HYPERBOLIC DYNAMICAL SYSTEMS AND GENERATION OF MAGNETIC FIELDS BY PERFECTLY CONDUCTING FLUIDS. (1)

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Abstract. We study the spectrum of the operator describing the growth of magnetic fields in an infinitely conducting, incompressible fluid. We show that, in many circumstances, it consists of an anulus and also give characterization of this spectrum by the Lyapunov exponents of periodic orbits of the Lagrangian flow.

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1. Introduction

In this paper we will describe some recent results about the infinitely conducting dynamo problem. That is, we assume that we are given a divergence free velocity field u and want to study the time evolution it induces on magnetic fields. Using Maxwell's equations, Ohm's law and incompressibility of the fluid, we obtain that the magnetic field evolves according to the equation:

$$\frac{\partial B}{\partial t} + u \cdot \nabla B = (B \cdot \nabla)u + \frac{1}{R_m}\nabla^2 B \tag{1.1}$$

In this paper, we will only consider the case when $1/R_m = 0$ which corresponds to the case that the fluid is perfectly conducting.

It is well known that in when $1/R_m = 0$, equation (1.1) has a purely geometric meaning and a purely geometric solution that can be written as

$$B_t = f_{t*}B_0$$

where f_t is the Lagrangian flow generated by u according to $\frac{d}{dt}f_t = u \circ f_t$ and the push-forward f_{t*} is defined by $\left(f_{t*}B\right)(x) \equiv Df_t(f_t^{-1}(x))B(f_t^{-1}(x))$.

We notice that this solution indicates that the magnetic field will grow when the term Df is large, that is, when the trajectories of particles suspended in the fluid exhibit a very sensitive dependence to initial conditions (See [BC] [FO].)

Since a trajectory could expand for a while and then start contracting, the substained rates of growth is a global problem that can be best formulated as studying $\lim_{t\to\infty} \frac{1}{t} \log ||f_{t*}||$ This reduces to studying the spectral properties of f_1.

The spectral properties of such push-forward operators have been studied in [Ma]. We recall some results from this paper:

Theorem 1.1. *Let f be a C^1 diffeomorphism of a manifold*

A) If aperiodic points of f are dense, then the spectrum of f_* acting on the complexification of continuous vector fields with the supremum norm is invariant under rotations in the complex plane around the origin.(That is, if a complex number belongs to the spectrum, all the complex numbers of the same modulus also do.)

B) In that case, the spectrum consists of a finite number of disjoint anuli. The number of such anuli is not larger than the dimension of the manifold.

C) f is an Anosov diffeomorphism if and only if, 1 does not belong to this spectrum.

We recall that we say that f is an Anosov diffeomorphism if we can find an splitting of the tangent bundle $T_xM = E^s_x \bigoplus E^u_x$ and constants $C > 0$, $\lambda < 1$ such that:

$$|Df^n(x)u| \le C\lambda^n |u| n \ge 0 \iff u \in E^s_x$$

$$|Df^n(x)u| \le C\lambda^{-n} |u| n \le 0 \iff u \in E^u_x.$$

That is, a system is Anosov if and only if all the infinitesimal displacements get exponentially stretched in the future or in the past.

We will refer to the spectrum of f_* on the complexification of continuous vector fields equipped with the sup norm as the *Mather spectrum of f* and denote it by $\Sigma_M(f)$. When there is no risk of confusion, we will suppress the f.

Actually, in [Ma], we can find a generalization of C) in Theorem 1.1, namely:

Theorem 1.2. *Assume that $\Sigma_M(f)$ is disjoint from $\{z \in \mathbb{C} \big| |z| = \rho\}$. Then, we can find an splitting of the tangent bundle $T_xM = E^<_x \bigoplus E^>_x$ and $C > 0$, $\epsilon > 0$ such that:*

$$|Df^n(x)u| \le C(\rho+\epsilon)^n |u| n \ge 0 \iff u \in E^<_x$$

$$|Df^n(x)u| \le C(\rho-\epsilon)^{-n} |u| n \le 0 \iff u \in E^>_x$$

Moreover, if we denote by $\Pi^>$, $\Pi^<$ the spectral projections of f_* associated to the two components of the complement of the circle of radius ρ, and by $\Pi^<_x$, $\Pi^>_x$ the projections in the tangent bundle associated to the splitting we have:

$$\left(\Pi^< v\right)(x) = \Pi^<_x v(x)$$
$$\left(\Pi^> v\right)(x) = \Pi^>_x v(x)$$

Notice that Theorem 1.2 implies immediately the consequence that there cannot be more anuli in the spectrum than the dimension of the manifold.

