SOME RIGOROUS RESULTS ABOUT 3D NAVIER-STOKES.

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Plan*

There are very few rigorous results on 3D Navier-Stokes equations. Here I select and review a few among the most remarkable:

1. Leray's local theorem
2. Scheffer's local theorem
3. Caffarelli, Kohn, Nirenberg regularity theorem.

I provide an interpretation of the scaling ideas in 1, 2, 3 with particular attention to the similarity of 3) with the theory of relevant variables in the renormalization group methods. The connection of 3) with Kolmogorov- Obuchov theory (quoted as (KO) below) is attempted.

Finally I present a comment inspired by Chorin's ideas on the vorticity filaments dynamics (3D). To simplify a subject, quite intricate in itself, only a fluid in a periodic container Ω is considered.

Key words: Fluid dynamics, Renormalization group, Kolmogorov Obuchov theory, vorticity filaments, Hausdorff dimension, Non linear PDE's.

§1. Leray's formulations of the theory.

The first disturbing and striking feature of 3DNS is that only a *non constructive* existence proof of a solution is available for general smooth data and forces. This amounts to saying that, unless one invokes statements about something being "physically obvious", the numerical experiments concerning turbulence phenomena are in fact experiments on the program generating them, rather than on the idealized NS fluid (this fact does not have, necessarily, negative implications, but it has to be kept in mind).

The NS equations are, with usual or obvious notations:

$$\begin{aligned}
&\underline{\dot u}+(\underset{\sim}{u}\cdot\underset{\sim}{\partial})\underline{u}=-\underline{\partial}p+\nu\Delta\underline{u}+\underline{f}\\
&\underline{\partial}\cdot\underline{u}=0\ ,\qquad\int_\Omega\underline{u}=\underline{0}\ ,\qquad\Omega=[0,L]^3_{\rm periodic}\\
&\underline{u}|_{t=0}=\underline{u}_0
\end{aligned}\tag{1.1}$$

* *Text of two lectures delivered at the Les Houches NATO-ASI meeting "Turbulence in spatially ordered systems", january, 1992.*

The force $\underline{f}$ will be supposed time independent and the force as well as the initial datum $\underline{u}_0$ will be supposed very smooth; $\underline{f}$, $\underline{u} \in C^\infty(\Omega)$, $\int \underline{f} = \underline{0}$, $\int \underline{u}_0 = \underline{0}$, $\underline{\partial} \cdot \underline{f} = 0$.

There is *nothing to gain nor to loose* in weakening such regularity assumptions (e.g. most of what follows could be adapted to $u_0 \in H_2(\Omega)$, *i.e.* to very wild $\underline{u}_0$'s, and one would just have to climb on mirrors to show that nothing really changes.

To the initial datum and to $\underline{f}$ we associate a few characteristic parameters:

$$E_0 = \int_\Omega |\underline{u}_0(\underline{x})|^2 \ , \quad D_1 = \int_\Omega |\underset{\sim}{\partial}\, \underline{u}|^2, \quad F_j = \int_\Omega |\underline{\partial}^j \underset{\sim}{f}|^2 \tag{1.2}$$

and:

$$\begin{array}{lll} T & = L^2\nu^{-1} & \text{(viscous time scale)} \\ V & = \nu L^{-1} & \text{(viscous velocity scale)} \\ V_1 & = (D_1 L^{-1})^{1/2} & \text{(initial velocity variation scale)} \\ W_1 & = (L F_1)^{1/4} & \text{(forcing velocity scale)} \\ R & = (V_1^2 + W_1^2)^{1/2}/V & \text{(Reynold's number)} \end{array} \tag{1.3}$$

Note that, if for instance $\underline{f} = \underline{0}$, the number $R = V_1/V = \nu^{-1}\Big(L \int |\underset{\sim}{\partial}\underline{u}|^2\Big)^{1/2}$ is the "usual" $(\delta u)L\nu^{-1}$ where $\delta u = V_1$ is a "velocity variation" on scale L, *i.e.* a velocity formed with an average of $\underset{\sim}{\partial}\, \underline{u}$ over the whole Ω.

Other possibilities could be considered to define the Reynold's number (for instance one could set: $R = L^{-2} \int_\Omega |\underset{\sim}{\partial}\underline{u}| d\underline{\xi}$) but, as we shall see, the above (1.3) is a good one as it allows us to make mathematically rigorous many statements formulated by using the concept of Reynold's number and usually believed to be heuristically true.

The (KO) length scale, frequency and number of degrees of freedom can be likewise given a precise definition in terms of the quantities in (1.3):

$$L_k = \frac{L}{R^{3/4}} \ , \quad \nu_k = \frac{V_1 + W_1}{L} R^{3/4} \ , \quad N_k = \frac{4\pi}{3} R^{9/4} \tag{1.4}$$

(see [LL], §31,§32, for a concise exposition of (KO)).

The mathematical problems of (1.1) are better illustrated if one uses the following auxiliary "regularized" equations:

$$\dot{\underline{u}} + \langle \underset{\sim}{u} \rangle_l \cdot \underset{\sim}{\partial}\, \underline{u} = \nu \Delta \underline{u} - \underline{\partial} p + \underline{f} + \text{supplementary and initial conditions as in (1.1)} \tag{1.5}$$

where the field $\underline{u}$ is transported by the field:

$$\langle \underset{\sim}{u} \rangle_l(\underline{x}) = l^{-3} \int_{R^3} \lambda(l^{-1}(\underline{x} - \underline{y}))\, \underline{u}(\underline{y}))d\underline{y} \tag{1.6}$$

i.e. by an "average velocity" on scale l, rather than by $\underline{u}$ itself. Here $\lambda \in C^\infty(R^3)$, $\lambda \geq 0$, $\int_{R^3} \lambda = 1$, and λ vanishes outside a finite sphere; the field $\underline{u}$ is extended periodically outside Ω.

We call (1.5) the "Leray's regularization" of the Navier-Stokes equations.

Remark:

If $\underline{u}(\underline{x}) = L^{-3/2} \sum_{k\neq 0} \underline{\gamma}_{\underline{k}} e^{i\underline{k}x}$; $\underline{\gamma}_{\underline{k}} = \overline{\underline{\gamma}}_{-\underline{k}}$, $\underline{k} \cdot \underline{\gamma}_{\underline{k}} = 0$, the (1.5) becomes:

$$\dot{\underline{\gamma}}_{\underline{k}} = -\nu \underline{k}^2 \underline{\gamma}_{\underline{k}} + \underline{f}_{\underline{k}} - iL^{-3/2} \sum_{\underline{k}_1+\underline{k}_2=\underline{k}} \hat{\lambda}(l\underline{k}_1)\, \underset{\sim}{\gamma}_{\underline{k}_1} \cdot \underset{\sim}{k}_2 \mathbf{\Pi}_{\underline{k}_2} \underline{\gamma}_{\underline{k}_2} \tag{1.7}$$

where $(\mathbf{\Pi}_{\underline{k}})_{ij} = (\delta_{ij} - k_i k_j \underline{k}^{-2})$ is the projection operator on the plane orthogonal to $\underline{k}$, and the pressure is given in terms of $\underline{u}$ by:

$$p = -\sum_{i,j=1}^{3} \Delta^{-1} \partial_i \partial_j (u_i u_j) = -L^{-3} \sum_{i,j} \frac{k_i k_j}{\underline{k}^2} e^{i\underline{k}\cdot\underline{x}} \sum_{\underline{k}_1+\underline{k}_2=\underline{k}} \gamma_{\underline{k}_1 i} \gamma_{\underline{k}_2 j} \tag{1.8}$$

The (1.7) is "different" from the "sharp momentum cut off" NS equation with cut off at l^{-1}: the latter corresponds to setting $\hat{\lambda} \equiv 1$ if $|\underline{k}_1| \leq l^{-1}$ and $\hat{\lambda} \equiv 0$ if $|\underline{k}| > l^{-1}$ and this means that λ is oscillating and slowly decaying so that in particular the key inequality:

$$|\langle \underset{\sim}{u} \rangle_l(\underline{x})| \leq \max_{\underline{y}} |\underset{\sim}{u}(\underline{y})| \qquad \text{for all } l \tag{1.9}$$

does not necessarily hold.

The main local result of Leray is the following:

THEOREM 1:

1) *Equation (1.5), i.e. the regularized NS equation on scale $l > 0$, with initial data $\underline{u}_0 \in C^\infty(\Omega)$, and stationary force $\underline{f} \in C^\infty(\Omega)$ admits a C^∞ solution, denoted $\underline{u}^l$, for all times t and verifies, for all $T_0 > 0$, the following "energy" and "vorticity" bounds:*

$$\int_\Omega |\underline{u}^l(\underline{x},t)|^2 d\underline{x} \leq \overline{E}_0 \;, \quad \nu \int_{[0,T_0]\times\Omega} d\vartheta\, d\underline{\xi}\, |\underset{\sim}{\partial u}|^2 \leq 2\overline{E}_0 + T_0 \sqrt{\overline{E}_0 F_0} \equiv \nu T_0 S(T_0) \tag{1.10}$$

with $\overline{E}_0 = \max\left(E_0, \left(\frac{L}{2\pi}\right)^2 \frac{F_0^{1/2}}{\nu}\right)$, $F_0 = \int_\Omega |\underline{f}|^2$, and $S(T_0)$ can be bounded in terms of the initial Reynolds number R_0 (e.g. $S(T_0) \leq \frac{TV^3}{2\pi^2} R_0 \left(R_0^2 + \left(\frac{L}{T_0 V}\right)^2\right)$, se (1.3)).

2) *There exist dimensionless constants $\mathcal{L}$, $\mathcal{L}_{pq}$ such that:*

$$|\partial_t^p \underline{\partial}^q \underset{\sim}{u}^l| \leq \mathcal{L}_{pq} V R \left(\frac{t}{T}\right)^{-\frac{1}{4}-p-2q}, \quad \text{for all } 0 < \frac{t}{T} \leq \mathcal{L} R^{-4} \tag{1.11}$$

where V, T are the characteristic scales in (1.3).

3) *The limit $\underline{u}^0$ of u^l, and of all its derivatives, as $l \to 0$, exists uniformly for $0 \le t \le T\mathcal{L}R^{-4}$ and defines a C^∞ solution of 3DNS.*

4) *If $\underline{u}_0$ is only assumed to be such that $\|\underline{u}_0\|_2 < \infty$, $\|\underset{\sim}{\partial}\underline{u}_0\|_2 < \infty$ the regularized equation admits a solution verifying:*

$$
\begin{array}{llll}
\text{a)} & \textit{bounds (1.11) for } t > 0 & l \ge 0 & \\
\text{b)} & \displaystyle\int_0^{T_0} \|\underline{u}^l(t)\|_2^2 dt < b(T_0) & l \ge 0 & \qquad (1.12)\\
\text{c)} & \|\underline{u}^l(t) - \underline{u}_0\|_2 \xrightarrow[t\to 0]{} 0 & l \ge 0 &
\end{array}
$$

where $\underline{u}^0$ means $\lim_{l\to 0} \underline{u}^l$. For $l = 0$, $\underline{u}$ is the only weak solution enjoying a, b, c.

