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Finitely Correlated Pure States †

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Abstract
We study a w*-dense subset of the translation invariant states on an infinite tensor product algebra $\bigotimes_{\mathbb{Z}} \mathcal{A}$, where $\mathcal{A}$ is a matrix algebra. These "finitely correlated states" are explicitly constructed in terms of a finite dimensional auxiliary algebra $\mathcal{B}$ and a completely positive map $\mathbb{E} : \mathcal{A} \otimes \mathcal{B} \to \mathcal{B}$. We show that such a state ω is pure if and only if it is extremal periodic and its entropy density vanishes. In this case the auxiliary objects $\mathcal{B}$ and $\mathbb{E}$ are uniquely determined by ω, and can be expressed in terms of an isometry between suitable tensor product Hilbert spaces.

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† Dedicated to Professor F. Cerulus on the occasion of his 65th birthday

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0. Introduction

In recent years impressive progress has been made in the statistical mechanics of classical systems. In comparison, the statistical mechanics of quantum systems has turned out to be surprisingly difficult. Even for one-dimensional lattice systems ("spin chains") important basic questions remain unanswered, whereas all classical models in one dimension are essentially trivial. These difficulties are intimately connected to the fundamental differences between quantum theory and classical probability in the description of composite systems. For example, it is a standard procedure in classical probability to condition a state of a composite system with respect to the configuration in a subsystem, i.e. to represent it as an integral over states whose restrictions to one subsystem are pure states. Many states in quantum theory, such as non-product pure states of a composite system, simply do not allow such representations. An example in the context of spin chains, first constructed by [**1**], is a state on a spin-1 chain which is translation invariant and pure, but has only impure local restrictions. This state occurs as the ground state of an antiferromagnetic interaction.

Due to the higher complexity of correlations in quantum systems it is often impossible to obtain an explicit representation of quantum states of large systems. Exceptions are convex combinations of product states, which play a crucial role in the mean-field approximation, and quasi-free states. In this paper we continue the study of a new class of states on quantum spin chains, the so-called C*-finitely correlated states, for which an explicit formula for all finite volume expectations can be given (see Section 1).

A special subclass of the C*-finitely correlated states was first introduced as a possible quantum generalization of Markov chains [**2**]. On the other hand, examples of such states appeared in the physics literature [**1**] in the study of the ground states of certain one-dimensional antiferromagnets. They were obtained in that context by a different construction, as so-called valence bond solid (VBS) states. In [**3**] and papers cited therein, we showed that suitable generalizations of these two constructions are, in fact, equivalent, and we began a systematic study of this class of states. We isolated a special subclass, the "purely generated" states such that each C*-finitely correlated state can be written as a local function of a purely generated one. Each such state has a unique convex decomposition into extremal periodic purely generated states. No further decomposition is possible: we showed that every extremal periodic purely generated state is the unique ground state of a suitable translation invariant finite range interaction, and, as a corollary, that these basic states are pure states of the infinite tensor product algebra. Thus in contrast to the classical case, where translation invariant pure states are necessarily homogeneous product states, our construction yields a large set (in fact, a w*-dense set [**4**]) of translation invariant pure states in the quantum case.

In this paper we show that the purity criterion used in [**3**] is also necessary, and establish several other necessary and sufficient conditions for the purity of a C*-finitely correlated state. At the same time we address the important question to what extent the objects from which we construct a C*-finitely correlated state are uniquely determined by the state.

The paper is organized as follows. In Section 1 we introduce the basic construction of C*-finitely correlated states in terms of a finite dimensional auxiliary algebra $\mathcal{B}$ and a completely positive map $\mathbb{E}$ and we formulate our main results. Section 2 is devoted to an analysis of the algebraic structure of the auxiliary algebra $\mathcal{B}$ and the map $\mathbb{E}$. In Section 3 we investigate the implications of vanishing mean entropy for a C*-finitely correlated state, and prove the results stated in Section 1.

1. Main Results

We now introduce some basic notation and terminology. For each $i \in \mathbb{Z}$ we consider a copy $\mathcal{A}_i$ of the algebra $\mathcal{A}$ of $d \times d$-matrices with complex entries. For a finite subset $\Lambda \subset \mathbb{Z}$, we put $\mathcal{A}_\Lambda = \bigotimes_{i\in\Lambda} \mathcal{A}_i$. The chain algebra $\mathcal{A}_{\mathbb{Z}}$ is then the C*-inductive limit of the local algebras $\mathcal{A}_\Lambda$, i.e. $\mathcal{A}_{\mathbb{Z}}$ is the C*-norm closure of $\bigcup_\Lambda \mathcal{A}_\Lambda$. A state ω on $\mathcal{A}_{\mathbb{Z}}$ is defined by its collection of correlation functions, i.e. its restrictions to the local subalgebras $\mathcal{A}_\Lambda$. It suffices to specify $\omega(A_m \otimes \cdots A_n)$ for any choice of $m \le n \in \mathbb{Z}$, and $A_i \in \mathcal{A}_i$, $i = m, \ldots n$. There is an evident action of the group $(\mathbb{Z}, +)$ as translation automorphisms on $\mathcal{A}_{\mathbb{Z}}$. We will denote the elementary translation automorphism by γ. A state ω is called translation invariant if it is invariant under this action. Obviously, a translation invariant state is completely determined by specifying the $\omega(A_1 \otimes \cdots A_n)$ for all $n = 1, 2, \ldots$, and $A_i \in \mathcal{A}$.

C*-finitely correlated states are now defined by giving an explicit formula for the correlation functions in terms of certain auxiliary objects. These are a finite dimensional C*-algebra $\mathcal{B}$ (i.e. a direct sum of matrix algebras), a positive element $e \in \mathcal{B}$, a positive functional ρ on $\mathcal{B}$, and a completely positive map $\mathbb{E} : \mathcal{A} \otimes \mathcal{B} \to \mathcal{B}$. We always assume that $\mathbb{E}$, e and ρ satisfy the *compatibility conditions* $\mathbb{E}(\mathbb{1} \otimes e) = e$ and $\rho\big(\mathbb{E}(\mathbb{1} \otimes B)\big) = \rho(B)$ for all $B \in \mathcal{B}$. Furthermore we assume that e and ρ satisfy the *normalization condition* $\rho(e) = 1$. We use the abbreviation $\mathbb{E}_A(B) = \mathbb{E}(A \otimes B)$ for $A \in \mathcal{A}$, $B \in \mathcal{B}$, i.e. for fixed $A \in \mathcal{A}$ we consider $\mathbb{E}_A$ as an operator on $\mathcal{B}$. The formula defining the "**C*-finitely correlated state generated by** $(\mathcal{B}, \mathbb{E}, e, \rho)$" is

$$\omega(A_1 \otimes \cdots A_n) = \rho\left(\mathbb{E}_{A_1} \circ \cdots \mathbb{E}_{A_n}(e)\right) \quad . \tag{1.1}$$

The compatibility and normalization conditions ensure that this state is well-defined and translation invariant. The positivity of ω is ensured by the positivity of ρ and e and by the complete positivity of $\mathbb{E}$.

The term *finitely correlated* refers to the following property: $\mathcal{A}_{\mathbb{Z}}$ can be written as $\mathcal{A}^L \otimes \mathcal{A}^R$, where $\mathcal{A}^L$ is generated by the $\mathcal{A}_i, i \leq 0$ and $\mathcal{A}^R$ by the $\mathcal{A}_i, i > 0$. For any $X \in \mathcal{A}^L$ we define a linear functional $\omega_X : \mathcal{A}^R \to \mathbb{C} : \omega_X(Y) = \omega(X \otimes Y)$. The state ω is called finitely correlated if the functionals $\{\omega_X \,|\, X \in \mathcal{A}^L\}$ form a finite dimensional linear space. In [**3**] it was shown that any finitely correlated state has a representation similar to that of formula (1.1). In general, however, $\mathcal{B}$ need not be a C*-algebra, but has to be taken a finite dimensional matrix ordered space [**5**]. The term "C*-finitely correlated" emphasizes that $\mathcal{B}$ is assumed to be a finite dimensional C*-algebra equipped with its natural matrix order.

In [**3**] it was shown that the objects $(\mathcal{B}, \mathbb{E}, e, \rho)$ generating a C*-finitely correlated state ω can always be chosen such that $e = \mathbb{I}_{\mathcal{B}}$ and ρ is faithful. In this situation we will say that $(\mathcal{B}, \mathbb{E}, \rho)$ is a **standard triple** for ω. In the sequel we are always considering standard triples.

The simplest kind of completely positive maps are those implemented by operators on the underlying Hilbert spaces. Specifically, we consider the case where $\mathcal{B} = \mathcal{M}_k$ is the algebra of $k \times k$-matrices, and $\mathbb{E}$ is of the form

$$\mathbb{E}(A \otimes B) = V^*(A \otimes B)V \quad , \tag{1.2}$$

where $V : \mathbb{C}^k \to \mathbb{C}^d \otimes \mathbb{C}^k$ is an isometry. (Recall that $\mathcal{A}$ has been chosen as the algebra $\mathcal{M}_d$ of $d \times d$ matrices.) This special form of $\mathbb{E}$ is equivalent to $\mathbb{E}$ being a "pure" map, which means that it has no non-trivial decompositions $\mathbb{E} = \mathbb{E}^{(1)} + \mathbb{E}^{(2)}$ into other completely positive maps $\mathbb{E}^{(i)}$ (see section 4 of [**3**]).

Definition 1.1. *A C*-finitely correlated state ω is called* **purely generated** *if there exists a standard triple $(\mathcal{B}, \mathbb{E}, \rho)$ generating ω such that $\mathbb{E}$ is of the form $X \in \mathcal{A} \otimes \mathcal{B} \mapsto \mathbb{E}(X) = V^*XV \in \mathcal{B}$, where $\mathcal{B}$ is a *-subalgebra of $\mathcal{M}_k$ containing the identity $\mathbb{I}_k$ of $\mathcal{M}_k$, and $V : \mathbb{C}^k \to \mathbb{C}^d \otimes \mathbb{C}^k$ is an isometry.*

We emphasize that a state is called purely generated if it is generated by *some* triple of the required form; there will always be many other triples generating the same state which are not of that form. The notion of purely generated states plays a central rôle in this paper, in which we address the following main questions:
Question 1:
Is there a canonical choice for the triple $(\mathcal{B}, \mathbb{E}, \rho)$ generating a given C*-finitely correlated state ω?

Question 2:
Which triples $(\mathcal{B}, \mathbb{E}, \rho)$ generate pure states ω?

About Question 1 (uniqueness):

As it is clear that one can always enlarge the algebra $\mathcal{B}$ by adding direct summands which are irrelevant for generating ω, one should avoid this trivial sort of non-uniqueness by imposing a minimality condition on $(\mathcal{B}, \mathbb{E}, \rho)$. In the following definition we describe the minimality condition we will use, and also the precise meaning of "isomorphism" up to which we will understand "uniqueness".