The possible applications of these theorems to the dynamo problem were suggested in [Vi],(in which, moreover, there is a thorough discussion of the problem of the asymptotics of the spectrum as $\frac{1}{R_m}$ approaches zero, which we will not discuss here).1

Unfortunately, there are some reasons why the previous results Theorem 1.1, Theorem 1.2 may not be directly applicable to dynamo theory. Let us just mention:

1)Since the f_t is generated by the vector field u, we always have $f_{t*}u = u$ so that it is not an Anosov map.

2) In realistic situations, one does not expect stretching in the whole manifold but only in a subset of it.

3) The vector field B satisfies the non-trivial constraint of having divergence zero. It is conceivable that considering spaces of vector fields with such constraint changes the spectral properties.

4) For applications in hydrodynamics, one would like to consider the spectrum in other spaces of more differentiable functions.

We just point that 1) and 2) are not very difficult objections. If we consider a time periodic vector field in an appropriate manifold, we could have an Anosov diffeomorphisms as the time-T map. Also, Mather's arguments only needs small adaptations to work for the cases of Anosov flows or for cases in which a closed invariant set takes the place of the manifold. We will not discuss such minor problems here.

Objection 3), however is rather substantial, and, as we will see later, it indeed changes the spectrum. For 4), we just present some bounds on how the spectrum could change when we consider spaces of differentiable vector fields.

Since there is considerable interest in computing the spectrum of the dynamo operator, we also present some characterization of Σ_M and of the spectrum of the dynamo operator in terms of periodic orbits. We just point out that periodic orbits have played an important role in the study of the dynamo problem in [Vi], [VF] [AG]. Of course, in the dynamical systems literature, there are many results relating properties of a dynamical system to properties of its periodic orbits. We also point out that the characterization we propose is quite easy to implement in the computer and we have found it rather reliable.

2. Statement of results

The arguments of [Ma] work in the case that spaces considered are closed under sup norms. So, our first task is to formulate precisely the Banach spaces under sup norm that may have divergence zero.

Here are several reasonable possibilities of spaces. Notice that if f is a volume preserving diffeomorphism, it preserves all of them.

χ^0 : The C^0 closure of all the C^1 vector fields with zero divergence.

χ^1 : The set of continuous vector fields with zero divergence in the sense of distributions.

χ^2 : The set of C^0 vector fields which have zero flux over any C^1 boundaryless surface.

$\chi^{1,b}$: The set of bounded vector fields with zero divergence in the sense of distributions.

$\chi^{2,b}$: The set of bounded vector fields with zero flux over ant C^1 boundaryless surface.

There are some obvious relations between those spaces. For example $\chi^0 \subset \chi^1$, $\chi^0 \subset \chi^2$ $\chi^1 \subset \chi^{1,b}$, $\chi^1 \neq \chi^{1,b}$, $\chi^2 \subset \chi^{2,b}$, $\chi^2 \neq \chi^{2,b}$. In the torus, it is easy to show by using convolutions that $\chi^0 = \chi^1 = \chi^2$ but I do not know how to prove the same on an arbitrary manifold. Even if it would be interesting to clarify their inclusions in general manifolds. for the concerns of these paper, all the results can be proved by the same methods for all the spaces.

We will start by stating some generalizations of characterizations of Mather spectrum that we will find useful later.

Theorem 2.1. *Let f be a C^1 diffeomorphism on M such that the set of aperiodic orbits is dense.*

A) The spectrum of f_* acting on the complexification of continuous vector fields is the same as the spectrum on the complexification of bounded vector fields.

B) If periodic points of f are dense, for any $z \in \partial\Sigma_M(f)$, we can find a sequence $\{x_n\}_{n=0}^{\infty}$ of periodic orbits $f^{N_n}(x_n) = x_n$ such that for some $\lambda_n \in \mathrm{spec}(Df^{N_n}(x_n))$ we have $|\lambda|_n^{1/N_n} \to |z|$.

C) Assume that the splittings in the tangent bundle induced by Theorem 1.2 are one-dimensional and that f is a transitive Anosov diffeomorphism. Then,

$$\Sigma_M(f) = \overline{\left\{\lambda e^{i\theta} \middle| \theta \in \mathbb{R}, \lambda^N \in \mathrm{spec}(Df^N(x)), f^N(x) = x\right\}}$$

Theorem 2.2. *Let f be a volume preserving diffeomorphism with dense aperiodic orbits.*

A) Then, the spectrum of f_* acting on any of the χ defined above is invariant under rotations. (We will denote it by $\Sigma_{ND}(f)$)

B) $\partial\Sigma_{ND} \subset \Sigma_M$, $\partial\Sigma_M \subset \Sigma_{ND}$ The inner and outer boundaries of Σ_{ND} are the same as the inner and outer boundaries of Σ_M.

C) If f is a transitive Anosov diffeomorphism with one-dimensional foliations, then Σ_{ND} consists of an anulus whose inner and outer boundaries are the inner and outer boundaries of $\Sigma_M(f)$.

