Renormalization Group and the Ginzburg-Landau Equation

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Abstract

We use Renormalization Group methods to prove detailed long time asymptotics for the solutions of the complex Ginzburg-Landau equations with initial data approaching, as $x \to \pm\infty$, different spiralling stationary solutions. A universal pattern is formed, depending only on this asymptotics at spatial infinity.

1. Introduction

Parabolic PDE's often exhibit universal scaling behaviour in long times: the solution behaves as $u(x,t) \sim t^{-\frac{\alpha}{2}} f^*(t^{-\frac{\beta}{2}}x)$ as $t \to \infty$, where the exponents α and β and the function f^* are *universal*, i.e. independent on the initial data and equation, in given classes. This fact has an explanation in terms of the Renormalization Group (RG) [5, 6, 1], very much like the similar phenomenon in statistical mechanics.

In [1] a mathematical theory of this RG was developed and here we would like to apply these ideas to a concrete situation, namely the complex Ginzburg-Landau equation

$$\dot{u} = \partial^2 u + u - |u|^2 u \tag{1}$$

where $u : \mathbf{R} \times \mathbf{R} \to \mathbf{C}$, $\partial = \frac{\partial}{\partial x}$ and dot denotes the time derivative. (1) has an infinite set of stationary solutions

$$u_{q\theta}(x) = \sqrt{1-q^2}\, e^{i(qx+\theta)} \tag{2}$$

and a natural question is to inquire about the time development of initial data $u(x)$ that approaches two such solutions at $\pm\infty$:

$$\lim_{x \to \pm\infty} |u(x) - u_{q_\pm \theta_\pm}(x)| = 0. \tag{3}$$

*Supported by NSF grant DMS-8903041

We prove in this paper a detailed long time asymptotics for such data. As a consequence, we shall show that, for any interval I,

$$\sup_{x\in I} |u(x,t) - e^{i\sqrt{t}\phi^*} u_{q^*\theta^*}(x)| \leq \frac{C_I}{\sqrt{t}} \tag{4}$$

where the constants q^*, ϕ^* and θ^* depend only on the boundary conditions (3). For the detailed asymptotics, see the Theorem in Section 3. This result amplifies those proved in [3].

2. The Renormalisation Group idea

Following [3], we write

$$u = (1-s)e^{i\phi}. \tag{1}$$

The equation (1.1) becomes in these variables

$$\begin{aligned} \dot{s} &= \partial^2 s - 2s + 3s^2 - s^3 + (\partial\phi)^2 - s(\partial\phi)^2 \equiv \partial^2 s - 2s + F(s,\partial\phi) \\ \dot{\phi} &= \partial^2\phi - \frac{2\partial\phi\partial s}{1-s} \equiv \partial^2\phi + G(s,\partial s,\partial\phi) \end{aligned} \tag{2}$$

with the initial data (it will be convenient to take the initial time as $t=1$)

$$\lim_{x\to\pm\infty} s(x,1) = s_\pm \ , \quad \lim_{x\to\pm\infty} |\phi(x,1) - \phi_\pm x - \theta_\pm| = 0. \tag{3}$$

where $2s_\pm = F(s_\pm, \phi_\pm)$. We will specify below the precise space of initial data (3). We will prove the following asymptotics as $t\to\infty$

$$\phi(x,t) = \sqrt{t}\phi^*(\frac{x}{\sqrt{t}}) + \theta^*(\frac{x}{\sqrt{t}}) + \mathcal{O}(\frac{1}{\sqrt{t}}) \tag{4}$$

$$s(x,t) = s^*(\frac{x}{\sqrt{t}}) + \frac{1}{\sqrt{t}}r^*(\frac{x}{\sqrt{t}}) + \mathcal{O}(\frac{1}{t}) \tag{5}$$

again in a certain Banach space. The functions ϕ^*, θ^*, s^*, r^* are *universal*, depending on the initial data only through the boundary conditions (3). They are smooth and therefore u will have the asymptotics (1.4), with

$$\phi^* = \phi^*(0), \quad q^* = \phi^{*'}(0), \quad \theta^* = \theta^*(0).$$

Before going to the proofs, we will explain in a heuristic fashion the RG idea behind the proof.

There are two mechanisms giving rise to the asymptotics (4), (5): the diffusive approach of ϕ to the scaling form (4) and the "slaving" of s to follow whatever ϕ is doing, due to the linear $-2s$ term in (2). We will now explain the first mechanism in the context of an equation for ϕ only. We will see in the next Section that this is precisely what the slaving mechanism produces: s in (2) will be effectively "slaved" to a function of $\partial\phi$ only.

Thus, we consider the equation

$$\dot{\phi} = (1 + a(\partial\phi, \partial^2\phi))\partial^2\phi \ , \quad \phi(x, 1) = f(x) \tag{6}$$

with the boundary conditions (3) for the initial data f. We assume that a is analytic around the origin.

The RG analysis of (6) proceeds as follows. We fix some (Banach) space of initial data $\mathcal{S}$. Next, we pick a number $L > 1$ and set

$$\phi_L(x, t) = L^\alpha \phi(Lx, L^2 t) \tag{7}$$

where α will be chosen later and ϕ solves (6) with the initial data $f \in \mathcal{S}$. The RG map $R : \mathcal{S} \to \mathcal{S}$ (this has to be proven!) is then

$$(Rf)(x) = \phi_L(x, 1). \tag{8}$$

Note that ϕ_L satisfies

$$\dot{\phi}_L = (1 + a_L(\partial\phi_L, \partial^2\phi_L))\partial^2\phi_L. \tag{9}$$

with $a_L(x, y) = a(L^{-1-\alpha}x, L^{-2-\alpha}y)$.