Remarks:

1) This means the existence of global smooth solutions for the regularized equation with global l-uniform bounds on the energy and on the energy dissipation.

2) And existence of local smooth solutions for the NS regularized equations.

3) And existence of local smooth solution *even for initial data which are not smooth* but just such that the initial dissipation, or "initial Reynolds number", is finite. Such initial data are "regularized" (as an effect of the viscosity) at positive time as the bounds (1.11) show.

4) The divergence of (1.11) as $t \to 0$ just reflects the (obvious) fact that smoothing can only occur at $t > 0$. If one wanted more uniform bounds one just could not obtain them in terms of R alone. In fact Leray proved a large number of corollaries of the technique of proof that he used: and among them a bound uniform as $t \to \infty$ of $\underline{u}^l$, in terms of R *as well as* of the maximum of $\underline{u}_0$ and $\underline{f}$, see the next remark.

5) If one added the assumptions that V_1 and $\max_{\underline{x}} |\underline{u}(\underline{x}_0)|$ are small enough then one could prove l-uniform bounds on $\underline{u}^l$ and its derivatives for *all* times. This yields a global existence, uniqueness and regularity theorem for (1.1). The drawback is that the conditions are so stringent (*i.e.* the dimensionless constants appearing in the bounds so big) that, whenever they hold, one would consider the fluid to be in a "very laminar" regime, hence uninteresting.

6) The proof of the above theorem is very simple. Such a theorem cannot, however, be proved with a sharp cut off in momentum, see comment to (1.7), because the validity of the inequality (1.9) plays a key role.

7) The Leray's cut off equations as well as the sharp cut off ones have solutions still verifying the energy conservation; this leads quite easily to (1.10). But *neither* verifies, in the 0 viscosity limit, the vorticity conservation laws formally valid for (1.1). We refer here to the vorticity, transported by a closed contour $\gamma(t)$ moving with the velocity field, $\Gamma(\gamma(t)) = \oint_{\gamma(t)} \underline{u} \cdot d\underline{l}$ which is a constant in the 0 viscosity limit: "Thomson's circulation theorem".

This is an extremely important point as no regularization appears to be known allowing a proof of the above theorem 1 and preserving the above circulation theorem.

8) The second *a priori* bound in (1.10) can be written as a property of the "instantaneous Reynolds number", $R(t)=(V_1(t)^2+W_1^2)^{1/2}/V$. One finds:

$$\begin{aligned}&\int_0^{T_0}\frac{d\vartheta}{T_0}R^2(\vartheta)\equiv\int_0^{T_0}\frac{d\vartheta}{T_0}\,(L^{-1}\int_\Omega|\partial\underset{\sim}{\underline{u}}|^2d^3\underline{\xi}+W_1^2)V^{-2}\le\\&\le\left(W_1^2+\left(\frac{2\overline{E}_0}{T_0}+\sqrt{\overline{E}_0F_0}\right)\frac{1}{L\nu}\right)\frac{1}{V^2}\equiv B^2<\infty\\&B^2\le\frac{1}{2\pi^2}R_0\left(R_0^2+(\frac{L}{VT_0})^2\right)\overset{\rm def}{=}R_0^2B_0^2\end{aligned}\tag{1.13}$$

directly from the (1.10), bounding everything in term of R for simplicity.

9) The above theorem is not as well known as its soft, "abstract nonsense", consequence which we proceed to discuss, after setting up a formal definition.

The following definition is so natural that some might be led to cede to the temptation of eliminating the qualifier "weak":

DEFINITION: *A "weak solution" of NS with initial condition $\underline{u}_0\in L_2(\Omega)$ is a solution of (1.7) with $\hat\lambda=1$ and with finite energy. In other words it is a sequence of functions $t\to\underline{\gamma}_{\underline{k}}(t)$, differentiable almost everywhere in t, and such that;*

$$\begin{aligned}&\underline{\gamma}_{\underline{0}}=0\;,\;\underline{\gamma}_{\underline{k}}(t)\equiv\overline{\underline{\gamma}_{-\underline{k}}(t)}\;,\quad\sum_{\underline{k}\ne\underline{0}}|\underline{\gamma}_{\underline{k}}(t)|^2\le\overline{E}_0\;,\quad\underline{k}\cdot\underline{\gamma}_{\underline{k}}\equiv0\\&\dot{\underline{\gamma}}_{k}=-\nu\underline{k}^2\underline{\gamma}_{\underline{k}}-iL^{-3/2}\sum_{\underline{k}_1+\underline{k}_2=\underline{k}}\underset{\sim}{\gamma}_{\underline{k}_1}\cdot\underset{\sim}{k}_2\,\mathbf{\Pi}_{\underline{k}}\underset{\sim}{\gamma}_{\underline{k}_2}+\underline{f}_{\underline{k}}\\&\underline{\gamma}_{\underline{k}}(0)\equiv\underline{\gamma}_{0\underline{k}}=\;\textit{Fourier transform of }\underline{u}_0\end{aligned}\tag{1.14}$$

for some $\overline{E}_0>0$.

Remarks:

1) We note that, if $\overline{E}_0<\infty$, the r.h.s. of the second of (1.14) makes sense and is bounded, (by using the Schwartz inequality) by:

$$\nu\underline{k}^2\sqrt{\overline{E}_0}+|\underline{k}|\,\overline{E}_0+\sqrt{F_0}\tag{1.15}$$

as one can see because $\underset{\sim}{k}_2\equiv\underset{\sim}{k}-\underset{\sim}{k}_1$ can be replaced by $\underset{\sim}{k}$ in $\underset{\sim}{\gamma}_{\underline{k}_1}\cdot\underset{\sim}{k}_2$ (by the incompressibility relation $\underline{k}\cdot\underline{\gamma}_{\underline{k}}=0$) so that (1.15) follows by bounding the transport term via the Schwartz inequality. Hence it always makes sense to write the NS equations, whenever an energy bound is assumed, in the weak form.

2) The remark 1) is not sufficient (hence the name "weak") to write the equations verified by a weak solution in the form (1.1) because the finiteness of the kinetic energy (or even of the dissipation) does not even imply the (pointwise) existence of the derivatives appearing in the equations.

3) In the case $\underline{f} = \underline{0}$ one expects that the fluid will slow down and stop, because of the dissipation. In fact (1.14) imply, easily:

$$||\underline{u}(t)||_2^2 \le e^{-\nu(2\pi L^{-1})^2 t} ||\underline{u}_0||_2^2 \tag{1.16}$$

Such a "physically obvious" result is not known to hold if $L = \infty$, *i.e.* in the case of a fluid filling the whole space: it is therefore a sign of how little we know about the solutions of the NS equations.

4) It is then easy to see that the regularized equation in weak form and with $\underline{u}_0 \in C^\infty(\Omega)$, or more generally with finite intial Reynolds number (*i.e.* $\int_\Omega |\underset{\sim}{\partial}\,\underline{u}|^2 < \infty$) admits a solution verifying theorem 1, part 1) or 3) (not 2) which is not as soft: see [L] or [G] for a proof). This implies that $\gamma^l_{\underline{k}}(t)$ exist and verify (1.7) and, hence, (1.15) (which also follows from (1.7) replacing $\hat\lambda$ by 1). But this means that $\gamma^l_{\underline{k}}(t)$ form a family of equicontinous equi-bounded functions (in the parameter l). Hence trivially there is a convergent subsequence converging to some $\underline{\gamma^0_{\underline{k}}}(t)$.

The vorticity bound can be used to check that, indeed, such a limit verifies (1.11) and to deduce the following:

THEOREM 2 (Leray):

Under the same assumptions of theorem 1 the 3DNS admits a weak solution, limit of regularized Leray solutions and global in time.

Remarks:

1) The "abstract nonsense" proof of theorem 2 is quite easy (see [L] or [G]). It is so soft that it does not even distinguish the Leray regularization from the sharp cut off regularization and, hence, weak solutions could also be constructed as limits of the sharp cut off regularization solutions.

2) Thus we have an "existence theorem": but totally useless and illusory. In fact *no* prescription has been provided to identify the convergent subsequence. This is a deep problem of principle as one *just cannot* write a computer program apt to tabulate the subsequence nor to compute $\underline{\gamma^0_{\underline{k}}}(t)$ for fixed $t > 0$ and with a prefixed precision ε (for $t \gg \mathcal{L}TR^{-4}$, (unless the initial data and external force are so small and regular to verify the laminarity conditions for global existence, see remark 5) to theorem 1, of course).

3) Another bad aspect of weak solutions is that the energy balance is no longer automatic. By multiplying both sides of (1.1) by $\varphi\underline{u}$, where $\varphi \ge 0$ is an arbitrary C^∞ function

of $\underline{x}$, t for $\underline{x} \in \Omega$, $t \in (0, s)$, one finds:

$$\begin{aligned}
&\frac{1}{2}\int_\Omega d\underline{\xi}|\underline{u}(\underline{\xi},t)|^2\varphi(\underline{x},s) + \nu\int_{t\le s}\int_\Omega \varphi(x,t)|\underset{\sim}{\underline{\partial}}\underline{u}|^2 d\underline{x}dt = \\
&= \int_{t\le s}\int_\Omega \Big[\frac{1}{2}(\varphi_t + \Delta\varphi)|\underline{u}|^2 + |\underline{u}|^2\langle\underset{\sim}{u}\rangle_l\cdot\underset{\sim}{\partial}\varphi + p^l\underline{u}\cdot\underline{\partial}\varphi + \underline{f}\cdot\underline{u}\varphi\Big]\, dt\, d\underline{\xi}
\end{aligned}\tag{1.17}$$

at least formally, if p^l is the pressure of the approximate solution given by the integral: $p^l = -\sum_{ij}\Delta^{-1}\partial_i\partial_j(u_i^l u_j^l)$.

But in general the interchanges of integrals and limits will not be allowed if the solution is just a weak solution: with the consequence that a weak solution will not necessarily obey the non regularized version of (1.17) (with $\underset{\sim}{u}$ instead of $\langle\underset{\sim}{u}\rangle_l$). It might, however, verify it with $\le$ replacing $=$: the latter form of (1.17) is called the "energy inequality". Thus we say:

DEFINITION: *A Leray solution of (1.1) is a weak solution of (1.1) verifying also the energy inequality defined (for all $\varphi \in C^\infty(\Omega\times(0,s])$, $\varphi(\underline{x},0) = 0$, $\varphi \ge 0$, and for all $s > 0$) by:*

$$\begin{aligned}
&\frac{1}{2}\int d\underline{\xi}\,|\underline{u}(\underline{\xi},s)|^2\varphi(\underline{x},s) + \nu\int_{t\le s}\varphi(x,t)|\underset{\sim}{\underline{\partial}}\underline{u}|^2 d\underline{x}dt \le \\
&\le \int_{t\le s}\int_\Omega \Big[\frac{1}{2}(\varphi_t + \Delta\varphi)|\underline{u}|^2 + |\underline{u}|^2(\underset{\sim}{u}\cdot\underset{\sim}{\partial}\varphi + p^0\,\underset{\sim}{u}\cdot\underset{\sim}{\partial}\varphi + \underline{f}\cdot\underline{u}\,\varphi)\Big]\, d\underline{x}\, dt
\end{aligned}\tag{1.18}$$

if p^0 is the pressure of the weak solution: $p^0 = -\sum_{ij}\Delta^{-1}\partial_i\partial_j(u_iu_j)$.

