Mass generation for an interface in the mean field regime

François Dunlop, Jacques Magnen, Vincent Rivasseau

Centre de physique théorique, CNRS, UPR14
Ecole Polytechnique, 91128 Palaiseau Cedex, France

ABSTRACT

We consider a two dimensional statistical mechanics model of an interface in three dimensional space with a weak potential tending to localize the interface near a preferred plane. For a number of different such potentials we prove that the two point function decreases exponentially in the mean field regime where the potential is very flat. We estimate the corresponding rate of decay.

A. 145 01 92 February 1992

I. Introduction

In [DMRR] we bounded the fluctuation of an interface for a gaussian model with an arbitrarily small attracting potential. In this paper we study the correlations of this interface and prove that in the mean field regime corresponding to a very flat potential well for which the quadratic approximation is valid over a wide range of values of the interface height, there is an exponential clustering property and the mass or rate of decay is given by the mean field value.

This result applies e.g. near a second order wetting transition in the case of long range forces (these transitions are usually of first order, but second order transitions have also been observed [TGVR]). For reviews see [D], [G].

We remark that our results are not optimal in the sense that when we vary the parameters of the potential we do not get in this paper exponential decay in the correct mass in the complete mean field region; we only prove the decay rigorously in a fraction of what should be this full mean field region. To improve on this point is presumably possible but requires a multiscale analysis together with e.g. the use of Sobolev inequalities. We postpone this to a future publication. Also it would be very interesting to investigate regimes in which non trivial exponents different from the mean field case appear [BHL][KZ]. This requires a rigorous multiscale renormalization group analysis to compute effective constants. It is presumably not out of reach of present mathematical techniques [R], but we postpone it also to future work.

In this paper the main technical tool is, in the language of field theory, a small field versus large field expansion which forces to use a non-translation invariant propagator. This technique is also necessary for several of the most difficult models in constructive field theory (e.g. [V][CMRV][MRS]), and the detailed version given in the simpler context of this paper can be also used as a pedagogical introduction to these more complicated constructions in field theory.

II. The gaussian well

Our interface model corresponds to a massless gaussian measure perturbed by a small interacting potential. We have to perform rigorously the thermodynamic limit. Therefore we want to consider first the massless gaussian measure in a finite volume , where for simplicity Λ is e.g. a large square in ^{2}: $\Lambda =^2 \cap[-L,L]^2$. Then the thermodynamic limit is simply $L \to \infty$. The massless gaussian measure is formally proportional to

$$\mathrm{e}^{-(1/2)\sum_{<x,y>}(h_x-h_y)^2} \prod_{x\in\Lambda} dh_x, \tag{II.1}$$

but such an expression is invariant under global translation of the variables $\{h_x, x \in\}$. To have a well defined measure we must break this global invariance, using some kind of boundary condition at the border of . A particularly convenient choice is to use free boundary conditions on the massive propagator C (with a value of the mass m which will be fixed below to precisely the value expected from the bottom of our gaussian well) and to make this propagator massless inside by insertion of the suitable "mass counterterm";

$$\mathrm{e}^{+(1/2)\sum_{x\in} m^2 h_x^2}. \tag{II.2}$$

This rule fits nicely with the Brydges-Battle-Federbush cluster expansion [B],[R] that we shall use below. Of course any set of bounded boundary conditions would in fact lead us to the same thermodynamic limit.

Therefore let us introduce $C(x,y)$, the ordinary massive lattice propagator with mass $m^2 << 1$ to be fixed later, which has the well known Fourier representation:

$$C(x,y) = \frac{1}{2\pi^2}\int_{-\pi}^{\pi} dk_1 \int_{-\pi}^{\pi} dk_2 \frac{e^{ik\cdot(x-y)}}{m^2 + 2(2-\cos k_1 - \cos k_2)} \tag{II.3}$$

This propagator has also a representation as a sum over random paths on the lattice which in particular proves that it is pointwise positive in x-space. It satisfies the estimate

$$C(x,y) \le K \cdot \log(1+m^{-1}) \cdot e^{-m'|x-y|} \tag{II.4}$$

for some positive constant K. m' is the optimal decay rate of C, defined by $\cosh m' = 1 + m^2/2$. For small m, $m' = m + O(m^3)$. Since we are on the lattice, there is in fact anisotropic decay and one can prove that the worst rate, m', occurs in the lattice directions (see (A.7)). This point is studied in detail in the Appendix.

In the rest of this paper we use often K as a generic name for such an m-independent large constant. Using free boundary conditions on C we define:

$$d\mu_\Lambda = \frac{1}{Z} \mathrm{e}^{+(1/2)\sum_{x\in} m^2 h_x^2} d\mu \quad ; \quad Z = \int \mathrm{e}^{+(1/2)\sum_{x\in} m^2 h_x^2} d\mu \tag{II.5}$$

where $d\mu$ is the normalized measure with propagator C (this measure can be defined directly in the infinite volume limit). In the rest of this paper expectation values such as $< \ >$ of an observable always refer to its mean value with respect to some normalized measure; subscripts are used to remind the reader of the particular measure considered. For instance it will be convenient to use the notation $< \ >$ instead of $< \ >_{d\mu_\Lambda}$.

By an easy gaussian computation the mean value $< h_x^2 >$ at any fixed site x diverges logarithmically as $\Lambda \to \infty$, i.e. as the thermodynamic limit is performed. We add now a small interacting potential which tends exponentially to a constant when h^2 tends to infinity but tends to confine h in a neighborhood of 0. This potential is

$$V(h) = -[e^{-\frac{h^2}{2a^2}} - 1] \tag{II.6a}$$

with a and ϵ both positive (see Fig.1).

Fig. 1

We define the normalized measure:

$$d\mu_{V,\Lambda} = \frac{1}{Z_{V,}} d\mu_\Lambda \prod_{x\in\Lambda} e^{-V(h_x)} \tag{II.6b}$$

and we will use the notation $< \ >_{V,}$ for the expectation value with respect to this measure $d\mu_{V,\Lambda}$.

The regime of parameters which we study is $a >> 1$ and $/a^2 << 1$. The rest of this paper is devoted to a proof that in this regime the two point function decreases exponentially and to an estimate of the corresponding mass gap. More precisely we prove:

Theorem II.1

Let $= \cap[-L,+L]^2$, and let $\{h_x\}_{x\in}$ be a family of real random variables distributed according to the probability measure (II.6b), i.e. the measure

$$\frac{1}{Z_{V,}} \mathrm{e}^{+(1/2)\sum_{x\in} m^2 h_x^2} \mathrm{e}^{-\sum_{x\in} V(h_x)} d\mu(\{h_x\}_{x\in})$$

where $d\mu$ is the gaussian measure of covariance $C(x,y)$ given by (II.3), $V(h)$ is given by (II.6) and $m = \sqrt{/}a$. Assume $0 <\leq 1$. We assume that the potential is such that

$$K \cdot \log(1+^{-1}) < \sqrt{a} \tag{II.7}$$

where K is a sufficiently large constant (this means that a is always large and that if $\to 0$, $a \to \infty$ in a certain way). Under these conditions the thermodynamic limit of the correlation functions exists and satisfy an exponential clustering property (the truncated correlation functions decrease exponentially). The decay rate or effective mass, can be computed in a systematic expansion around the decay rate m' of C (which itself by (A.7) is of order $m = \sqrt{/}a$ for small m). For instance there exist positive constants K and c such that:

$$lim_{\to^2} < h_x h_y >_{V,} \leq K \log(a/\sqrt{})e^{-c\sqrt{}|x-y|/a} \tag{II.8}$$

and c tends to 1 if is fixed and $a \to \infty$.

Let us remark first that the technical condition (II.7) under which the theorem is proved is not expected to be the optimal one under which we should have exponential clustering within the mean field regime (i.e. in this model a mass of order $/a^2$). We expect that this theorem in fact holds under the weaker assumption $K \cdot \log(1+^{-1}) < a$. The attentive reader will trace the necessity for a square root in (II.7) to the third case C) in the proof of Lemma II.5. Here the estimate that we perform is quite loose. To improve on it we must find a better upper bound on the L^4 norm of h in terms of a quadratic norm. This is provided e.g. by the regular Sobolev inequality in two dimensions $\|h\|_4 \leq K\|h\|_{H_1}$ where the H_1 norm is $\sqrt{\|h\|_2^2 + \|\nabla h\|_2^2}$. However we do not use this kind of inequality here since it seems to require the use of a multiscale analysis, so that the size of the gradients is adapted to the size of the boxes in which the inequality is used.