We may now iterate R to study the asymptotics of (6). R depends, besides α, on L and a. Let us denote this by $R_{L,a}$. We have then the "semigroup property"

$$R_{L^n,a} = R_{L,a_{L^{n-1}}} \circ \ldots \circ R_{L,a_L} \circ R_{L,a}. \tag{10}$$

Each R on the RHS involves a solution of a fixed time problem and the long time problem on the LHS is reduced to an iteration of these. Letting $t = L^{2n}$, we have

$$\phi(x, t) = t^{-\frac{\alpha}{2}}(R_{L^n,a}f)(xt^{-\frac{1}{2}}). \tag{11}$$

Now one tries to show that there exists an α such that

$$a_{L^n} \to a^* \ , \quad R_{L^n,a}f \to f^* \tag{12}$$

where

$$R_{L,a^*}f^* = f^* \tag{13}$$

is the fixed point of the RG, corresponding to the scale-invariant equation $\dot{\phi} = (1 + a^*)\partial^2\phi$. Then, rescaling x, the asymptotics of the original problem is given by

$$\phi(xt^{1/2}, t) \sim t^{-\frac{\alpha}{2}} f^*(x). \tag{14}$$

What are α and f^*? To understand this, consider first the trivial case $a = 0$. This is just the diffusion equation with initial conditions increasing linearly at infinity. This problem is of course exactly soluble. We have

$$\begin{aligned} t^{-\frac{1}{2}}\phi(\sqrt{t}x, t + 1) &= \frac{1}{\sqrt{4\pi t}} \int e^{-\frac{1}{4}(y-x)^2} f(\sqrt{t}y) dy, \\ &\to_{t\to\infty} \phi_- x + (\phi_+ - \phi_-)(x + 2\partial)e(x) \equiv \phi_0^*(x) \end{aligned} \tag{15}$$

where $e(x) = \int_{-\infty}^{x} e^{-\frac{1}{4}y^2} \frac{dy}{\sqrt{4\pi}}$. In RG terminology, we have the "Gaussian" fixed point ϕ_0^* corresponding to this $\phi_\pm$ boundary condition problem. It is easy to check that ϕ_0^* is a fixed point for the map (7) with $\alpha = -1$.

How about $a \neq 0$? We take again $\alpha = -1$ whence the fixed point equation is

$$\dot{\phi} = (1 + a^*(\partial\phi))\partial^2\phi \tag{16}$$

where $a^*(\cdot) = a(\cdot, 0)$ and the fixed point is the scale invariant solution

$$\phi(x,t) = \sqrt{t}\phi^*(\frac{x}{\sqrt{t}}). \tag{17}$$

We get (replacing $\frac{x}{\sqrt{t}}$ by z)

$$(1 + a^*)\frac{d^2}{dz^2}\phi^* + \frac{1}{2}z\frac{d}{dz}\phi^* - \frac{1}{2}\phi^* = 0 \tag{18}$$

with $a^* = a^*(\partial\phi^*)$ and, for small $\phi_\pm$, we look for a solution

$$\phi^* = \phi_0^* + \rho \tag{19}$$

where $\rho(\pm\infty) = 0$ and ϕ_0^* is the Gaussian solution (15), which solves (18) with $a^* = 0$. This is easy to solve (see Proposition 1 below or the Proposition in Section 4 of [1]).

Consider then θ^*. We set

$$\phi(x,t) = \phi^*(x,t) + \theta(x,t) \tag{20}$$

where, with an abuse of notation, $\phi^*(x,t) = \sqrt{t}\phi^*(\frac{x}{\sqrt{t}})$ and ϕ^* is given by (19); ϕ solves (6). Then,

$$\dot{\theta} = \partial^2\theta + (a\partial^2\phi - a^*\partial^2\phi^*) \tag{21}$$

with $\theta(\pm\infty) = \theta_\pm$. Now we set

$$\theta_L(x,t) = \theta(Lx, L^2t) \tag{22}$$

which, using $\partial^l\phi^*(Lx, L^2t) = L^{1-l}\partial^l\phi^*(x,t)$, satisfies the equation

$$\begin{aligned} \dot{\theta}_L \;&=\; \partial^2\theta_L + L(a(\partial\phi^* + L^{-1}\partial\theta_L, L^{-1}\partial^2\phi^* + L^{-2}\partial^2\theta_L)(\partial^2\phi^* + L^{-1}\partial^2\theta_L) \\ &\quad -a^*(\partial\phi^*)\partial^2\phi^*). \end{aligned} \tag{23}$$

Thus, reasoning as above, we expect

$$\theta_{L^n} \to \theta^* \tag{24}$$

where $\theta^*(x,t) = \theta^*(\frac{x}{\sqrt{t}})$ satisfies the $L \to \infty$ form of (23):

$$\dot{\theta}^* = \partial^2\theta^* + a^*\partial^2\theta^* + (a_x(\partial\phi^*, 0)\partial\theta^* + a_y(\partial\phi^*, 0)\partial^2\phi^*)\partial^2\phi^*. \tag{25}$$

This is a linear equation, easy to solve, whose solution is a small perturbation of the "Gaussian" solution (which solves (25) with $a = 0$),

$$\theta_0^*(x) = \theta_- + (\theta_+ - \theta_-)e(x). \tag{26}$$

Actually, $\theta_0^*(x) = \phi_0^{*'}(x)$, where ϕ_0^* is given by (15), and $\theta_0^*(x)$ is a fixed point of the map

$$(R_L\theta)(x) = (e^{(L^2-1)\partial^2}\theta)(Lx). \tag{27}$$

i.e. of (7), (8) with $\alpha = 0$ and $a = 0$.

Finally, one sets

$$\phi(x,t) = \phi^*(x,t) + \theta^*(x,t) + \psi(x,t) = \sqrt{t}\phi^*(\frac{x}{\sqrt{t}}) + \theta^*(\frac{x}{\sqrt{t}}) + \psi(x,t) \tag{28}$$

and one studies the asymptotics of ψ in the same manner. The last two steps (θ and ψ) were carried out in [1] in great detail. We will now combine this RG approach with the slaving principle to arrive at (4), (5).