Remarks:

1) If φ is chosen to be the characteristic function of a sphere $|\underline{x} - \underline{x}_0| < r$ times a characteristic function of a time interval $0 < \varepsilon \le t \le s$ then, if $\underline{u}$ is smooth, one easily finds that (1.18) means that the variation of the kinetic energy in the sphere does not exceed the balance of 1) energy dissipated by friction inside, 2) energy produced inside by the work of the external force, 3) energy flowing out of the box by transport and 4) energy put into the box by the work of the pressure. A general φ just allows us to consider energy balances in more general boxes, even with moving boundaries.

2) The above inequality says that, by considering "weak" solutions of Leray's type, we may introduce some uncontrolled dissipation, but at least no "uncontrolled" creation of energy. Hence Leray's solution have at least the physical property of not making it possible to use them for the purpose of creating a perpetuum mobile. This is not a joke as it seems that for Euler's equations, in fact, one can exhibit solutions "producing" energy (*i.e.* verifying the energy inequality with the $>$ sign, from time to time), (Scheffer).

The most well known theorem of Leray, on the above weak solutions, is:

THEOREM 3:

One can find a weak solution of the (1.1), i.e. a solution to (1.14), such that:

1) *It is a limit, as $l \to 0$, of a subsequence of solutions of the Leray's regularized NS equation, (1.7).*
2) $\underset{\sim}{\partial}\underline{u}(t) \in L_2(\Omega \times [0, T_0])$ *(for all T_0) and verifies the second of (1.10) as well as (1.18);*
3) *For almost all t's the velocity $\underline{u}(t, \underline{x})$ is C^∞ (in t and $\underline{x}$) to the right of t for a short enough (t-dependent) time.*

Remarks:

1) In fact all the weak limits of sequences of solutions of the Leray's regularized equations are "Leray' solutions", *i.e.* they verify the above properties 1,2 of theorem 3.

2) Apart from some logical exercises necessary to deal with "subsequences" (*i.e.* with non constructed objects, which therefore in some sense do not really exist), the part of the theorem not dealing with the energy inequality is a trivial consequence of the vorticity bound in (1.10). Recalling that the Reynolds number R is $(V_1^2 + W_1^2)^{1/2}/V$, see (1.3), can be written in the form (1.13):

$$\int_0^{T_0} R(t)^2 dt < \infty \quad \text{for all } T_0 > 0 \tag{1.19}$$

where $R(t) = (V_1(t)^2 + W_1^2)^{1/2}/V$.

Hence, heuristically, (1.10) implies that for almost all times the Reynold's number is finite: $R(t) < \infty$ and the (constructive part of) Leray's theorem (points 1,2), implies that $\underline{u}(t)$ is C^∞ for $t \in (t_0, t_0 + \mathcal{L}TR(t_0)^{-4})$ for any t_0 for which $R(t_0) < \infty$; *i.e.* for almost all t's. See below for a more precise analysis.

3) The reason why the Leray's approximations play a special role is that theorems 1,2 above hold only for Leray's regularized solutions: therefore the weak solution in the above theorem has to be constructed from Leray's method (not a serious limitation as, in any case, we are talking of objects that, in principle, cannot be constructed). This might be confusing as even the weak solutions obtained as limits of the equations regularized with the sharp cut off verify (1.14): but in general they might not have the other properties.

4) The difficulty in making rigorous the heuristic argument in remark 1 is that, strictly speaking, from the above argument we do not know that the weak solution after a time t_0 when $R(t_0) < \infty$ coincides with the smooth solution with the same $\underline{u}(t_0)$ constructed by Leray's local theorem. An abstract argument is necessary, however, to insure this property: see remark 1) after the proof of theorem 3' below. Such a delicate point arises simply because we have essentially lost control on the construction of solutions and we are proceeding on abstract grounds (in the axiom of choice realm) where every subtlelty becomes possible and important.

5) The proof of (1.18) follows from (1.17) by passage to the limit on a convergent subsequence $\underline{u}^{l_j}$. The l.h.s. of (1.17) is (obviously, being a norm) upper semicontinous if $\underline{u}^{l_j} \to \underline{u}^0$ weakly (*i.e.* componentwise in the Fourier transform components). And the r.h.s. is in general continous: hence the inequality (1.18). The latter continuity is a non trivial but standard consequence of two important inequalities that will play a major role below, namely the Sobolev and the Calderon- Zygmund inequalities, see (2.10), (2.11) below. The proof of the continuity property is quite interesting as it shows a few general regularity properties of the weak solutions of the NS equations and it is reported in appendix B, for completeness.

Most reasonable people will, *however*, refuse to talk about properties of solutions for which no constructive algorithm is provided. For such people one can just avoid introducing the theorems 2, 3 above and formulate them as uniform properties of the approximating solutions.

For instance theorem 3 can be rephrased in the apparently less elegant but far clearer form:

THEOREM 3':

For all $l > 0$, $T_0 > 0$ the solution $\underline{u}^l$ of Leray's regularized equation with initial Reynold's number R_0 and the initial quantity B_0^2 (see [1.11]), verifies:

$$|\partial_t^p \underline{\partial}^q \underline{u}^l(\underline{x}, t)| \le \mathcal{L}_{pq} \frac{R_0 V n}{((t - t_0)/T)^{1/4+p+q/2}}, \quad 0 \le t - t_0 \le \frac{\mathcal{L} T}{n^4} R_0^{-4} \tag{1.20}$$

for a set Δ_n of times $t_0 \le T_0$ with complement in $[0, T_0]$ with measure:

$$\int_{\Delta_n} \frac{d\tau}{T_0} \le \frac{B_0}{n^2} \tag{1.21}$$

Remarks:

1) This is much better; it shows that the set of times where the solution does not exceed by a factor n the bounds it verifies in the happy initial interval (where it can be really constructed by theorem 1, part 2)) has a large measure as $n \to \infty$

2) The above form has the advantage of making it clearer what we still do not understand: no information is given about the location of Δ_n, (*it is an open problem*).

3) It also shows clearly that the Leray's solutions play a privileged role.

They are such that the boundedness of the Reynolds number implies a bound on $\underline{u}$ and its derivatives. A property which is not proved to hold for weak solutions constructed starting with the sharp cut off equations.

4) The proof of theorem 3' is trivial and this shows that the aura of mistery surrounding theorem 3 is entirely due to the logical problems which usually accompany any use the axiom of choice (to construct evanescent converging subsequences). Here is the proof:

Proof (of theorem 3'): From theorem 1, (1.10), (1.13) we see that:

$$\int_0^{T_0} R_l^2(\tau)\frac{d\tau}{T_0} \equiv \int_0^{T_0}\frac{d\tau}{T_0}\left(\int_\Omega |\underset{\sim}{\partial}\underline{u}^l|^2\frac{d^3\underline{x}}{L}+W_1^2\right)\frac{1}{V^2} \le R_l^2 B_0^2 \tag{1.22}$$

hence the set Δ_n of the points τ where $R_l(\tau) < nR_0$ has complement with measure $\le B_0/n^2$. If $t_0 \in \Delta_n$ then, by theorems 1, 2, the (1.20) holds! Had we used sharp cut off regularizations the above argument would not have worked (because 2) of theorem 1 might not hold).

Remarks:

1) The above proof is essentially all there is to say to get theorem 3 as well. In fact to prove theorem 3 it will be sufficient, if $R_l(t)$ is the Reynold's number of $\underline{u}^l(t)$, to interpret $R(t)^2$ as $R(t)^2 = \liminf_{l\to 0} R_l(t)^2$ and note that $R(t)^2 \ge \int_\Omega |\underset{\sim}{\partial}\,\underline{u}(t)|^2$ (by Fatou's lemma).

One has, however, to worry about the following problem: how can one be sure that, if $R(\tau)^2 < \infty$, the weak solution limit of the approximating $\underline{u}^l$ as $l \to 0$ is in fact converging to the strong (*i.e.* C^∞- smooth) solution existing in the interval $(\tau, \tau + \mathcal{L}TR(\tau)^{-4})$? for completeness we discuss this (minor) point in appendix A.

We conclude the introduction with an important theorem, corollary of the proof of Leray's theorem:

THEOREM 4' (Serrin):

Let $t \to \underline{u}^l(\underline{x}, t)$ be a solution of the Leray's regularized equations (1.5), (1.7). Then if in the r-vicinity of $(\underline{x}, t_0)$ it is $|u^l(\underline{x}, t) \le C$ (i.e. this holds for $|\underline{x} - \underline{x}_0| < r$, $|t - t_0|\nu < r^2$) then in a $r/2$-vicinity of $(\underline{x}_0, t_0)$, u^l can be bounded togheter with all its derivatives by l-independent (computable) functions of C alone.

For a proof see [L] or [G]. This theorem has the abstract nonsense companion:

THEOREM 4:

If $t \to \underline{u}(\underline{u}, t)$ is a Leray's solution of (1.1), see theorem 3, and if (apart from a 0 measure set of $(\underline{x}, t)$'s):

$$|\underline{u}(\underline{x}, t)| \le C \quad \textit{for} \quad (\underline{x}, t) \in \{(\underline{x}, t)|\ |\underline{x} - \underline{x}_0| < r,\ \nu|t - t_0| < r^2\} = Q_r \tag{1.23}$$

then $\underline{u}$ is in $C^\infty(Q_{r/2})$.

The above theorem 4' says that a (weak) Leray's solution of 3DNS can only be singular near points where the velocity diverges. Or "the singularities of 3DNS correspond to velocity divergences": they cannot come from singularities in the velocity gradient alone. "No shock waves" are possible.

§2 Scheffer's theory of the singularities fractal dimension.

After Leray's work no substantial progress has been made towards the construction of solutions (weak or not) with periodic boundary conditions. At the cost of a repetition let me stress once more that a construction of a solution (weak or not) is also a uniqueness theorem; and no such thing exists (even for weak solutions). For non periodic boundary conditions, of course, the situation is no better.

Progress has, instead, been achieved in showing that a singularity in a solution cannot be as serious as one could fear. Such results must be viewed as steps towards a proof of *non existence* of singularities in 3DNS, or more concretely as intermediate steps towards the construction of a solution algorithm.

Of course we do not know whether singularities exist or not: but *there is not the least evidence* for them, so far.

A first step is the theorem 3 (or 3') of Leray showing that singularities, if existent, can occurr at most on a set of times of zero measure. For instance the solution cannot be singular on a line in space time. It could, however, be singular on a set of times which is countable or even uncountable like a Cantor set.

We would like to exclude also the above types of singularities as well as singularities occurring at fixed time in various points of Ω.