To prove Theorem II.1 we want to analyze the theory with respect to a lattice $\mathbf{D}$ which is a regular paving of by squares of side $l = a/\sqrt{}$, namely the inverse of the expected mass. In the squares where the average value of h is less than $\sqrt{a}$, which we call the small field region, the quadratic approximation to the potential which gives a mass $m = \sqrt{}/a$ is valid. In the rest of this paper the parameter m introduced in (II.2-3) is therefore fixed to this value $\sqrt{}/a$. In the other squares, called the large field region, the potential is strictly above its absolute minimum by a value about $/a$. Taking into account the number $a^2/$ of sites in a square, we remark that large field squares are rare in probability; they have a suppressing factor e^{-a}. In order to combine these observations into a proof of the theorem, we are going to perform a cluster expansion with respect to the lattice $\mathbf{D}$. Here we go.

For each square we will write

$$\begin{aligned} 1 &= \chi(\frac{1}{||}\sum_{x\in} h_x^4/a^2) + (1 - \chi(\frac{1}{||}\sum_{x\in} h_x^4/a^2)) \\ &= \chi(\sum_{x\in} h_x^4/a^4) + (1 - \chi(\sum_{x\in} h_x^4/a^4)) \end{aligned} \tag{II.9}$$

where is a fixed C^∞ function with support in [0,1], which is one on [0,1/2]. We require also a rather mild technical condition on :

$$\sup_{h\in} \frac{d^n}{dh^n}\chi(h) \;\leq\; K.(n!)^q \tag{II.10}$$

for some fixed numbers $K, q > 0$ (this is e.g. true with $q = 2$ for a standard shape such as $e^{-1/h}$).

We expand and call the set of large field squares

$$1 = \sum_{\subset} (h) \;;\; (h) \equiv \prod_{\notin} \chi(\sum_{x\in} h_x^4/a^4) \prod_{\in}(1 - \chi(\sum_{x\in} h_x^4/a^4)) \tag{II.11}$$

We insert this expansion (II.11) in the numerator and denominator of the normalized two point function. Furthermore we develop the potential in the small field region and combine its quadratic piece with the counterterm (II.2). The potential of the theory after this manipulation becomes

$$V = -\Big(\sum_{x\in}[e^{-\frac{h_x^2}{2a^2}} - 1] + \sum_{x\notin}[e^{-\frac{h_x^2}{2a^2}} - 1 + \frac{h_x^2}{2a^2}]\Big)$$

$$= \Big((\sum_{x\in}\int_0^1 dt\frac{h_x^2}{2a^2}e^{-t\cdot\frac{h_x^2}{2a^2}}) - (\sum_{x\notin}\int_0^1 (1-t)dt\frac{h_x^4}{4a^4}e^{-t\cdot\frac{h_x^2}{2a^2}})\Big) \tag{II.12}$$

(by some slight abuse of notations we will also call the set of sites in the squares of). The two point function is then given by:

$$< h_x h_y >_{V,} = \frac{\sum_{\subset}\int h_x h_y (h) e^{-V} \mathrm{e}^{+(1/2)\sum_{x\in} m^2 h_x^2} d\mu}{\sum_{'\subset}\int_{'}(h)e^{-V_{'}} \mathrm{e}^{+(1/2)\sum_{x\in'} m^2 h_x^2} d\mu} \tag{II.13}$$

(remark that in this formula it is the gaussian measure $Zd\mu = \mathrm{e}^{+(1/2)\sum_{x\in} m^2 h_x^2} d\mu$ which naturally appears rather than $d\mu$).

We decompose into connected components $_{1,..,n}$ in the following way. We consider a large factor $M = K \cdot a^{1/4}$, and we say that two squares of $\mathbf{D}$ are close if their minimal distance is smaller than M/m'. When two squares are close in this sense we draw a link joining them which we call a distance link. Then a connected component $_i$ is a maximal set of squares of connected through such distance links (hence such that two of them can be linked together through a chain of squares of $_i$, each of which is close to the next one in the sense above).

The cluster expansion has to be performed with some care because the local term $\mathrm{e}^{+(1/2)\sum_{x\in} m^2 h_x^2}$ cannot be treated as a small perturbation, when we stay inside a given connected region $_i$ of . However when we change of connected region, because M/m' is large compared to the decay length $1/m'$ typical of C, we do get a small factor. We shall perform an expansion which exploits this fact to factorize the connected components $_i$. We call $_S$ the characteristic function of a set of sites S and C the propagator corresponding to the normalized gaussian measure $d\mu$.

We want now to compare systematically the covariance C to C by means of the resolvent identity:

$$C(x,y) = C(x,y) + \sum_z C(x,z)m^2(z)C(z,y)$$

$$= C(x,y) + \sum_z C(x,z)\sum_{i=1}^n m_{_i}^2(z)C(z,y) \tag{II.14}$$

¿From now on let us forget the summation over intermediate points z. We define new objects $\mathbf{C}^{j,k}$, $j,k = 0,1,...n$, called "chains", through the formulas:

$$\mathbf{C}^{0,0} = C + \sum_{p\geq 1}\sum_{\substack{i_1,\ldots,i_p\in[1,n]\\ i_q\neq i_{q+1}, q=1,\ldots,p-1}} C\prod_{q=1}^p m_{_{i_q}}^2 C \tag{II.15}$$

$$\mathbf{C}^{0,k} = C + \sum_{p\geq 1}\sum_{\substack{i_1,\ldots,i_p\in[1,n]\\ i_q\neq i_{q+1}, q=1,\ldots,p-1;\ i_p\neq k}} C\prod_{q=1}^p m_{_{i_q}}^2 C \tag{II.16}$$

$$\mathbf{C}^{j,0} = C + \sum_{p\geq 1}\sum_{\substack{i_1,\ldots,i_p\in[1,n]\\ i_q\neq i_{q+1}, q=1,\ldots,p-1;\ i_1\neq j}} C\prod_{q=1}^p m_{_{i_q}}^2 C \tag{II.17}$$

$$\mathbf{C}^{j,k} = C + \sum_{p\geq 1} \sum_{\substack{i_1,\ldots,i_p \in[1,n] \\ i_q\neq i_{q+1},q=1,\ldots,p-1;\ i_1\neq j, i_p\neq k}} C\prod_{q=1}^{p} m^2_{i_q} C \quad \text{if } 0\neq j\neq k\neq 0 \tag{II.18}$$

$$\mathbf{C}^{j,j} = \sum_{p\geq 1} \sum_{\substack{i_1,\ldots,i_p \in[1,n] \\ i_q\neq i_{q+1},q=1,\ldots,p-1;\ i_1\neq j, i_p\neq j}} C\prod_{q=1}^{p} m^2_{i_q} C \quad \text{if } j\neq 0. \tag{II.19}$$

We apply the identity (II.14) repeatedly and we obtain, in the sense of operators:

$$C = \mathbf{C}^{0,0} + \sum_{p\geq 1}\sum_{j_1,\ldots,j_p\in[1,n]} \mathbf{C}^{0,j_1}\prod_{k=1}^{p}\Big((m^2_{j_k} C_{j_k} m^2_{j_k}) \ \ \mathbf{C}^{j_k,j_{k+1}}\Big) \tag{II.20}$$

with the convention that $j_{p+1}\equiv 0$. When is made of a single connected component $_1$ these formulas simplify a lot; we have $C = C_1$ and

$$\mathbf{C}^{0,0} = C + Cm^2_1 C \tag{II.21}$$

$$\mathbf{C}^{0,1} = \mathbf{C}^{1,0} = C \ ; \ \mathbf{C}^{1,1} = 0 \tag{II.22}$$

$$C = \mathbf{C}^{0,0} + \mathbf{C}^{0,1}(m^2_1 C_1 m^2_1) \ \mathbf{C}^{1,0} \tag{II.23}$$

Then we define h_0 and h_i, $i = 1,..,n$ as independent gaussian random variables with respective covariance $\mathbf{C}^{0,0}$ and C_i. The corresponding normalized gaussian measures are called respectively $d\mu_0(h_0)$ and $d\mu_i(h_i)\equiv d\mu_i(h_i)$, $i = 1,...,n$. If we perform the substitution

$$h(x) = h_0(x) + \sum_{i=1}^{n}\sum_{y} \ \mathbf{C}^{0,i}(x,y)m^2_i(y)h_i(y) \tag{II.24}$$

we have

$$d\mu(h) = \frac{1}{P}\prod_{i=0}^{n} d\mu_i(h_i)e^{Chains} \ ; \ P = \int\prod_{i=0}^{n} d\mu_i(h_i)e^{Chains} \tag{II.25}$$

where

$$Chains \equiv (1/2)\sum_{1\leq i\leq n, 1\leq j\leq n} m^2_i h_i \ (\mathbf{C}^{i,j})m^2_j h_j \ . \tag{II.26}$$

(II.25) means that the integral of any function f of the variable $h = \{h_x, x\in\}$ with respect to the measure $d\mu$ is equal to the same function integrated with respect to the right hand side measure (II.25) if the substitution (II.24) is made in f (we used the fact that $\mathbf{C}^{0,i}(x,y) = \ \mathbf{C}^{i,0}(y,x)$).