3. The full model

We may summarize the previous discussion in a more conventional language: we look for solutions of (2.2) of the form

$$\begin{aligned} \phi(x,t) &= \phi^*(x,t) + \theta^*(x,t) + \psi(x,t) \\ s(x,t) &= s^*(x,t) + r^*(x,t) + v(x,t) \end{aligned} \tag{1}$$

with

$$\begin{aligned} \phi^*(x,t) = \sqrt{t}\phi^*(\frac{x}{\sqrt{t}}),\ \ \theta^*(x,t) = \theta^*(\frac{x}{\sqrt{t}}), \\ s^*(x,t) = s^*(\frac{x}{\sqrt{t}}),\ \ r^*(x,t) = \frac{1}{\sqrt{t}}r^*(\frac{x}{\sqrt{t}}) \end{aligned} \tag{2}$$

(we stick to this abuse of notation in order not to proliferate symbols, but we shall use $\frac{d}{dz}$ or prime for the derivative with respect to the single argument of ϕ^*, θ^*, etc...,as opposed to ∂ for the partial derivative with respect to x). The boundary conditions are

$$\begin{aligned} \lim_{x\to\infty} |\phi^*(x) - \phi_\pm x| = 0\ ,\ \lim_{x\to\infty} |\theta^*(x) - \theta_\pm| = 0\ , \\ \lim_{x\to\infty} |s^*(x) - s_\pm| = 0\ ,\ \lim_{x\to\infty} r^*(x) = 0\ , \end{aligned} \tag{3}$$

ϕ^* and θ^* are solved by a fixed point method near the Gaussian solutions

$$\begin{aligned} \phi_0^*(z) &= \phi_- z + (\phi_+ - \phi_-)(z + 2\frac{d}{dz})e(z) \\ \theta_0^*(z) &= \theta_- + (\theta_+ - \theta_-)e(z) \end{aligned} \tag{4}$$

and s^* and r^* are "slaved" to them.

Thus, let us set

$$s^* = s^*(\partial\phi^*), \tag{5}$$

where s^* solves the algebraic equation $-2s^* + F(s^*, \partial\phi^*) = 0$, which is equivalent to $1 - s^* = \sqrt{1 - (\partial\phi^*)^2}$, so that s^* is of order $(\partial\phi^*)^2$ for $(\partial\phi^*)$ small. Note that, by (2), $\partial\phi^*(x,t) = \phi^{*'}(\frac{x}{\sqrt{t}})$, so $s^*(\frac{x}{\sqrt{t}}) = s^*(\phi^{*'})$. Then ϕ^* is the solution of $\dot{\phi}^* = \partial^2\phi^* + G(s^*, \partial s^*, \partial\phi^*)$, i.e., using (2),

$$B\phi^* \equiv (-\frac{d^2}{dz^2} - \frac{1}{2}z\frac{d}{dz} + \frac{1}{2})\phi^* = -2\frac{\frac{d}{dz}s^*\frac{d}{dz}\phi^*}{1 - s^*} = -2\frac{\frac{d}{dz}\phi^*\frac{d^2}{dz^2}\phi^*}{1 - (\frac{d}{dz}\phi^*)^2}. \tag{6}$$

We set

$$\phi^* = \phi_0^* + \rho \tag{7}$$

and solve the fixed point equation (6) in the space of C^N functions equipped with the norm

$$\|\rho\|_N = \max_{0\leq m\leq N}\sup_z |\frac{d^m}{dz^m}\rho(z)|e^{\frac{z^2}{8}}. \tag{8}$$

Then we have

Proposition 1. *For any N, there exists an $\epsilon > 0$ such that, for $|\phi_\pm| \leq \epsilon$ in* (3), (6) *has a unique solution with*

$$\|\rho\|_N \leq C\epsilon^3. \tag{9}$$

and then

$$\|s^*\|_N \leq C\epsilon^2 \tag{10}$$

Next, we turn to θ^* and r^*. Set

$$\phi = \phi^* + \theta\ ,\ s = s^* + r \tag{11}$$

whereby

$$\dot{\theta} = \partial^2\theta + G(s, \partial s, \partial\phi) - G(s^*, \partial s^*, \partial\phi^*) \tag{12}$$

and

$$\dot{r} = \partial^2 r - 2r + F(s, \partial\phi) - F(s^*, \partial\phi^*) + H_1 \tag{13}$$

where

$$H_1 = \partial^2 s^* - \partial_t s^* = t^{-1}(s^{*''}(\frac{x}{\sqrt{t}}) + (\frac{x}{2\sqrt{t}})(s^{*'}(\frac{x}{\sqrt{t}})) \tag{14}$$

Note that $H_1(x,t) = t^{-1}H_1(\frac{x}{\sqrt{t}})$. r^* is now taken (recall (2)) to be the part of (13) proportional to $t^{-\frac{1}{2}}$: we solve

$$2r^* = DF_{s^*,\partial\phi^*}(r^*, \partial\theta^*) \tag{15}$$

for $r^*(\partial\theta^*(x,t)) = \frac{1}{\sqrt{t}}r^*(\theta^{*'}(\frac{x}{\sqrt{t}}))$, where DF is the derivative of F, i.e.

$$DF_{s^*,\partial\phi^*}(r^*, \partial\theta^*) = 6s^*r^* - 3s^{*2}r^* + 2\partial\phi^*\partial\theta^* - 2s^*\partial\phi^*\partial\theta^* - (\partial\phi^*)^2r^*$$

Thus, $r^*(\partial\theta^*)$ is of order $\epsilon\partial\theta^*$, and we let θ^* solve

$$\dot{\theta}^* = \partial^2\theta^* + DG_{s^*,\partial s^*,\partial\phi^*}(r^*, \partial r^*, \partial\theta^*) \tag{16}$$

where DG is the derivative of G, i.e.