The notion of Hausdorff dimension is useful at this point. We consider in fact the set S_0 of times at which a Leray's solution is singular and the set S of $(\underline{x}, t)$'s at which, in space time, the solution is singular. By the above theorem of Serrin (theorem 4);

$$
\begin{aligned}
S_0 &= \{t \big| \, 0 \le t < \infty, \quad \text{such that } \sup_{\Omega} |\underline{u}(\underline{x}, t)| \text{ is unbounded near } t\} \\
S &= \{(t, \underline{x}) \big| \, 0 < t < \infty, \, \underline{x} \in \Omega \quad \text{such that } |\underline{u}(\underline{x}', t')| \text{ is unbounded near } \underline{x}, t\} \ .
\end{aligned}
\tag{2.1}
$$

We know already that $\int_{S_0} dt = 0$, $\int_S d\underline{x} dt = 0$.

Let us recall the notion of *Hausdorff dimension* and *Hausdorff measure* of a set $A \subset R^d$. Given $\alpha > 0$, let $\delta > 0$ and consider the coverings C_δ of A with sets F with positive diameter $\le \delta$. A "best covering" is one which minimizes $\sum_{F \in C_\delta}$ (diameter $F)^\alpha$ and one sets:

$$
\mu_\alpha(A) = \lim_{\delta \to 0} \inf_{C_\delta} \sum_{F \in C_\delta} (\text{diameter } F)^\alpha
\tag{2.2}
$$

It can be shown that the limit exists and for any α the function $A \to \mu_\alpha(A)$ is countably additive on the smallest family $\mathcal{B}$ of sets containing the closed sets and invariant with respect to the operations of complement and countable union, see [DS], III.9.47.

It is easy to check that given $A \in \mathcal{B}$ there is $\alpha_c \ge 0$ such that:

$$
\begin{aligned}
\mu_\alpha(A) &= +\infty \quad \text{if} \quad \alpha < \alpha_c \\
\mu_\alpha(A) &= 0 \qquad \text{if} \quad \alpha > \alpha_c
\end{aligned}
\tag{2.3}
$$

and α_c is called the "Hausdorff dimension", while, $\mu_{\alpha_c}(A)$ is the "Hausdorff measure".

The dimension of a smooth k- dimensional surface is $\alpha_c = k$. The subsets of the interval $[0,1]$ consisting of the points which, when expanded in base $q = 2,3\ldots$, do *not* contain the digit 1 (such sets are the "Cantor sets" C_q) have Hausdorff dimension $\alpha_c = \log_q(q-1)$ and α_c-Hausdorff measure 1, as well as length zero (*i.e.* zero Lebesgue measure).

There are sets C in $[0,1]$ with Hausdorff dimension 1 but zero length as well as zero Hausdorff measure: *e.g.* $C = \bigcup_{q\geq 3} C_q$. And, therefore, the Hausdorff dimension and measure can provide interesting properties of zero measure sets.

A simple, but conceptually important for the methods set up by the proof, theorem is the following:

THEOREM 5, (Scheffer):

The Hausdorff dimension of S_0, the time singularity set of a Leray's solution of Navier Stokes equation, is $\alpha_c \leq \frac{1}{2}$ and $\mu_{1/2}(S) = 0$.

Remarks:

1) The proof is given below as it illustrates a general method used to convert *a priori* estimates into Hausdorff dimension estimates. The very same method will be used, later, to study the space time singularities.

2) Clearly we are very far from $\alpha_c = 0$ (if true).

3) Of course one could formulate theorem 5 without talking of the non algorithmically constructible Leray's solutions. It would become a theorem on Leray's approximations. We regard theorem 5 just as a short way to formulate such results; the results on the approximations can be easily set up from the following proof, but we omit the discussion (bowing to social pressure, requiring easy to remember, possibly shocking, statements).

4) To simplify the situation we take, below, $\underline{f} = \underline{0}$: nothing is really different when $\underline{f} \neq \underline{0}$.

Proof: Let $t \in [0, T_0]$ (T_0 being an arbitrary length of time), and suppose that for some $\sigma > 0$:

$$\left(\frac{\sigma}{T}\right)^{-\alpha} \int_{t-\sigma}^{t} d\vartheta R(\vartheta)^2 < A \tag{2.4}$$

where $R(t)$ is Reynold's number defined in (1.3), verifying the *a priori* estimate (1.13); and T is the natural time unit defined in (1.3).

The (2.4) can be rewritten $\int_{t-\sigma}^{t} \frac{\delta\vartheta}{\sigma} R(\vartheta)^2 < \frac{A}{T}\left(\frac{\sigma}{T}\right)^{-(1-\alpha)}$; so that it implies that there is at least one $\vartheta \in (t-\sigma, t)$ such that $R(\vartheta)^2 < \frac{A}{T}\left(\frac{\sigma}{T}\right)^{-(1-\alpha)}$.

Thus from Leray's local theorem $\underline{u}$ has to be bounded in an interval of $(\vartheta, \vartheta+\delta)$ with $\delta = \mathcal{L}TR(\vartheta)^{-4} \geq \mathcal{L}T\left(\frac{T}{A}\right)^2 \left(\frac{\sigma}{T}\right)^{2(1-\alpha)}$.

Hence if $\alpha = \frac{1}{2}$ and $\left(\frac{A}{T}\right)^2 \geq \mathcal{L}$ the interval has lenght $\delta \geq \sigma$ and, therefore, it covers the point t. Thus $\underline{u}$ is C^∞ in the vicinity of t (and any $\underline{x}$).

We see that the singularity set S_0 can be covered with intervals $F=(t-\sigma,t]$ of diameter σ smaller than any profixed $\delta_0>0$ and such that (2.4) with $A=\mathcal{L}^{1/2}T$ is violated, *i.e.*:

$$\limsup_{\sigma\to 0}\left(\frac{\sigma}{T}\right)^{-1/2}\int_{t-\sigma}^{t} d\vartheta R(\vartheta)^2\ge T\mathcal{L}^{1/2} \tag{2.5}$$

otherwise t would be a "regularity time".

We fix $\delta_0>0$ and consider the covering $\mathcal{F}$ of S_0 with *all* the intervals F violating (2.4). By (2.5) *every point of S_0 is covered by such intervals with arbitrarily small diameter.*

In the latter situation a simple "covering theorem" (by Vitali) provides the existence of a countable family of *pairwise disjoint* elements of $\mathcal{F}$, denoted F_1, $F_2,\ldots$, such that the family $4F_1$, $4F_2,\ldots$ of the intervals, obtained from $F_1,F_2,\ldots$ by dilating each by a factor 4 about its center, is still a covering of S_0.

We have, therefore, a covering $\{4F_j\}_{j=1}$ of S_0 with sets whose diameter is $\le 4\delta_0$. Then (2.5) implies:

$$\begin{aligned}\sum_j\left(\frac{4}{T}\text{ diameter }F_j\right)^{1/2}&=2\sum_j\left(\frac{\text{diam}F_j}{T}\right)^{1/2}\le \\ \le\frac{2}{T\mathcal{L}^{1/2}}\sum_j\int_{F_j}R(\vartheta)^2d\vartheta&\le\frac{2}{\mathcal{L}^{1/2}T}\int_{S_0^{4\delta_0}\cap[0,T_0+4\delta_0]}R(\vartheta)^2d\vartheta<\infty\end{aligned} \tag{2.6}$$

where $S_0^{4\delta_0}$ denotes the set of points within a distance $4\delta_0$ from S_0, and because we have the estimate (1.13) for any T_0. This shows that $\alpha_i(S_0)\le\frac{1}{2}$.

We could even replace $S_0^{4\delta_0}$ with an open set G_{δ_0} containing S_0 and *with measure* within, say, $4\delta_0$ from that of S_0 (which is zero): in fact we could consider an open set with measure $4\delta_0$ containing S_0 and use as covering the sets $4F_j\cap G_{\delta_0}$. And we see that the last integral in (2.6) could be replaced by the integral to the set $G_{\delta_0}\cap[0,T_0]$ which, letting $\delta_0\to 0$, converges to the integral over $S_0\cap[0,T_0]$ of $R(\vartheta)^2$, which is zero because S_0 has zero measure. Hence $\mu_{1/2}(S_0)=0$.

The analysis of the set S of the space time singularities is much more interesting. The main result of Scheffer is a new local regularity theorem, which we formulate in terms of a family of "observables" or "operators" describing the local properties of $\underline{u}$ relevant for our singularity theory.

The attention of the reader should be focused on the analogy of what follows with the "renormalization group" methods for the analysis of a singularity. In fact the reader familiar with the methods can think of the words "operators" and "observables" to have the "same" meaning (see [G2]).

Our operators corresponding to a "box $Q_r(\underline{x},t)$ of scale r" around $(\underline{x},t)$, or "describing an eddie of scale r" with:

$$Q_r(\underline{x},t)=\big\{(\underline{\xi},\vartheta)|\ |\underline{\xi}-\underline{x}|<r,\ t-r^2\nu^{-1}<\vartheta<t\big\}=B_r\times\Delta_r \tag{2.7}$$

where $B_r = \{\underline{\xi} |\ |\underline{\xi} - \underline{x}| < r\}$, $\Delta_r = \{\vartheta |\ t - \frac{r^2}{\nu} < \vartheta < t\}$, will be the following functional of the velocity field in the eddie $Q_r(\underline{x}, t)$:

$$\begin{aligned}
A_r &= \max_{\vartheta \in \Delta_r} \frac{1}{\nu^2 r} \int_{B_r} |\underline{u}(\underline{\xi}, \vartheta)|^2 d^3\underline{\xi}, \quad \delta_r = \frac{1}{\nu^2 r} \int_{Q_r} d\vartheta d\underline{\xi}\, |\underset{\sim}{\partial}\, \underline{u}|^2 \\
G_r &= \frac{1}{\nu^2 r^2} \int_{Q_r} d\vartheta d\underline{\xi}\, |\underline{u}|^3, \qquad J_r = \frac{1}{\nu^2 r^2} \int_{Q_r} |\underline{u}|\, |p| d^3\underline{\xi} d\vartheta \\
K_r &= \frac{r^{-13/4}}{\nu^{3/2}} \Big(\int_{\Delta_r} d\vartheta \big(\int_{B_r} |p|\, d\underline{\xi} \big)^{5/4} \Big) \\
S_r &= \nu^{-4/3} r^{-5/3} \int_{Q_r} (|\underline{u}|^{10/3} + |p|^{5/3}) d\underline{\xi} d\vartheta
\end{aligned} \tag{2.8}$$

where the "dimensionless" operators A, δ, G, K, S, J are defined supposing that the pressure p is espressed in terms of $\underline{u}$ via:

$$p = -\int_\Omega \sum_{i,j=1}^{3} \partial_i \partial_j \Delta^{-1}(u_i u_j) \tag{2.9}$$

It is by no means obvious that they have a finite value if $\underline{u}$ is a Leray's solution of the NS equations: this can be proved (see appendix C for details) to be a consequence of the *a priori* bounds (1.10) by making use of the Hölder inequality together with the Calderon–Zygmund inequality, which below we shall call (CZ)-inequality:

$$\int_\Omega |\sum_{i,j} (\Delta^{-1} \partial_i \partial_j)(u_i u_j)|^q d\underline{\xi} \le C_q^L \int_\Omega |\underline{u}|^{2q} d\underline{\xi}\,, \quad 1 < q < \infty \tag{2.10}$$

and the Sobolev inequality, called below (S)-inequality, in the form:

$$\begin{aligned}
\int_{B_r} |\underline{u}|^q d\underline{\xi} \le \tilde{C}_q \Big[\big(\int_{B_r} (\underset{\sim}{\partial} \underline{u})^2 d\underline{\xi} \big)^a \big(\int_{B_r} |\underline{u}|^2 d\underline{\xi} \big)^{\frac{q}{2} - a} + \\
+ r^{-2a} \big(\int_{B_r} |\underline{u}|^2 d\underline{\xi} \big)^{q/2} \Big], \qquad 2 \le q \le 6, \quad a = \frac{3}{4}(q-2)
\end{aligned} \tag{2.11}$$

with the second term in the r.h.s. missing if $\underline{u}$ has zero average over B_r.