Since what appears in (II.13) is the measure

$$\mathrm{e}^{+(1/2)\sum_{x\in} m^2 h_x^2} d\mu = Z d\mu \tag{II.27}$$

we have to compute the normalizing ratio Z/P. This is done using the following Lemma:

Lemma II.1
We have

$$Z/P = \prod_{i=1}^{n} Z_i \tag{II.28}$$

Proof We rewrite P as

$$P = \int e^{Chains} d\mu_0 \prod_{i=1}^{n} d\mu_i(h_i)$$

$$= \int e^{Chains} \prod_{i=1}^{n} \Big(Z_{i}^{-1}\ e^{(1/2)\sum_{x\in_i} m^2 h_i^2(x)}\ d\mu(h_i)\Big) \tag{II.29}$$

using the fact that the factor $Chains$ does not depend on h_0 and (II.27). It remains therefore to prove that

$$\int e^{(1/2)\sum_{1\le i\le n, 1\le j\le n} m_i^2 h_i\ (\mathbf{C}^{i,j}) m_j^2 h_j} \prod_{i=1}^{n} \Big(e^{(1/2)\sum_{x\in_i} m^2 h_i^2(x)}\ d\mu(h_i)\Big)$$

$$= \int e^{(1/2)\sum_{x\in} m^2 h^2(x)}\ d\mu = Z. \tag{II.30}$$

This is just an exercise in expanding both sides of (II.30) into power series in m^2, integrating by Wick's theorem and identifying the cycles of propagators on both sides. Indeed on both sides of (II.30) we get cycles made of insertions m^2 joined by propagators C, but on the left hand side of (II.30) these cycles are simply decomposed according to whether or not successive insertions of are of the type ${}_i$-${}_i$ or ${}_i$-${}_j$ with $j \neq i$. Identity (II.30) is therefore quite the analogue for cycles of the resolvent identity (II.20).

Using this lemma, we can rewrite (II.13) as

$$< h_x h_y >_{V,} = \frac{\sum_{\subset} \prod_{i=1}^{n} Z_i \int h_x h_y (h) e^{-V(h)} \prod_{i=0}^{n} d\mu_i(h_i) e^{Chains}}{\sum_{'\subset} \prod_{i'=1}^{n'} Z_{i'}' \int_{'} (h) e^{-V_{'}} \prod_{i'=0}^{n'} d\mu_{i'}(h_{i'}) e^{Chains'}} \tag{II.31}$$

where

$$Chains' \equiv (1/2) \sum_{1\le i'\le n', 1\le j'\le n'} m_{i'}^2 h_{i'}\ (\mathbf{C}^{i',j'}) m_{j'}^2 h_{j'}. \tag{II.32}$$

is the analogue of (II.26) for the decomposition of $'$ as the union of connected components ${}'_{i'}$, $i' = 1, ..., n'$.

The reason for all this rewriting is that the various propagators $\mathbf{C}^{j,k}$ $j,k = 0,...n$ as defined by (II.15-19) are well defined through absolutely convergent series and have good decay properties. More precisely:

Lemma II.2
For any $' > 0$ (arbitrarily small) there exists $K > 0$ (depending on $'$ but not on m) such that

$$\mathbf{C}^{j,k}(x,y) \le K \log(1+m^{-1}) e^{-m'(1-')|x-y|} \qquad \forall j \in [0,n], \forall k \in [0,n] \tag{II.33}$$

Proof We use the fact that $dist({}_{i,j}) \ge M/m'$ together with the estimate (II.2). The conditions that two consecutive indices in (II.15-19) have to be different ensure that for M large enough we can extract a small factor for each of these terms, and keep an exponential decrease $e^{-m'(1-')|u-v|}$ between the ends u and v of each C piece. The triangular inequality and the convergence of geometric series with ratio smaller than one completes the proof, if we ensure that $\log(1+m^{-1})e^{-M(1-')} << 1$, a condition which is satisfied by our choice $M = K \cdot a^{1/4}$ and the condition (II.7), which ensures $\log(1+m^{-1}) < a^{1/2}$ at large a.

We shall also need a lemma to control the normalization factors Z_i^{-1} which appear in (II.31):

Lemma II.3
We have, for some constant c independent of m

$$Z^{-1} \ \le\ e^{c\cdot N() \log(1+m^{-1})} \tag{II.34}$$

where $N() = m^2||$ is the number of of squares in , if $||$ is the number of sites in .

Proof We have, using the explicit formula for the normalization of a gaussian integral:

$$Z^{-1} = \Big(\det(1-m^2C)\Big)^{1/2} = e^{(1/2)\sum_{n\ge1}\frac{1}{n}Tr\left(m^2C\right)^n}$$

$$\le e^{(1/2)||\sum_{n\ge1}\frac{1}{n}\sup_{x\in}\left(m^2C\right)^n(x,x)}$$

$$= e^{(1/2)|| \sum_{n\geq 1} \frac{1}{n} \frac{1}{2\pi^2} \int_{-\pi}^{\pi} dk_1 \int_{-\pi}^{\pi} dk_2 \left(\frac{m^2}{m^2+2(2-\cos k_1 - \cos k_2)} \right)^n}$$

$$\leq e^{cm^2 || \log(1+m^{-1})} \tag{II.35}$$

In the first inequality we have used the fact that the propagator (II.3) is pointwise positive to increase the sums, which were restricted to to the full volume ^{2} (except for one, which fixes translation invariance and gives the volume factor $||$). Then we used Fourier analysis. The last inequality is easy; the term with $n = 1$ gives explicitly the logarithmic factor, and the other terms are uniformly bounded by $m^2 \sum_{n\geq 2} O(1)/n^2$.

It remains to perform a cluster expansion on (II.31). This is done both on the numerator and denominator of (II.31). We use the formalism of Brydges-Battle-Federbush (see [B],[R] for reviews). We will first describe this cluster expansion and give an outline of the main important points to understand its structure and the reasons for its convergence. Then we state the main result in the form of Lemma II.4 below for which we give a more detailed proof of convergence.

We consider e.g. the numerator of (II.31). First we list in an arbitrary order the set U made of all squares of the small field region plus the n elements $_1, ..., _n$. Then we introduce a parameter s which in every propagator C in any of the $\mathbf{C}^{j,k}$ terms of (II.15-19) decouples the first element of U from the rest. For instance if this first element is called we write:

$$C(s) = C + (1-)C(1-)$$

$$+s\big(C(1-) + (1-)C\big) \tag{II.36}$$

When we insert this interpolated covariance into (II.31), the measure $d\mu_0$, the factor e^{Chains} and the definition of h through (II.24) in the factor $h_x h_y (h) e^{-V(h)}$ change. Remark that the measures $d\mu_i$ or the normalization factor $\prod Z_i$ do not change; indeed these factors are already factorized over the connected large filed regions.

We have to check that inserting $C(s)$ instead of C in the definition (II.15) of $\mathbf{C}^{0,0}$ gives still a measure of positive type, so that we have a well defined functional integral. This is obvious. Then we expand the numerator of (II.31) at first order around $s = 0$ using the Taylor formula with integral remainder. The term at $s = 0$ decouples from the rest; the remainder term couples to some other element of U, $'$ (which again can be a small field square or some $_i$) by means of some explicit propagator C inside the definition of some chain $\mathbf{C}^{j,k}$. Then we consider $\cup'$ (or $\cup_i$) as a single new entity in U and iterate. This cluster expansion is described in detail in [B],[R]. Rather than to repeat all the details here, we will simply insist on all the differences with the standard case treated in [B],[R] of a gaussian measure with propagator C perturbed by a polynomial interaction (such as h^4). The differences are the non polynomial nature of the "interaction" factor $(h)e^{-V(h)}$ and the particular structure of the chains $\mathbf{C}^{j,k}$ which do not reduce to a single propagator C.

