ASYMPTOTIC DISTRIBUTION OF ENTRANCE TIMES FOR EXPANDING MAPS OF AN INTERVAL

by

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Abstract. We prove that the entrance time into a small interval is asymptotically exponentially distributed. We also prove that under suitable hypothesis the sequence of visits to a small interval converges to a Poisson point process.

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I. Introduction

In a dynamical system given by a transformation f of a space X with an ergodic invariant measure μ, there is a natural time scale associated to a set of positive measure, namely the inverse of it's measure. For example. the ergodic theorem tells us that the asymptotic fraction of time that a typical trajectory spends into a given set is precisely the measure of that set.

It is therefore natural to consider the events associated to a set K (contained in X) at this natural time scale $1/\mu(K)$. Examples of such events are the first entrance time in K of a trajectory, or the number of returns to K in a given time interval. For a general set K of positive measure, one can only expect to derive rather simple results. For example, if the set K has positive measure, ergodicity implies that the entrance time in K is almost surely finite. One can hope however to obtain more precise information by looking at the asymptotic when the set K is getting small, or in some large deviation regime (see [Ke.] for such results).

In this paper we will derive several asymptotic results fro small intervals in the case of piecewise regular expanding maps of the interval. We recall that such maps f of the interval $[0,1]$ are defined on a finite number r of disjoint open intervals $]a_j, a_{j+1}[$ whose union

$$\bigcup_{j=0}^{r-1}]a_j, a_{j+1}[$$

is dense in $[0,1]$. The restriction of the map f to each of these intervals is monotone, regular and extends to a regular map on each closed interval. Moreover, the map is expanding, that is to say that there is a number $\rho > 1$ such that the slope of each branch of f is in modulus larger than ρ.

It is well known that expanding maps of the interval has an absolutely continuous invariant measure and we will assume that this measure is ergodic and mixing. We will always denote by μ this measure and by h it's density with respect to the Lebesgue measure denoted by l. This function h is known to be of bounded variation. We refer to [C.] or [H.K.] and [L.Y.] for proofs of the fundamental properties of these dynamical systems. In particular, the mixing property implies the following exponential mixing (also called exponential decay of correlations). There is a positive constant C and a positive number $\gamma < 1$ such that if u_1 is a function of bounded variation, and u_2 an integrable function, we have

$$\left| \int u_1(x) u_2(f^n(x)) d\mu(x) - \int u_1(x) d\mu(x) \int u_2(x) d\mu(x) \right|$$

$$\leq C\gamma^n \left(\vee u_1 + \int |u_1(x)| dl(x) \right) \int |u_2|(x) dl(x) \; , \qquad (D.C)$$

where $\vee u_1$ denotes the total variation of u_1.

The results of the present paper can be proven under more general (or slightly different) hypothesis, for example in cases with a countable number of branches under a uniform distortion hypothesis, or for the case of Gibbs measures over a subshift of finite type (see [W.], and [H.K.] for the definitions and basic results about such systems). The proofs in these general cases are almost word for word translations of the proofs presented below, but we refrain to present the more general setting which is often obscured by unnatural abstract details. We also conjecture that the present results are true for unimodal maps under the standard hyperbolicity hypothesis of the critical orbit (see [K.N.] and [L.S.Y.] for the mixing results).

We now introduce the definition of entrance time and recurrence time in a set K.

Definition. *The first entrance time of the trajectory of a point x into the set K is the integer $\tau_K(x)$ given by*

$$\tau_K(x) = \min\{n \geq 0 \,|\, f^n(x) \in K\} \ .$$

As explained above, since μ is ergodic the integer valued function τ_K is μ almost surely finite. Our first result can be formulated as follows.

Theorem 1. *Let $(K_j)_{j\in\mathbb{N}}$ be a sequence of intervals such that $\mu(K_j) > 0$ for any integer j and $\lim_{j\to\infty} l(K_j) = 0$. Then there is a diverging sequence of positive numbers T_j such that for any positive real number t, we have*

$$\lim_{j\to\infty} \mu(\{\tau_{K_j} > tT_j\}) = e^{-t} \ .$$

In other words, the sequence of laws of the random variables τ_{K_j}/T_j converges in the weak* topology to an exponential distribution with parameter 1. From now on we will simply say that a sequence of random variables converges in law instead of saying that the sequence of laws converge in the weak* topology. We recall that the exponential distribution with parameter $\alpha > 0$ is the measure on $\mathbb{R}^+$ whose density is $\alpha e^{-\alpha x}$.