The (CZ) inequality holds for $\Omega = R^3$ or for $\Omega = [0, L]^3$ periodic, the (CZ)-inequality can be found in [SW], p. 225, while the (S)-inequality is, for $q = 6$, the theorem 2 or p. 57 of [S] (with $q^* = 6$, $s = n = 4$, $p = 2$, $l = 1$: valid for $q^* = 6$ because $n = s = 3$ and by the footnote (20) on page 43); for the other q's it follows by trivial interpolation between $q^* = 6$ an d $q = 2$.

The (CZ) and (S) inequalities immediately imply (see appendix C) the existence of a constant C_L such that, for any $T_0 > 0$:

$$\int_0^{T_0} \int_\Omega d\vartheta\, d\underline{\xi} (|\underline{u}|^{10/3} + |p|^{5/3}) \le C_L L V^2 E_0^{2/3} T_0 \int_0^{T_0} \frac{d\vartheta}{T_0} R(\vartheta)^2 < \infty \tag{2.12}$$

with the notations (1.3), giving the best known *a priori* bounds on the L_q-norms of $\underline{u}$ and p. It also implies (by Hölder inequality) the finiteness of J_r, K_r, G_r.

In fact (by Hölder inequality, see appendix C), there is a constant $C_0 > 0$ such that:

$$S_r \geq (C_0 J_r),\ (C_0 K_r),\ (C_0 G_r) \tag{2.13}$$

and S_r is obviously bounded because of (2.12). Also A_r, S_r are, obviously finite because of the *a priori* bounds on energy and dissipation (1.10), (1.13).

The basic theorem of Scheffer is the following:

THEOREM 6 (Scheffer):

There exists a dimensionless constant ε_s such that $(G_r + J_r + K_r) < \varepsilon_s$ implies that $\underline{u}$ is regular in the box $Q_{r/2}(\underline{x}, t)$; more precisely $\underline{u}$ is uniformly bounded there; by $C_s \varepsilon_s^{1/3} r^{-1}$ for some dimensionless constant C_s.

Remarks:

1) This theorem is a consequence of $\underline{\partial} \cdot \underline{u} = 0$ and (1.18) with p given by (2.9). No other properties are used. Hence one can say that it is a consequence of the energy conservation alone. In particular no use is made of Thomson's theorem, *i.e.* of the property that in the $\nu = 0$ limit the vorticity of a closed loop transported by the fluid is conserved.

2) Let $Q^*_r(\underline{x}^*, t^*)$ be the box of scale r such that $(\underline{x}, t)$ is the center of $Q_{r/2}(\underline{x}^*, t^*)$: this is the box with $\underline{x} = \underline{x}^*$, $t^* = t + r^2/8\nu$. Denote with a $*$ superscript the (2.8) evaluated in $Q^*_r(\underline{x}^*, t^*)$. Clearly if $(\underline{x}, t) \in S$ it must be $J_r + G_r + K_r \geq \varepsilon_s$ for all $r > 0$: otherwise the above theorem would imply regularity in $Q^*_{r/2}(\underline{x}^*, t^*) \ni (\underline{x}, t)$. Hence $S \geq \varepsilon_s C_0$, or:

$$\int_{Q^*_r} (|\underline{u}|^{10/3} + |\rho|^{5/3}) \geq \varepsilon_s C_0 \nu^{4/3} r^{5/3} \tag{2.14}$$

We can therefore repeat the proof of theorem 5 by replacing the intervals $(t - \sigma, t]$ with the boxes Q^*_r. A covering theorem of the type of Vitali's theorem can be proved also for such sets. This means that there is a omothety constant $\lambda > 0$, such that out of a covering *with boxes $F = Q^*_r$ of S with boxes of arbitrarily small diameter covering each single point of S* one can extract a countable family of pairwise disjoint sets $F_1, F_2, \ldots$ such that diam $F_j \leq \delta_0$, for any prefixed δ_0, and such that $\{\lambda F_j\} \ldots$, is a covering of S. One can check that the omothety (about the center) factor λ can be taken $\lambda = 6$. Therefore we deduce, proceeding as in the proof of theorem 5, that $\alpha_c(S) \leq \frac{5}{3}$ and $\mu_{5/3}(S) = 0$:

COROLLARY 7 (Scheffer):

The singularity set S has $\alpha_c(S) \leq \frac{5}{3}$, $\mu_{5/3}(S) = 0$.

3) This is remarkable as the previous estimate, theorem 5, would have allowed the possibility that, at a given time, *all* the points $\underline{x} \in \Omega$ became singular (*i.e.* a Hausdorff

dimension for S equal to 3) or even worse (up to the formation of a singularity set of dimension 3.5, taking theorem 5 into account).

4) As we shall see this is not "optimal" as far as Hausdorff dimension is concerned.

5) We consider a regularized Leray's solution at scale l, and the set of points $(\underline{x}, t)$ for which one cannot find a small enough vicinity $Q_r^*(\underline{x}, t)$ with $r > \eta_s$ with η_s a suitable function of l/L infinitesimal as $l \to 0$, and such that inside $Q_r^*(\underline{x}, t)$ the velocity $\underline{u}$ *cannot* be bounded by $C_s \varepsilon_s^{1/3} r^{-1}$. Then one can prove that the set S_l of such points has Hausdorff dimension $\alpha_c(S_l) \le \frac{5}{3}$, $\mu_{5/3}(S_l) = 0$. This is a constructive form of corollary 7. The computation of η_s requires some extra work (without new ideas), with respect to the following description of the proof.

The description of the proof of theorem 6 is as follows (see [CKN]).

Given a Leray's solution and a point $(\underline{x}, t)$ we see that if $(\underline{x}, t) \notin S$, *i.e.* if $\underline{u}$ is C^∞ near $(\underline{x}, t)$, then A_r, S_r, G_r, J_r, $K_r \xrightarrow[r \to 0]{} 0$ at speed r^2, r^4, r^3, r^3, $r^{5/2}$, respectively.

Hence we try to show that $A_r \to 0$ at least fast as r^2: this will imply that $\underline{u}(\underline{x}, t)$ is bounded (at least, to be precise, if $(\underline{x}, t)$ is a Lebesgue point* for $\underline{u}$).

Furthermore the argument will be "uniform" for all points $(\underline{x}', t') \in Q_{r/2}(\underline{x}, t)$, hence $\underline{u}(\underline{x}, t)$ will be almost everywhere bounded in $Q_{r/2}(\underline{x}, t)$, (*i.e.* in all Lebesque points of $Q_{r/2}(\underline{x}, t)$), hence C^∞ in the interior of $Q_{r/2}(\underline{x}, t)$ by Serrin's theorem.

Scheffer's idea is that we can infer that $A_r \to 0$ as r^2 simply by showing that A_r verifies a recursive relation linking its value on a certain scale to the values on the "previous scales".

We fix a sequence $r_n = L2^{-n}$ of length scales, and we denote $G_n = G_{r_n}$, $A_n = A_{rn}$, etc.

At this point one can try several paths. We can, for instance, try to find a relation linking A_n, δ_n or, to simplify, $T_n = A_n + \delta_n$ to T_{n-1} and to T_{n_0}, if $r_{n_0} =$ scale where the assumption $G_{n_0} + J_{n_0} + K_{n_0} < \varepsilon_s$ is verified.

This amounts to say, from the renormalization group viewpoint, that we regard $T_n = A_n + \delta_n$ as the "operator" describing the phenomenon (of absence or presence of a singularity in $\underline{x}, t$) and we look for a "beta functional" $\mathcal{B}_n$:

$$T_n = \mathcal{B}_n(T_{n-1}, \ldots, T_{n_0};\ G_{n_0},\ J_{n_0},\ K_{n_0}), \qquad n \le n_0 \tag{2.15}$$

defined on small enough scales (*i.e.* for $n \le n_0$ for some n_0 large enough). The property of theorem 6 should be expressed by the fact that (2.15) implies $T_n \le \bar{\varepsilon} r_n^2$ for all $n \le n_0$ if G_{n_0}, J_{n_0}, K_{n_0} are small enough, and for a suitable $\bar{\varepsilon} > 0$.

The tools to build, or estimate, the beta functional $\mathcal{B}_n$ (assumed to exist) are very limited.

---

* Recall that a Lebesgue point of $f \in L$, (R^3) is a point $\underline{x}$ such that $|B_r^{-1} \int_{B_r} f(\xi) d\underline{\xi} \to f(\underline{x})$ as the radius r of the sphere B_r around $\underline{x}$ tends to 0: this happens necessarily almost everywhere "Lebesgue theorem".

A first tool is to rely on some simple relations between quantities on various scales which are just of "kinematical nature", *i.e.* independent on the fact that $\underline{u}$ verifies NS equations. They are just expressing (S) or (CZ) or Hölder inequalities (denoted (H) below).

For instance (see lemma 3.1 in [CKN]):

$$G_{n+1} \le C G_n \ , \quad A_{n+1} \le C A_n$$

$$G_n \le C(A_n^{3/2} + A_n^{3/4}\delta_n^{3/4}) \le C(A_n+\delta_n)^{3/2} \equiv C T_n^{3/2} \tag{2.16}$$

are either trivial or consequences of (S) inequality on the integration over $\underline{\xi} \in B_n$ followed by (H) inequality for the $\vartheta \in \Delta_n$ integration. Some details on the techniques to obtain such inequalities from the (S) and (CZ) inequalities are illustrated in appendix C). The (2.16) permits us to eliminate, from any relation linking the T_j's, the quantities G_n by bounding them in terms of T_n.

A second tool is to derive relations between T_n and T_j, G_j by using the "dynamical relation" (1.18) expressing the energy conservation: it is however clear that not only G_j but J_j and perhaps K_j as well will appear in any relation linking T_n to $T_{n-1}, \ldots$. Hence the attempt to find a kinematical relation between J_n, K_n and J_j, K_j, $j<n$ leads necessarily to a relation involving many scales, as the pressure is "non local", see (2.9).

