge of $Q_{L-1,L}$, this occurs with probability $\frac{1}{|\Lambda|}\exp\{-(2J_2 - h)\beta\}$. After that follows a sequence of spin flips, decreasing energy, on contiguous sites adjacent from the exterior to Q starting near the proturberance and leading to $Q_{L,L}$. The rest of the time up to T_0 is spent in $Q_{L,L}$.

Clearly

$$\begin{aligned} \mathcal{E}_{L-1,L} \supset \bigcup_{\sigma \in \mathcal{B}(Q_{L-1,L})} &\{\sigma_0 = Q_{L-1,L}, \sigma_s \in \mathcal{B}(Q_{L-1,L}) \\ &\text{for every } s \leq t_{L-1,L} - T_0, \sigma_{t_{L-1,L}-T_0} = \sigma\} \bigcap \{G_{t_{L-1,L}-T_0} \bar{\mathcal{E}}_{L-1,L}(\sigma)\}. \end{aligned} \tag{68}$$

Here G_s is the time translation operation by s acting in a natural way on paths. Since also, directly from the definition (67), one gets

$$P(\bar{\mathcal{E}}_{L-1,L}(\sigma)) \geq \frac{1}{|\Lambda|^{T_0}} \exp\{-(2J_2 - h)\beta\} \tag{69}$$

for every $\sigma \in \mathcal{B}(Q_{L-1,L})$, the bound (50) implies the bound (66). In a similar way one obtains the bound (66). Directly from the definitions (45′), (45″), (45‴), (47), (47′), and (47″) it is seen that the events $\tilde{\mathcal{E}}_{n_{2,2},\ldots,n_{L^*-1,L^*}}$ are mutually disjoint. Hence, using (65), (66), and the Markov property, we get

$$\begin{aligned} P(\mathcal{E}^{(4)}) \geq \sum_{n_{2,2}=1}^{\bar{n}_{2,2}} \cdots \sum_{n_{L^*-1,L^*}=1}^{\bar{n}_{L^*-1,L^*}} \big(1 - e^{(\varepsilon-h)\beta}\big)^{n_{2,2}T_0} \frac{1}{|\Lambda|^{T_0}} \exp\{-(2J_1 - h)\beta\} \ldots& \\ \ldots \big(1 - e^{\varepsilon\beta} e^{-hL^*-2}\big)^{n_{L^*-1,L^*}T_0} \exp\{-(2J_2 - h)\beta\}& \end{aligned} \tag{70}$$

for every sufficiently small $\varepsilon > 0$, and all β sufficiently large. Given the choice (47″) of the constants $\bar{n}$ and the values of the quotients in the geometric series above, the sums in (70) turns out to run effectively to ∞. Given δ in the equation (47″), we can choose ε sufficiently small to get

$$P(\mathcal{E}^{(4)}) \geq \exp\bigl\{-[H(\mathcal{P}) - H(Q_{2,2}) - 2\delta]\beta\bigr\}. \tag{71}$$

Now, since the events $\mathcal{E}^{(1)}_{\sigma,t_1}$ with different t_1's are mutually disjoint, and similarly for $\mathcal{E}^{(2)}_{t_1,t_2}$, we have, for β sufficiently large,

$$P(\mathcal{E}_\sigma) \geq \sum_{t_1=1}^{\bar{t}_1} \sum_{t_2=t_1+1}^{t_1+\bar{t}_2} P(\mathcal{E}^{(1)}_{\sigma,t_1} \cap \mathcal{E}^{(2)}_{t_1,t_2} \cap \mathcal{E}^{(3)}_{t_2}) \exp\bigl\{-[H(\mathcal{P}) - H(Q_{2,2}) - 2\delta]\beta\bigr\}. \tag{72}$$

Suppose that, for all $\varepsilon > 0$ and all β sufficiently large, we are able to prove that

$$\inf_{\sigma\in\mathcal{A}} \sum_{t_1=1}^{\bar{t}_1} P(\mathcal{E}^{(1)}_{\sigma,t_1}) \geq e^{-\varepsilon\beta}. \tag{73}$$