Definition 1.2. *A standard triple $(\mathcal{B}, \mathbb{E}, \rho)$ generating a C*-finitely correlated state ω is called* **minimal** *if there is no proper subalgebra of $\mathcal{B}$ containing $\mathbb{1}$ which is invariant under all operators $\mathbb{E}_A$, $A \in \mathcal{A}$.*
Two standard triples $(\mathcal{B}^{(1)}, \mathbb{E}^{(1)}, \rho^{(1)})$ and $(\mathcal{B}^{(2)}, \mathbb{E}^{(2)}, \rho^{(2)})$ are called **isomorphic** *if there is a C*-isomorphism $\tau : \mathcal{B}^{(1)} \to \mathcal{B}^{(2)}$ such that*

$$\mathbb{E}^{(2)} \circ \left(\mathrm{id}_{\mathcal{A}} \otimes \tau\right) = \tau \circ \mathbb{E}^{(1)} \quad ,$$

and $\rho^{(2)} \circ \tau = \rho^{(1)}$.

It is easy to check that every C*-finitely correlated state is generated by a minimal triple (see Lemma 2.5 of [**3**]), and also that isomorphic triples generate the same state. Unfortunately, minimal triples are not unique in general. The following example shows that even the stronger condition of "minimal dimension of $\mathcal{B}$" is not enough to force a unique generating triple. Consider any minimal triple $(\mathcal{B}, \mathbb{E}, \rho)$, and let $\mathbb{F} : \mathcal{B} \to \mathcal{B}$ denote the operator $\mathbb{F}(B) = \lambda B + (1-\lambda)\rho(B)\mathbb{1}$ for some fixed $\lambda \in (0,1)$. Since $\mathbb{F}(\mathbb{1}) = \mathbb{1}$ and $\rho \circ \mathbb{F} = \rho$ the inverse of this operator is of the same form with λ replaced by $1/\lambda$. Of course, $\mathbb{F}^{-1}$ is not positive. We define $\mathbb{E}^{(1)}(A \otimes B) = \mathbb{E}(A \otimes \mathbb{F}(B))$ and $\mathbb{E}^{(2)}(A \otimes B) = \mathbb{F}(\mathbb{E}(A \otimes B))$. Then both triples $(\mathcal{B}, \mathbb{E}^{(i)}, \rho)$ generate the same state. However, in spite of the fact that $\mathbb{E}^{(1)}_A$ and $\mathbb{E}^{(2)}_A$ are similar as linear operators via the invertible transformation $\mathbb{F}$ there is in general no transformation with a positive inverse (i.e. no C*-isomorphism) with this property. Hence the two triples generate the same state, but are not isomorphic in the sense of Definition 1.2. This lack of uniqueness is due to the constraints on the positivity structure of $\mathcal{B}$ imposed by taking this space as a C*-algebra. In particular, a properly ordered subspace of a C*-algebra cannot in general be given an algebraic structure, which induces the positive cone inherited from the algebra. Thus we cannot pass from an algebra to the minimal $\mathbb{E}_{\mathcal{A}}$-invariant linear subspace without leaving the category of C*-algebras. In the category of general matrix ordered spaces and their natural isomorphisms, we can obtain uniqueness of minimal triples, precisely by demanding the minimality of $\mathcal{B}$ as a (matrix-) ordered linear space. The problem with this approach, however, is that far less is known about the structure of matrix ordered spaces than is known about C*-algebras.

On the other hand, in the case of purely generated states we do have uniqueness in the sense of Definition 1.2:

Theorem 1.3. *Let ω be a purely generated C*-finitely correlated state. Then up to isomorphism there is only one minimal triple $(\mathcal{B}, \mathbb{E}, \rho)$ which generates ω.*

The proof of this result is given at the end of Section 3. It plays an important rôle in the analysis of symmetries **[6]** of C*-finitely correlated states: by uniqueness every automorphism of $\mathcal{A}$ under which a purely generated state ω is invariant can be lifted to a symmetry of the generating triple.

About Question 2 (purity):

We will establish several different ways of characterizing the C*-finitely correlated pure states of the chain algebra $\mathcal{A}_{\mathbb{Z}}$. An essential ingredient in the proof of the conditions we will obtain, is the notion of entropy density of a translation invariant state ω on $\mathcal{A}_{\mathbb{Z}}$, which we now review briefly. For each finite segment $\Lambda = [m, n] \subset \mathbb{Z}$ the restriction $\omega \restriction \mathcal{A}_\Lambda$ is a state on a finite matrix algebra, which is of the form $\omega(A) = \operatorname{tr}(D_\omega^\Lambda A)$ for a unique density matrix $D_\omega^\Lambda \in \mathcal{A}_\Lambda$. We can thus define the usual von Neumann entropy $S(\omega \restriction \mathcal{A}_\Lambda) = -\operatorname{tr}(D_\omega^\Lambda \ln D_\omega^\Lambda)$. Being a concave function, S is a measure of the mixedness of a state. S is always non-negative and it vanishes precisely for pure states. The entropy density, or "mean entropy" of ω is then defined as

$$S_M(\omega) = \lim_{\Lambda \nearrow \mathbb{Z}} \frac{1}{|\Lambda|} S(\omega \restriction \mathcal{A}_\Lambda) \quad , \tag{1.4}$$

where the limit is over all segments $\Lambda = [m, n]$ such that $m \to -\infty$ and $n \to +\infty$. Here the division by the number of sites in Λ makes S_M a measure of the "mixedness per site". It turns out **[7]** that S_M is an affine function of ω. Thus the set of states ω with $S_M(\omega) = 0$ is a convex set, and even a face of the simplex of translation invariant states of $\mathcal{A}_{\mathbb{Z}}$, i.e. all translation invariant convex components of a state with vanishing entropy density have again this property.

If $\mathbb{E}$ has a decomposition into two completely positive summands, we may substitute this decomposition into formula (1.1) at any given site, thus obtaining a convex decomposition of ω. We therefore expect ω to have positive "mixedness per site", i.e. positive entropy density. This intuition is borne out by the following Theorem, which is one of our main technical tools in this paper.

Theorem 1.4. *The entropy density of a C^*-finitely correlated state vanishes if and only if it is purely generated.*

This Theorem is also proven at the end of section 3, together with the following one, which summarizes our results regarding the purity of C*-finitely correlated states.

Theorem 1.5. *Let ω be a C^*-finitely correlated state. Then the following are equivalent:*

(i) ω *is pure*

(ii) $S_M(\omega) = 0$ *and* ω *is spatially clustering, i.e. for* $A, A' \in \mathcal{A}_\Lambda$*:*
$$\lim_{n\to\infty} \omega\big(A \otimes \underbrace{\mathbb{1} \otimes \cdots \mathbb{1}}_{n \text{ times}} \otimes A'\big) = \omega(A)\omega(A') \quad .$$

(iii) ω *is purely generated and the operator* $\mathbb{E}_{\mathbb{1}}$ *has trivial peripheral spectrum, i.e. the only eigenvector of* $\mathbb{E}_{\mathbb{1}}$ *with eigenvalue of modulus 1, is* $\mathbb{1}$

(iv) ω *is generated by a minimal triple* $(\mathcal{B}, \mathbb{E}, \rho)$*, with* $\mathcal{B} = \mathcal{M}_k$*, and* $\mathbb{E}(X) = V^*XV$ *with an isometry* $V : \mathbb{C}^k \to \mathbb{C}^d \otimes \mathbb{C}^k$.

Using this Theorem it is straightforward to construct an abundance of translation invariant pure states on $\mathcal{A}_\mathbb{Z}$. We briefly address the question how the set of states constructed in this way is positioned inside the space of translation invariant $\mathcal{A}_\mathbb{Z}$. We showed in [**3**] that the C*-finitely correlated states form a dense face in that space. Here we show (in Proposition 2.7) that the purely generated states, too, form a dense face. In a forthcoming paper [**4**], we show that the minimality assumption in Theorem 1.5.(4) is satisfied for generic V. Therefore the pure translations invariant states are w*-dense in the translation invariant states of the chain.

2. The central flow of a C*-finitely correlated state

In this section we study the basic decompositions of C*-finitely correlated states, and their generating triples. Every unity preserving completely positive map satisfies the "2-positivity"-inequality $\mathbb{E}(X^*X) \geq \mathbb{E}(X)^*\mathbb{E}(X)$. The following Lemma [**8**] gives a very useful characterization of the case of equality in this operator inequality. We will usually apply this Lemma to the map $\mathbb{E} : \mathcal{A} \otimes \mathcal{B} \to \mathcal{B}$.

Lemma 2.1. *Let $\mathbb{E} : \mathcal{A} \to \mathcal{B}$ be a unity preserving completely positive map between unital C*-algebras, and let $X \in \mathcal{A}$ be such that $\mathbb{E}(X^*X) = \mathbb{E}(X)^*\mathbb{E}(X)$. Then $\mathbb{E}(YX) = \mathbb{E}(Y)\mathbb{E}(X)$ for all $Y \in \mathcal{A}$.*

Proposition 2.2. *Let $(\mathcal{B}, \mathbb{E}, \rho)$ be a minimal triple in the sense of Definition 1.2 and let $\mathcal{Z}_1(\mathbb{E}) = \{B \in \mathcal{B} \mid \mathbb{E}_{\mathbb{1}}(B) = B\}$ denote the eigenspace of $\mathbb{E}_{\mathbb{1}}$ for eigenvalue 1. Then $\mathbb{E}\big(Y(\mathbb{1} \otimes B)\big) = \mathbb{E}(Y)B$ and $\mathbb{E}\big((\mathbb{1} \otimes B)Y\big) = B\mathbb{E}(Y)$ for all $B \in \mathcal{Z}_1(\mathbb{E})$ and $Y \in \mathcal{A} \otimes \mathcal{B}$. Moreover, $\mathcal{Z}_1(\mathbb{E})$ is a subalgebra contained in the center of $\mathcal{B}$.*

Proof : Let $B \in \mathcal{Z}_1(\mathbb{E})$, i.e. $\mathbb{E}_{\mathbb{1}}(B) = B$. We then find by the complete positivity of $\mathbb{E}$:

$$X = \mathbb{E}\big((\mathbb{1} \otimes B^*)(\mathbb{1} \otimes B)\big) - \mathbb{E}(\mathbb{1} \otimes B^*)\mathbb{E}(\mathbb{1} \otimes B) = \mathbb{E}(\mathbb{1} \otimes B^*B) - B^*B \geq 0 \quad .$$

Since ρ is invariant under $\mathbb{E}_{\mathbb{1}}$, $\rho(X) = \rho(B^*B - B^*B) = 0$, and since ρ is faithful this implies $X = 0$. Hence by Lemma 1 we have $\mathbb{E}\big(Y(\mathbb{1} \otimes B)\big) = \mathbb{E}(Y)\mathbb{E}(\mathbb{1} \otimes B) = \mathbb{E}(Y)B$. This relation implies immediately that $\mathcal{Z}_1(\mathbb{E})$ is a subalgebra. Because $\mathbb{E}$ preserves adjoints $\mathcal{Z}_1(\mathbb{E})$ is a *-subalgebra, and the second equation follows from the first by taking adjoints.