Note that at this point we have no estimate on the time scale T_j. We will need another hypothesis to ensure that it behaves like $1/\mu(K_j)$ (see below for the precise conditions, and [H.] for a counter example).

We refer to [G.S.] for a version of this result adapted to large deviation sets, and to [C.G.] for a related result under different hypothesis.

The proof of this result can be extended to the case of sequences of finite union of intervals provided the number of intervals does not grow too fast.

Our second result will give more precise information but this requires stronger hypothesis. We will look at the sequence of successive times at which the orbit of a point x belongs to an interval K. This is conveniently described by an atomic measure on the time axis which gives a weight equal to one to the instants where the orbit is in K. We will of course consider this measure on the time axis scaled by the natural time scale $1/\mu(K)$, and it will be denoted by $N(x, K)$. In other words, we have

$$N(x, K) = \sum_{n=0}^{\infty} \chi_K \circ f^n(x) \delta_{n\mu(K)}$$

where δ_t denotes as usual the Dirac measure at the point t. We now observe that if we choose x according to the invariant measure μ, we obtain a random atomic measure on the positive real line, that is to say a point process on $\mathbb{R}^+$ (we refer to [N.] for the definitions and basic properties). In other words, we have defined a measure $\mathcal{M}_K$ on the positive σ-finite measures over $\mathbb{R}^+$. By considering a sequence of intervals (K_j) we construct a sequence of such measures $(\mathcal{M}_{K_j})$ and we can ask if this sequence converges. This is indeed the case under the hypothesis of the following theorem.

Theorem II. *Let $(K_j)_{j\in\mathbb{N}}$ be a sequence of intervals satisfying the assumptions of Theorem I and moreover, assume that there is a non decreasing diverging sequence of integers $(m_j)_{j\in\mathbb{N}}$ such that for any j we have*

$$f^m(K_j)\cap K_j=\emptyset \qquad \text{for} \quad 0<m\le m_j \ ,$$

then

$$\lim_{j\to\infty} T_j\mu(K_j)=1 \ .$$

If we have also

$$\liminf_{j\to\infty}\inf_{K_j} h>0 \ ,$$

then the sequence of measures $(\mathcal{M}_{K_j})$ is tight, and the limit is the random measure of a Poisson point process of parameter 1.

We recall that a Poisson point process of parameter $\alpha>0$ on $\mathbb{R}^+$ is an increasing sequence of random variables whose increments are independent and have exponential distribution with parameter α. We refer to [N.] for definitions and basic properties of point processes, and to [S.] for some other motivations and applications.

We observe that the previous hypothesis are typical for the measure μ. Indeed μ almost every point x is a point of continuity of h such that $h(x)>0$ which is our second hypothesis. We can also assume moreover that the point x is not periodic (since the number of periodic points is countable). For such points it is easy to construct a decreasing sequence of intervals which converges to x and satisfies the hypothesis of Theorem II.

Note also that in our first hypothesis, we have not put any constraint on the rate of divergence of the sequence $(m_j)_{j\in\mathbb{N}}$. Since the map is expanding, m_j can be at most of the order of $-\log\mu(K_j)$.

Part 1) of Theorem II gives the precise and natural time scale for the events we are considering. We see that this natural time scale holds almost everywhere, but not everywhere as follows from an example of [H.].

Several interesting results follow easily from the above theorem, some particular cases have already been reported in the literature with more complicated proofs. One can for example consider the succession of entrance times in a set K which can be defined recursively as follows. If $\tau_K^1=\tau_K$, $\tau_K^2,\cdots,\tau_K^l$ have already been defined, we define τ_K^{l+1} by

$$\tau_K^{l+1}(x)=\min\{n>0\,|\,f^{n+\sum_{j=1}^{l}\tau_k^j(x)}(x)\in K\} \ .$$

As for the first return time, all these numbers are μ almost surely finite. We recall that for the measure $\mu|_K/\mu(K)$ these numbers form a stationary sequence (see [B.]), however we are concerned here with the measure μ itself. An easy consequence of Theorem II is the following result (we refer to [N.] for the proof using Theorem II).