$$DG_{s^*,\partial s^*,\partial\phi^*}(r^*, \partial r^*, \partial\theta^*) = -\frac{2\partial s^*\partial\phi^*}{(1-s^*)^2}r^* - \frac{2\partial s^*}{1-s^*}\partial\theta^* - \frac{2\partial\phi^*}{1-s^*}\partial r^*.$$

and we get

$$(-\frac{d^2}{dz^2} - \frac{1}{2}z\frac{d}{dz})\theta^* = -\frac{2s^{*'}\phi^{*'}}{(1-s^*)^2} - \frac{2s^{*'}\theta^{*'}}{1-s^*} - \frac{2\phi^{*'}r^{*'}}{1-s^*} \tag{17}$$

This is solved by setting

$$\theta^* = \theta_0^* + \tau \tag{18}$$

and we have

Proposition 2. *For any N, there exists an $\epsilon > 0$ such that, for $|\theta_\pm| \le \epsilon$ in* (3)*, and s^*, ϕ^* as in Proposition 1,* (16) *has a unique solution with*

$$\|\tau\|_N \le C\epsilon^3. \tag{19}$$

and then

$$\|r^*\|_N \le C\epsilon^2. \tag{20}$$

We can now write the final equations that we will study. In accordance with (1), we set

$$\theta = \theta^* + \psi\ , \quad r = r^* + v \tag{21}$$

whence, from (12) and (13) we get the equations

$$\begin{aligned}
\dot{\psi} &= \partial^2\psi + [G(s,\partial s,\partial\phi) - G(s^*+r^*, \partial(s^*+r^*), \partial(\phi^*+\theta^*))] + \\
&\quad [G(s^*+r^*, \partial(s^*+r^*), \partial(\phi^*+\theta^*)) - G(s^*,\partial s^*,\partial\phi^*) - DG_{s^*,\partial s^*,\partial\phi^*}(r^*,\partial r^*,\partial\theta^*)] \\
&= \partial^2\psi + A(v,\partial v,\partial\psi) + D \qquad (22)\\
\dot{v} &= \partial^2 v - 2v + [F(s,\partial\phi) - F(s^*+r^*, \partial(\phi^*+\theta^*))] + \\
&\quad [F(s^*+r^*, \partial(\phi^*+\theta^*)) - F(s^*,\partial\phi^*) - DF_{s^*,\partial\phi^*}(r^*,\partial\theta^*)] + H \\
&= \partial^2 v - 2v + B(v,\partial\psi) + E \qquad (23)
\end{aligned}$$

where $H = H_1 + H_2$ and

$$\begin{aligned} E &= F(s^* + r^*, \partial(\phi^* + \theta^*)) - F(s^*, \partial\phi^*) - DF_{s^*,\partial\phi^*}(r^*, \partial\theta^*) + H \\ H_2 &= \partial^2 r^* - \partial_t r^* = t^{-\frac{3}{2}}(r^{*''}(\frac{x}{\sqrt{t}}) + (\frac{x}{2\sqrt{t}})(r^{*'}(\frac{x}{\sqrt{t}})) \end{aligned} \tag{24}$$

and we have separated the inhomogenous terms in (22), (23) for future convenience.

We will now specify the initial data for (2.2) that we will consider. We take a perturbation of a "Gaussian" data satisfying the boundary conditions.

$$\phi(x) \equiv \phi(x,1) = \phi_0^*(x) + \theta_0^*(x) + \psi(x) \tag{25}$$

$$s(x) \equiv s(x,1) = s^*(\partial\phi_0^*(x)) + r^*(\partial\theta_0^*(x)) + v(x) \tag{26}$$

and describe now a Banach space of ψ and v.

Due to the fact that the equations (22),(23) involve second derivatives in the nonlinear terms (v is coupled to $\partial\psi$ and ψ to ∂v), it is useful to use Fourier transform to control smoothness. Due to the fact that they involve functions like ϕ^* that do not decay at infinity, we need to be careful about the spatial behaviour at infinity. We need norms that combine these two aspects, i.e. the *phase space* behaviour of ψ and v.

Thus, let χ be a non negative C^∞ function on $\mathbf{R}$ with compact support on the interval $(-1,1)$, such that its translates by $\mathbf{Z}$, $\chi_n = \chi(\cdot - n)$, form a partition of unity on $\mathbf{R}$. For $f \in C^2$, we then introduce the norms, for $j = 1, 2$

$$\|f\|^{(j)} = \sup_{n\in\mathbf{Z}, k\in\mathbf{R}, i\le j} (1+n^4)(1+k^2)|\widehat{\chi_n \partial^i f}(k)|. \tag{27}$$

Roughly, $\|f\|^{(j)} < \infty$ means that f falls off at least as x^{-4} at infinity and $\hat{f}(k)$ as k^{-2}. Note that r^*, the derivatives of θ^* and s^* and $\partial^2\phi^*$ have a norm bounded by ϵ (actually, by ϵ^2 for s^* and r^*). To check this, use the explicit form (2.15), (2.26) of ϕ_0^*, θ_0^* and the bound $\|f\|^{(j)} \le C\|f\|_N$ for $N \ge j+2$. The n^4 could be changed to anything increasing faster than n^γ, for $\gamma > 1$, but not faster than $e^{\frac{n^2}{8}}$ (coming from (8)). These norms for different choices of χ may be shown to be equivalent. The reason we need different norms for v and ψ has to do with the slaving and will be explained below.

Our main theorem may now be stated.

Theorem. *For any $\delta > 0$ there is an $\epsilon > 0$ such that for $|\phi_\pm|$, $|\theta_\pm|$, $\|\psi\|^{(2)}$, $\|v\|^{(1)} \le \epsilon$, the equations* (2.2) *have a unique solution satisfying*

$$\begin{aligned} \lim_{t\to\infty} t^{\frac{1}{2}(1-\delta)} \|\phi(\sqrt{t}\cdot, t) - \sqrt{t}\phi^*(\cdot) - \theta^*(\cdot)\|^{(2)} = 0 \\ \lim_{t\to\infty} t^{1-\delta} \|s(\sqrt{t}\cdot, t) - s^*(\phi^{*'}(\cdot)) - \frac{1}{\sqrt{t}} r^*(\theta^{*'}(\cdot))\|^{(1)} = 0 \end{aligned} \tag{28}$$

where ϕ^* and θ^* are given in Propositions 1 *and* 2 *and s^* and r^* in* (5) *and* (15).