In order to prove the centrality of $\mathcal{Z}_1(\mathbb{E})$, define $\mathcal{D}_{k+1}$ as the C*-algebra generated by $\mathcal{D}_k \cup \mathbb{E}(\mathcal{A} \otimes \mathcal{D}_k)$ with $\mathcal{D}_0 = \mathbb{C}\mathbb{1}$. Clearly, every $B \in \mathcal{Z}_1(\mathbb{E})$ commutes with $\mathcal{D}_0$, and for $D \in \mathcal{D}_k,\ A \in \mathcal{A}$ we have

$$[B, \mathbb{E}(A \otimes D)] = \mathbb{E}([\mathbb{1} \otimes B,\ A \otimes D]) = \mathbb{E}(A \otimes [B, D]) = 0$$

by induction on k. By minimality of $\mathcal{B}$, the sets $\mathcal{D}_k$ generate $\mathcal{B}$, so $\mathcal{Z}_1(\mathbb{E})$ is in the center.

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As $\mathcal{Z}_1(\mathbb{E})$ is a finite dimensional abelian algebra all the information about its structure is contained in the set of its minimal projections. For any minimal projection P we consider the algebra $\mathcal{B}_P = P\mathcal{B}P$. Clearly $\mathcal{B} = \bigoplus \mathcal{B}_P$, where the direct sum runs over all minimal projections $P \in \mathcal{Z}_1(\mathbb{E})$. Then by Proposition 2.2 $\mathbb{E}(\mathcal{A} \otimes \mathcal{B}_P) \subset \mathcal{B}_P$, and we can define the restriction $\mathbb{E}_P = \mathbb{E}\restriction\mathcal{A} \otimes \mathcal{B}_P$ and $\mathbb{E} = \bigoplus \mathbb{E}_P$.

With $\rho_P = \rho(P)^{-1}\rho\restriction\mathcal{B}_P$ we thus have a triple $(\mathcal{B}_P, \mathbb{E}_P, \rho_P)$ generating a C*-finitely correlated state ω_P. A direct expression for ω_P is

$$\omega_P(A_1 \otimes \cdots A_n) = \rho(P)^{-1}\rho\left(\mathbb{E}_{A_1} \circ \cdots \mathbb{E}_{A_n}(P)\right) \quad . \tag{2.1}$$

All states ω_P are translation invariant and it is obvious that $\omega = \sum \rho(P)\omega_P$. Since P is minimal in $\mathcal{Z}_1(\mathbb{E})$ it is the unique eigenvector of $(\mathbb{E}_P)_{\mathbb{1}}$, i.e. $\mathcal{Z}_1(\mathbb{E}_P) = \mathbb{C}P = \mathbb{C}\mathbb{1}_{\mathcal{B}_P}$. From this one readily concludes that ω_P is clustering in the mean, and hence **[7, 4.3.10, 4.3.11]** *ergodic*, i.e. extremal in the convex set of translation invariant states. The decomposition of a translation invariant state into ergodic components is unique, hence the states ω_P are uniquely determined by ω. However, we are not assured in general that the ω_P are all different for different P. This has to do with the lack of uniqueness in the representation of a state in terms of $(\mathcal{B}, \mathbb{E}, \rho)$.

Lemma 2.1 can be applied in yet another situation. Rather than looking at fixed points of $\mathbb{E}_{\mathbb{1}}$, we can study fixed points of $\mathbb{E}_U$ for other elements $U \in \mathcal{A}$. Since each $\mathbb{E}_U$ is the direct sum of the corresponding operators for the ergodic components of ω, and since there is no intrinsic connection between the ergodic components, it is best to study this problem for one ergodic component at a time, i.e. for ergodic C*-finitely correlated states.

Proposition 2.3. *Let $(\mathcal{B}, \mathbb{E}, \rho)$ be minimal. Assume that $\mathcal{Z}_1(\mathbb{E}) = \mathbb{C}\mathbb{1}$, and that the one-site restriction of the C*-finitely correlated state generated by $(\mathcal{B}, \mathbb{E}, \rho)$ is faithful. Let $U \in \mathcal{A}$ with $\|U\| \leq 1$, and $\widetilde{U} \in \mathcal{B}$ with $\widetilde{U} \neq 0$, and*

$$\mathbb{E}(U \otimes \widetilde{U}) = \widetilde{U} \quad . \tag{2.2}$$

Then U is unitary, and $\widetilde{U}$ is uniquely determined by U and (2.2) up to a scalar factor, which may be chosen so that $\widetilde{U}$ becomes unitary as well. Moreover,

$$\mathbb{E}\big(Y(U \otimes \widetilde{U})\big) = \mathbb{E}(Y)\widetilde{U} \qquad \text{for all } Y \in \mathcal{A} \otimes \mathcal{B}. \tag{2.3}$$

Let $G(\mathbb{E})$ denote the set of unitary $U \in \mathcal{A}$ for which a $\widetilde{U}$ satisfying (2.2) exists. Then $G(\mathbb{E})$ is a subgroup of the unitary group of $\mathcal{A}$. For every $U \in G(\mathbb{E})$, $\widetilde{U}$ commutes with ρ, i.e. for all $B \in \mathcal{B}$, $\rho(\widetilde{U}^* B \widetilde{U}) = \rho(B)$.

Proof : Let $X = U \otimes \widetilde{U}$. Then

$$\begin{aligned} 0 &\leq \rho\big(\mathbb{E}(X^*X) - \mathbb{E}(X)^*\mathbb{E}(X)\big) = \rho\big(\mathbb{E}(U^*U \otimes \widetilde{U}^*\widetilde{U}) - \widetilde{U}^*\widetilde{U}\big) \\ &\leq \rho\big(\mathbb{E}(\mathbb{1} \otimes \widetilde{U}^*\widetilde{U}) - \widetilde{U}^*\widetilde{U}\big) = \rho\big(\widetilde{U}^*\widetilde{U} - \widetilde{U}^*\widetilde{U}\big) = 0 \quad . \end{aligned}$$

Hence (2.3) follows from the faithfulness of ρ and Lemma 1. The above string of inequalities also implies that $\mathbb{E}(\mathbb{1} \otimes \widetilde{U}^*\widetilde{U}) = \widetilde{U}^*\widetilde{U}$ so that $\widetilde{U}^*\widetilde{U} \in \mathcal{Z}_1(\mathbb{E})$ is a non-zero multiple of the identity. We choose $\widetilde{U}$ unitary from now on. Since $\omega(U^*U) = \rho(\mathbb{E}(U^*U \otimes \mathbb{1})) = \rho(\mathbb{E}(U^*U \otimes \widetilde{U}^*\widetilde{U})) = \rho(\widetilde{U}^*\widetilde{U}) = 1$ the faithfulness of the one-site restriction of ω implies $U^*U = \mathbb{1}$. By taking the adjoint of (2.2) we find that $G(\mathbb{E})$ is closed under taking inverses, and equation (2.3) implies that it is closed under products. Finally we check that $\rho(\widetilde{U}^* B\widetilde{U}) = \rho(B)$, for all $B \in \mathcal{B}$. Using (2.3) and the invariance of ρ under $\mathbb{E}_{\mathbb{1}}$ we have: $\rho(\widetilde{U}^* B\widetilde{U}) = \rho(\mathbb{E}_{\mathbb{1}}(\widetilde{U}^* B\widetilde{U})) = \rho(\widetilde{U}^*\mathbb{E}_{\mathbb{1}}(B)\widetilde{U})$ and hence $\rho(\widetilde{U}^* \cdot \widetilde{U})$ is an invariant state for $\mathbb{E}_{\mathbb{1}}$ and as $\mathcal{Z}_1(\mathbb{E}) = \mathbb{C}\mathbb{1}$ this implies $\rho(\widetilde{U}^* \cdot \widetilde{U}) = \rho$.

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For the decomposition theory we are especially interested in the subgroup of elements with $U = \zeta^{-1}\mathbb{1}$, i.e. in the solutions of the eigenvalue equation $\mathbb{E}_{\mathbb{1}}(\widetilde{U}) = \zeta\widetilde{U}$ with $|\zeta| = 1$. Since $\|\mathbb{E}_{\mathbb{1}}\| = 1$ these eigenvalues are on the rim of the spectral circle, and will be called *peripheral eigenvalues* of $\mathbb{E}_{\mathbb{1}}$. In the Perron-Frobenius theory of matrices of transition probabilities, such eigenvalues determine the periodic decomposition of a stationary Markov chain. Similar results in the non-commutative case have been obtained in [**9**]. However, these results only cover the case $\mathcal{A} = \mathbb{C}$ (in our notation). In the following Proposition we apply Proposition 2.3 to obtain the necessary refinements. It also yields the periodic decompositions of ω, and the fine structure of the minimal projections of $\mathcal{Z}_1(\mathbb{E})$.

Proposition 2.4. *Let* $(\mathcal{B}, \mathbb{E}, \rho)$ *be minimal, and*

$$\mathcal{Z}(\mathbb{E}) = \{B \in \mathcal{B} |\, \exists n \in \mathbb{N}\ \mathbb{E}_{\mathbb{1}}^n(B) = B\} \quad .$$

Then $\mathcal{Z}(\mathbb{E})$ *is spanned by the eigenvectors of* $\mathbb{E}_{\mathbb{1}}$ *of modulus one, and is an algebra contained in the center of* $\mathcal{B}$. *Let* Π *denote the set of minimal projections in* $\mathcal{Z}(\mathbb{E})$. *Then* $\eta(P) = \mathbb{E}_{\mathbb{1}}(P)$ *defines a bijective map* $\eta : \Pi \to \Pi$, *and, more generally,*

$$\mathbb{E}\big(Y(\mathbb{1}\otimes P)\big) = \mathbb{E}(Y)\eta(P) \qquad \text{for all } Y \in \mathcal{A}\otimes\mathcal{B} \text{ and } P \in \Pi.$$

Moreover, if $\eta^n(P) = P$ *the operator* $\mathbb{E}_{\mathbb{1}}^n\restriction P\mathcal{B} \to P\mathcal{B}$ *has no eigenvalues of modulus one except the simple eigenvalue* 1.

Proof : By the observations following the proof of Proposition 2.2, $\mathbb{E}$ is the direct sum of maps $\mathbb{E}_P : \mathcal{A}\otimes\mathcal{B}_P \to \mathcal{B}_P$ with P a minimal projection in $\mathcal{Z}_1(\mathbb{E})$. It suffices to consider one of these summands, i.e. we may assume that $\mathcal{Z}_1(\mathbb{E}) = \mathbb{C}\mathbb{1}$. Let $\zeta\mathbb{1} \in G(\mathbb{E})$ for some ζ with modulus one and let $\widetilde{U} \in \mathcal{B}$ be such that $\mathbb{E}(\zeta\mathbb{1}\otimes\widetilde{U}) = \widetilde{U}$. Then since $G(\mathbb{E})$ is a group we have $\mathbb{E}_{\mathbb{1}}(\widetilde{U}^n) = \zeta^{-n}\widetilde{U}^n$. Since $\mathcal{B}$ is finite dimensional, the spectrum of $\mathbb{E}_{\mathbb{1}}$ is a finite set, so that we must have $\zeta^p = 1$ for some $p \in \mathbb{N}$. By adjusting the phase of $\widetilde{U}$ we can also make $\widetilde{U}^p = \mathbb{1}$. Now let $\widetilde{U} = \sum_{m=0}^{p-1}\zeta^m P_m$ be the spectral resolution of $\widetilde{U}$, where $\zeta = \exp(2\pi i/p)$. Then for $r = 0, \ldots, p-1$:

$$\sum \zeta^{mr}\mathbb{E}_{\mathbb{1}}(P_m) = \mathbb{E}_{\mathbb{1}}(\widetilde{U}^r) = \zeta^{-r}\widetilde{U} = \sum \zeta^{mr}P_{m+1} \quad ,$$

where $P_p \equiv P_0$. Hence, as $\det(\zeta^{ij}) \neq 0$, we can conclude that $\mathbb{E}_{\mathbb{1}}(P_m) = P_{m+1}$.Thus η is the cyclic shift on $\Pi \cong \{0, \ldots, p-1\}$. Inserting the resolution into $\mathbb{E}\big(Y(\mathbb{1} \otimes \widetilde{U})\big) = \zeta \mathbb{E}(Y)\widetilde{U}$ gives the relation stated in the Proposition.