Theorem III. *Let $(K_j)_{j\in \mathbb{N}}$ satisfying the assumptions of Theorem II. Then the process $\mu(K_j)\tau_{K_j}$ converges to a Poisson point process with parameter 1.*

This was proven in [H.] for the case of maps possessing the topological Markov property using the zeta function formalism. In our more general case which is not Markov, this follows at once from Theorem II whose proof is more direct and simpler.

An easy corollary gives the limit distribution of the number of visits to the sets (K_j) in time intervals of the order of the natural time scale. For any set K, and any positive number t, we define the number of visits to K during the period $[0,t]$ as the random number

$$N_K(t) = \sum_{0\le j\le t} \chi_K \circ f^j \ .$$

We can now state our result.

Corollary IV. *If the sequence of intervals (K_j) satisfies the hypothesis of Theorem II, then for any fixed positive number s, the sequence of random variables $(N_{K_j}(s/\mu(K_j)))$ converges in law to a Poisson random variable with parameter e^{-s} .*

This result was proven in [P.] for the particular case of Markov chains with a finite number of states using different techniques.

We mention that it is also possible to derive using our techniques generalized versions of Theorem II and of it's consequences. One can for example weight the visits according to the point of landing in the set K. This leads to the definition of the following point process.

Let u be a real function of bounded variation with compact support. For any point $x\in[0,1]$ and any positive real number ϵ, we define an atomic measure $N_\epsilon(x)(\cdot)$ by

$$N_\epsilon(x) = \sum_{n=0}^{\infty} u\left(\frac{f^n(y)-x}{\epsilon}\right)\delta_{n\epsilon} \ .$$

The proof of Theorem II can be copied almost word for word to prove that when $\epsilon\to 0$ the family of random measures $N_\epsilon(\cdot)$ is x almost surely tight, and the limit distribution is that of a random variable X with characteristic function

$$\mathbb{E}\left(e^{itX}\right) = e^{h(x)\int (e^{itu(y)}-1)dy} \ .$$

Finally we mention also that all our results can be proven if we start with a non equilibrium measure which is absolutely continuous with respect to μ and with a density of bounded variation. This is due to the exponential decay of correlations ([H.K.], [C.]).

The rest of this paper is devoted to the proofs of Theorem I, which is given in section II, while the proof of Theorem II is given in section III.

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II Proof of Theorem I.

The proof relies on a perturbation argument for the transfer operator P. We recall that this operator is defined by

$$Pu(x) = \sum_{f(y)=x} u(y)/|f'(y)| \ .$$

The main basic result about the operator P is the following estimate due to Lasota and Yorke [L.Y.]. Under our hypothesis on the map f, there are two positive numbers Γ and $\alpha < 1$ such that for any integer q large enough (independently of g), we have

$$\bigvee P^q(g) \leq \alpha^q \vee g + \Gamma \|g\|_{L^1(dl)} \ . \qquad (L.Y.)$$

This is the basic estimate used in [L.Y.] to prove the existence of the invariant and absolutely continuous measure μ. It is also the main ingredient for the spectral theory of the operator P developed in [H.K.] (or [C.]). We recall that under our mixing condition, in the Banach space $\mathcal{B}$ of functions of bounded variation equipped with the norm

$$|||g||| = \vee g + \|g\|_{L^1(dl)} \ ,$$

the operator P has a simple eigenvalue equal to 1 with eigenvector h and whose associated linear form is the integration over the Lebesgue measure l. The rest of the spectrum is contained in a disk of radius $\rho < 1$.

We now briefly explain how perturbation theory comes into play. Let K be a fixed interval, then we have for any positive integer n

$$\chi_{\{\tau_K > n\}} = \prod_{j=0}^{n} \chi_{K^c} \circ f^j \ ,$$

where K^c denotes the complement of a set K, and χ_K is the characteristic function of that set.