If one introduces the average pressure, in the sphere B_n around $\underline{x}$, defined by the integral $p_n(\vartheta) = |B_n|^{-1}\int_{B_n} p(\underline{\xi},\vartheta) d^3\underline{\xi}$ and the auxiliary (dimensionless) "operator":

$$L_n = r_n^{-2}\int_{Qn} |\underline{u}|\ |p-\overline{p}_n(\vartheta)| d\underline{\xi} d\vartheta \tag{2.17}$$

then one finds, easily, that the (CZ), (S), (H) inequalities imply:

$$\begin{aligned} L_n \le C\Big[\Big(\frac{r_{n-1}}{r_{n_0}}\Big)^{7/5} A_{n-1}^{1/5} G_{n-1}^{1/5} K_{n_0}^{4/5} + \Big(\frac{r_{n-1}}{r_{n_0}}\Big)^{5/3} G_{n-1}^{1/3} G_{n_0}^{2/3} + G_{n-1} + \\ + r_{n-1}^3 G_{n-1}^{1/3} \sum_{k=n_0}^{n-1} r_k^{-3} A_k \Big] \ . \end{aligned} \tag{2.18}$$

for any $n \ge n_0 \ge 1$, (see lemma 3.2 in [CKN]).

The energy inequality, togheter with (2.16) gives a relation between T_n and $T_{n_1} \ldots T_{n_0}$, $G_{n-1} \ldots, J_{n_0}$ which, if combined with (2.16), (2.18) becomes, after same algebra:

$$\begin{aligned} T_n \le 2^{-2n} \Big(& 2^{2n_0}\varepsilon^{3/2} + \sum_{n_0+1}^{n-1} 2^{2k} T_k^{3/2} + 2^{7n_0/5}\varepsilon \sum_{k=n_0+1}^{n-1} 2^{-3k/5} T_k^{1/2} + \\ & + \varepsilon 2^{5n_0/3} \sum_{n_0+2}^{n-1} 2^{k/3} T_k^{1/2} + \sum_{n_0+1}^{n-1} 2^{2k} T_k^{3/2} + \varepsilon 2^{2n_0} + \sum_{n_0+2}^{n-1} 2^{-k} T_k^{1/2} + \sum_{p=n_0+1}^{k} 2^{2p} T_p \Big) \end{aligned} \tag{2.19}$$

$$\varepsilon \equiv C \max(G_{n_0}^{2/3}, K_{n_0}^{4/5}, J_{n_0})$$

The above relation immediately implies that, *if ε is small enough*, then the operator T_n is bounded by $T_n \le \varepsilon^{2/3} 2^{-2(n-n_0)}$, which is what we wanted to show, as it implies $r_n^{-2} A_n \le \varepsilon^{2/3} r_{n_0}^{-2}$, *i.e.*:

$$|\underline{u}(\underline{x}, t)| \le \varepsilon^{1/3} r_{n_0}^{-1} \tag{2.20}$$

if $(\underline{x}, t)$ if a Lebesgue point for the function $\underline{u}$.

The argument can be repeated for all $(\underline{x}', t') \in Q_{r/2}(\underline{x}, t)$. In fact it is clear that $Q^1_{r/2} = Q_{r/2}(\underline{x}', t') \subset Q_n(\underline{x}, t)$ and that requiring that $J^1_{r/2} + K^1_{r/2} + G^1_{r/2} < \varepsilon_0$ is implied by requiring the $J_r + K_r + G_r < \varepsilon_0 C_0^{-1}$ with C_0 a large enough dimensionless constant. This shows that (2.20) holds not only for $(\underline{x}, t)$ but for all $(\underline{x}', t') \in Q_{r/2}(\underline{x}, t)$ provided $J_r + K_r + G_r < \varepsilon_s = C_0^{-1}\varepsilon_0$.

But we have mentioned that other strategies could be followed. This leads to the Caffarelli, Kohn, Nirenberg result discussed in §3.

§3 Kolmogorov-Obuchov scale and CKN dimension 1 bound.

The alternative approach mentioned in §3 is more attractive as it is more ambitious and a simple physical meaning can be associated with its results. We start from the physical interpretation. The theory of developed turbulence due to (KO) rests on several assumptions which do not have a precise mathematical formulation, being in some sense of rather qualitative nature.

One of the most important is that it should be possible to define a "local" Reynolds number which is defined on each scale r, *i.e.* it is associated with the velocity field $\underline{u}$ restricted to a box B_r of radius r around (any) selected point.

We gave in §1 a precise definition of Reynolds number. In the situation that we are considering (for simplicity) of absence of external forces $R = \delta u L/\nu \equiv \delta u / V$ is a dimensionless number formed out of the container size L, the viscosity ν and a velocity δu associated with a global average of $\underset{\sim}{\underline{\partial u}}$, giving a characteristic "velocity variation"; the definition was (see (1.3)):

$$R^2 = L^{-1} \int |\underset{\sim}{\underline{\partial\, u}}|^2 d^3\underline{x} \frac{L^2}{\nu^2} \tag{3.1}$$

The reason why this is identified with Reynold's number (usually defined only empirically) is that we have seen that such a quantity plays a key role in the theory and essentially all our bounds were formulated in terms of it.

If we now consider a box B_r we can define the "local" Reynolds number as:

$$R_r^2 = \frac{1}{\nu r} \int_{B_r} |\underset{\sim}{\underline{\partial\, u}}|^2 d^3\underline{x} \frac{r^2}{\nu} \tag{3.2}$$

The above can be interpreted, in a smooth flow as:

$$\delta_r = \frac{1}{\nu^2 r} \int_{Qr} |\underset{\sim}{\underline{\partial u}}|^2 d^3\underline{x} d\vartheta \tag{3.3}$$

where Q_r is $B_r \times \Delta_r$, $\Delta_r = \left(t - \frac{r^2}{2\nu}, t + \frac{r^2}{2\nu}\right)$.

Our claim here is that the "correct" definition of Reynold's number on scale r in the box Q_r is in fact (3.3) (different from (3.2) in non smooth flows or in turbulent eddies of scale r, as in such cases the integral over ϑ cannot be replaced by the multiplication by $r^2\nu^{-1}$).

Basically the motivation for the claim is that if (3.3) is accepted as definition, then δ_r has at least one of the key properties it is expected to have in (KO) theory.

The (KO) theory says, among other (perhaps much more) interesting things, that the "inertial range" ends when the Reynold's number becomes of order 1. When the local Reynold's number reaches $O(1)$ nothing interesting happens any more as the viscosity dominates and the flow becomes very smooth.

The following theorem, CKN theorem, says in a precise sense that the above statement is correct, (recall the definition of $Q^*_r(\underline{x}, t)$ given before (2.14)):

THEOREM 8, (CKN):

a) there is a dimensionless constant ε_{ckn} such that if for all boxes $Q^*_r(\underline{x}, t)$ around $(\underline{x}, t)$ it is:

$$\delta^*_r \equiv \frac{1}{\nu^2 r} \int_{Q^*_r(\underline{x},t)} (\underset{\sim}{\partial} \underline{u})^2 d^3\underline{\xi} d\vartheta \le \varepsilon_{ckn} \tag{3.4}$$

then $\underline{u}$ is C^∞ near $(\underline{x}, t)$, i.e. $\underline{x}, t$ is a regular point.

b) The size of $\underline{u}$ can be estimated explicity in terms of the global a priori estimates and of the size of the largest r_0 such that (3.4) holds for all $r < r_0$.

Remark:

By the covering arguments of §3 we see that the singularity set S can be covered by infinitely many sets Q^*_r violating (3.4) and since $\int_0^{t_0} \int_\Omega |\underset{\sim}{\partial} \underline{u}|^2 d^3\underline{x}\, d\vartheta < \infty$ by the *a priori* estimates (1.10), this means that $\alpha_c(S) \le 1$, $\mu_1(S) = 0$, [CKN]:

COROLLARY 9 (CKN): *It is $\alpha_c(S) \le 1$, $\mu_1(S) = 0$.*

Remark:

Clearly also this theorem is susceptible of being formulated as a property of the regularized approximants of Leray's regularized solutions. And it yields a quite remarkable and explicit formulation, because of the statement b) of existence of bounds in terms of the *a priori* estimates.

The proof rests on the more ambitious program of trying to find a beta function connecting all the operators $(A_n, \delta_n, J_n, G_n, K_n)$ rather than only the operator $A_n + \delta_n = T_n$.

The program fails because no useful recursion can be found for δ_n. One finds however a recursion of the form, if $\underline{X}_n \equiv (A_n, G_n, J_n, K_n)$:

$$\underline{X}_n = \mathcal{B}_{n,p}(\underline{X}_{n-p}, \delta_{n-p}) \tag{3.5}$$

for $p \geq 1$.

As in §3 the relations (3.5) are partly "purely kinematical", *i.e.* just consequences of the (H), (S), (CZ) inequalities to which is convenient to add Poincare's inequality (denoted (P) below):

$$\int_B d^d\underline{x}\,|f-F|^\alpha \leq (\mathrm{diam}B)^\alpha |B|^{1-\alpha}\Big(\int_\Omega |\underline{\partial} f|d^d\underline{x}\Big)^\alpha \tag{3.6}$$

where F is the average of f over the sphere B (used in particular for $\alpha = d(d-1)^{-1}$). One easily finds, for all $p \geq 1$:

$$\begin{aligned} J_n &\leq C\,(2^{-p/5}A^{1/5}_{n-p}G^{1/5}_n K^{4/5}_{n-p} + 2^{2p}A^{1/2}_{n-p}\delta_{n-p})\\ K_n &\leq C\,(2^{-p/2}K_{n-p} + 2^{5p/4}A^{5/8}_{n-p}\delta^{5/8}_{n-p})\\ G^{2/3}_n &\leq C\,(2^{-2p}A_{n-p} + 2^{2p}A^{1/2}_{n-p}\delta^{1/2}_{n-p}) \end{aligned} \tag{3.7}$$

where, here and below $C > 0$ denotes a suitable positive constant (not necessarily the same everywhere), (see lemmata 5.3, 5.4, 5.2 in [CKN]).

To such kinematical inequalities one can add one more "dynamical" inequality:

$$A_n \leq C\,(2^p G^{2/3}_{n-p} + 2^p A_{n-p}\delta_{n-p} + 2^p J_{n-p}) \tag{3.8}$$

which follows from the energy inequality (1.18) combined with (H), (S), (CZ), (P) inequalities (see lemmata 5.5, 5.1 in [CKN] recalling that we took the density of external force $\underline{f} = \underline{0}$ for simplicity). The (3.7), (3.8) yield (3.5).