Clearly, $P_i \in \mathcal{Z}(\mathbb{E})$. Conversely, let $C \in \mathcal{Z}(\mathbb{E})$, i.e. $\mathbb{E}_{\mathbb{1}}^n(C) = C$ for some n. It is clear from the construction that apart from the p^{th} roots of unity all eigenvalues of $\mathbb{E}_{\mathbb{1}}$ have modulus strictly less than one. Hence $C = \mathbb{E}_{\mathbb{1}}^{pn}(C)$ must be in the eigenspace of $\mathbb{E}_{\mathbb{1}}^p$ for eigenvalue 1, hence must be a linear combination of the P_m. It follows that $\mathcal{Z}(\mathbb{E})$ is generated by the P_m, and that they are precisely the minimal projections of $\mathcal{Z}(\mathbb{E})$.

■

We will call $\eta : \Pi \to \Pi$ the *central flow* associated with $\mathbb{E}$. From this flow the periodic components of the C*-finitely correlated state ω can be obtained in the same way as the ergodic components were defined via equation (2.1). The only difference is that the resulting states are now no longer translation invariant, so one has to distinguish the translates of a periodic state by keeping track of the location (modulo a period) of the observables on which the state is evaluated. Thus we set, for every $P \in \Pi$:

$$\omega_P \left(A_1 \otimes \cdots A_n\right) = \rho(P)^{-1} \rho\left(\mathbb{E}_{A_1} \circ \cdots \mathbb{E}_{A_n}(\eta^{-n} P)\right) \quad . \tag{2.4}$$

By construction, the flow η applied to P produces the translates $\omega_P \circ \gamma = \omega_{\eta^{-1}(P)}$. Since $\rho(\eta(P)) = \rho(P)$, the average over the p translates of ω_P (where $\eta^p(P) = P$) is the ergodic state associated to $P' = \sum_{k=0}^{p-1} \eta^k(P) \in \mathcal{Z}_1(\mathbb{E})$ via equation (2.1).

Each of the states ω_P can be considered as a translation invariant state of a regrouped chain algebra, where the single site algebra is then the $p-$fold tensor product of $\mathcal{A}$. In [**3**] it is shown that these states are C*-finitely correlated on the regrouped chain. Since $\mathbb{E}_{\mathbb{1}}^p$, restricted to $P\mathcal{B}$, has only one eigenvalue of modulus 1, the powers of $P\mathbb{E}_{\mathbb{1}}^p$ contract exponentially to the subspace of invariant elements, i.e. to the projection onto the multiples of P. The states ω_P are therefore clustering on the regrouped chain and so are extremal periodic on the original chain. We thus obtain a decomposition of ω into a convex combination of periodic states, which is uniquely determined by $(\mathcal{B}, \mathbb{E}, \rho)$. It is determined by ω only up to repetitions, as there is no guarantee that states determined by different minimal projections in $\mathcal{Z}(\mathbb{E})$ are different. In fact, one can construct examples in the same spirit of the one given after Definition 1.2, that show that minimality of the generating triple is in general not enough to guarantee that all states in (2.1) or (2.4) be distinct. Non-uniqueness arises from the occurrence of non-equivalent minimal representations of a component.

For the discussion of regrouped chains it is convenient to introduce the following notations. For all integers $n \geq 1$ we define inductively the "iterates" $\mathbb{E}^{(n)} : \mathcal{A}^{\otimes n} \otimes \mathcal{B} \to \mathcal{B}$ of $\mathbb{E}$ by

$$\mathbb{E}^{(1)} = \mathbb{E} \qquad \text{and} \qquad \mathbb{E}^{(n+1)}(A \otimes A_n \otimes B) = \mathbb{E}\big(A \otimes \mathbb{E}^{(n)}(A_n \otimes B)\big) \quad .$$

Note that $\mathbb{E}^{(n)}(A_1 \otimes \cdots A_n \otimes B) = \mathbb{E}_{A_1} \circ \cdots \mathbb{E}_{A_n}(B)$ as needed in formula (1.1), and that $\mathbb{E}^{(p)}$ is the operator needed to generate ω as a state on the regrouped chain. We will therefore use the analogue of the notation $\mathbb{E}_A(B) = \mathbb{E}(A \otimes B)$ also for the iterates $\mathbb{E}^{(n)}$. $\mathbb{E}^{(n)}$ is completely positive since composing and tensoring with identity maps preserves complete positivity. As completely positive unity preserving maps all iterates $\mathbb{E}^{(n)}$ have norm one. Hence the norm estimates

$$\left\|\mathbb{E}_A^{(n)}(B)\right\| \leq \|A \otimes B\| = \|A\|\,\|B\| \qquad \text{and} \qquad \left\|\mathbb{E}_A^{(n)}\right\| \leq \|A\| \tag{2.5}$$

hold for all $A \in \mathcal{A}^{\otimes n}$ and $B \in \mathcal{B}$.

We will also need the linear subspace $\mathcal{B}_0 \subset \mathcal{B}$ which is the smallest subspace of $\mathcal{B}$ containing $\mathbb{1}$ and invariant under all operators $\mathbb{E}_A, A \in \mathcal{A}$. Obviously, this space is given by $\mathcal{B}_0 = \left\{\mathbb{E}_A^{(n)}(\mathbb{1}) \middle| n \geq 1, A \in \mathcal{A}^{\otimes n}\right\}$. By the finite-dimensionality of $\mathcal{B}$, and because $\mathbb{E}_{\mathbb{1}}(\mathbb{1}) = \mathbb{1}$, there exists an integer m such that actually

$$\mathcal{B}_0 = \left\{\mathbb{E}_A^{(m)}(\mathbb{1}) \middle| A \in \mathcal{A}^{\otimes m}\right\} \quad . \tag{2.6}$$

In the following Lemma we give a criterion in terms of of $\mathcal{Z}_1(\mathbb{E})$, $\mathcal{Z}(\mathbb{E})$, and $\mathcal{B}_0$ for the absence of multiple representations of periodic components by projections $P \in \Pi$. We will see in Proposition 3.7 that this condition is automatically satisfied for purely generated states.

Lemma 2.5. *Let $(\mathcal{B}, \mathbb{E}, \rho)$ be a minimal triple generating a C^*-finitely correlated state ω.*

(1) $\mathcal{Z}_1(\mathbb{E}) \subset \mathcal{B}_0$ if and only if all ω_Q from (2.1) with minimal projections $Q \in \mathcal{Z}_1(\mathbb{E})$ are distinct.

(2) $\mathcal{Z}(\mathbb{E}) \subset \mathcal{B}_0$ if and only if all ω_P from (2.4) with minimal projections $P \in \mathcal{Z}(\mathbb{E})$ are distinct.

Proof : Let p denote a common multiple of all periods appearing in the central flow η. If we perform a regrouping of the chain by lumping together intervals of p

sites and generate ω, considered as a C*-finitely correlated state on the new chain by the triple $(\mathcal{B}, \mathbb{E}^{(p)}, \rho)$, we see that all $\omega_{Q,q}$ now become ergodic components, that $\mathcal{Z}_1(\mathbb{E}^{(p)}) = \mathcal{Z}(\mathbb{E})$, and that $\mathcal{B}_0$ is the same as before. Therefore (2) follows from (1), and we need only consider (1).

"$\Leftarrow$": Assume that $\mathcal{Z}_1(\mathbb{E}) \subset \mathcal{B}_0$. Then there exists a $n \geq 1$ such that for any minimal projection $Q \in \mathcal{Z}_1(\mathbb{E})$ we have an element $A_Q \in \mathcal{A}^{\otimes n}$ for which $\mathbb{E}^{(n)}_{A_Q}(\mathbb{1}) = Q$. In the decomposition $\mathcal{B} = \bigoplus \mathcal{B}_Q$, where $\mathcal{B}_Q = Q\mathcal{B}Q$, each of the algebras $\mathcal{B}_Q$ is invariant under all operators $\mathbb{E}^{(m)}_A$. In particular $\mathbb{E}^{(n)}_{A_{Q'}}(Q) \in \mathcal{B}_Q$ for all $Q' \in \mathcal{Z}_1(\mathbb{E})$, and hence $\mathbb{E}^{(n)}_{A_{Q'}}(Q) = \delta_{Q,Q'}Q$. Using this relation one immediately gets that $\omega_Q(A_{Q'}) = \delta_{Q,Q'}$, proving that indeed all ω_Q are distinct.

"$\Rightarrow$": Suppose now that all ω_Q are distinct. Then the ω_Q are mutally disjoint, and hence for any $\varepsilon > 0$ there exists an integer m_ε and projections $P^{\varepsilon,Q} \in \mathcal{A}^{\otimes m_\varepsilon}$ such that

$$|\omega_Q(P^{\varepsilon,Q'}) - \delta_{Q,Q'}| \leq \varepsilon \tag{2.7}$$

Define

$$Y_N^{Q,\varepsilon} = \frac{1}{N}\sum_{k=0}^{N-1} \gamma^k(P^{\varepsilon,Q})$$

It is then obvious that $\|Y_N^{\varepsilon,Q}\| \leq 1$ and $\|Y_N^{\varepsilon,Q} - \gamma(Y_N^{\varepsilon,Q})\| \leq 2/N$.
Let $X_N^{\varepsilon,Q} = \mathbb{E}^{(m_\varepsilon+N-1)}_{Y_N^{\varepsilon,Q}}(\mathbb{1})$. Using (2.5) we get:

$$\begin{aligned}\|X_N^{\varepsilon,Q} - \mathbb{E}_{\mathbb{1}}(X_N^{\varepsilon,Q})\| &= \|\mathbb{E}^{(m_\varepsilon+N)}_{Y_N^{\varepsilon,Q}-\gamma(Y_N^{\varepsilon,Q})}(\mathbb{1})\| \\ &\leq \|Y_N^{\varepsilon,Q} - \gamma(Y_N^{\varepsilon,Q})\| \\ &\leq \frac{2}{N}\end{aligned}$$

and hence any cluster point of $(X_N^{\varepsilon,Q})_{N\in\mathbb{N}}$ must belong to $\mathcal{Z}_1(\mathbb{E}) \cap \mathcal{B}_0$. In fact, using

$$X_N^{\varepsilon,Q'}Q = \mathbb{E}^{(m_\varepsilon+N-1)}_{Y_N^{\varepsilon,Q'}}(\mathbb{1})Q = \mathbb{E}^{(m_\varepsilon+N-1)}_{Y_N^{\varepsilon,Q'}}(Q) = QX_N^{\varepsilon,Q'} \quad ,$$

we obtain

$$\rho(X_N^{\varepsilon,Q'}Q) = \rho(\mathbb{E}^{(m_\varepsilon+N-1)}_{Y_N^{\varepsilon,Q'}}(Q)) = \rho(Q)\omega_Q(Y_N^{\varepsilon,Q'}) = \rho(Q)\omega_Q(P^{\varepsilon,Q'})$$

and by (2.7) and the faithfulness of ρ this implies

$$\|(\delta_{Q,Q'}\mathbb{1} - X_N^{\varepsilon,Q})P_{Q'}\| \leq \varepsilon \quad .$$

For any sequence of $\varepsilon_N > 0$ such that $\lim_N \varepsilon_N = 0$ we will therefore have:
$\lim_N X_N^{\varepsilon_N,Q} = Q$, which proves the assertion.