Remark 1. The convergence in the norm (27) implies convergence in L^1 and in L^∞, see equations (47) and (48) below, applied to $i = 0$. Thus, using $1 - s^* = \sqrt{1 - (\partial\phi^*)^2}$, we have, for any interval I,

$$\sup_{x\in I} |u(x,t) - \sqrt{1 - q^{*2}} e^{i(xq^* + \sqrt{t}\phi^*(0) + \theta^*(0))}| \le C_I t^{-\frac{1}{2}+\delta} \tag{29}$$

The "anomalous" term $\phi^*(0)$ equals $\pi^{-\frac{1}{2}}(\phi_- - \phi_+) + \mathcal{O}(\epsilon^3)$ and $q^* = (\partial\phi^*)(0) = \frac{1}{2}(\phi_+ - \phi_-) + \mathcal{O}(\epsilon^3)$

Remark 2. The Theorem does not depend much on the specific form of F and G in (2.2). For a more general application of the slaving principle, see [2]. One obtains easily more refined asymptotics, as in [1], namely the error in the first equation in (28) is $t^{-\frac{1}{2}}\psi^*(\cdot) + \mathcal{O}(t^{-1+\delta})$.

Proof. Using the Propositions (where we always assume that we have N as large as needed), we can replace in (25), (26), ϕ_0^*, θ_0^* by the true fixed points ϕ^*, θ^*, and still denote by ψ, v the remainder. We consider the equations (22,23) and solve them by the RG method. Thus, let

$$\begin{aligned} \psi_n(x,t) &= \psi(L^n x, L^{2n} t), \quad \Psi_n(x) = \psi_n(x,1) \\ v_n(x,t) &= L^n v(L^n x, L^{2n} t), \quad V_n(x) = v_n(x,1). \end{aligned} \tag{30}$$

We have

$$\Psi_{n+1}(x) = \psi_n(Lx, L^2)\ , \quad V_{n+1}(x) = L v_n(Lx, L^2) \tag{31}$$

and ψ_n, v_n are the solutions of the equations

$$\dot{\psi}_n = \partial^2\psi_n + A_n(v_n, \partial v_n, \partial\psi_n) + D_n \tag{32}$$

$$\dot{v}_n = \partial^2 v_n - L^{2n} v_n + B_n(v_n, \partial\psi_n) + E_n \tag{33}$$

obtained from (22,23) upon the scaling (30). Concretely, let $s_n \equiv s^* + L^{-n}r^* + L^{-n}v_n$ and $\partial\phi_n \equiv \partial\phi^* + L^{-n}\partial(\theta^* + \psi_n)$. Then, using $s^*(L^n x, L^{2n}t) = s^*(x,t)$ and the corresponding scaling properties of $r^*, \partial\phi^*, \partial\theta^*$, we get

$$\begin{aligned} A_n &= L^{2n}[G(s_n, L^{-n}\partial s_n, \partial\phi_n) - (\ldots)|_{\psi_n = v_n = 0}] \\ &= \sum_{|\mathbf{p}|>0} L^{n d_{\mathbf{p}}} a_{\mathbf{p}} v_n^{p_1} \partial v_n^{p_2} \partial\psi_n^{p_3} \end{aligned} \tag{34}$$

where p_2, $p_3 = 0, 1$, $|\mathbf{p}| = p_1 + p_2 + p_3$ and

$$a_{\mathbf{p}} = 2(-)^{1+p_1} \frac{(\partial s^* + L^{-n}\partial r^*)^{1-p_2}(\partial\phi^* + L^{-n}\partial\theta^*)^{1-p_3}}{(1 - s^* - L^{-n}r^*)^{1+p_1}} \tag{35}$$

$$d_{\mathbf{p}} = 1 - p_1 - p_2 - p_3.$$

In the same way

$$D_n = \sum_{|\mathbf{p}|>1} L^{n d_{\mathbf{p}}} \tilde{a}_{\mathbf{p}} r^{*p_1} \partial r^{*p_2} \partial\theta^{*p_3} \tag{36}$$

$\tilde{a} = a|_{r^* = \theta^* = 0}$. Note that in (34) each term has $d_{\mathbf{p}} \le 0$, while in (36), we have $d_{\mathbf{p}} \le -1$. B_n has a similar expansion

$$\begin{aligned} B_n &= L^{3n}[F(s_n, \partial\phi_n) - (\ldots)|_{v_n = \psi_n = 0}] \\ &= \sum_{|\mathbf{p}|>0} L^{n(3-p_1-p_2)} b_{\mathbf{p}} v_n^{p_1} \partial\psi_n^{p_2} \end{aligned} \tag{37}$$

(where in fact $|\mathbf{p}| \leq 3$, see (2.2)). Finally,

$$E_n = \sum_{|\mathbf{p}|>1} L^{n(3-p_1-p_2)} \tilde{b}_{\mathbf{p}} r^{*p_1} \partial \theta^{*p_2} + L^n H_1 + H_2 \tag{38}$$

Note that here we have $3 - p_1 - p_2 \leq 2$ in (37) and $3 - p_1 - p_2 \leq 1$ in (38). We will show inductively in n that

$$\|\Psi_n\|^{(2)}, \|V_n\|^{(1)} \leq (\frac{C}{L})^n \epsilon \equiv \epsilon_n. \tag{39}$$

Here and below, C denotes a generic constant, which may change from place to place.