The (3.4) and (3.5) are quite remarkable: they induce us to consider the "operators":

$$\alpha_n = A_n\ , \quad \kappa_n = K^{8/5}_n \tag{3.9}$$

and we see that the operators J, K can be eliminated from (3.7), (3.8) and it follows:

$$\begin{aligned} \alpha_n &\leq 2^{-p}C\alpha_{n-p} + 2^{3p}C(\alpha^{1/2}_{n-p}\delta^{1/2} + \alpha_{n-p}\delta) + C2^{p/5}\alpha^{1/2}_{n-2p}\kappa^{1/2}_{n-2p}\\ \kappa_n &\leq 2^{-8p/15}C\kappa_{n-p} + 2^{3p}C\delta^{5/8}\alpha_{n-p} \end{aligned} \tag{3.10}$$

where δ denotes a common bound or δ_{n-p}, δ_{n-2p}.

By using the inequality $(ab)^{1/2} \leq \frac{1}{\sqrt{2}}(\mu a + \mu^{-1}b)$, valid for any $\mu > 0$, we can "linearize" the two terms $(\alpha_{n-p}\delta)^{1/2}$ and $(\alpha_{n-2p}\kappa_{n-2p})^{1/2}$. We take μ such that $2^{-1/2}\mu 2^{3p}C = 2^{-p}C$ for the first and $2^{-1/2}\mu 2^{p/5}C = 2^{-p}C$ for the second. We find:

$$\begin{aligned} \alpha_n &\leq (22^{-p}C + 2^{3p}C\delta)\alpha_{n-p} + 2^{-p}C\alpha_{n-2p} + C2^{7/5p}\kappa_{n-2p} + 2^{4p}C\delta\\ \kappa_n &\leq 2^{-8p/15}C\kappa_{n-p} + 2^{3P}C\delta^{5/8}\alpha_{n-p} \end{aligned} \tag{3.11}$$

for all $p > 0$, and δ denotes a common bound or δ_{n-p}, δ_{n-2p}.

Given $\varepsilon < 0$ we can fix first p large (so that $2^{-p}C$ is as small as needed) and then δ small, say $p = p_0$, $\delta = \delta_0$, so that the r.h.s. of (3.11) can be bounded by:

$$\begin{aligned} \alpha_n &\le (2\,2^{-p}C + 2^{3p}C\delta)\alpha_{n-p} + 2^{-p}C\alpha_{n-2p} + C2^{7/5}\kappa_{n-2p} + 2^{4p}C\delta \\ \kappa_n &\le 2^{-8p/15}C\kappa_{n-p} + 2^{3p}C\delta^{5/8}\alpha_{n-p} \end{aligned} \tag{3.12}$$

with the coefficients of $\alpha.,\kappa.$ in the r.h.s. all smaller than ε except the one equal to $\tilde C = C2^{7/5}$ which, of course, will be (very) large:

$$\begin{aligned} \alpha_n \le &\varepsilon\alpha_{n-p} + \varepsilon\alpha_{n-2p} + \tilde C\kappa_{n-2p} + \bar C\delta \\ \kappa_n \le &\varepsilon\kappa_{n-p} + \varepsilon\alpha_{n-p} \end{aligned} \tag{3.13}$$

with $\bar C = 2^{4p}C$ and δ denotes a common bound or δ_{n-p}, δ_{n-2p}.

Thus, introducing the vectors:

$$\begin{aligned} \underline{a}' &= \underline{a}_n = (\alpha_n,\kappa_n,\alpha_{n-p},\kappa_{n-p}) = (\alpha',\kappa',\eta',\beta') \\ \underline{a} &= \underline{a}_{n-p} = (\alpha_{n-p},\kappa_{n-p},\alpha_{n-2p},\kappa_{n-2p}) = (\alpha,\kappa,\eta,\beta) \\ \underline{\rho} &= (1,0,0,0) \end{aligned} \tag{3.14}$$

we can rewrite (3.13) as:

$$\underline{a}_n \le M\underline{a}_{n+p} + \overline{C}\delta\underline{\rho} \tag{3.15}$$

where the matrix M can be read from (3.13) and it has all its matrix elements non negative and bounded by ε except one of them, which is off diagonal. Such a 4×4 matrix has norm less than 1 if ε is small enough and therefore (3.15) implies:

$$\limsup_{n\to\infty}\underline{a}_n \le \overline{C}\delta\frac{M}{1-M}\underline{\rho} \tag{3.16}$$

componentwise.

Hence we see that if δ_n is small enough for all $n \ge n_0$, the (3.5) generates a trajectory which "flows' within $O(\delta)$ of the origin. In fact one should remark that as soon as $\delta_n < \varepsilon_{\rm ckn}$, *i.e.* for $n \ge n_0$, the recursion relation (3.5) (or, better, its simplified form (3.16)) starts contracting. At the scale r_{n_0} the $G_{n_0} + J_{n_0} + K_{n_0} + A_{n_0}$ can be bounded in terms of the *a priori* bounds: the bounds will be huge but explicit.

Therefore, since the contraction rate is bounded by that of M, within a large but explicitly controlled number of steps the values of such observables shrink to become $< \varepsilon_s$. In other words knowing r_{n_0}, the scale at which the Reynolds number becomes $< \varepsilon_{\rm ckn}$ (*which is a quantitative way of saying that it becomes "of* $O(1)$*"*) we can find, in term of the *a priori* bounds, on which lower scale the velocity field becomes bounded by an *a priori* controlled bound (via theorem 6).

Hence if $\delta < \varepsilon_{\rm ckn}$ we see that α_n, κ_n and, by (3.7), also $G_n + J_n + K_n$ become smaller than ε_s, the constant in Scheffer's theorem (theorem 6). Clearly this means that $\underline{u}$ is C^∞ in $Q^*_{\bar r/2}(\underline{x}, r)$ if $\bar r$ is the value of r_n for which $G_n + J_n + K_n < \varepsilon_s$ for $r_n = \overline{r}$.

This is a very small information, but at least it makes clear the mathematical meaning and the validity of one of the assumption of (KO) theory. We also see that, from (KO) theory, we expect that the local Reynold's number δ_n should decrease according to a well defined law, at least if on the scale L it is very large. The law is, (see [LL], p. 153):

$$\delta_n \approx R^2 \left(\frac{r_n}{L}\right)^{8/3} = R^2 2^{-n8/3} \tag{3.17}$$

which would suggest that one should look for a relation of the δ_n on various scales of the form:

$$\delta_n \le 2^{-p8/3} \delta_{n-p} + ? \ . \tag{3.18}$$

It is however probable that any relation of this type can only come from taking appropriately into account the Thomson's law (valid in the limit of $\nu = 0$) and, possibly, by putting some restriction on the initial data so that they look like "typical" turbulent data.

(Un)fortunately the problem is completely open.

§4 Comments on filament dynamics in the Euler regime.

Some remarkable numerical experiments have been performed by Chorin who studies the filaments evolution under Euler and Navier Stokes equations.

He proposes a method of integration in which the filaments are considered very irregular to start with: this allows to bypass effectively the *a priori* obstacle due to the well known divergence of the velocity of smooth filaments of zero width.

Another way to avoid the problem is to give a finite width to the filaments or to form sheets of dimension 2. The disadvantage is first of all of numerical nature as one has to evolve two or three dimensional objects with the obvious storage and memory problems.

Here we present a mathematical analysis of the velocity at a given time for a filament of vorticity concentrated on a *one dimensional curve* γ. By this we mean that γ is a one parameter continuous closed curve everywhere tangent to the vorticity field $\underline{\omega} = \mathrm{rot}\, \underline{u}$.

The main problem is to understand if one can find a way around the fact that if γ is smooth then the Biot–Savart formula gives, for the velocity $\underline{u}(\underline{x})$ in a point $\underline{x} \in \gamma$:

$$\underline{u}(\underline{x}) = \frac{\Gamma}{4\pi} \oint_\gamma \frac{d\underline{\rho} \wedge (\underline{x} - \underline{\rho})}{|\underline{x} - \underline{\rho}|^3} \tag{4.1}$$

where $d\underline{\rho}$ is the line element on γ, and Γ is the vortex "intensity".

The (4.1) gives $\underline{u}$ a logarithmically infinite value at every point $\underline{x}$ of γ where the curvature $C(\underline{x})$ does not vanish: this means that if γ is replaced by a tube of width δ then $\underline{u}$ is proportional to $C(\underline{x})\log C(\underline{x})\delta$: this velocity is directed orthogonally to the osculating plane (determined by the tangent vector to γ and the principal normal vector, pointing towards the center of curvature).

The idea, which seems to be a possible interpretation of the continuous version of Chorin's lattice fluids, is that a filament can exist and move at finite speed only if it is so "rugged" that it has *no* well defined curvature. This is also suggested by the fact that a really thin filament will be deformed in dramatically different ways and directions, because of the fact that the velocity is proportional to the curvature; and the larger the thinner is the filament. Hence even if initially smooth it will be immediately deformed and look "like a brownian sample path": unless the filament is a circle: a case in which it would zoom to infinity (disappearing together with our problem, so we could forget about it).

The conjecture is, thus, that (4.1) gives finite velocity to the points of the filaments when the filaments are so rugged to be a sample of a (suitable class of) stochastic processes.

To put the conjecture in the form of a theorem we must give a meaning to (4.1). We therefore begin by considering a plane $z = 0$ and, on it, a curve γ which is roughly a circle centered at the origin and with radius R. The filament γ parametric equations in cylindrical coordinates will be, for $\alpha \in [0, 2\pi]$:

$$r(\alpha) = R(1 + \varepsilon(\alpha)), \quad z = R\vartheta(\alpha) \tag{4.2}$$

with $\varepsilon(\alpha)$, $\vartheta(\alpha)$, for the time being, arbitrary but periodic.

Then we can calculate the three components of the line element $d\underline{\rho}$:

$$d\underline{\rho} = (-R(1 + \varepsilon)s + R\varepsilon' c,\ R(1 + \varepsilon)c + R\varepsilon' s, \quad R\vartheta')$$

$$\underline{\rho} - \underline{\rho}_0 = (R(1 + \varepsilon)c - R(1 - \varepsilon_0),\ R(1 + \varepsilon)s,\ R(\vartheta - \vartheta_0)) \tag{4.3}$$

where $c \equiv \cos\alpha$, $s \equiv \sin\alpha$, $\varepsilon_0 \equiv \varepsilon(0)$, $\vartheta_0 \equiv \vartheta(0)$, $\underline{\rho}_0 = \underline{\rho}(0)$, $\underline{\rho} = \underline{\rho}(\alpha)$ and ε', ϑ' are the derivatives with respect to α of ε, ϑ.

Setting $\mu = \vartheta - \vartheta_0$, $\eta = \varepsilon - \varepsilon_0$ we get:

$$|\underline{\rho} - \underline{\rho}_0| = \Big(2(1 - c)(1 + \varepsilon + \varepsilon_0 + \varepsilon\varepsilon_0) + \eta^2 + \mu^2\Big)R^2 \tag{4.4}$$

and the three components of $d\underline{\rho} \wedge (\underline{\rho} - \underline{\rho}_0)$ are:

$$\frac{d\underline{\rho}}{d\alpha} \wedge (\underline{\rho} - \underline{\rho}_0) = \begin{cases} R^2[(1 + \varepsilon)c\mu + \varepsilon' s\mu - (1 + \varepsilon)\vartheta' s] \\ R^2[(1 + \varepsilon)(c - 1)\vartheta' + \vartheta'\eta + (1 + \varepsilon)\mu s - \mu\varepsilon' c] \\ R^2[-(1 + \varepsilon)\eta + (1 + \varepsilon_0)(1 - c) + (1 + \varepsilon)\varepsilon' cs + (1 + \varepsilon_0)\varepsilon' s] \end{cases} \tag{4.5}$$

To simplify the discussion we set $(1 + \varepsilon) \sim 1$ as we imagine ε, ϑ to be very small (but we could carry them around without this assumption).