Using the elementary properties of the operator P, it is easy to see that the probability of the event $\{\tau_K > n\}$ is given by

$$\mu(\{\tau_K > n\}) = \int \chi_{K^c} \left(P\chi_{K^c}\right)^n h \, dl \ .$$

It is intuitively clear that for large integer n, the largest eigenvalue of $P\chi_{K^c}$ (if there is one) will give the asymptotic behavior. Moreover since $P\chi_{K^c} = P - P\chi_K$, one is tempted to try to argue that the operator $P\chi_K$ is small. This seems to be reasonable in the present context since K will be a small interval. However this is not so simple because a direct estimate involves the variation norm of the function χ_K which is always equal to 2. In the next section we will indeed develop this idea but using first a remark on the preimages of small intervals, and then a change of norm.

This spectral problem is related to several questions about the exit time of the neighborhood of a repeller, and in particular to the exponential tail of its distribution. We refer to [E.R.], [L.Y.2] and [Ke.] for more details on these other topics.

As explained above, the main tool in the proofs will be a perturbative control of the operator

$$P\chi_{K^c} = P - P\chi_K \ .$$

In order to control some constants we will instead consider the operator

$$(P\chi_{K^c})^q$$

where q will be a fixed large integer chosen once for all independently of K for small K. Let us call Δ_q the operator satisfying

$$(P\chi_{K^c})^q = P^q + \Delta_q \ ,$$

and we will prove that the operator Δ_q is small in some sense. We start by a Lemma which gives a detailed control of the operator Δ_q.

Lemma 5. *There is a positive constant M such that for any fixed integer q and any small enough interval K, we have for any function g of bounded variation*

$$\vee(\Delta_q g) \le (6q^2\alpha^q + Mql(K)) \vee g + Mq(1+ql(K))\|g\| \ ,$$

and

$$\|\Delta_q g\| \le Mq^2 l(K) |||g||| \ .$$

Proof. At first sight, the number of terms in Δ_q is $2^q - 1$, each of them of order one. Our first goal will be to prove that this large sum can be rearranged in a sum with a much smaller number of terms.

After some easy algebra we get

$$(P\chi_{K^c})^q = P^q - \sum_{j=0}^{q-1} P^j P\chi_K P^{q-j-1} + \sum_{i+j\le q-2} P^i P\chi_K \left(P\chi_{K^c}\right)^{q-i-j-2} P\chi_K P^j \ .$$

If we assume K small enough (i.e. the length smaller than the diameter of the smallest interval of monotonicity of f^q and also smaller than the minimal non zero distance between the points of the orbits up to time q of the points (a_n)), then for any integer $l \le q$, and for any point x of the interval, there are at most two preimages of order l of x in K.

This implies that for K small enough, the set

$$f^{-l}(K) \cap K$$

is an interval (eventually empty) or the union of two intervals (this special case can occur if two branches of f^l of opposite slope have an extreme point in common which is a periodic point of period at most q). By looking at the orbit up to l of these intervals, we conclude that

$$f^{-l}(K)\cap_{j=1}^{l-1} f^{-j}(K^c)\cap K$$

is also an interval or the union of two intervals.

This immediately implies that

$$P^i P\chi_K \left(P\chi_{K^c}\right)^{q-i-j-2} P\chi_K P^j = P^{q-j}\chi_{\tilde{K}_{i,j}} P^j$$

where $\tilde{K}_{i,j}$is an interval or the union of two intervals. This implies that the above sum contains at most $2q^2$ terms which can be written

$$P^{l+1}\chi_{\tilde{K}} P^{q-l-1}$$

where l is an integer smaller than q, and $\tilde{K}$ is an interval which depends on l and with length at most the length of K. Moreover, for each fixed l, there is at most $2q$ such terms.