■

As a first application of Lemma 2.5, we prove in the following Proposition that the situation described in 2.5.(1) indeed occurs in some of the minimal triples generating ω.

Proposition 2.6. *For any C^*-finitely correlated state ω there is a triple $(\mathcal{B}, \mathbb{E}, \rho)$ such that $\mathcal{Z}_1(\mathbb{E}) \subset \mathcal{B}_0$.*

Proof : Let $(\mathcal{B}', \mathbb{E}', \rho')$ be any minimal triple generating ω and consider the direct sum decomposition $\mathcal{B}' = \bigoplus \mathcal{B}'_P$ as described in the paragraph following Proposition 2.2. Denote by Π' the set of minimal projections in $\mathcal{Z}_1(\mathbb{E}')$. We can then select a subset $\Pi_0 \subset \Pi'$, such that all ω_P, $P \in \Pi_0$, are distinct, and $\{\omega_P \mid P \in \Pi_0\} = \{\omega_P \mid P \in \Pi'\}$. A triple with the desired property is then obtained by putting: $\mathcal{B} = \bigoplus_{P\in\Pi_0} \mathcal{B}'_P$, $\mathbb{E} = \bigoplus_{P\in\Pi_0} \mathbb{E}'_P$ and $\rho(\bigoplus_{P\in\Pi_0} B_P) = \sum_{P\in\Pi_0} \sum_{P'\in\Pi';\ \omega_P=\omega_{P'}} \rho'(P')\rho'_P(B_P)$.

■

In the sequel we are especially interested in purely generated states. By Definition 1.1 we may generate such states with an $\mathbb{E}$ of the form $\mathbb{E}(X) = V^*XV$, and it is clear that in this case we may take the the auxiliary algebra $\mathcal{B}$ as the full matrix algebra $\tilde{\mathcal{B}} = \mathcal{M}_k$. Although the algebra $\tilde{\mathcal{B}}$ has no center, the central flow of a purely generated state may be non-trivial, because $\tilde{\mathcal{B}}$ is in general *not* minimal. It is useful to remark that the ergodic and periodic components of a purely generated state (considered as C*-finitely correlated states on a regrouped algebra), are still purely generated. This is proven in the following Proposition.

Proposition 2.7. *The set of purely generated states is a w*-dense face in the set of translation invariant states on $\mathcal{A}_\mathbb{Z}$.*

Proof : The density of the purely generated states follows from the stronger statement that even the C*-finitely correlated pure states are w*-dense in the set of translation invariant ones [**4**].

In order to show that the set of purely generated states is convex let, for each $i = 1, \ldots, N$, $\mathcal{K}_i$ be a finite dimensional Hilbert space, $V_i : \mathcal{K}_i \to \mathbb{C}^d \otimes \mathcal{K}_i$ an isometry and ρ_i a state on $\mathcal{B}(\mathcal{K}_i)$ generating a purely generated state ω_i. Then for any choice of coefficients $\lambda_i \geq 0$ with $\sum_i \lambda_i = 1$ we set $\mathcal{K} = \bigoplus_i \mathcal{K}_i$, and $V := \bigoplus_i V_i : \mathcal{K} \to \mathbb{C}^d \otimes \mathcal{K}$, where we identify $\varphi \otimes \bigoplus_i \psi_i$ with $\bigoplus_i (\varphi \otimes \psi_i)$ for $\varphi \in \mathbb{C}^d$ and $\psi_i \in \mathcal{K}_i$. The state ρ on $\mathcal{B}(\mathcal{K})$ is defined as $\bigoplus(\lambda_i \rho_i)$. Then it is elementary to check that these objects generate the state $\sum_i \lambda_i \omega_i$ on $\mathcal{A}_\mathbb{Z}$.

Finally, we show that the convex components of a purely generated state ω are again purely generated. Let $V : \mathcal{K} \to \mathbb{C}^d \otimes \mathcal{K}$ be the isometry with $\mathbb{E}(A \otimes B) = V^*(A \otimes B)V$. Let $\mathcal{B}$ be the smallest subalgebra of $\mathcal{B}(\mathcal{K})$ containing $\mathbb{1}$ which is closed under all operators $\mathbb{E}_A$. The ergodic components of ω are labeled by the minimal projections P in $\mathcal{Z}_1(\mathbb{E}\restriction \mathcal{A} \otimes \mathcal{B})$, and the component ω_P is generated by

$\mathcal{B}_P = P\mathcal{B}P \subset \mathcal{B}(P\mathcal{K})$, and the suitable restrictions of $\mathbb{E}$ and ρ to this algebra. We set $V_P\psi = (\mathbb{1}\otimes P)VP\psi$ for $\psi \in P\mathcal{K}$. Then for $B \in \mathcal{B}_P$: $\mathbb{E}(A\otimes B) = P\mathbb{E}(A\otimes PBP)P = PV^*(A\otimes PBP)VP = V_P^*(A\otimes B)V_P$. Hence $\mathbb{E}\restriction\mathcal{A}\otimes\mathcal{B}_P$ is pure.

■

3. States with vanishing entropy density

The main purpose of this section is the proof of Theorems 1.3, 1.4, and 1.5. The stated propositions (with the exception of 3.7) are partial results to this end, and are later superseded by the corresponding theorems. We begin with two criteria for a C*-finitely correlated state to have vanishing entropy density. In both cases we will later also establish the conditions for the converse.

Proposition 3.1. *Let ω be a C^*-finitely correlated state, which is either pure (i.e. extremal in the state space of $\mathcal{A}_{\mathbb{Z}}$) or purely generated. Then $S_M(\omega) = 0$.*

Proof : The idea of the proof is to show that for a C*-finitely correlated state ω which is either pure or purely generated, and generated by a triple $(\mathcal{B}, \mathbb{E}, \rho)$ where $\mathcal{B} \subset \mathcal{M}_k$, the entropy of the local restrictions of ω satisfies: $S(\omega\restriction\mathcal{A}_{[m,n]}) \le 2\log k$.

Suppose first that ω is pure. If $\mathcal{B} \subset \mathcal{M}_k$ we can extend $\mathbb{E}$ to $\mathcal{A}\otimes\mathcal{M}_k$ by projecting first to the subalgebra $\mathcal{B}$. The extended $\mathbb{E}$ will then generate the same state, although the new triple will no longer be minimal. Anyhow, we can suppose $\mathcal{B} = \mathcal{M}_k$. Then the completely positive map $\mathbb{E}$ is of the form $\mathbb{E}(X) = \sum_\alpha V_\alpha^* X V_\alpha$ where $V_\alpha : \mathbb{C}^k \to \mathbb{C}^d \otimes \mathbb{C}^k$ and the index α takes values in a finite set.

The maps $\mathbb{E}^\alpha(X) = V_\alpha^* X V_\alpha$ are also completely positive although not unity preserving. For any finite interval $[m,n] \subset \mathbb{Z}$ and any choice of indices $\alpha_m, \ldots, \alpha_n$ we can therefore define a non-negative functional $\omega_{\alpha_m,\ldots,\alpha_n}$ on $\mathcal{A}_{\mathbb{Z}}$ by
$\omega_{\alpha_m,\ldots,\alpha_n}(A_{m'}\otimes\cdots\otimes A_{n'}) = \rho(\mathbb{E}_{A_{m'}}\circ\cdots\mathbb{E}_{A_{m-1}}\circ\mathbb{E}^{\alpha_m}_{A_m}\circ\cdots\circ\mathbb{E}^{\alpha_n}_{A_n}\circ\mathbb{E}_{A_{n+1}}\circ\cdots\mathbb{E}_{A_{n'}}(\mathbb{1}))$
for all $m' \le m$, $n' \ge n$ and $A_{m'}, \ldots A_{n'} \in \mathcal{A}$. As $\sum_{\alpha_m,\ldots,\alpha_n} \omega_{\alpha_m,\ldots,\alpha_n} = \omega$, at least for one set of indices, say $\beta_m, \ldots, \beta_n$ the functional $\omega_{\beta_m,\ldots,\beta_n} \neq 0$. As we have assumed that ω is pure there must exist a constant $C > 0$ such that $\omega = C\omega_{\beta_m,\ldots,\beta_n}$. The entropy of the state $C\omega_{\beta_m,\ldots,\beta_n}\restriction\mathcal{A}_{[m,n]}$ can be estimated by looking at its support. The density matrix $D_\omega^{[m,n]}$ of $\omega\restriction\mathcal{A}_{[m,n]}$ can be computed using the defining formula (1.1). We find

$$D_\omega^{[m,n]} = D^{[m,n]}_{\omega_{\beta_m,\ldots,\beta_n}} = \mathrm{tr}_{\mathbb{C}^k}\, K\rho K^* \quad , \tag{3.1}$$

where $K : \mathbb{C}^k \to (\mathbb{C}^d)^{\otimes n-m+1}\otimes\mathbb{C}^k$, $K = (\mathbb{1}_d\otimes\cdots\mathbb{1}_d\otimes V_{\beta_n})\cdots(\mathbb{1}_d\otimes V_{\beta_{m+1}})V_{\beta_m}$, and where $\mathrm{tr}_{\mathbb{C}^k}$ denotes the partial trace with respect to the first tensor factor. Writing out the sum defining this trace, and also the spectral resolution of ρ in terms of one-dimensional eigenprojections we obtain a representation of $D_\omega^{[m,n]}$ as the sum of k^2 rank one operators. It then immediately follows that

$$S(\omega\restriction\mathcal{A}_{[m,n]}) \le 2\log k \quad .$$

This concludes the proof of the Proposition for a pure C*-finitely correlated state ω.

The case of a purely generated state can be treated along the same lines. One also has an expression like in (3.1) for the local density matrices, except that now all the V_β coincide with the V defining the purely generated state (see (1.2)). The rest of the argument remains the same.

■

For studying the converses of this result, it turns out that it is especially useful to study the state $\omega\otimes\omega$ on the chain with one-site algebra $\mathcal{A}\otimes\mathcal{A}$. We caution the reader that the ergodicity of ω does not imply the ergodicity of $\omega\otimes\omega$, as is readily seen for convex combinations of periodic states. Indeed, let ω be an ergodic state on $\mathcal{A}_\mathbb{Z}$ and suppose that ω has a non-trivial decomposition into two extremal 2-periodic states: $\omega = \frac{1}{2}(\omega_1+\omega_2)$. The state $\omega\otimes\omega$ is then no longer ergodic: it has a decomposition $\omega\otimes\omega = \frac{1}{4}\big(\omega_1\otimes\omega_1+\omega_2\otimes\omega_2\big)+\frac{1}{4}\big(\omega_1\otimes\omega_2+\omega_2\otimes\omega_1\big)$ into two translation invariant states.