Notice that a corollary is that the rate of growth of the magnetic field evolving under (1.1) ($1/R_m = 0$) can be computed by taking the supremum of the Lyapunov exponent at periodic orbits. This is a rather simple algorithm to implement numerically and preliminary numerical calculations by the author suggest it should be quite effective. The relation of this exponent of growth of the magnetic field and the behaviour at periodic orbits has been studied numerically by [AG] even for the case when $1/R_m \neq 0$. The work of [Vi], [VF] also provides with some mathematically rigorous results relating the behaviour at periodic orbits and the behaviour of the exponent of growth of (1.1).

Theorem 2.3. *Let f be a C^{r+1} diffeomorphism, $r \in \mathbb{N}$. Assume that $\Sigma_M \subset \{z \in \mathbb{C} | \lambda_- \leq |z| \leq \lambda_+\}$ then the spectrum of f_* acting on the Banach space of C^r vector fields is contained in $\{z \in \mathbb{C} | \lambda_- \lambda_+^r \leq |z| \leq \lambda_+ \lambda_-^{-r}\}$ (If f is volume preserving there are similar results)*

Remark. We suspect that the conditions on the spectrum on C^r are optimal for most diffeomorphisms. For Hölder regularities of sufficiently small exponent, there are more detailed results that we will not discuss here.

3. Sketches of proofs.

In this section, we will just present sketches of proofs. Full details as well as some extensions will appear elsewhere.

As in the arguments in [Ma] we will have to work simultaneously on the level of the functional analysis properties of the operator f_* and on the level of the dynamics of f.

It is an standard result in functional analysis that if A is a bounded operator acting on a Banach space $\mathcal{X}$ and $z \in \mathbb{C}$ is in the boundary of the spectrum of A, we can find a sequence $\{u_n\}_{n=0}^{\infty} \subset \mathcal{X}$ such that $||u_n|| = 1$, $||Au_n - zu_n|| \to 0$. Conversely, if such a sequence can be found, we may conclude that z is in the spectrum of A (it may not be in the boundary). We well refer to the elements of such a sequence as approximate eigenfunctions. We also note that in general it is not true that all the spectrum of a linear operator can be obtained through approximate eigenfunctions, but all the spectrum which is not residual can. We will refer to the part of the spectrum that can be characterized by the existence of approximate eigenfunctions as the *non-residual spectrum*. A large part of the argument will be to show that if we have approximate eigenfunctions in one space, we can use the dynamica properties of the system to construct approximate eigenfunctions in another.

Let us start by discussing the proof of Theorem 2.1.

Clearly, if we can find a sequence of approximate eigenfunctions which are continuous, we can use it as a sequence of bounded approximate eigenfunctions. Hence $\partial\Sigma_{C^0} \subset \Sigma_{\text{bounded}}$. If we have a bounded approximate eigenfunction, it is possible to obtain an approximate eigenfunction of the form $v = \sum_{j=0}^{N} v_j$, where v_j has support in $f^j(x)$. This can be extended in a continuous fashion to a small neighborhood of those points so that it is still an approximate eigenfunction. We conclude that $\partial\Sigma_{\text{bounded}} \subset \Sigma_{C^0}$. We can use the argument in Theorem 1.2 to conclude that the gaps in the spectrum induce splittings of the tangent bundle characterized by the rates of growth of individual vectors. So that the circles have to agree.

If periodic points of f are dense, given an approximate eigenfunction of f_*, we can restrict it to a periodic orbit and hence, find an orbit whose return maps make a vector grow at an exponential rate close to the modulus of the point in the spectrum we are considering. This implies that there is an eigenvalue of, roughly, the same exponent.

For a transitive Anosov diffeomorphism, any pair of periodic orbits has a homoclinic connections. By Smale's analysis of the horseshoe, we can find periodic orbits that spend any proportion (up to an arbitrarily small error) of their time in any neighborhood of the two periodic orbits. By the invariance of the splitting, and the fact that the splittings are one-dimensional, we know that the eigenvectors of the points in the *intermediary* periodic orbit will be quite close to the eigenvectors of the original periodic orbits. The multiplication factors will be as close as we like to those of the original periodic orbits for any proportion of the time that we want (up to arbitrarily small errors). ¿From that, it follows that the closure of the set of Lyapunov exponents at periodic orbits is a convex set in the real line. Since we showed that it can get as close as desired to the edges of Σ_M this finishes the argument for Theorem 2.1

We notice that a consequence of part C is that, under the hypothesis of the theorem, all of Σ_M is not residual, since we can take as approximate eigenfunctions vectors supported on the periodic orbit in the case that we consider bounded vector fields or supported in an small balls near the periodic orbit for the case of continuous vector fields.

We will only prove Theorem 2.2 in the case that we consider the spectrum on χ^0. the other cases are very similar.