Thus, let us assume (39) for n and prove it for $n+1$. We need to solve (32,33) with initial data $\Psi_n(x)$, $V_n(x)$ up to time L^2. This is done by writing (32,33) as integral equations :

$$\begin{aligned}
\psi_n(t) &= (e^{(t-1)\partial^2}\Psi_n + \int_0^{t-1} ds e^{s\partial^2} D_n(t-s)) + \int_0^{t-1} ds e^{s\partial^2} A_n(t-s) \equiv \psi_n^0 + \mathcal{A}_n(\psi_n, v_n) \\
v_n(t) &= (e^{(t-1)(\partial^2-2L^{2n})} V_n + \int_0^{t-1} ds e^{s(\partial^2-2L^{2n})} E_n(t-s)) + \int_0^{t-1} ds e^{s(\partial^2-2L^{2n})} B_n(t-s) \\
&\equiv v_n^0 + \mathcal{B}_n(\psi_n, v_n).
\end{aligned} \tag{40}$$

with obvious notations, and where ψ_n^0 , v_n^0 regroup the first two terms ; as in [4], we use the contraction mapping principle with the norms

$$\|w\|_L^{(j)} = \sup_{t\in[1,L^2]} \|w(\cdot,t)\|^{(j)}. \tag{41}$$

We shall show that $T(\psi, v) = (\psi_n^0, v_n^0) + (\mathcal{A}_n(\psi, v), \mathcal{B}_n(\psi, v))$ maps the ball $B = \{(\psi, v) | \|\psi - \psi_n^0\|_L^{(2)} + \|v - v_n^0\|_L^{(1)} \leq \epsilon_n\}$ into itself and is a contraction there.

We need to estimate the norms of $\mathcal{A}$ and $\mathcal{B}$. Consider $\mathcal{A}$ first, and a generic term in (34):

$$\alpha(x, \mathbf{p}) = \int_0^{t-1} ds \int dy e^{s\partial^2}(x-y) a_{\mathbf{p}}(y) F_{\mathbf{p}}(y). \tag{42}$$

where $F_{\mathbf{p}}(y) = (v^{p_1} \partial v^{p_2} \partial \psi^{p_3})(y, t-s)$. We localize the y variable:

$$\alpha(x, \mathbf{p}) = \sum_{m\in\mathbf{Z}} \int_0^{t-1} ds \int dy e^{s\partial^2}(x-y) \chi_m(y) a_{\mathbf{p}}(y) F_{\mathbf{p}}(y) \equiv \sum_{m\in\mathbf{Z}} \alpha_m(x, \mathbf{p}). \tag{43}$$

We want to bound, for $i \leq 2$,

$$\beta_{ml}(\mathbf{p}) = \sup_k |(1+k^2) \chi_l \widehat{\partial^i \alpha_m}(k, \mathbf{p})|. \tag{44}$$

We distinguish between $|l - m| \geq 2$ and $|l - m| < 2$.

(A) Let first $|m - l| \geq 2$. Then χ_m and χ_l have disjoint supports and $e^{s\partial^2}(x-y)$ is smooth uniformly in s. We write

$$\beta_{ml}(\mathbf{p}) = \sup_k |\int dx e^{-ikx} \int_0^{t-1} ds \int dy (1-\partial_x^2)(\chi_l(x) \partial_x^i e^{s\partial^2}(x-y)) \chi_m(y) a_{\mathbf{p}}(y) F_{\mathbf{p}}(y)| \tag{45}$$

and estimate the various factors on the RHS.

First, we use, for $j \leq 4$, $s \leq L^2 - 1$,

$$|\partial^j e^{s\partial^2}(x-y)| \leq C_L e^{-|m-l|} \tag{46}$$

where C_L denotes a constant depending on L; (46) holds on the support of χ_m and χ_l. To bound $F_{\mathbf{p}}(y)$, note first that our norms (27) imply the following L^∞ and L^1 bounds. First, for any function w,

$$\begin{aligned} |\partial^i w(x)| &= \sum_l |\chi_l(x)\partial^i w(x)| \leq \sum_l \int dk |\widehat{\chi_l \partial^i} w(k)| \\ &\leq \sum_l (1+l^4)^{-1} \int dk (1+k^2)^{-1} \|w\|^{(j)} \leq C\|w\|^{(j)} \end{aligned} \tag{47}$$

for $i \leq j$, and secondly, by Schwartz

$$\int |\chi_m \partial^i w| =\leq \left(2\int |\chi_m \partial^i w|^2\right)^{\frac{1}{2}} = C\left(2\int |\widehat{\chi_m \partial^i} w|^2\right)^{\frac{1}{2}} = \frac{\sqrt{2}\|w\|^{(j)}}{1+m^4}. \tag{48}$$

We will see below that

$$\|\psi_n^0\|_L^{(2)},\ \|v_n^0\|_L^{(1)} \leq C\epsilon_n \tag{49}$$

so that, for (ψ, v) in B,

$$\|\psi\|_L^{(2)} + \|v\|_L^{(1)} \leq (2C+1)\epsilon_n. \tag{50}$$

Finally, from (35), (2.15), (2.26) and the Propositions, we have

$$\|a_{\mathbf{p}}\|_\infty \leq C^{1+p_1}\epsilon^{3-2p_2-p_3}. \tag{51}$$

Now we bound (45): the x integral is controlled by $\chi_l(x)$ or its derivatives, the s integral is less than L^2 and we use (47) with $w = v$ or ψ for all factors in $F_{\mathbf{p}}(y)$ except one, for which we use (48):

$$\sum_{|\mathbf{p}|>0} L^{-nd_{\mathbf{p}}}\beta_{ml}(\mathbf{p}) \leq C_L\epsilon(\|v\|_L^{(1)} + \|\psi\|_L^{(2)})e^{-|m-l|}(1+m^4)^{-1} \tag{52}$$

We get ϵ from (51) if p_2 or $p_3 = 0$, or, otherwise, from (50), because we have then a nonlinear term in $\|v\|_L^{(1)}, \|\psi\|_L^{(2)}$.