As a first step we will convert the information about the entropy into a property of the state $\omega\otimes\omega$. By $\mathrm{F}_\mathcal{A}$ we denote the *flip* in $\mathcal{A}$, i.e. the unitary operator in $\mathcal{A}\otimes\mathcal{A}$ implementing the automorphism $A_1\otimes A_2\mapsto A_2\otimes A_1$. Representing $\mathcal{A}$ as the set of linear operators on $\mathbb{C}^d$, and $\mathcal{A}\otimes\mathcal{A}$ as the operators on $\mathbb{C}^d\otimes\mathbb{C}^d$ we can define $\mathrm{F}_\mathcal{A}\,\varphi_1\otimes\varphi_2 = \varphi_2\otimes\varphi_1$, $\varphi_i\in\mathbb{C}^d$.

Lemma 3.2. *Let ω be a state on $\mathcal{A}_\mathbb{Z}$ with vanishing entropy density. Then*

$$\limsup_{n\to\infty}\frac{-1}{n}\ln\omega\otimes\omega\big(\underbrace{\mathrm{F}_\mathcal{A}\otimes\cdots\mathrm{F}_\mathcal{A}}_{n\text{ factors}}\big) = 0 \quad . \tag{3.2}$$

Proof : For a translation invariant state ω with associated local density matrices D_ω^Λ and any $q>1$ the Rényi-entropy $R^q(\omega\restriction\mathcal{A}_\Lambda)$ is defined by

$$R^q(\omega\restriction\mathcal{A}_\Lambda) = -\frac{1}{q-1}\log\mathrm{tr}(D_\omega^\Lambda)^q \quad ,$$

and the mean Rényi-entropy $R_M^q(\omega)$ is then defined by

$$R_M^q(\omega) = \limsup_{n-m\to\infty}\frac{1}{n-m}R^q(\omega\restriction\mathcal{A}_{[m,n]}) \quad .$$

In **[10]** it is shown that for $1<q_1<q_2$:

$$0\le R_M^{q_2}(\omega)\le R_M^{q_1}(\omega)\le S_M(\omega) \quad .$$

Therefore the proof of the Lemma will be complete if we show that the LHS of (3.2) is exactly $R^2_M(\omega)$. This can be seen using the following observation:

Let A_1 and A_2 be two trace class operators on an Hilbert space $\mathcal{H}$ and let $\{e_i \mid i = 1, 2, \ldots\}$ be an orthonormal basis for $\mathcal{H}$. Then:

$$\begin{aligned}
\mathrm{tr}_{\mathcal{H}}\, A_1 A_2 &= \sum_{i_1, i_2} \langle e_{i_1}\, ,\, A_1 e_{i_2} \rangle \langle e_{i_2}\, ,\, A_2 e_{i_1} \rangle \\
&= \sum_{i_1, i_2} \langle e_{i_1} \otimes e_{i_2}\, ,\, A_1 \otimes A_2\; e_{i_2} \otimes e_{i_1} \rangle \\
&= \mathrm{tr}_{\mathcal{H}\otimes\mathcal{H}}((A_1 \otimes A_2)\; \mathrm{F}_{\mathcal{H}}) \quad ,
\end{aligned} \tag{3.3}$$

where $\mathrm{F}_{\mathcal{H}} : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H} \otimes \mathcal{H} : \varphi_1 \otimes \varphi_2 \mapsto \varphi_2 \otimes \varphi_1$. Applying (3.3) with $\mathcal{H} = (\mathbb{C}^d)^{\otimes n-m+1}$ and $A_1 = A_2 = D_\omega^{[m,n]}$, we get:

$$\begin{aligned}
R^2(\omega \restriction \mathcal{A}_{[m,n]}) &= \omega \otimes \omega(\mathrm{F}_{\mathcal{H}}) \\
&= \omega \otimes \omega(\quad \underbrace{\mathrm{F}_{\mathcal{A}} \otimes \cdots \mathrm{F}_{\mathcal{A}}}_{n-m+1 \text{ factors}} \quad) \quad .
\end{aligned}$$

Taking the lim sup the desired result follows.

■

Note that this result does not use the fact that ω is a C*-finitely correlated state. However, if we do have a C*-finitely correlated $(\mathcal{B}^{(2)}, \mathbb{E}^{(2)}, \rho^{(2)})$ generating $\omega \otimes \omega$, expectations as the one in equation (3.2) are particularly easy to compute. One can easily see that $\omega \otimes \omega$ is C*-finitely correlated iff ω is C*-finitely correlated. We then have

$$\omega \otimes \omega\big(\mathrm{F}_{\mathcal{A}} \otimes \cdots \mathrm{F}_{\mathcal{A}}\big) = \rho\big(\mathbb{E}^{(2)}_{\mathrm{F}_{\mathcal{A}}} \circ \cdots \mathbb{E}^{(2)}_{\mathrm{F}_{\mathcal{A}}}(\mathbb{1})\big) = \rho\big(\mathbb{F}^n(\mathbb{1})\big) \quad ,$$

where

$$\mathbb{F}(B) = \mathbb{E}^{(2)}_{\mathrm{F}_{\mathcal{A}}}(B) = \mathbb{E}^{(2)}(\mathrm{F}_{\mathcal{A}} \otimes B) \quad .$$

Thus the asymptotics of the left hand side of (3.2) is governed by the spectral properties of $\mathbb{F}$. Note that

$$\mathbb{F} : B \mapsto \mathrm{F}_{\mathcal{A}} \otimes B \mapsto \mathbb{E}^{(2)}(\mathrm{F}_{\mathcal{A}} \otimes B) = \mathbb{F}(B)$$

is a contraction as the product of two contractions. We may thus apply the following Lemma to $\mathbb{F}$ as an operator on the Banach space $X \equiv \mathcal{B}^{(2)}$ with $x = \mathbb{1}$, and $\xi = \rho$.

Lemma 3.3. *Let X be a finite dimensional Banach space, and $\mathbb{F}$ a contraction in X. Suppose that there are $x \in X$, $\xi \in X^*$, such that $\langle \xi, \mathbb{F}^n x \rangle$ is positive for all $n \in \mathbb{N}$, and that $\limsup_n \frac{-1}{n} \ln\langle \xi, \mathbb{F}^n x \rangle = 0$. Then 1 is an eigenvalue of $\mathbb{F}$.*

Proof : Let $\mathbb{F} = \sum_\lambda (\lambda P_\lambda + N_\lambda)$ be the spectral decomposition of $\mathbb{F}$ with

$$P_\lambda P_\mu = \delta_{\lambda,\mu} P_\lambda$$
$$P_\lambda N_\mu = N_\mu P_\lambda = \delta_{\lambda,\mu} N_\lambda$$

and N_λ nilpotent. Then, for the eigenvalues with $|\lambda| = 1$, the nilpotent part N_λ must vanish. Otherwise let k be such that $N_\lambda^k \neq 0,\ N_\lambda^{k+1} = 0$ and pick $x' \in P_\lambda X, \xi' \in X^*$ such that $\langle \xi',\, N_\lambda^k x'\rangle \neq 0$. Then

$$\langle \xi', \mathbb{F}^n x'\rangle = \lambda^n \sum_{\kappa=1}^{k} \binom{n}{\kappa} \lambda^{-\kappa} \langle \xi', N_\lambda^\kappa x'\rangle$$

Apart from the factor in front of the sum, which is of constant modulus, this is a polynomial of degree k in n. This contradicts the assumption that $\mathbb{F}$ is a contraction. Hence we have

$$\langle \xi, \mathbb{F}^n x\rangle = {\sum_\lambda}' \lambda^n \langle \xi, P_\lambda x\rangle + R_n \quad ,$$

where the prime indicates a sum only over the eigenvalues of modulus 1, and $|R_n| \leq c\mu^n$ for some $0 \leq \mu < 1$. If there are no eigenvalues of modulus 1, then $\limsup_n \frac{-1}{n} \ln\langle \xi, \mathbb{F}^n x\rangle \geq \limsup_n (\frac{-1}{n}(\ln c + n \ln \mu)) = -\ln \mu > 0$, contradicting the assumption. Now for large n the matrix element $\langle \xi, \mathbb{F}^n x\rangle$ becomes equal to an almost periodic function in n. This can be positive only if the mean of this function (which is equal to the constant term in the Fourier decomposition) is positive. Consequently, $\langle \xi, P_1 x\rangle$ is non-zero, which proves the assertion.

■

If $\mathcal{Z}_1(\mathbb{E}^{(2)})$ is trivial so that $G(\mathbb{E}^{(2)})$ is defined, the Lemma just states that $\mathrm{F}_\mathcal{A} \in G(\mathbb{E}^{(2)})$. If we take the two factors of $\omega \otimes \omega$ to be generated by a priori different triples this flip induces a symmetry exchanging the two representations. This is the basic idea of the following uniqueness result. However, as we have noted above $\omega \otimes \omega$ is not necessarily ergodic, and the argument becomes slightly more involved. Combined with Proposition 3.1 the following Proposition proves Theorem 1.3 under the extra assumption that ω is ergodic.

Proposition 3.4. *Let ω be an ergodic C^*-finitely correlated state with vanishing mean entropy. Then, up to isomorphism in the sense of Definition 1.2, there is a unique minimal triple $(\mathcal{B}, \mathbb{E}, \rho)$ generating ω. For this triple, $\mathcal{Z}_1(\mathbb{E}) = \mathbb{C}\mathbb{1}$.*

Proof : Every standard triple $(\mathcal{B}, \mathbb{E}, \rho)$ generating ω is the direct sum of triples $(\mathcal{B}_P, \mathbb{E}_P, \rho_P)$ satisfying the additional condition $\mathcal{Z}_1(\mathbb{E}_P) = \mathbb{C}\mathbb{1}$. By ergodicity each of these triples generates the same state ω. Suppose we have shown that all triples with the additional property $\mathcal{Z}_1(\mathbb{E}_P) = \mathbb{C}\mathbb{1}$ are isomorphic. Then applying an appropriate isomorphism in each summand we find that $(\mathcal{B}, \mathbb{E}, \rho)$ is isomorphic to a direct sum of *equal* triples. It is easy to see that the subalgebra $\mathcal{B}' \subset \mathcal{B}$ of this direct sum, which consists of the direct sum of equal elements, contains 1 and is invariant under $\mathbb{E}_{\mathcal{A}}$. By minimality we conclude that $\mathcal{B}' = \mathcal{B}$, i.e. there can be only one summand.