Let us address these differences now. Each derivation d/ds which couples some element $\in U$ to $' \in U$ creates an explicit propagator C in some chain with its two ends in the prescribed objects and $'$. This is called a cluster link between and $'$. In the standard case [R], using integration by parts, we have at the two ends of the propagator a functional derivation $\frac{}{h}$ acting either on the sources or on the exponential of the interaction, which creates a so called "derived" vertex (such as $4h^3$ in the case of an h^4 Ginzburg-Landau model) localized at this end. In our case let us also call a derived vertex the result of a functional derivative $\frac{}{h}$ applied to the interaction term which in our case is $(h)e^{-V(h)}$.

Then a cluster link appears as a propagator C created by a d/ds derivation, with its ends in and $'$, which lies in some chain $\mathbf{C}^{j,k}$. We have to decsribe the factors which lie at the end of this chain. An end of chain corresponding to an index j in $[1, n]$ has simply a factor $m_j^2 h_j$ hooked to it (this is true both for the two ends of the chains in the exponential term e^{Chains}, or for one end of the chains of the type $\mathbf{C}^{0,k}$ or $\mathbf{C}^{j,0}$, $j, k \in [1, n]$ which appear in the replacement of h in (II.28) by formula (II.24)). An end of chain corresponding to a 0 index (such as both ends of a $\mathbf{C}^{0,0}$ chain or one end of a $\mathbf{C}^{0,k}$ or $\mathbf{C}^{j,0}$, $j, k \in [1, n]$ chain) instead has a "derived vertex" $\frac{}{h}(h)e^{-V(h)}$ hooked to it (after the functional derivative has been computed we have to replace h by its value (II.24) to reexpress this derived vertex in terms of the fields h_0 and h_i, $i = 1, ...n$).

Indeed this is directly the result in the case of the end with 0 index of a $\mathbf{C}^{0,k}$ or $\mathbf{C}^{j,0}$, $j, k \in [1, n]$ chain and in the case of the two ends of a $\mathbf{C}^{0,0}$ chain this is the result of the functional derivative $\frac{}{h}$ hooked to the end, after an integration by parts on h_0. Indeed from (II.24) we see that the action of a functional derivative $\frac{}{h}$ on a function of h is the same as the action of $\frac{}{h_0}$ on this function, reexpressed in terms of h_0 and h_i, $i = 1, ...n$.

This description of the cluster link is correct up to two further remarks. In a small number of cases (at most two for a two point function) the derivation at the end of a chain instead of producing a derived vertex may hit the source fields h_x or h_y. Also a derived vertex may be hit by further derivations, hence in the most general case what lies at the end of a chain is really a multiply derived vertex common to several links, or a source. This is completely standard, although remark that in the case of a polynomial interaction such as h^4, a derived vertex can be rederived only at most a fixed number of times. In our case the intercation factor is non-polynomial and a derived vertex can be hit an arbitrary number of times. However remark that these derivations must be associated to farther and farther squares and the longer and longer distance factors in (II.33) will control the associated combinatoric problem.

A slight difference with the standard case which may worry the reader is that a chain $\mathbf{C}^{j,k}$ is made of a sum over n of terms containing an arbitrarily large number n of propagators C. How to control the combinatoric of choosing on which of these n terms a d/ds derivation may act? This is easy. Because of the inductive rules of the Battle-Brydges-Federbush cluster expansion, only one propagator C in a given chain sandwiched between two characteristic functions of two given large field region $_i$ and $_j$ can be explicitly derived (indeed later these two regions are treated as a single block). Therefore paying a factor 2 per such sandwiched propagator we may decide whether it will be derived or not. This sloppy bound is then easy to control because each such sandwiched propagator gives an arbitrarily small factor (see the proof of Lemma II.2).

Finally there is a subtlety here that we have to take into account which is caused by our definition of distance links between large field cubes. Because of this definition, a large field region has a halo of radius M/m' where other large field cubes cannot enter. This is an analogue of the hardcore condition which is familiar in cluster expansions.

In the end of the cluster expansion we have sets of squares connected together through explicit cluster links, and the connected large field regions $_i$ which are connected together by distance links. Taking all connections into account (both cluster links and distance links) we have a collection of connected sets of squares E_1, ...,E_r called clusters or polymers (whose union must be all of). We can discard the trivial clusters made of empty small field squares, whose amplitude is 1, and consider the non-trivial ones, whose union is no longer . We claim now that our functional integral is factorized over these sets. Indeed the only reason for non-factorization may come now from the normalizing factor in (II.28): but this factor is $\prod_{i=1}^n Z_{\ i}^{-1}$, and since each $_i$ is contained entirely in a single E_k, we conclude that the E_k are factorized. Applying this process to the numerator and the denominator of (II.31) we obtain

$$< h_x h_y >_{V,} = \frac{\sum_{\{E_j\}, E_j \text{ generalized disjoint family}} A_{x,y}(E_1) \prod_{j \geq 2} A_{\emptyset}(E_j)}{\sum_{\{E'_k\}, E'_k \text{ generalized disjoint family}} \prod_k A_{\emptyset}(E'_k)} \tag{II.37}$$

where $A_{x,y}(E_1)$ is the amplitude of the connected cluster E_1 containing the sources x and y (by parity they have to lie in the same cluster if the interaction is even), and $A_{\emptyset}(E_j)$ or $A_{\emptyset}(E_k)$ is a vacuum amplitude associated to the non-trivial cluster E_j, $j \geq 2$ or E_k. The condition E'_k generalized disjoint family means that the E_j are disjoint in the ordinary sense *plus* the fact that any large field square in E_j is separated by at least M/m' from any large field square in $E_{j'}$, $j \neq j'$.

More generally one can derive a formula which generalizes (II.37) to the computation of any correlation function S of N external sources $h_{i_1}, ..., h_{i_N}$, as a sum of products of amplitudes, where in the numerator the union of all amplitudes has to contain all external sources.

To complete the result of the cluster expansion it is standard that we need only to prove that non-trivial clusters are small so that they can be resummed. Let us introduce a measure $N(E)$ of the size of the cluster E which is equal to the number of small field squares plus M^2 times the number of large field squares in E. Then we will prove below:

Lemma II.4

Each vacuum amplitude is bounded so that (if O is an arbitrary origin)

$$\sum_{E/O\in E} A_{\emptyset}(E)e^{N(E)} \leq 1 \tag{II.38}$$

Furthermore the amplitude containing the two external sources at sites x and y has exponential decay so that

$$\sum_{E/x,y\in E} A_{x,y}(E)e^{N(E)} \leq Ke^{-(1-'')m'|x-y|} \tag{II.39}$$

where $''$ tends to 0 if $a \to \infty$.

We could establish more general "tree decay" between sources for amplitudes containing more than two sources. They are not necessary however for the proof of our main theorem. In fact we will limit ourselves to study vacuum amplitudes and prove as usual a result stronger then (II.38), namely one in which the precise constant e can be replaced by any other fixed constant, if a is large enough and $/a^2$ is small enough. The result (II.39) follows easily by evaluating $A_{x,y}(E)$ in the same manner than a vacuum amplitude, but taking into account that x and y have to be connected through chains of cluster or distance links. The cluster links give directly exponential decay of the form (II.39) and each distance link give a small factor which tends to 0 as $a \to \infty$ (Lemma II.6 below). Combining both effects we get (II.39).

Assuming (II.38-39) the proof of the theorem is achieved by a standard Mayer expansion on (II.37) which removes the constraints of generalized disjointness in (II.37) and accordingly computes the normalized functions. This is completely standard (see e.g. [B],[R]). The only subtlety has to do with the nature of the generalized disjointness condition. Because of the halo in this condition one gets an exponential of the number of squares in the region E plus the surrounding halo in the process of resummation of clusters linked to E through Mayer links (or overlapping conditions). This factor is then compensated thanks to our definition of $N(E)$, so that Lemma II.4 does indeed ensure the convergence of the Mayer expansion.