We will now estimate the variation of a function $P^r\chi_{\tilde{K}}P^{q-r}g$. Using formula (L.Y.) and the classical estimate

$$\vee(g_1g_2)\le \vee g_1\|g_2\|_\infty + \vee g_2\|g_1\|_\infty$$

for the variation of the product of two functions, we deduce

$$\vee(P^r\chi_{\tilde{K}}P^{q-r}g)\le \alpha^r\vee(\chi_{\tilde{K}}P^{q-r}g)+\Gamma\|\chi_{\tilde{K}}P^{q-r}g\|\le$$

$$\alpha^r(\vee(P^{q-r}g)+2\|P^{q-r}g\|_\infty)+\Gamma\|\chi_{\tilde{K}}P^{q-r}g\|\le \alpha^r\vee(P^{q-r}g)+(2\alpha^r+\Gamma l(K))\|P^{q-r}g\|_\infty\ .$$

Using again formula (L.Y.) and the bound

$$\|g\|_\infty\le \vee g+\|g\|$$

we obtain

$$\vee(P^r\chi_{\tilde{K}}P^{q-r}g)\le (3\alpha^q+\Gamma l(K)\alpha^{q-r})\vee g+(\alpha^r+(1+\Gamma)(2\alpha^r+\Gamma l(K))\|g\|\ .$$

The first result follows by summing over r and multiplying by $2q$. The second result is even easier to obtain. We have

$$\|P^r\chi_{\tilde{K}}P^{q-r}g\|=\|\chi_{\tilde{K}}P^{q-r}g\|\le l(\tilde{K})\|P^{q-r}g\|_\infty\le l(K)(\alpha^{q-r}\vee g+(1+\Gamma)\|g\|)\ ,$$

and the result follows by summation over r and multiplication by $2q$.

The main point in the above result is that the two estimates are very dissymetric. The only large constant Mq appears so to speak in an off diagonal term. This is reminiscent of the numerical problems associated to ill balanced matrices, and suggests a natural change of norms. We introduce a one parameter family of norms $|||\,|||_\theta$ defined by

$$|||g|||_\theta=\theta\vee g+\|g\|\ ,$$

where θ is a positive number to be chosen later on. We will denote by $\mathcal{B}_\theta$ the associated Banach space. Of course all these norms are equivalent and we have in particular the obvious inequalities

$$||| \; |||_\theta \leq ||| \; ||| \leq \theta^{-1} ||| \; |||_\theta \; .$$

The advantage of the new norm can be seen in the following Lemma which is the technical ingredient needed for the perturbation theory. We first recall the main spectral result on the transfer operator P (see [H.K.] or [C.]). Under the hypothesis of the previous section on the map f, in the Banach space $\mathcal{B}$ (and also in $\mathcal{B}_\theta$), the operator P has a spectral decomposition

$$P = P_0 + R \; ,$$

where P_0 is a rank one operator with eigenvalue 1, and whose eigenvector is the density h of the absolutely continuous invariant measure μ. The associated linear form ϕ is integration with respect to the Lebesgue measure. There is also a positive number C and a positive number $\gamma < 1$ such that for any integer n

$$|||R^n||| \leq C\gamma^n \; .$$

Lemma 6. *Under the above hypothesis we have for any fixed integer q and K small enough the estimates*

$$|||\Delta_q|||_\theta \leq 6q^2\alpha^q + Mq[\theta + (q\theta^{-1} + 1)l(K)] \; ,$$

$$|||R^q|||_\theta \leq C\theta^{-1}\gamma^q \; ,$$

$$|||P_0|||_\theta \leq 1 + \Gamma\theta \; .$$

Proof. The first inequality is an immediate consequence of Lemma 5. The second inequality follows at once from the equivalence of the norms. For the third inequality, we observe that for any integer s, we have

$$P_0 = P^s - R^s \; .$$

Therefore from the definition of the new norm and equation (L.Y.) we get

$$|||P_0|||_\theta \leq |||P^s|||_\theta + C\gamma^s\theta^{-1} \leq \alpha^s + 1 + \theta\Gamma + C\theta^{-1}\gamma^s \; ,$$

and the result follows by letting s diverge.

We can now harvest the advantages of the change of norm. We can take for example $\theta = \gamma^{q/2}$ and then q large enough such that the number $6q^2\alpha^q + q\gamma^{q/2}$ is small enough. Then we take K small enough accordingly and the above Lemma shows that the perturbation is small. We collect the results from perturbation theory in the following proposition.