Now, since for all $\varepsilon > 0$, from Lemma 1 one has

$$\lim_{\beta\to\infty} P_{-\underline{1}}(\tau_{Q_{1,1}} < \exp(2J_1 + 2J_2 - h - \varepsilon)\beta) = 0 \tag{74}$$

and

$$P(\mathcal{E}^{(3)}) \geq \frac{1}{|\Lambda|^4} \exp\{-(H(Q_{2,2}) - H(-\underline{1}))\beta\}. \tag{75}$$

¿From (72), (74), (75), and (48) one gets

$$\sum_{t_2=t_1+1}^{t_1+\bar{t}_2} P(\mathcal{E}^{(2)}_{t_1,t_2} \cap \mathcal{E}^{(3)}_{t_2}) \geq \exp\{\beta h(L^* - 2) - \beta(H(Q_{2,2}) - H(-\underline{1}) - \delta)\} \tag{76}$$

and then, from (73) and (76), one has

$$P(\mathcal{E}_\sigma) \geq e^{-3\delta\beta} \exp\{-\Gamma\beta + \beta h(L^* - 2)\}. \tag{77}$$

To get (73) we use the following argument:
in a time shorter than T_0 and with a probability larger than $\frac{1}{|\Lambda|^{T_0}}$ we go, starting from any $\sigma \in \mathcal{A}$ to a configuration given by a set of noninteracting subcritical rectangles. Then, from Lemmas 2, 3, and the definition of $\bar{t}_2$ (see the equation (47′)), with large probability one goes to $-\underline{1}$ before $\bar{t}_2$. We leave the details of this argument to the reader.

Now let

$$T_1 = \exp\{\beta[h(L^* - 2) + \delta_1]\} \tag{78}$$

and

$$T_2 = \exp\{\beta(\Gamma + \delta_2)\}. \tag{79}$$

Further, let us divide the time interval T_2 into m subintervals of length T_1 with $m = \frac{T_2}{T_1}$ supposed to be an integer. Let

$$U_i = iT_1, \quad i = 1, \dots, m-1.$$

We have ($\mathcal{E}^c$ denotes the complementary set of $\mathcal{E}$)

$$P_{-\underline{1}}(\tau_{\partial\mathcal{A}} > T_2) = \sum_{\bar\sigma_1,\dots,\bar\sigma_{m-1}\mathcal{A}} P_{-\underline{1}}(\tau_{\partial\mathcal{A}} > T_2, \sigma_{U_i} = \bar\sigma_i, i = 1, \dots, m-1) \leq$$

$$\leq \sum_{\bar\sigma_1,\dots,\bar\sigma_{m-1}\mathcal{A}} P_{-\underline{1}}(\sigma_{U_i} = \bar\sigma_i, i = 1, \dots, m-1, \mathcal{E}^c(-\underline{1}) \cap G_{U_1}\mathcal{E}^c(\bar\sigma_1) \cap \dots \cap G_{U_{m-1}}\mathcal{E}^c(\bar\sigma_{m-1})) \leq$$

$$\leq \big(1 - \inf_{\sigma\mathcal{A}} P(\mathcal{E}(\sigma))\big)^m \leq \exp\big\{e^{-3\delta\beta-\Gamma\beta+h(L^*-2)\beta} e^{\beta(\Gamma+\delta_2-\delta 1)} e^{-h(L^*-2)}\big\}. \quad (80)$$

If $\delta_2 > \delta_1 + 3\delta$, we get

$$\lim_{\beta\to\infty} P_{-\underline{1}}(\tau_{\partial\mathcal{A}} > T_2) = 0.$$

Since δ, δ_1 are arbitrarily small, this concludes the proof of the inequality (32) and thus also of Theorem 2. □

Theorem 1 is now a corollary of Theorem 2 — it follows from the properties 2) and 3) of the set $\mathcal{A}$ whose existence was established during the proof of Theorem 1.

Finally, Theorem 3 directly follows from the results in [S], Lemmas 2,3,4, and Theorem 1.

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