If the amplitudes in (II.22) have exponential tree decay between the sources, this exponential tree decay follows for the truncated correlation functions computed by the Mayer expansion. In particular in the case of the two point function this argument achieves the proof of our main theorem.

We concentrate therefore on the summation of vacuum amplitude which contains the origin, and give a detailed proof of (II.38).

The cluster expansion has produced a certain number of explicit fields h_i, $i = 0, 1, ...n$, hooked to the ends of $C^{j,k}$ chains to which the derived cluster links C belong, or produced by functional derivatives $/h$ acting on e^{-V}. Let us explicit the structure of all these terms.

The outcome of the Brydges-Battle-Federbush expansion is indexed by a tree T [2,3] of cluster links between nodes called N_1,..., N_{n+1}, which can be the large field regions $_1, ..., _r$ or the small field squares $_1, ..., _{r'}$ which together form the support of E. We have $n+1 = r+r'$. The tree T is therefore made of $n = r+r'-1$ links L_l, $l = 1, ..., r+r'-1$, each of which contains a C propagator with ends in two different nodes S_{j_l} and S_{k_l}. (By the Brydges-Battle-Federbush process one has $n \geq 1$ except for trivial clusters whose values is 1). This propagator is in fact part of a chain C_l which together with C may contain other propagators and has in addition some attached factors at the end. Let us describe this in more detail.

The l-th chain C_l has its ends in squares $_{e_l}$ and $_{f_l}$ which are in E but can be of course different from S_{j_l} and S_{k_l}. If the chain is of type $C^{j,k}$ with $j, k > 0$ we have fields h_j and h_k attached to these ends and this is the end of the story. The set of all fields h_i, $i > 0$, attached to such chains form a monomial which we call S.

But if the chain is of type $C^{0,0}$, $C^{0,j}$ or $C^{k,0}$ we have instead of explicit fields h_0 functional derivatives $\frac{}{h(x)}$ e^{-V} attached to the ends with 0 index, which have to be applied to the factor e^{-V}. We have to perform these functional derivatives, and in the corresponding fields produced, we have to replace h by formula (II.24), which again may or may not produce eventually chains which link $_{e_l}$ and $_{f_l}$ to a set of final squares $^s_{g_l}$ and $^s_{h_l}$ that contain the final fields h_j, $j > 0$ associated to the chains $C^{j,0}$ which occur in (II.24). ¿From the explicit form (II.9) and (II.12) of e^{-V} we conclude below that at most five fields can be produced by a functional derivative, hence the index s takes at most five values (see Fig.2).

Fig. 2

If we put together all the functional derivatives $\frac{}{h(x)}$ which act in a given square they must be of the form $\left(\sum_{x\in}\frac{}{h(x)}\right)^n$, with $\sum_{\in E} n \le 2n-2$ (because there are at most two ends per chain C_l, hence at most $2n-2$ such derivatives).

We compute the action of these derivatives and obtain:

$$\prod(\sum_{x\in}\frac{}{h(x)})^n e^{-V} = \prod \sum_{\substack{n_{,1},n_{,2}\\ n_{,1}+n_{,2}=n}} \binom{n}{n_{,1}}$$

$$\sum_{p_{,1}\le n_{,1}} Q_{p_{,1}}(\{h(x)\},x\in)^{(p_{,1})}(\frac{1}{||}\sum_{x\in} h_x^4/a^2)R_{n_{,2}}(\{h(x)\},x\in)e^{-V}. \tag{II.40}$$

where $=$ if $\notin$ and $=1-$ if $\in$. Q is a polynomial of order $4p_{,1}-n_{,1}$ and R is a polynomial of order at most $5n_{,2}$ times an exponential of a negative quadratic form in the variables $(\{h(x)\},x\in)$. This result is obtained using the form (II.12) of V. The exact formulas for Q and R are tedious to write down, because the Leibniz formulas to derive products after many iterations become complicated. However to bound the outcome of (II.41) we need only to keep track of the general structure of Q and R. First we bound by 1 the exponential of the negative quadratic form in the variables $(\{h(x)\},x\in)$ remaining in R, and we bound all the factors t or $1-t$ and the integrals $\int_0^1 dt$ coming from (II.12) by 1. Second we use the condition (II.10) to bound $\prod\binom{n}{n_{,1}}^{(p_{,1})}(\frac{1}{||}\sum_{x\in}h_x^4/a^2)$ by $K^n\prod(n\ !)^q\Omega(h)$, where we use that $\sum n\le 2n-2$, and we define $\Omega(h)=\prod_{\notin}{}_1(\frac{1}{||}\sum_{x\in}h_x^4/a^2)\prod_{\in}({}_2(\frac{1}{||}\sum_{x\in}h_x^4/a^2))$, where $_1$ is the characteristic function of [0,1] and $_2$ the characteristic function of $[1/2,+\infty]$.

Finally in Q and R we replace h by its value (II.24). This is the step which generates the squares $^s_{g_l}$ and $^s_{h_l}$; we see that as announced there is only at most 5 such fields per functional derivative.

In this way up to a numerical factor

$$K^n\prod(n\ !)^q \tag{II.41}$$

and up to the explicit value of the chains $C^{j,k}$, the functional integral that remains to be bounded has the form of a certain function $F=P\cdot\Omega e^{-V}$ of the fields, where $P=\prod_{i=0}^n\prod_{j=1}^{N(i)}h_i(x_j)$ is the explicit product of all the fields produced in all the polynomials Q, R, and in the first monomial S directly attached to the chains in (II.31) and considered above.

We want now to show that to each cluster link L_l is associated a small factor. This factor will come either from the spatial decay of the links in Lemma II.2 or to some small factors attached to the fields produced by functional derivatives when they act in the small field region. Let us explain this point in detail, considering again the particular form (II.11-12) of e^{-V}. The outcome of a functional derivative $\frac{}{h(x)}$ in a square $_{e_l}$ or $_{f_l}$ depends on whether this square is a large or small field square. In the case of a small field square every derivation on e^{-V} acts either on (h^4/a^4) or on $(h^4/a^4)e^{-h^2/a^2}$ and produces either (h^3/a^4) or $(h^3/a^4)e^{-h^2/a^2}$ or $(h^5/a^6)e^{-h^2/a^2}$. In (II.40) we said that we bound the negative exponentials by one and keep only the produced fields. Therefore the outcome of one functional derivation is a set of at most 5 fields, which have then to be expanded according to (II.24). If the square is a large field square, i.e. belongs to , the functional derivative acts either on (h^4/a^4) or on $(h^2/a^2)e^{-h^2/a^2}$ and produces either (h^3/a^4) or $(h/a^2)e^{-h^2/a^2}$. Again recall that the negative exponential is bounded by one.

First we remark that from the form of these vertices to each independent summation over x in a square we can associate a factor $/a^2=1/||$ coming either from the factors and a^{-2} or a^{-4} in (II.12) or from (II.11). In this sense every such summation is properly normalized. Then in addition to these factors we have *in the case of a vertex produced in a small field square* an additional factor a^{-2} for an h^3 vertex and a^{-4} for an h^5 vertex. In order to take correctly into account these additional factors let us introduce a new notion.

We say that a cluster link is a small field link if S_{j_l} and S_{k_l} are both in the small field region. In the other case we call it a large field link. For every large field link, by our rule we must have $\mathrm{dist}\{S_{j_l},S_{k_l}\}\ge M/m'$. Therefore we can extract a factor at least $e^{-M/5}=e^{-Ka^{1/4}/5}$ from every such link, still keeping four fifths of the spatial exponential decay in the l chain for other purposes.

In the case of a small field link both S_{j_l} and S_{k_l} have to be at a distance at least M/m' from any large field square of . We call the link a regular link if both squares $_{e_l}$ and $_{f_l}$ are in the small field region and all the fields produced by formula (II.24) are of the type h_0. In the converse case we call the link irregular. For an irregular link at least one of the squares $_{e_l}$, $_{f_l}$, $^{s}_{g_l}$ or $^{s}_{h_l}$ has to belong to . Therefore using the triangular inequality plus one fifth of the spatial decrease of the $C^{j,k}$ chains we can again extract a factor at least $e^{-M/5} = e^{-Ka^{1/4}/5}$ for such an irregular link.

The regular links have all their produced fields of the h_0 type, hence these fields belong to the production square $_{e_l}$ or $_{f_l}$. For these regular links we can use the additional powers of a^{-1} described above to attribute a small factor $1/\sqrt{a}$ to each end of the link and a small factor $1/\sqrt{a}$ to each of the produced fields.