Proposition 7. *For q large enough and K small enough (with $\mu(K) > 0$) accordingly, the operator $P\chi_{K^c}$ has a spectral decomposition in the space $\mathcal{B}_\theta$ (and also in $\mathcal{B}$)*

$$P\chi_{K^c} = P_K + R_K \ ,$$

where P_K is a rank one operator with a real positive eigenvalue $\lambda(K) < 1$, the corresponding eigenvector h_K is a function with uniformly bounded variation and it can be taken positive of integral 1. The associated linear form ϕ_K is then positive. There is a positive constant C_1 and a number $\eta < 1$ such that uniformly in K small enough, we have

$$|1 - \lambda(K)| \leq C_1 l(K) \ ,$$

and for any integer n

$$|||R_K^n|||_\theta \leq C_1 \eta^n \ .$$

Finally, when $l(K) \to 0$, h_K converges to h in L^1, and ϕ_K converges weakly to ϕ (which is the integration over dl) in $\mathcal{B}'_\theta$.

Proof. The fact that there is a simple largest eigenvalue for small enough K and the uniform estimate on the spectral decomposition is a direct consequence of perturbation theory in the new norm (see [K.]). This is however not enough to control the convergence when $l(K) \to 0$, and we do this directly.

We first observe that if g is a non negative function, we have $0 \leq P\chi_K g \leq Pg$, and therefore the spectrum of $P\chi_K$ must be contained in the unit disk. Therefore, if g is of bounded variation, and u is an integrable positive function, we have

$$\phi_K(g) \int u h_K dl = \lim_{n\to\infty} \lambda(K)^{-n} \int u \left(P\chi_K\right)^n g dl \ ,$$

which implies that $\lambda(K)$ is real and positive, that h_K is a non negative function, and that ϕ_K is a non negative linear form.

¿From perturbation theory we have also

$$h_K = h + r_K$$

with r_K uniformly bounded for K small. The eigenvalue relation $P\chi_K h_K = \lambda(K) h_K$ implies immediately the relation

$$R^q r_K + \Delta_q(h + r_K) = (\lambda(K) - 1)h + \lambda(K)^q r_K \ .$$

If we integrate over dl, and use $\phi \circ R^q = 0$ and $\phi(r_K) = 0$, we obtain

$$1 - \lambda(K)^q = -\int dl \Delta_q(h + r_K) \ .$$

Since Δ_q is a finite sum, we get

$$|1 - \lambda(K)| \leq \mathcal{O}(1) l(K) \ .$$

¿From perturbation theory, it follows that when $l(K) \to 0$, h_K is uniformly bounded in $\mathcal{B}$, and therefore precompact in L^1. Similarly, ϕ_K is uniformly bounded and therefore weakly precompact. Let h_* and ϕ_* be two respective accumulation points. If g is of bounded variation and u is in L^∞, we have for any integer n

$$\int u(P\chi_{K^c})^n g = \phi_K(g)\lambda(K)^n \int uh_K dl + \mathcal{O}(\eta^n \|u\|_\infty |||g|||) .$$

For a fixed n, we let $l(K)$ tend to zero in such a way that h_K converges to h_* in L^1 and ϕ_K converges weakly to ϕ_*. We obtain

$$\int uP^n g = \phi_*(g) \int uh_* dl + \mathcal{O}(\eta^n \|u\|_\infty |||g|||) ,$$

and by letting n diverge we conclude that $h_* = h$ and $\phi_* = \phi$. Moreover, since we have only one accumulation point, we have convergence. This ends the proof of the proposition.

It is easy to verify that the linear functional

$$u \longrightarrow \phi_K(uh_K)$$

initially defined on functions of bounded variation extends to a positive measure which is invariant and ergodic by f. This measure is supported by the invariant set of points whose orbit does not meet K. It is often called (rather unfortunately) a quasi invariant measure. In that direction one may wonder how large can be the set K in order to expect that the same results hold (see [Ke.]). One can however easily construct examples where the situation is completely different. Consider the map of the interval

$$f(x) = \begin{cases} 2x, & \text{if } x < 1/2, \\ 2 - 2x & \text{if } x \geq 1/2. \end{cases}$$

It is easy to verify that if K is the set $[0, 2/3]$, then either $x \in K$, or $f(x) \in K$. In other words,

$$(P\chi_K)^2 = 0 .$$

We now have all the necessary tools for the proof of Theorem I.