First of all, we observe that since approximate eigenfunctions in χ^0 are also approximate eigenfunctions in the space of continuous functions we have $\partial\Sigma_{ND} \subset \Sigma_M$.

We also observe that if we have an approximate eigenfunction in χ^0 or in C^0 we can also produce an approximate eigenfunction for z $v = \sum_{j=0} v_j$ with v_j has support on $f^j(x)$. We observe that $\sum_{j=0}^{N} v_j e^{ij\theta}$ is an approximate eigenfunction for $ze^{i\theta}$. As soon as we show that such finitely supported approximate eigenfunctions can be turned into approximate eigenfunctions in χ^0 we would have shown that $\partial\Sigma_{ND}$ is rotation invariant and also that $\partial\Sigma_M \subset \Sigma_{ND}$. Since the outer edges of the spectrum of f_* are characterized by the rates of growth of $||f_*^n||$ and this quantity is larger is we consider larger spaces, we obtain that the Σ_{ND} is contained in the anulus bounded by the inner and outer edges of Σ_M.

The basic idea of this construction of divergence free approximate eigenfunctions is that it is possible to construct vector fields of zero divergence and arbitrarily small support that, nevertheless, have well defined directions. Of course, it is quite impossible to make divergence zero vector fields that have small support and that their vector values are well defined. Nevertheless, we can observe that for the rates of growth of vectors, the orientation of the vectors is irrelevant. If we pick a direction, we can imagine two very thin pipes running along it joined to form a closed circuit by much thinker pipes. If we make a fluid go around this circuit, it will have appreciable velocity only in the thin segments of pipe, which have the direction we want. In the segments that the velocity has the wrong direction, it has so small modulus that it does not affect the property of being an approximate eigenfunction.

This finishes the proof of part A) and B) of Theorem 2.2.

Actually, we have proved something more. We have proved that all the points in Σ_M that belong to the non-residual spectrum, are in the spectrum of Σ_{ND}. For transitive Anosov diffeomorphisms with one-dimensional splittings, we have shown that all of Σ_M is non-residual, so that, for such systems, $\Sigma_M \subset \Sigma_{ND}$.

Under these circumstances, we see that Σ_{ND} should consist of anuli that cover the anuli of Σ_M. The only possible gaps in Σ_{ND} are the gaps in Σ_M.

We want to derive a contradiction with the fact that there actually some gaps. If some of these gaps existed, we could find spectral projections on χ^0 associated with them. These spectral projections would be characterized by rates of growth, of $f_*^n u$. Hence, they should be the restrictions to the space of vector fields of zero divergence of the spectral projections of continuous vector fields. By Theorem 1.2, we know that such projections can only be projections on invariant subbundles, which is easy to see that never preserve divergence free vector fields. Hence, we conclude that the existence of such projections is impossible. Hence, there are no gaps in Σ_{ND}.

Notice that, part B) alone does not allow us to reach the same conclusion we reached here. If B) was the only restriction, it could happen that Σ_{ND} had gaps but they could be contained in one of anuli of Σ_M. This looks rather implausible, but we do not know at the moment how to exclude it nor how to present examples.

This finishes the arguments for Theorem 2.2.

To prove Theorem 2.3, it suffices to show that if $|z| > \lambda_+ \lambda_-^{-r}$ then $-\sum_{i=0}^{\infty} z^{-i-1} {f_*}^i$ converges uniformly as operators in C^r and that if $|z| < \lambda_- \lambda_+^{-r}$ so does $\sum_{i=0}^{\infty} z^i f_*^{-i-1}$.

Once we prove uniform convergence, one can justify formal rearrangements so that those sums are the resolvent of the operator in C^r.

To prove the desired uniform convergence for the first series we consider $L_1, \cdots, L_k$, $k \le r$, fist order differential operators with C^∞ coefficients and show that

$$||L_1 \cdots L_k \left(f_*^i v \right) ||_{C^0} \le C(\lambda_+ \lambda_-^{-k})^i i^k ||v||_{C^k}$$

Such estimates can be obtained by applying the chain rule and estimating the number of terms. Similar calculations are done in [LMM].

The reason to conjecture that these estimates are optimal is that it is easy to check that they are optimal for linear maps since then, a simple calculation can get the rate of groth of C^r norm in terms of the eigenvalues. One could imagine localizing examples in a neighborhood of a periodic points that has eigenvalues whose moduli are close to the edges of the spectrum simultaneously. It is not clear that all diffeomorphisms have periodic orbits whose Lyapunov exponents are simultaneously close to both edges – we have proved that there are orbits whose Lyapunov exponents are close to each of the edges –, but using the homoclinic connections, it seems plausible that a generic one does.

For α small enough, it is possible to find regions arbitrarily close to Σ_M that enclose the spectrum on C^α vector fields.

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5. References

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