For irregular links or large field links we have for large a $e^{-Ka^{1/4}/5} << a^{-16}$. Therefore we can attribute also a small factor $1/\sqrt{a}$ to each of the produced fields, and a factor $1/a$ to the link l and to each of the $C^{0,j}$ links produced by formula (II.24) (there are at most five of them per end of l hence at most ten of them). Finally we use condition (II.7) which ensures that $\log(1+m^{-1}) < \sqrt{a}$. Each link l and each of the links $C^{0,j}$ produced has therefore in this way an associated factor $\sqrt{a}/a = 1/\sqrt{a}$, instead of the factor $\log(1+m^{-1})$ of Lemma II.2, and each vertex produced is normalized by $/a^2$ and each field produced is normalized by a factor $1/\sqrt{a}$.

It remains to extract, using (II.4) and Lemma II.2 the remaining exponential decay from the explicit propagators. We call d_l the distance factor which is the sum of the minimum distance between S_{j_l} and S_{k_l}, plus if necessary, the minimum distance between S_{j_l} and $_{e_l}$, S_{k_l} and $_{f_l}$, and between $_{e_l}$ and each $^{s}_{g_l}$, $_{f_l}$ and each $^{s}_{h_l}$. We can extract, using (II.4) and Lemma II.2 the remaining four fifths of the corresponding exponential decay from the explicit propagators associated to the l-th chain in Fig. 2.

In this way we can bound the explicit cluster links by a factor

$$(K/\sqrt{a})^n \prod_{l=1}^{n} e^{-(1-')(4m'/5)\cdot d_l} \tag{II.42}$$

where K is again some constant independent of m, and each vertex produced is normalized by $/a^2$ and each field produced is normalized by a factor $1/\sqrt{a}$.

The exponential decrease in (II.40) will be used later both to sum over the various locations of the squares and regions which form E and also to control "local factorials" generated by integration of the fields produced by the functional derivatives, and by the combinatoric of Leibniz's formula for derivations of products.

The sums over all combinatoric factors associated with the various choices in the Leibniz formula are certainly bounded again by a factor $K^n \prod (n!)^q$ for K and q large enough. This simplifies our bound; taking (II.41) and (II.42) into account we collect a multiplicative factor

$$(K/\sqrt{a})^n \prod (n!)^{2q} \prod_{l=1}^{n} e^{-(1-')(4m'/5)\cdot d_l} \tag{II.43}$$

(with some enlarged value for K) and we have still to bound the supremum over functional integrals of functions $F' = P' \cdot \Omega e^{-V}$ in which P' is now a monomial (without sums and prefactors) $\prod_x (h_i(x)/\sqrt{a})^{p_i(x)}$.

The functional integration over F' is bounded using a Schwartz inequality to separate the fields in P' from the rest. This means that we write

$$\int F'(\{h(x)\}) \prod_{i=0}^{n} d\mu_i$$
$$\leq \Big(\int \prod_x (h(x)/\sqrt{a})^{2p(x)} \prod_{i=0}^{n} d\mu_i\Big)^{1/2} \Big(\int \Omega e^{-2V} \prod_{i=0}^{n} d\mu_i\Big)^{1/2} \tag{II.44}$$

(since $\Omega = \Omega^2$).

We bound first the second functional integral in (II.44).

Lemma II.5

The second functional integral in (II.41) satisfies the bound:

$$\int \Omega e^{-2V} \prod_{i=0}^{n} d\mu_i \leq K^n \prod_{\in E\cap} e^{-\sqrt{a}\cdot\log a} \tag{II.45}$$

where K is a positive constant.

Proof We remark first that the positive hence potentially dangerous piece of e^{-2V} corresponding to the small field squares of E (see (II.12)) can be exactly bounded, using the small field condition $_1$ in Ω by $e^{2r'}$, where we recall that r' is the number of small field squares in E. This factor can be absorbed in the constant K in (II.45).

Let be a square of the large field region , and K be some large constant. Either

A)

$$\sup_{x\in} |h_x| \leq 2K\sqrt{a\log a} \tag{II.46}$$

or

B) There is a site $x \in$ such that

$$|h_x| \geq 2K\sqrt{a\log a} \tag{II.47}$$

and

$$\inf_{x\in} |h_x| \geq K\sqrt{a\log a} \tag{II.48}$$

or

C) There is a site $x \in$ and a site $y \in$ such that

$$|h_x| \geq 2K\sqrt{a\log a} \tag{II.49}$$

$$|h_y| \leq K\sqrt{a\log a} \tag{II.50}$$

In the first case we write, using the large field condition $_2$:

$$1/2 \leq \frac{1}{||}\sum_{x\in} h_x^4/a^2 \leq \left(\frac{1}{||}\sum_{x\in} h_x^2/a^2\right)(\sup_{x\in}|h_x|)^2 \tag{II.51}$$

Therefore

$$\left(\sum_{x\in} h_x^2/a^2\right) \geq \frac{||}{8K^2a(\log a)} = \frac{a}{8K^2(\log a)} \tag{II.52}$$

The potential then gives the small factor. Indeed we can assume $K^2(\log a)/a \leq 1$, since a is large; then $|h_i| \leq 1$ for $x \in$ and since $e^{-t}-1 \leq -t/e$ for $t \leq 1$, we conclude that

$$e^{-V} = e^{\sum_{x\in} e^{-h_x^2/a^2}-1} \leq e^{-\sum_{x\in} h_x^2/e.a^2} \leq e^{-\frac{a}{8eK^2(\log a)}} \leq e^{-\sqrt{a}\cdot\log a} \tag{II.53}$$

if a is big enough.

In the case B) we use the fact that $\inf_{x\in}|h_x| \geq K\sqrt{a\log a}$ to obtain directly that

$$e^{-V} = e^{\sum_{x\in} e^{-h_x^2/a^2}-1} \leq e^{-\sum_{x\in}(1/e)\inf\{h_x^2/a^2,1\}}$$

$$\leq e^{-(a/e)\inf\{K^2(\log a),a\}} \leq e^{-\sqrt{a}\cdot\log a} \tag{II.54}$$

In the case C) we use the fact that the gaussian measure gives a small factor when two sites not too far apart have very different values. More precisely we write

$$\int d\mu \sum_x \sum_y (|h_x| \geq 2K\sqrt{a\log a})(|h_y| \leq K\sqrt{a\log a})$$

$$\leq (a^2/)^2 e^{-2\sqrt{a}\cdot\log a} \leq e^{-\sqrt{a}\cdot\log a} \tag{II.55}$$

Indeed the gaussian distribution corresponding to two sites x and y after integrating on the others behaves as $e^{-c(h_x-h_y)^2/2\log|x-y|}$, where c is some constant, and $\log|x-y| \leq \log a/\sqrt{\ } \leq \sqrt{a}$, if a is large, using the condition (II.7) in the Theorem. If we take $cK^2 > 2$ (by increasing K) the first inequality in (II.55) follows. The second inequality is again obtained using (II.7) since $a^4/^2 < a^4 e^{(2/K)\sqrt{a}}$ is beaten by the small factor $e^{-\sqrt{a}\cdot\log a}$. In every case the proof of (II.45) is achieved.

Let us return to the first functional integral in (II.44). We perform this gaussian integration explicitly. Remark that all fields h_i produced for $i = 1, ..., n$ are in fact of the type $m^2_{i}(x)h_i(x)$, that is they are localized inside $_i$ and they are multiplied by a factor m^2. The result of Wick's theorem is a certain number of graphs with propagators $C^{0,0}$ or C_{i}. We use first a Schwartz inequality again to bound $C^{0,0}(x,y)$ or $C_{i}(x,y)$ respectively by $(C^{0,0}(x,x))^{1/2}(C^{0,0}(y,y))^{1/2}$ or $(C_{i}(x,x))^{1/2}(C_{i}(y,y))^{1/2}$. Then we use the following bound

Lemma II.6

$$C^{0,0}(x,x) \leq K \cdot \log(1+m^{-1}) \tag{II.56}$$

$$C_{i}(x,x) \leq K \cdot \sup\{\log(1+m^{-1}), \log(1+d(x,\partial_i)\} \tag{II.57}$$

Proof (II.56) follows from (II.33). For (II.57) using the random path expansion of the propagator it is easy to bound $C_{i}(x,x)$ by $C_{^2-\{y\}}(x,x)$, where y is the point closest to x in the complement of . We can consider again that the distribution for the two sites in a massless gaussian measure after integration of the others is $e^{-c(h_x-h_y)^2/2\log|x-y|}$. The distribution for $C_{^2-\{y\}}(x,x)$ is massless except for a factor $e^{-(1/2)m^2h_y^2}$. Using this factor and integrating on y we obtain the distribution for h_x and achieve the proof of (II.57).