Proof (of Theorem I). Let $T_j = -\log \lambda(K_j)$, then if $[a]$ denotes the integer par of a positive real number a, we have for any $t > 0$ as explained in the introduction

$$\mu(\{\tau_{K_j} > tT_j\}) = \int dl\chi_{K_j^c} \left(P\chi_{K_j^c}\right)^{[tT_j]} h .$$

Using proposition 7, we obtain immediately

$$\mu(\{\tau_{K_j} > tT_j\}) = \lambda^{[tT_j]}\phi_{K_j}(h) \int \chi_{K_j^c} h_{K_j} dl + \mathcal{O}(\eta^{tT_j}) .$$

The result follows immediately from Proposition 7 since $\lim_{j\to\infty} T_j = \infty$.

Note that the above proof works also with any sequence of numbers (T_j) such that

$$\lim_{j\to\infty} -T_j / \log \lambda(K_j) = 1 .$$

III. Proof of Theorem II

We will denote by K one of the intervals of the sequence (K_j) with j large enough. Let $\tilde{m}_K = \inf\{m_K, 1/\sqrt{\mu(K)}\}$, where we have denoted by m_K the number m_j if $K = K_j$. Using again the decomposition $h_K = h + r_K$ as in the proof of proposition 7, the eigenvalue relation $P\chi_{K^c}h_K = \lambda(K)h_K$, and the hypothesis $f^n(K) \cap K = \emptyset$ for $0 < n \leq \tilde{m}_k$ imply

$$R^{\tilde{m}_K} r_K + \Delta_{\tilde{m}_K}(h + r_K) = (\lambda(K)\tilde{m}_K - 1)h + \lambda(K)^{\tilde{m}_K} r_K \ .$$

If we integrate this relation over dl and use the relations $\phi \circ R^{\tilde{m}_K} = 0$ and $\phi(r_K) = 0$, we obtain

$$1 - \lambda(K)^{\tilde{m}_K} = -\int dl \Delta_{\tilde{m}_K}(h + r_K) = \sum_{n=0}^{\tilde{m}_K - 1} \int dl \ P^{\tilde{m}_K - n} \chi_K P^n(h + r_K)$$

$$= \sum_{n=0}^{\tilde{m}_K - 1} \int dl \chi_K P^n(h + r_K) = \tilde{m}_K \mu(K) + \sum_{n=0}^{\tilde{m}_K - 1} \int dl \chi_K P^n r_K \ .$$

Since r_K has bounded variation, we have using formula (L.Y.)

$$\|P^n r_K\|_\infty \leq \alpha^n \vee r_K + (1 + \Gamma)\|r_K\|_1 \ .$$

¿From proposition 7, we have for j large enough a uniform bound on the numbers $\vee r_{K_j}$, and also the sequence $(\|r_{K_j}\|_1)$ converges to zero. Therefore, we have

$$1 - \lambda(K_j)^{\tilde{m}_j} = \tilde{m}_j \mu(K_j) + \mu(K_j)\mathcal{O}(1)(1 + \tilde{m}_j \|r_{K_j}\|_1) \ ,$$

and the first result follows by letting j diverge.