Let $(\mathcal{B}_1, \mathbb{E}_1, \rho_1)$ and $(\mathcal{B}_2, \mathbb{E}_2, \rho_2)$ be two minimal triples generating ω such that $\mathcal{Z}_1(\mathbb{E}_1) = \mathbb{C}\mathbb{1}$ and $\mathcal{Z}_1(\mathbb{E}_1) = \mathbb{C}\mathbb{1}$. It remains to be shown that these triples are isomorphic. $\omega \otimes \omega$ is again a C*-finitely correlated state and as generating triple we take

$$(\mathcal{B}^{(2)}, \mathbb{E}^{(2)}, \rho^{(2)}) \equiv (\mathcal{B}_1 \otimes \mathcal{B}_2, \mathbb{E}_1 \otimes \mathbb{E}_2, \rho_1 \otimes \rho_2) \quad .$$

In the definition of $\mathbb{E}^{(2)}$ we implicitly reshuffle the tensor factors, i.e.

$$\mathbb{E}^{(2)}\big((A_1 \otimes A_2) \otimes (B_1 \otimes B_2)\big) = \mathbb{E}_1(A_1 \otimes B_1) \otimes \mathbb{E}_2(A_2 \otimes B_2) \quad ,$$

so that $\mathbb{E}^{(2)} : (\mathcal{A} \otimes \mathcal{A}) \otimes \mathcal{B}^{(2)} \to \mathcal{B}^{(2)}$ as required. It is elementary to check that we have indeed constructed a standard triple generating $\omega \otimes \omega$. The algebra $\mathcal{B}^{(2)}$ in this triple is minimal. Indeed a minimal algebra by definition must be generated from the identity by algebraic operations and operators $B \in \mathcal{B}^{(2)} \mapsto \mathbb{E}^{(2)}(A \otimes B)$ for all $A \in \mathcal{A} \otimes \mathcal{A}$. If we use only elements $A = A_1 \otimes \mathbb{1}$ we generate the algebra $\mathcal{B}_1 \otimes \mathbb{1}$ since $\mathcal{B}_1$ was assumed to be minimal. By the same token $\mathbb{1} \otimes \mathcal{B}_2$ lies in the minimal algebra, hence $\mathcal{B}^{(2)} = \mathcal{B}_1 \otimes \mathcal{B}_2$ is minimal.

Combining Lemma 3.2 and Lemma 3.3 we conclude that the equation $\mathbb{E}^{(2)}(\mathrm{F}_{\mathcal{A}} \otimes S) = S$ has a nonzero solution $S \in \mathcal{B}^{(2)}$. As $\omega \otimes \omega$ is not necessarily ergodic it will be useful to focus our attention on a suitable ergodic component of $\omega \otimes \omega$. So let P be some minimal projection in $\mathcal{Z}_1(\mathbb{E}^{(2)})$ such that $PS \neq 0$. Then PS again satisfies the equation for S. We shall focus on the ergodic component $\tilde{\omega} \equiv (\omega \otimes \omega)_P$ associated to P which is generated by $\tilde{\mathcal{B}} = P\mathcal{B}^{(2)} \subset \mathcal{B}^{(2)}$, $\tilde{\mathbb{E}} = \mathbb{E}^{(2)}\restriction(\mathcal{A} \otimes \mathcal{A}) \otimes \tilde{\mathcal{B}}$, and the state $\tilde{\rho} = \rho(P)^{-1}\rho\restriction\tilde{\mathcal{B}}$. We take as S its component in $\tilde{\mathcal{B}}$, i.e. $S \equiv PS$. We thus have $\mathrm{F}_{\mathcal{A}} \in G(\tilde{\mathbb{E}})$ with associated unitary $S \in \tilde{\mathcal{B}}$. Since ${\mathrm{F}_{\mathcal{A}}}^2 = \mathbb{1}$ we get $S^2 \in \mathcal{Z}_1(\tilde{\mathbb{E}})$ so that we may adjust S with a scalar factor to satisfy $S^2 = S^*S = P\mathbb{1}_{\tilde{\mathcal{B}}}$. It is clear that

$$\tau(B) = SBS \qquad \text{for } B \in \tilde{\mathcal{B}} \tag{3.4}$$

defines a "flip" automorphism on $\tilde{\mathcal{B}}$. This automorphism will be used below. It is elementary to check that

$$\tau \circ \tilde{\mathbb{E}}_{A_1 \otimes A_2} = \tilde{\mathbb{E}}_{A_2 \otimes A_1} \circ \tau \quad .$$

We will now compare the two generating triples $(\mathcal{B}_1, \mathbb{E}_1, \rho_1)$ and $(\mathcal{B}_2, \mathbb{E}_2, \rho_2)$ using the automorphism τ.

Consider the homomorphism $\pi_1 : \mathcal{B}_1 \to \tilde{\mathcal{B}}$ with $\pi_1(B_1) = P(B_1 \otimes \mathbb{1})$, and an analogously defined π_2. Then a straightforward computation shows that

$$\pi_1\big(\mathbb{E}_1(A_1 \otimes B_1)\big) = \tilde{\mathbb{E}}\big((A_1 \otimes \mathbb{1}) \otimes \pi_1(B_1)\big) \quad .$$

We claim that both π_1 and π_2 are faithful. Indeed, consider the ideal $\ker\pi_1 \subset \mathcal{B}_1$. Then it is obvious from the above relation between $\mathbb{E}_1$ and $\tilde{\mathbb{E}}$ that $\ker\pi_1$ is invariant under $\mathbb{E}_1(A_1 \otimes \cdot)$ for all $A_1 \in \mathcal{A}$. Since $\mathcal{B}_1$ is a direct sum of finite matrix algebras, the only ideals in $\mathcal{B}_1$ are direct summands. The central projection $z \in \mathcal{B}_1$ with

$z\mathcal{B}_1 = \ker\pi_1$ is thus invariant under $\mathbb{E}_1(\mathbb{1}\otimes\cdot)$, and by the assumption $\mathcal{Z}_1(\mathbb{E}_1) = \mathbb{C}\mathbb{1}$, and $\mathbb{E}_1(\mathbb{1}\otimes\mathbb{1} = \mathbb{1}$, we must have $z = 0$.

As the last step in the proof of the Proposition we construct an isomorphism $\tilde{\tau}$ between the triples $(\mathcal{B}_1, \mathbb{E}_1, \rho_1)$ and $(\mathcal{B}_2, \mathbb{E}_2, \rho_2)$. We first show that

$$\tau\pi_1(\mathcal{B}_1) = \pi_2(\mathcal{B}_2), \tag{3.5}$$

where τ is the automorphism of $\tilde{\mathcal{B}}$ defined in (3.4). Then, since π_1 and π_2 are faithful, the isomorphism $\tilde{\tau} = \pi_2^{-1}\tau\pi_1 : \mathcal{B}_1 \to \mathcal{B}_2$ is well defined. In order to prove (3.5), let $\mathcal{D}_k, k \geq 0$ be the increasing family of C*-algebras generating $\mathcal{B}_1$ as in the proof of Proposition 2.2. We shall show inductively that $\tau\pi_1(\mathcal{D}_k) \subset \pi_2(\mathcal{B}_2)$. Since τ is a homomorphism it suffices to show that if $A \in \mathcal{A}$, and $\tau\pi_1(D) = P(\mathbb{1}\otimes D_2) \in \pi_2(\mathcal{B}_2)$, then $\tau\pi_1(D') = P(\mathbb{1}\otimes D_2')$ for $D' = \mathbb{E}_1(A\otimes D)$. But

$$\begin{aligned}
\tau\pi_1(D') &= S^3(D'\otimes\mathbb{1})S = S\Big(\mathbb{E}_1\otimes\mathbb{E}_2\Big)\big((A\otimes\mathbb{1})\otimes(D\otimes\mathbb{1})\big)S \\
&= \tau\tilde{\mathbb{E}}((A\otimes\mathbb{1})\otimes\pi_1(D)) = \tilde{\mathbb{E}}((\mathbb{1}\otimes A)\otimes\tau\pi_1(D)) \\
&= \tilde{\mathbb{E}}((\mathbb{1}\otimes A)\otimes\pi_2(D_2)) = P(\mathbb{1}\otimes\mathbb{E}_2(A\otimes D_2)) \\
&= P(\mathbb{1}\otimes D_2') \quad .
\end{aligned}$$

Hence $\tau\pi_1(\mathcal{B}_1) \subset \pi_2(\mathcal{B}_2)$ and in reversing the roles of 1 and 2 we conclude that $\tau\pi_1(\mathcal{B}_1) = \pi_2(\mathcal{B}_2)$. The same computation shows that $\tau\pi_1\mathbb{E}_1(A\otimes B_1) = \pi_2\mathbb{E}_2(A\otimes \tilde{\tau}(B_1))$, which implies $\tilde{\tau}\circ\mathbb{E}_1 = \mathbb{E}_2\circ\mathrm{id}\otimes\tilde{\tau}$. The relation for the ρ's which is needed to get the isomorphism between triples follows from the uniqueness of the $\mathbb{E}$-invariant state in a standard triple with $\mathcal{Z}_1(\mathbb{E}) = \mathbb{C}\mathbb{1}$.

■

For a general C*-finitely correlated state ω generated by a minimal triple $(\mathcal{B}, \mathbb{E}, \rho)$ the algebra $\mathcal{Z}(\mathbb{E})$, defined in Proposition 2.4, is a proper subalgebra of the center of $\mathcal{B}$. However, this possibility is excluded if we impose the condition $S_M(\omega) = 0$.

Proposition 3.5. *Let ω be an ergodic C*-finitely correlated state with vanishing mean entropy. Then in the minimal triple $(\mathcal{B}, \mathbb{E}, \rho)$ generating ω the center of $\mathcal{B}$ coincides with $\mathcal{Z}(\mathbb{E})$.*

Proof : By Proposition 3.4 we can assume that $\pi_2 = \tau\pi_1$. Suppose that $\mathcal{B} = \bigoplus_\alpha \mathcal{B}_\alpha$ is the decomposition of $\mathcal{B}$ into irreducible summands. Denote the corresponding central projections by p_α. Since p_α is central in $\pi_1(\mathcal{B})$, and $[\pi_1(\mathcal{B}), \pi_2(\mathcal{B})] = 0$, $\pi_1(p_\alpha)$ is central in $\tilde{\mathcal{B}}$. Since τ is an inner automorphism in $\tilde{\mathcal{B}}$, we have $\pi_1(p_\alpha) = \tau\pi_1(p_\alpha) = \pi_2(p_\alpha)$.