(B) Let now $|m-l| < 2$. The difficulty is that we do not have (46) for s close to zero. We will use Fourier transforms. Let $\phi_m \in C_0^\infty(\mathbf{R})$ be such that $\phi_m\chi_m = \chi_m$ and denote $\phi_m a_{\mathbf{p}}$ by f_m. Then

$$\widehat{\chi_l \partial^i}\alpha_m(k, \mathbf{p}) = \int dsdpdq\hat{\chi}_l(k-p)(ip)^i e^{-sp^2}\hat{f}_m(p-q)\widehat{\chi_m F_{\mathbf{p}}}(q). \tag{53}$$

Let us consider the various factors on the RHS. Since χ is C^∞ with compact support, we have

$$|\hat{\chi}_l(k-p)| = |e^{-i(k-p)l}\hat{\chi}(k-p)| \leq C_r(1+|k-p|^r)^{-1} \tag{54}$$

for any r.

For $\hat{f}_m$, note that due to (2.15), (2.26) and the Propositions,

$$\int |(1+(-\partial^2)^r)\phi_m a_{\mathbf{p}}(x)|dx \le C^{1+p_1}\epsilon^{3-2p_2-p_3} \tag{55}$$

for all $r \le \frac{N-1}{2}$, whence, taking N large enough in Propositions 1,2,

$$|\hat{f}_m(p-q)| \le CC^{1+p_1}\epsilon^{3-2p_2-p_3}(1+(p-q)^{2r})^{-1}. \tag{56}$$

for any r. Also, $\int ds|p|^i e^{-sp^2} \le CL^2$ if $i \le 2$, so, provided we can show

$$|\widehat{\chi_m F_{\mathbf{p}}}(q)| \le C^{|\mathbf{p}|}(\|v\|_L^{(1)})^{p_1+p_2}(\|\psi\|_L^{(2)})^{p_3}(1+m^4)^{-1}(1+q^2)^{-1} \tag{57}$$

we can perform the convolutions in (53) to get

$$\sum_{|\mathbf{p}|>0} L^{+nd_{\mathbf{p}}}\beta_{ml}(\mathbf{p}) \le C_L\epsilon(\|v\|_L^{(1)}+\|\psi\|_L^{(2)})(1+m^4)^{-1}. \tag{58}$$

(52) and (58) yield, combined with (50),

$$\|\mathcal{A}_n(\psi,v)\|^{(2)} \le C_L\epsilon\epsilon_n. \tag{59}$$

To prove (57), use

$$|\widehat{\chi_m\partial^i}w| \le (1+m^4)^{-1}(1+k^2)^{-1}\|w\|^{(j)}$$

and

$$|\widehat{\partial^i w}(k)| = |\sum_n \chi_n\widehat{\partial^i}w(k)| \le \frac{C\|w\|^{(j)}}{1+k^2}$$

for $w=v$ or ψ, and perform the convolutions using

$$\int dp(1+(k-p)^2)^{-1}(1+p^2)^{-1} \le C(1+k^2)^{-1}. \tag{60}$$

For $\mathcal{B}_n$ the analysis is similar, but we have to take advantage of the "mass" term, $-2L^{2n}$, in (40), in order to control the terms with $3-p_1-p_2=2$ in (37) or $3-p_1-p_2=1$ in (38). It is also here that the difference in the norms (27) enters.

The only change in (A) above is in (46):

$$|\partial^j e^{s(\partial^2-L^{2n})}(x-y)| \le C_L e^{-sL^{2n}}e^{-|m-l|} \tag{61}$$

(for $|l-m|\ge 2$).

For (B), we integrate by parts $\partial_x=-\partial_y$ in (45) (recall $i\le 1$ here), and we get $\partial_y(\chi_m(y)\alpha_{\mathbf{p}}(y)F_{\mathbf{p}}(y))$. After going to Fourier transform, as in (53), we see that we can use (54), (56) when $\chi_m(y)$ or $\alpha_{\mathbf{p}}(y)$ is replaced by its derivative. We also have (57) for $F_{\mathbf{p}}(y)$ replaced by its derivative, because F does not involve derivatives of v and contains only the first derivative of ψ. It is here that we use a different norm on v and ψ. Hence proceeding as before, we end up estimating

$$\sup_k (1+k^2)\int_0^{t-1} dse^{-s(k^2+L^{2n})}(1+k^2)^{-1} \leq CL^{-2n} \tag{62}$$

We also get L^{-2n} from the integral over s of (61). Therefore,

$$\|\mathcal{B}_n(\psi, v)\|^{(1)} \leq C_L \epsilon\epsilon_n. \tag{63}$$

The bounds (49) for the inhomogenous terms are proved in the same way, remembering that $d_{\mathbf{p}} \leq -1$ in (36) and $3 - p_1 - p_2 \leq 1$ in (38). For H_1, H_2, it is easy to see that their norm is bounded by $C\epsilon^2$, because derivatives of s^*, r^* are bounded using (10), (20).

Thus, we have shown that T maps B into itself (first, choose L so that, in (39), $C \leq L^\delta$; then, take ϵ small enough, depending on L). The contraction is proved in the same way :

$$\begin{aligned} \|\mathcal{A}(\psi_1, v_1) - \mathcal{A}(\psi_2, v_2)\|^{(2)} &\leq C_L\epsilon(\|\psi_1 - \psi_2\|^{(2)} + \|v_1 - v_2\|^{(1)}) \\ \|\mathcal{B}(\psi_1, v_1) - \mathcal{B}(\psi_2, v_2)\|^{(1)} &\leq C_L\epsilon(\|\psi_1 - \psi_2\|^{(2)} + \|v_1 - v_2\|^{(1)}) \end{aligned} \tag{64}$$

To complete the induction for (39), we need to study the inhomogenous terms in (40).