Then we remark that in the product (II.44) each field to integrate has an associated normalizing factor $1/\sqrt{a}$. Using the fact that at large a $\log(1+m^{-1}) < \sqrt{a} << a$ we can use these normalizing factors to compensate again all the factors $\log(1+m^{-1})$ produced by gaussian integration and Lemma II.6.

Using an other fifth of the exponential decrease $\prod_{l=1}^n e^{-(1-')(4m'/5)\cdot d_l}$ we can beat the product of all factors $\log(1+d(x,\partial_i))$ generated by (II.57). Indeed each field $h_i(x)$ is at the end of a chain whose last propagator has length at least equal to $d(x,\partial_i)$.

It remains to bound the local factorials generated by Wick's theorem in the gaussian integration of the fields. Our final fields to contract are localized in squares of the type $^s_{g_l}$ or $^s_{h_l}$, and a naive bound would involve factorials of the number of fields localized in such squares. But using an other fifth of the remaining exponential decrease $\prod_{l=1}^n e^{-(1-')(3m'/5)\cdot d_l}$ it is easier to choose, in the Wick's contraction process, the squares $_{e_l}$ or $_{f_l}$ which contain the initial vertex from which the chain to the contracted field emanate. In this way the factorials of Wick's contractions is bounded by $K^n \prod (n!)^5$ (since there are at most five fields produced per vertex).

Using a standard volume argument we know that from a remaining fifth of the exponential decay $\prod_{l=1}^n e^{-(1-')(2m'/5)\cdot d_l}$ we can extract a factor $K^n \prod (n!)^{-q'}$ where q' can be made as large as we want [3, Lemma III.1.3]. This is because n can become large only when we have more and more distant squares $'$ hooked to by the cluster expansion. In particular we can take $q' > 2q+5$. In this way we can compensate the local factorials $\prod (n!)^{2q+5}$.

Finally the sums over the positions of the squares in E is also made using the last fifth of the exponential decay $\prod_{l=1}^n e^{-(1-')(m'/5)\cdot d_l}$; the summation is made according to the natural tree structure T which is the outcome of the Battle-Brydges-Federbush process. We have also to sum over the regions $_i$, $i = 1, .., r$, knowing one of their squares. This is done using the distance links, and there is therefore an associated factor $\prod_{\in E\cap} M^2$. This factor is compensated by the one of Lemma II.5, since $M = K \cdot a^{1/4}$.

In order to have the small factor in $N(E)$ in (II.38) we must also extract a factor e^{-M^2} from each small field square in E. This can be extracted from a fraction of the factor $\prod_{\in E\cap} e^{-\sqrt{a}\cdot\log a}$ in (II.45), since $e^{M^2} = e^{K^2\sqrt{a}} << e^{+\sqrt{a}\cdot\log a}$.

After having extracted this factor we get a final geometric sum over the number n of elements in E of a term certainly bounded by

$$K^n a^{-n/2} \tag{II.58}$$

where K is independent of a. Taking a large enough we get a geometric series with ratio as small as we want. This proves in particular the condition (II.38), hence achieves the proof of Lemma II.4 and of the convergence of our cluster expansion.

III. The wetting potential with a wall

In order to model the presence of the wall in a wetting problem we consider now different potentials which are not symmetric, but for which the method of the last section applies with minor modifications. We shall not detail as much the proof in these asymmetric cases.

A) The linear exponentials

The first example that can be considered is a potential made of two competing linear exponentials (Fig.3):

$$V(h) =^2 /2+e^{-h}-(1/2)e^{-2h} \tag{III.1}$$

Fig. 3

The "wall" is the region $h<0$, which is exponentially suppressed. On the side $h>0$ we have fast asymptoticity of the potential to a constant, just as in the preceding model (but the asymptotic value is reach in a linear instead of quadratic exponential way). We find that the minimum is at h^* such that $e^{-h^*} =$. If we write $\hat{h}=h-h^*$, the analogue of the Taylor formula (II.12) is

$$e^{-V(h)} = \exp\{-\frac{2}{2}[e^{-\hat h}-1]^2\} \tag{III.2a}$$

$$= \exp\{-\frac{{}^{22}\hat h^2}{2} +^{2\ 3}\hat h^3 \int_0^1 (-\frac{e^{-t\hat h}}{6}+\frac{2e^{-2t\hat h}}{3})\frac{(1-t)^2}{2}dt\}\ . \tag{III.2b}$$

The first form will be used in the large field region out of the well, the second form is suited for a cluster expansion in the small field region (inside the well).

Therefore the mass, in the regime where the gaussian well is very flat, is expected to be $m =$. For a fixed value of , we can make the mass very small by letting $\to 0$. In this sense the parameter plays the rôle of a^{-2} in the previous section.

Here we need a slightly more complicated small field condition which states both that the field is approximately inside the well, and that the exponential in the potential is under control. For instance we can bound the Taylor remainder in (III.2b) using a Schwartz inequality:

$$\begin{aligned} &{}^{23}\hat h^3 \int_0^1 (-\frac{e^{-t\hat h}}{6}+\frac{2e^{-2t\hat h}}{3})\frac{(1-t)^2}{2}dt\} \\ &\leq \left(\frac{1}{||}\int^2 \hat h^6\right)^{1/2} . \left(\frac{1}{||}\int \sup\{1,e^{-4h}\}\right)^{1/2} \end{aligned} \tag{III.3}$$

Therefore we can choose as the small field condition for a square of side m^{-1}:

$$(\frac{1}{||}\int^2 \hat{h}^6)(\frac{1}{||}\int \sup\{1, e^{-4h}\}) \ . \tag{III.4}$$

so that in the small field region we control the Taylor remainder. We need again as technical condition analogue of (II.7) a bound which is not optimal:

$$K \log(1+^{-1}) <^{-1/3} \tag{III.5}$$

Remark again that for technical reasons we cannot cover with our a single scale analysis the complete mean field region, which we expect here to be given by a condition of the type

$$K \log(1+^{-1}) <^{-2/3} \ . \tag{III.6}$$

Indeed the vertices produced in the small field region are $\frac{}{h}^{23}\hat{h}^3$ times exponentials which are controlled by the second part of the small field condition. Using some simple power counting, such a vertex is evaluated by $\log^{3/2}(1+^{-1})$ after integration in a square . In order to correspond to a small factor, condition (III.6) is enough and (III.5) is more than sufficient. However in the large field region we need to gain a factor small enough to compensate the normalization. The worst case for the large field region is when the first function in (III.4) is replaced by $(1-)$, the other case giving a much smaller factor. But performing an analysis similar to that of Lemma II.5, in case C) we obtain a small factor in $e^{-^{-2/3}/\log(1+^{-1})}$, which has to beat a normalization factor similar to (II.35), hence condition (III.5) is necessary.

We can state:

Theorem III.1

Theorem II.1 holds true if $V(h)$ in (II.6a) is taken as in (III.1) with condition (III.5) and $m =$.

B) The compact wall

A slight modification of the exponential wall (III.1) is to add a cutoff function $\prod_x (h_x)$ where is a C^∞ function which is 0 for $h_x < 0$ and 1 if $h_x \geq 1$, in order to model better the fact that the interface cannot penetrate the wall (see Fig.4).

Fig. 4

The rules of the expansion are unchanged and the only additional technical problem is when a functional derivative hits an function. This produces vertices of the type $'(h) = \hat{h} + h^*$. Such fields correspond to the large field region, and do not change the range of validity of the expansion. Therefore Theorem III.1 also holds for this model.