Recall that $\tilde{\mathcal{B}}$ was defined as $P\mathcal{B}\otimes\mathcal{B}$ for a suitable minimal projection $P \in \mathcal{Z}_1(\mathbb{E}\otimes\mathbb{E})$. Since P is in the center of $\mathcal{B}\otimes\mathcal{B}$, it is a sum of projections of the form $p_\alpha\otimes p_\beta$. By definition of π_i we have $P(\mathbb{1}\otimes p_\alpha) = \pi_2(p_\alpha) = \pi_1(p_\alpha) = P(p_\alpha \otimes \mathbb{1})$. Hence $p_\alpha \otimes p_\beta$ can be contained in P only if $\alpha = \beta$. On the other hand π_i is faithful so that each p_α has to appear in the first factor at least once. Consequently,

$$P = \sum_\alpha p_\alpha \otimes p_\alpha \quad .$$

The operator $\mathbb{E}$ can be decomposed into blocks along the direct sum decomposition of $\mathcal{B}$ as $\mathbb{E} = \sum_{\alpha\beta} \mathbb{E}_{\alpha\beta}$ with $\mathbb{E}_{\alpha\beta}(A\otimes B) = p_\alpha \mathbb{E}(A\otimes(p_\beta B))$. Suppose that for some fixed β and two different α_1, α_2 we have $\mathbb{E}_{\alpha_i\beta} \neq 0$. Then

$$\begin{aligned}
0 \neq\ & (\mathbb{E}_{\alpha_1\beta}(\mathbb{1})) \otimes (\mathbb{E}_{\alpha_2\beta}(\mathbb{1})) \\
&= (p_{\alpha_1}\otimes p_{\alpha_2})\mathbb{E}\otimes\mathbb{E}(\mathbb{1}_{\mathcal{A}\otimes\mathcal{A}} \otimes (p_\beta \otimes p_\beta)) \\
&\leq (p_{\alpha_1}\otimes p_{\alpha_2})\mathbb{E}\otimes\mathbb{E}(\mathbb{1}\otimes P) \\
&= (p_{\alpha_1}\otimes p_{\alpha_2})\ P = 0 \quad .
\end{aligned}$$

This is a contradiction, so that for each β there must be a unique α with $\mathbb{E}_{\alpha\beta} \neq 0$. In other words there is a permutation σ such that $\mathbb{E}(\mathbb{1}\otimes p_\alpha) = p_{\sigma(\alpha)}$. Hence each p_α is invariant under a suitable power of $\mathbb{E}_{\mathbb{1}}$ and p_α is a minimal projection in $\mathcal{Z}(\mathbb{E})$. Comparing with Proposition 2.4 it is clear that we have reconstructed the central flow of $\mathbb{E}$.

■

In Proposition 3.1 we have shown that a purely generated C*-finitely correlated state has zero mean entropy. We now prove the converse under the extra assumption of ergodicity of the C*-finitely correlated state ω. Combining both results we obtain therefore Theorem 1.4 for the ergodic case.

Proposition 3.6. *Any ergodic C*-finitely correlated state ω with vanishing mean entropy is purely generated.*

Proof : Using the same notation as in the proof of Proposition 3.5, let $\mathbb{E}_\alpha \equiv \mathbb{E}_{\alpha,\sigma(\alpha)} : \mathcal{A}\otimes\mathcal{B}_\alpha \to \mathcal{B}_{\sigma(\alpha)}$ be one of the non-zero summands of $\mathbb{E}$. This operator is "pure" in the terminology of Davies **[11]** i.e. a pure state φ on $\mathcal{B}_{\sigma(\alpha)}$ is mapped by $\mathbb{E}_\alpha$ in a pure state on $\mathcal{A}\otimes\mathcal{B}_\alpha$. In order to prove this we will use the fact that a state φ on a matrix algebra is pure iff $\varphi\otimes\varphi(\mathrm{F}) = 1$, where F denotes the flip operator. This is an immediate consequence of the trace formula (3.3). Let S_α be the component of $S \in \tilde{\mathcal{B}} = \bigoplus_\alpha \mathcal{B}_\alpha \otimes \mathcal{B}_\alpha$ in $\mathcal{B}_\alpha\otimes\mathcal{B}_\alpha$. Since for $B\in\mathcal{B}_\alpha$ we have $S_\alpha(B\otimes\mathbb{1})S_\alpha = \mathbb{1}\otimes B$, S_α must be equal to the flip $\mathrm{F}_\alpha \in \mathcal{B}_\alpha\otimes\mathcal{B}_\alpha$ up to a phase ε_α. Since $S_\alpha^2 = p_\alpha$ this phase is further determined as $\varepsilon_\alpha = \pm 1$. Now let φ be a pure state on $\mathcal{B}_{\sigma(\alpha)}$. By Proposition 3.5 $\varphi' \equiv \varphi \circ \mathbb{E}_\alpha$ is supported by $\mathcal{A} \otimes \mathcal{B}_\alpha$. By assumption

$$
\begin{aligned}
1 = \varphi \otimes \varphi(\mathrm{F}_{\sigma(\alpha)}) &= \varepsilon_{\sigma(\alpha)} \varphi \otimes \varphi(S) = \varepsilon_{\sigma(\alpha)} \varphi \otimes \varphi\big(\mathbb{E} \otimes \mathbb{E}(\mathrm{F}_\mathcal{A} \otimes S)\big) \\
&= \varepsilon_{\sigma(\alpha)} \varphi' \otimes \varphi'(\mathrm{F}_\mathcal{A} \otimes S) = \varepsilon_{\sigma(\alpha)} \varepsilon_\alpha \varphi' \otimes \varphi'(\mathrm{F}_\mathcal{A} \otimes \mathrm{F}_\alpha) \quad .
\end{aligned}
$$

Since $\varphi' \otimes \varphi'(\mathrm{F}_\mathcal{A} \otimes \mathrm{F}_\alpha)$ cannot be -1 it must be $+1$, hence φ' is pure. In passing we have also shown that $\varepsilon_{\sigma(\alpha)} = \varepsilon_\alpha$, i.e. $S_\alpha = \pm \mathrm{F}_\alpha$ with a sign independent of α.

It is shown in Theorem 3.1 of **[11]** that a positive map between matrix algebras is "pure" iff it is either (i) implemented by a linear map, (ii) implemented by an antilinear map, or (iii) of rank one. Of these possibilities the second is ruled out because $\mathbb{E}_\alpha$ is not only positive but completely positive. The third would map any operator into a multiple of the identity, which is incompatible with the property $\mathbb{E}_\alpha(\mathrm{F}_\mathcal{A} \otimes \mathrm{F}_\alpha) = \mathrm{F}_{\sigma(\alpha)}$ except in the trivial case where $\dim \mathcal{B}_\alpha = 1$ for all α which is covered by (i) anyway. Hence $\mathbb{E}_\alpha$ is linearly implemented. More explicitly, if we set $\mathcal{B}_\alpha = \mathcal{B}(\mathcal{K}_\alpha)$, and $\mathcal{A} = \mathcal{B}(\mathcal{H})$ for some finite dimensional Hilbert spaces $\mathcal{K}_\alpha, \mathcal{H}$ we find that there are isometries

$$
V_\alpha : \mathcal{H} \otimes \mathcal{K}_\alpha \to \mathcal{K}_{\sigma(\alpha)} \quad ,
$$

which are unique up to a phase, such that $\mathbb{E}_\alpha(X) = V_\alpha^* X V_\alpha$. We can also introduce the space $\mathcal{K} = \bigoplus_\alpha \mathcal{K}_\alpha$, and take $\mathcal{B}$ to be represented on $\mathcal{K}$ in the obvious way. One easily checks that $V : \mathcal{H} \otimes \mathcal{K} \to \mathcal{K}$ defined by $V(\varphi \otimes \bigoplus_\alpha \psi_\alpha) = \sum_\alpha V_\alpha(\varphi \otimes \psi_\alpha)$ is an isometry, and that for $A \in \mathcal{A}, B \in \mathcal{B}$ we have $\mathbb{E}(X) = V^* X V$. This formula can be taken as the definition of an extension of $\mathbb{E}$ from the minimal algebra $\mathcal{A} \otimes \mathcal{B}$ to $\mathcal{A} \otimes \mathcal{B}(\mathcal{K})$. This extension is then a pure map generating ω.

∎

Most of the results in this section were proven under the additional assumption that the C*-finitely correlated state ω was ergodic. The following Proposition shows that for (not necessarily ergodic) purely generated states the equivalent conditions of Lemma 2.5(2) hold automatically, thus eliminating the last remaining source for non-uniqueness in the representation of such states. It also extends Proposition 3.5 to the non-ergodic case.

Proposition 3.7. *Let $(\mathcal{B}, \mathbb{E}, \rho)$ be a minimal triple generating a purely generated state ω. Then $\mathcal{Z}(\mathbb{E}) \subset \mathcal{B}_0$, and $\mathcal{Z}(\mathbb{E})$ is equal to the center of $\mathcal{B}$.*

Proof : We show that, for distinct minimal projections $P, P' \in \mathcal{Z}_1(\mathbb{E})$, the states ω_P and $\omega_{P'}$ are distinct. By Proposition 2.5 ω_P is again purely generated as an ergodic components of ω, and by Proposition 3.4 it has up to isomorphism a unique generating triple. Hence if two summands ω_P and $\omega_{P'}$ coincide, the $\mathbb{E}$ acts on corresponding summands like $\mathbb{E}_A(B \oplus B') = (\mathbb{E}_A(B)) \oplus \tau(\mathbb{E}_A(\tau^{-1}B'))$, where τ is the map implementing the isomorphism. The minimal algebra generated from $\mathbb{1} \oplus \mathbb{1}$

and the successive images of $\mathbb{E}_A$'s therefore contains only elements with $\tau B = B'$. This contradicts the assumption that $\mathcal{B}$ was minimal in the first place. It follows that to each ergodic component corresponds a unique minimal projection in $\mathcal{Z}_1(\mathbb{E})$, and by Lemma 2.7 we conclude that $\mathcal{Z}_1(\mathbb{E}) \subset \mathcal{B}_0$. Applying the same arguments to a regrouped chain, we find also that $\mathcal{Z}(\mathbb{E}) \subset \mathcal{B}_0$. The last statement follows by decomposing the center of $\mathcal{B}$ into summands with respect to $\mathcal{Z}_1(\mathbb{E})$, and applying Proposition 3.5 to each summand.

■

Proof of Theorems 1.3, 1.4, and 1.5: Suppose ω is a C*-finitely correlated state with vanishing entropy density. Then since the entropy density is a positive affine functional on the set of translation invariant states the same is true for the ergodic components of ω. Applying Proposition 3.6 to each component and using the convexity of the set of purely generated states (Proposition 2.5) we find that ω is purely generated. This, together with Proposition 3.1, proves Theorem 1.4.

Now suppose that ω is purely generated, and let $(\mathcal{B}, \mathbb{E}, \rho)$ be a minimal triple generating ω. Then by the previous Proposition each ergodic component corresponds to a unique minimal projection in $\mathcal{Z}_1(\mathbb{E})$. Furthermore, the ergodic decomposition is uniquely determined by the state ω alone. Since by Proposition 3.4 the minimal triple generating each ergodic component is unique, the minimal triple for ω is uniquely determined as the direct sum of the triples representing the components, which proves Theorem 1.3.

Finally it remains to prove Theorem 1.5. If ω is pure, its mean entropy vanishes by Proposition 3.1. By Theorem 1.4 this implies that ω is purely generated. Thus if any one of the conditions in Theorem 1.5 is satisfied ω is purely generated, and hence has a unique minimal generating triple by Theorem 1.3. We will argue now that any one of the conditions also implies that ω has no non-trivial decomposition into periodic components. Since condition (i) excludes *any* convex decomposition this is trivial for (i). From (iv) we find that $\mathcal{Z}(\mathbb{E}) = \mathbb{C}\mathbb{1}$ for a minimal triple, which implies by Proposition 2.4 that $\mathbb{E}_{\mathbb{1}}$ has trivial peripheral spectrum. This implies exponential decay of correlation functions, i.e. spatial clustering of ω. By standard theorems [**7**] this implies that ω is ergodic, and furthermore has no periodic decomposition. We have thus shown (iv) $\iff$ (iii) $\iff$ (ii)$\Leftarrow$(i). The remaining direction (iv)$\Rightarrow$(i) was shown in Proposition 5.8 of [**3**] by demonstrating that ω is the unique ground state of a finite range Hamiltonian.

■

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