The main task is to show

$$\|R_0\Psi_n\|^{(2)} = \|(e^{(L^2-1)\partial^2}\Psi_n)(L\cdot)\|^{(2)} \leq \frac{C}{L}\|\Psi_n\|^{(2)}. \tag{65}$$

It is now easier to work in the x representation. We have

$$R_0 g = \int G(x,y)g(y)dy \tag{66}$$

with

$$G(x,y) = L^{-1}(4\pi(1-L^{-2}))^{-\frac{1}{2}} e^{-\frac{1}{4}(1-L^{-2})^{-1}(x-L^{-1}y)^2}. \tag{67}$$

We need to bound

$$\|R_0 g\|^{(2)} = \sup_{k,l,i\leq 2}(1+l^4)|\int dxe^{ikx}\int dy(1-\partial_x^2)(\chi_l(x)\partial_x^i G(x,y))g(y)| \leq CL^{-1}\|g\|^{(2)}. \tag{68}$$

We consider first $|l| \geq C\log L$. Then

$$\begin{aligned} |\ldots| &\leq \sum_m \int dxdy|(1-\partial_x^2)(\chi_l(x)\partial_x^i G(x,y))\chi_m(y)g(y)| \\ &\leq CL^{-1}\sum_m e^{-|l-L^{-1}m|}(1+m^4)^{-1}\|g\|^{(2)} \leq CL^{-4}\frac{\|g\|^{(2)}}{1+l^4} \end{aligned} \tag{69}$$

for L large enough. We used (48) (with $\partial^i w$ replaced by g) and (67).

For $|l| \leq C \log L$, and the terms with $|m| \leq L$, we use, for $j \leq 4$,

$$|\partial_x^j G(x,y)| \leq CL^{-1} e^{-|l|} \tag{70}$$

and thus, using (48) again,

$$\sum_{|m|\leq L} \leq CL^{-1} \sum_{|m|\leq L} e^{-|l|} \frac{\|g\|^{(2)}}{1+|m|^4} \leq CL^{-1} \frac{\|g\|^{(2)}}{1+l^4}. \tag{71}$$

For the terms with $|m| \geq L$, by (48) and $|\partial_x^j G(x,y)| \leq CL^{-1}$

$$\sum_{|m|\geq L} \leq CL^{-1} \sum_{|m|\geq L} \frac{\|g\|^{(2)}}{1+m^4} \leq CL^{-4} \|g\|^{(2)} \leq CL^{-1} \frac{\|g\|^{(2)}}{1+l^4}. \tag{72}$$

for L large, since $|l| \leq C \log L$. Hence, (68) follows.

The corresponding term for v in (40) is estimated similarly (it is of course superexponentially small), and the other contributions to ψ_n^0 and v_n^0 are $\mathcal{O}(\epsilon\epsilon_n)$, as are those of $\mathcal{A}$ and $\mathcal{B}$, see (59), (63). (39) is proved. □

Proof of the Propositions. Consider first (9). Using (6), (7), and $B\phi_0^* = 0$, we have the equation

$$B\rho = G(s^*(\frac{d}{dz}(\phi_0^* + \rho)), \frac{d}{dz} s^*(\frac{d}{dz}(\phi_0^* + \rho)), \frac{d}{dz}(\phi_0^* + \rho)) \equiv g(\frac{d}{dz}\rho, \frac{d^2}{dz^2}\rho) \tag{73}$$

Put

$$h = B\rho \tag{74}$$

and solve

$$h = g(\frac{d}{dz} B^{-1} h, \frac{d^2}{dz^2} B^{-1} h) \equiv h_0 + N(h) \tag{75}$$

in the Banach space defined by (8). h_0 is the value of g at $h = 0$, i.e.

$$h_0 = G(s^*(\frac{d}{dz}\phi_0^*), \frac{d}{dz} s^*(\frac{d}{dz}\phi_0^*), \frac{d}{dz}\phi_0^*).$$

In $N(h)$ we encounter terms like $\frac{d}{dz} B^{-1} h$ and $\frac{d^2}{dz^2} B^{-1} h$. We write the latter as $\frac{d^2}{dz^2} B^{-1} h = (-B - \frac{1}{2} z \frac{d}{dz} + \frac{1}{2}) B^{-1} h = -h + (\frac{1}{2} - \frac{1}{2} z \frac{d}{dz}) B^{-1} h$. Thus,

$$N(h) = R(B^{-1}h, \frac{d}{dz} B^{-1} h, z \frac{d}{dz} B^{-1} h) \tag{76}$$

with $R : \mathbf{C}^3 \to \mathbf{C}$ analytic near zero. Thus all we now need is

Lemma. *The operators B^{-1}, $\frac{d}{dz} B^{-1}$ and $z \frac{d}{dz} B^{-1}$ are continous in the norm* (8).

This Lemma was proved in [1] for $A = B - \frac{1}{2}$, but the proof holds for B as well. ¿From the explicit form of G, of ϕ_0^* in (2.15), and of $s^*(\frac{d}{dz}\phi_0^*)$ in (5), we have $\|h_0\|_N \leq C\epsilon$

while $\|N(h)\|_N \leq C(\epsilon^2\|h\|_N + \|h\|_N^2)$. So, we can use the contraction mapping principle in a ball whose radius is of order ϵ^3 around h_0. This proves (9). The bound (10) holds because s^* is of order $(\frac{d}{dz}\phi^*)^2$ and $\frac{d}{dz}\phi_0^*$ is of order ϵ.

Proposition 2 is proven in the same way. It is enough to note that θ_0^* solves (17) with the RHS equal to zero and, setting $h = A\tau$, with A as in [1], we get an equation like (75), with $N(h)$ linear in h and where $h_0 =$ the RHS of (17) with θ^* replaced by θ_0^*; we have the bounds $\|h_0\|_N \leq C\epsilon^3$ for $N = N' - 1$ where N' is the N of Proposition 1, and $\|N(h)\|_N \leq C\epsilon^2\|h\|_N$. □

Acknowledgments

We thank P. Collet for explaining to us the problems discussed in this paper. J.B. benefited ¿from the hospitality of Rutgers University and A.K. from the one of the University of Louvain. This work was supported in part by NSF Grant DMS-8903041.

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