We now come to the proof of the second part. From Theorem I.16 in [N.], the result follows from the convergence of the Laplace transform of the measures $(\mathcal{M}_{K_j})$. In other words, it is enough to verify that for any function g in $C_0^+(\mathbb{R}^+)$, we have

$$\lim_{j \to \infty} \int e^{-\sum_{n=0}^\infty \chi_{K_j}(f^n(x)) g(n\mu(K_j))} d\mu(x) = e^{-\int (1 - e^{-g(y)}) dy} \ .$$

We observe that this is of course a statement about the sequence of random variables

$$N_j(g)(x) = \sum_{n=0}^\infty \chi_{K_j}(f^n(x)) g(n\mu(K_j)) \ ,$$

and we will prove that the sequence of characteristic functions of these random variables converges. More precisely, we claim that for any complex number z we have

$$\lim_{j \to \infty} \int e^{-z\sum_{n=0}^\infty \chi_{K_j}(f^n(x)) g(n\mu(K_j))} d\mu(x) = e^{-\int (1 - e^{-zg(y)}) dy} \ .$$

We now observe that on both sides of the above formula we have entire functions. This is obvious for the function in the right hand side, for the function on the left hand side this follows from the fact that the sum over the integer n is finite since g has compact support. This implies that the convergence of the characteristic functions follows from the convergence of the moments (we refer to [B.] for a proof).

We have for the moment of order r of $N_j(g)$

$$\mathbb{E}(N_j(g)^r) = \sum_{0\le n_1,\cdots,0\le n_r} \int h(x)dl(x) \prod_{l=1}^{r} g(\mu(K_j)n_l)\chi_{K_j}(f^{n_l}(x)) \;.$$

We now observe that from our hypothesis,

$$\chi_{K_j}\;\chi_{K_j}\circ f^n = 0 \qquad \text{if} \qquad 1\le n\le m_j \;.$$

This suggests to rearrange the sum in the expression of $\mathbb{E}(N_j(g)^r)$ into a sum over different indices, and by the above remark these different indices must differ by at least m_j. We get

$$\mathbb{E}(N_j(g)^r) =$$
$$\sum_{l=1}^{r} \sum_{\substack{0<t_1\cdots,0<t_l\\ t_1+\cdots+t_l=r}} \frac{r!}{l!\prod_{s=1}^{l} t_s!} \sum_{\substack{0\le n_1,\cdots,0\le n_l\\ |n_p-n_q|>m_j \text{ for } p\neq q}} \int h(x)dl(x) \prod_{s=1}^{l} g^{t_s}(n_s\mu(K^j))\chi_{K_j}(f^{n_s}(x)) \;.$$

At this point we would like to apply recursively the decay of correlations (D.C.). This is where we use our second hypothesis

$$\lim_{j\to\infty}\inf \inf_{K_j} h > 0 \;.$$

Let a be a positive number such that for j large enough $\inf_{K_j} h > a$. If B is a measurable set contained in K_j, the decay of correlation (D.C) implies since $\vee\chi_{K_j} = 2$

$$\left|\int \chi_{K_j}(x)\chi_B(f^n(x))d\mu(x) - \mu(K_j)\int \chi_B(x)d\mu(x)\right| \le 3C\gamma^n a^{-1}\int \chi_B(x)d\mu(x) \;,$$

or in other words

$$(1-3C\gamma^n a^{-1})\int \chi_B(x)d\mu(x) \le \int \chi_{K_j}(x)\chi_B(f^n(x))d\mu(x) \le (1+3C\gamma^n a^{-1})\int \chi_B(x)d\mu(x) \;.$$

Using recursively this estimate and the fact that for any integer p

$$\lim_{\epsilon\to 0^+} \epsilon\sum_{n=0}^{\infty} g^p(n\epsilon) = \int g^p(y)dy \;,$$

we derive that the moment of order r of the random variable $N_j(g)$ converges when j diverges to the number

$$\sum_{l=1}^{r} \sum_{\substack{0<t_1,\cdots,0<t_l,\\ t_1+\cdots+t_l=r}} \frac{r!}{l!\prod_{n=1}^{l} t_n!} \prod_{n=1}^{l} \int g^{t_n}(y)dy \ .$$

It is easy to verify using formal power series that these numbers are also the moments of a random variable X with characteristic function given by

$$\mathbb{E}(e^{zX}) = e^{-\int (1-e^{zg(y)})dy} \ ,$$

and the Theorem is proved.

We finally remark that one can easily adapt the above proof to the case where instead of the invariant measure μ one uses the sequence of measures $(\mu|_{K_j}/\mu(K_j))$ which is the sequence of stationary measures for the processes (τ_{K_j}).

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