C) The Van-der-Waals potential

The Van der Waals potential for molecular attraction ([D],[G]) leads to the consideration of wetting potentials of the type:

$$V = -\frac{}{2h^2} + \frac{1}{3h^3} + \frac{^3}{6} \quad \text{if} \quad h \geq 0$$
$$V = +\infty \quad \text{if} \quad h < 0 \tag{III.7}$$

The minimum of the potential is at $h^* = {}^{-1}$. The expected mass is $m = {}^{5/2}$ and goes to 0 with . There is here a single parameter. We can put again $\hat{h} = h - h^*$ and write (for $h \geq 0$):

$$V = +\frac{^5}{2}\hat{h}^2 + \int_0^1 (\frac{12}{t^5(\hat{h}+{}^{-1})^5} - \frac{20}{t^6(\hat{h}+{}^{-1})^6})\frac{(1-t)^2}{2}dt\} \ . \tag{III.8}$$

We can write again large and small field conditions using (III.8). However the situation is simpler here since there is only one parameter. Our cluster expansion therefore does apply simply for small enough:

Theorem III.2

Theorem II.1 holds true if $V(h)$ in (II.6a) is taken as in (III.7) with small enough and $m = {}^{5/2}$.

The case of a Lennard-Jones potential is exactly similar, but with different values of the exponents in (III.7).

D) More general potentials

¿From the discussion of the specific examples above it is clear that our method generalizes to any sufficiently smooth potential with a single absolute minimum strictly below all othe local minima, in the regime where the corresponding well is very flat, i.e. the gaussian approximation is a good approximation for a rather large range of values of the interface height. The exact limits of validity of the cluster expansion depends of the shape and parametrization of the curve giving the potential, so that we do not state a precise general theorem. Smoothness of the potential everywhere is presumably not physically essential but for our method it is a useful technical ingredient, even in the “large field region”, because it allows analytic computation of the functional derivatives, which in our cluster expansion can act in this large field region. To treat the case of non-smooth potential is presumably possible but certainly requires some additional technical work.

Appendix: Estimate of the lattice covariance

The inverse covariance of mass m on ^{2} is defined as

$$C^{-1}(x,y) = (4+m^2)_{x,y} - _{|x-y|,1} \tag{A.1}$$

so that

$$\frac{1}{2}\sum_{x,y\in^2} (x)C^{-1}(x,y)(y) = \frac{1}{2}\sum_{|x-y|=1} \big((x)-(y)\big)^2 + \frac{m^2}{2}\sum_{x\in^2}(x)^2 \tag{A.2}$$

where the sum is over *pairs* of nearest neighbors and

$$<(x)(y)>= C(x,y) = \frac{1}{4\pi^2}\int_{-\pi}^{\pi} dk_2 \int_{-\pi}^{\pi} dk_1 \frac{e^{ik_1(x_1-y_1)+ik_2(x_2-y_2)}}{m^2+4-2\cos k_1 - 2\cos k_2} \tag{A.3}$$

Lemma

Let $m \le 1$ and $x_1 \ge x_2 \ge 0$. Let m_1 and m_2 be the functions of m, x_1, x_2 defined through

$$m_1x_1 + m_2x_2 = \sup_{\cosh m_1' + \cosh m_2' = 2+m^2/2} \{m_1'x_1 + m_2'x_2\} \tag{A.4}$$

Then

$$C\big((x_1,x_2),(0,0)\big) = O\big(\log(m(1+x_1))^{-1}\big) \quad \text{if } mx_1 \le 1$$

$$C\big((x_1,x_2),(0,0)\big) \;=\; O(1)\frac{e^{-m_1x_1-m_2x_2}}{\sqrt{mx_1}} \quad \text{if } mx_1 \ge 1 \tag{A.5}$$

which implies $\forall x, y$

$$C(x,y) \le c(\log m^{-1})\cdot e^{-m'|x-y|} \tag{A.6}$$

where $|x-y|$ is the Euclidean distance and m' is defined by

$$\cosh m' = 1 + m^2/2 \tag{A.7}$$

Remark The slowest decay, at $\pi/4$, is given by $e^{-m''|x-y|}$ with

$$\cosh\frac{m''}{\sqrt{2}} = 1 + m^2/4 \tag{A.8}$$

Proof $x_1 = 0$ is easy. We suppose $x_1 \ge 1$ and begin by shifting the contour of integration in k_2. For any $m_2 < m'$ we have

$$\int_{-\pi}^{\pi} dk_2 e^{ik_2x_2} \int_{-\pi}^{\pi} dk_1 \frac{e^{ik_1x_1}}{m^2+4-2\cos k_1 - 2\cos k_2}$$

$$= \int_{-\pi}^{\pi} dk_2' e^{-m_2x_2+ik_2'x_2} \int_{-\pi}^{\pi} dk_1 \frac{e^{ik_1x_1}}{m^2+4-2\cos k_1 - 2\cos(im_2+k_2')} \tag{A.9}$$

We then integrate by residue over k_1. The location of the relevant pole is given by $k_1 = im_1(k_2') + k_1'(k_2')$ with

$$\cosh m_1(k_2')\cos k_1'(k_2') = 2 + m^2/2 - \cosh m_2 \cos k_2' \tag{A.10}$$

$$\sinh m_1(k_2')\sin k_1'(k_2') = \sinh m_2 \sin k_2' \tag{A.11}$$

so that

$$C\big((x_1,x_2),(0,0)\big) = \frac{1}{2\pi}\int_{-\pi}^{\pi} dk_2' \frac{e^{-m_2x_2+ik_2'x_2-m_1(k_2')x_1+ik_1'(k_2')x_1}}{2\sinh\big(m_1(k_2')-ik_1'(k_2')\big)} \tag{A.12}$$

We now choose m_2 so that

$$m_1(0)x_1 + m_2x_2 = \sup_{\cosh m'_1 + \cosh m'_2 = 2 + m^2/2} \{m'_1x_1 + m'_2x_2\} \tag{A.13}$$

The assumption $x_1 \geq x_2 \geq 0$ implies $m_1(0) \geq m_2$ and $m_2 < m'$ as required, and also $m_1(0) = O(m)$. Equation (A.10) implies $|k'_1(k'_2)| < \pi/2$ and $m_1(k'_2)$ even in k'_2. One can then compute $\frac{dm_1(k'_2)}{dk'_2}$ and check that it is positive for $k'_2 \geq 0$. We then write

$$m_1(k'_2) = m_1(0) +_1 (k'_2) \tag{A.14}$$

which, inserted into (A.10), using (A.11), gives

$$\cosh_1(k'_2) - 1 + \tanh m_1(0) \cdot \sinh_1(k'_2) = O((k'_2)^2) \tag{A.15}$$

so that

$$_1(k'_2) = O(\frac{(k'_2)^2}{m_1(0)}) \quad \text{if } k'_2 \leq m_1(0) \qquad \text{(A.16}$$

$$_1(k'_2) \;=\; O(k'_2) \quad \text{if } k'_2 \geq m_1(0) \tag{A.17}$$

We split accordingly the integral over k'_2:

$$C\big((x_1,x_2),(0,0)\big) \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} dk'_2 \frac{e^{-m_2x_2 - m_1(0)x_1 -_1(k'_2)x_1}}{\sqrt{2}\sinh\big(m_1(0) +_1 (k'_2)\big)} \tag{A.18}$$

First

$$\int_0^{m_1(0)} dk'_2 \frac{e^{-_1(k'_2)x_1}}{\sinh\big(m_1(0) +_1 (k'_2)\big)} = m_1(0)\int_0^1 dk_2 \frac{e^{-O(k_2^2)m_1(0)x_1}}{\sinh\big(m_1(0)(1+O(k_2^2))\big)}$$

$$= \;O(1) \qquad\qquad \text{if } m_1(0)x_1 \leq 1 \tag{A.19}$$

$$= O(\frac{1}{\sqrt{m_1(0)x_1}}) \quad \text{if } m_1(0)x_1 \geq 1 \tag{A.20}$$

Then

$$\int_{m_1(0)}^{\pi} dk'_2 \frac{e^{-_1(k'_2)x_1}}{\sinh\big(m_1(0) +_1 (k'_2)\big)} = \int_{m_1(0)x_1}^{\pi x_1} dk_2 \frac{e^{-O(k_2)}}{O(k_2)}$$

$$= O\big(1 + \log\big(m_1(0)x_1\big)^{-1}\big) \quad \text{if } m_1(0)x_1 \leq 1 \tag{A.21}$$

$$< \;\frac{e^{-O(m_1(0)x_1)}}{m_1(0)x_1} \qquad\qquad \text{if } m_1(0)x_1 \geq 1 \tag{A.22}$$

Putting everything together, we obtain the lemma.

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