The three components of the velocity at $\underline{\rho}_0$ are then given by:

$$\underline{u} = \frac{\Gamma}{4\pi R}\int_0^{2\pi}\frac{d\alpha}{(2(1-c)+\eta^2+\mu^2)^{3/2}}\cdot \\ \cdot\Big(\mu c+\varepsilon'\mu s-\vartheta' s,\ (c-1)\vartheta'+\vartheta'\eta+\mu s-\mu\varepsilon' c,\ -\eta+(1-c)+2\varepsilon' cs\Big) \tag{4.6}$$

The quantities ε', ϑ' present interpretation problems because we want, eventually, to suppose that $\alpha\to\varepsilon(\alpha)$, $\alpha\to\vartheta(\alpha)$ are "brownian paths", *i.e.* samples of a gaussian process generated on the circle by the operator $-D\partial_\alpha^2$ acting on the functions with zero average. This is locally the same as the unconstrained brownian motion, and one can think of it as an independent increments process (the corrections due to the periodicity of the boundary conditions can be easily introduced).

The formal expression $\varepsilon'(\alpha)=(\varepsilon(\alpha+\delta)-\varepsilon(\alpha))\,\delta^{-1},\delta\to 0$, shows that, by the independence[1] of the increments of ε,ϑ and by the smoothness of ε,ϑ (as they will typically be Hölder continuous of order $\sim 1/2$), the terms with the derivatives give zero contribution to the average of (4.6) over the samples.

Hence the proposal is to define:

$$\underline{u} = \frac{\Gamma}{4\pi R}\int_0^{2\pi}\frac{d\alpha}{(2(1-c)+\eta^2+\mu^2)^{3/2}}\quad(\mu c,\mu s,-\eta+(1-c)) \tag{4.7}$$

setting to zero the terms with the derivatives, which average to zero.

One can then check easily that $\langle\underline{u}\rangle$, $\langle\underline{u}^2\rangle$ are finite. For instance the only non zero component of $\underline{u}$ is in the z direction and has value:

$$v=\frac{\Gamma}{8\pi R}\int d\alpha d\eta d\mu\frac{1-\cos\alpha}{(1-\cos\alpha+\eta^2+\mu^2)^{3/2}}f_\alpha(\eta)g_\alpha(\mu) \tag{4.8}$$

where $f_\alpha(\eta)$, $g_\alpha(\mu)$ are the gaussian[2] distribution functions of $\eta=\varepsilon(\alpha)-\varepsilon(0)$ and $\mu=\vartheta(\alpha)-\vartheta(0)$. The integral (4.8) is finite as the gaussians f_α, g_α have dispersion $|\alpha|\,D$, for some $D>0$, assuming ε and ϑ to be the same process, and one easily finds:

$$V=\frac{\Gamma}{RD}C \tag{4.9}$$

where C is a finite computable constant.

To formalize the above heuristic analysis we consider a filament:

$$\gamma:\qquad \alpha\to\underline{\rho}(\alpha)+\underline{\xi}(\alpha) \tag{4.10}$$

[1] neglecting the periodic boundary conditions corrections
[2] neglecting the periodic boundary conditions corrections

where $\underline{\rho} \in C^\infty$ and $\underline{\xi}$ is a sample of a stochastic process, *e.g.* a brownian motion on the circle. The evolution equation for the filaments then is:

$$\begin{aligned} \frac{d}{dt}(\underline{\rho}_0 + \underline{\xi}_0) =\text{''}\ \frac{\Gamma}{4\pi} \oint \frac{(d(\underline{\rho} + \underline{\xi}) \wedge (\underline{\rho} + \underline{\xi} - \underline{\rho}_0 - \underline{\xi}_0)}{|\underline{\rho} - \underline{\rho}_0 + \underline{\xi} - \underline{\xi}_0|^3}\ \text{''} \overset{\text{def}}{\equiv} \\ \overset{\text{def}}{\equiv} \frac{\Gamma}{4\pi} \oint \frac{d\underline{\rho} \wedge (\underline{\rho} + \underline{\xi} - \rho_0 - \xi_0)}{|\underline{\rho} - \underline{\rho}_0 + \underline{\xi} - \underline{\xi}_0^3} = \underline{u}_0(\underline{\rho}_0), \qquad \underline{\rho}_0 \in \gamma \end{aligned} \tag{4.11}$$

where the $d\underline{\xi}$ terms are *eliminated*, by definition. This is in some sense the analogue of the elimination of the self interaction terms in the vortex dynamics in $d = 2$; and it is justified empirically by the fact that everything in the integrand in (4.11) is smooth except $d\underline{\xi}$, which however has zero average. Note, however, that in two dimensions the physical and mathematical meaning of the elimination of the self interaction terms is *perfectly* understood, see [MP].

The theorem is:

THEOREM 10:

The function $\underline{u}_0(\underline{\rho})$ in the r.h.s. of (4.11) is a well defined random variable if the $\underline{\xi}$ are samples of brownian motion on the circle. Such is also $|\underline{u}_0|^2$.

Continuing in a heuristic fashion we see that (4.11) generates two equations, one for the average shape $\underline{\rho}$ and one for its fluctuations $\underline{\xi}$:

$$\begin{aligned} \frac{d\underline{\rho}_0}{dt} &= \langle \frac{\Gamma}{4\pi} \oint \frac{d\underline{\rho} \wedge (\underline{\rho} + \underline{\xi} - \rho_0 + \xi_0)}{|\underline{\rho} + \underline{\xi} - \underline{\rho}_0 - \underline{\xi}_0^3} \rangle \\ \frac{d\underline{\xi}_0}{dt} &= \frac{r}{4\pi} \oint \frac{d\underline{\rho} \wedge (\underline{\rho} + \underline{\xi} - \rho_0 + \xi_0)}{|\underline{\rho} + \underline{\xi} - \underline{\rho}_0 - \underline{\xi}_0^3} - \frac{d\underline{\rho}_0}{dt} \end{aligned} \tag{4.12}$$

where $\underline{\xi} \equiv \underline{\xi}(\underline{\rho})$, $\underline{\xi}_0 \equiv \underline{\xi}(\underline{\rho}_0)$, $\underline{\rho}, \underline{\rho}_0 \in \gamma$ and the average $\langle \cdot \rangle$ is over the distribution of the fluctuation field at time $t = 0$. $\underline{\xi}$.

It would be very nice if one could say that the evolution (4.12) does not change the statistics of the fluctuations (*i.e.* that the statistics of the flutuation field is time independent), at least for some initial fluctuation distribution. This would decouple the two equations and one would have a well defined time evolution of the (very smooth) average filament: the velocity scale will be set by the dispersion D of the distribution of $\underline{\xi}$.

This invariance of the fluctuations distributions might be approximately true, or even exact in some limit situations (*e.g.* $C \to 0, D \to 0$ with $D^a/C =$ constant for some suitable a). And one can imagine to check the following analogue of a well known $2D$ vortex dynamics results.

Let us suppose that $\underline{\omega}$ be a vorticity field with flux line *all closed.*

We approssimate $\underline{\omega}$ by a finite family of smooth filaments (which is "weakly" possible by our closed flux lines assumption) γ_j, $j = 1, 2\ldots$ separated by a distance of order $e > 0$. Then we attribute a fluctuations field $\underline{\xi}_j^l$ to each filament with (say) brownian distribution with dispersion D_j^l.

Is it possible to fix D_j^l so that as $l \to 0$ the evolution (via the first of (4.12)) of the local average velocity field approaches a limit $\underline{u}(\underline{x}, t)$, with the corresponding vorticity field approaching a limit $\underline{\omega}(\underline{x}, t)$, and $\underline{u}(\underline{x}, t)$ is a solution of Euler's equations? Of course the problem can be modified by allowing D_j^l to depend on t or by avoiding the assumption of invariance of the fluctuations and by using the two (4.12) as a coupled system of equations.

But it is clear that it is perhaps too early to draw any conclusions: we think that the above analysis can suggest several numerical experiments which could perhaps help to a better understanding of the important works of Chorin.

Appendix A: An abstract argument.

The following is a version of the abstract argument necessary to prove theorem 3. Below $R(t)^2 \equiv \liminf_{l\to 0} \nu^{-2} L \int_\Omega |\underset{\sim}{\partial} \underline{u}^l(t)|^2$.

Let $\mathcal{T}_n$ be the set of times in the prefixed interval $[0, T_0]$ where $R(t)^4 < n$. Consider the time intervals $(t, t + \mathcal{L}Tn^{-1})$ with $t \in \mathcal{T}_n$. Let $\{t_j^{(n)}\}$ be a sequence of times such that:

$$\Theta_n = \bigcup_{j=1}^{\infty} (t_j^{(n)}, t_j^{(n)} + \mathcal{L}Tn^{-1}) = \bigcup_{t\in\mathcal{T}_n} (t, t + \mathcal{L}Tn^{-1}) \tag{A1}$$

Let l_p be a sequence such that $\underline{u}^{l_p}$ converges to a weak solution and such that:

$$\int |\underset{\sim}{\partial} \underline{u}^{l_p}(\underline{\xi}, t_j^{(n)})\, d\underline{\xi} \underset{n}{\rightarrow} {}^{1/2} \tag{A2}$$

for all j.

Then $\underline{u}^{l_p}$ verifies, in Θ_n, the bounds of theorem 1, (1.10), (1.11), eventually in p. Hence the weak solution that is constructed as a limit of a subsequence of $\{l_p\}$ verifies the same bounds, and *in particular* it is C^∞ in Θ_n.

Since Θ_n, $n = 1, 2, \ldots$ is a countable family we can choose the subsequence $\{l_p\}$ to be "good for all n's", simoultaneously. And since the set $\bigcup_n \Theta_n$ has complement of measure zero (because $\int R(t)^2 dt < \infty$ by the *a priori* estimate) the wished conclusion is reached (with no effort other than the above logical exercise).

Appendix B: Energy inequality.

As mentioned in the remark 5) after theorem 3 we have to check the convergence of the r.h.s. of (1.17) to the r.h.s. of (1.18) when $\underline{u}^l \to \underline{u}^0$ weakly (*i.e.* componentwise in the Fourier transform components), under the hypothesis that $\underline{u}^l, \underline{u}^0$ verify the *a priori* bounds (1.10) in theorem 1.

The tools for the proof of (1.18) are the (S) and (CZ) inequalities (2.10),(2.11) applied to the functions $\underline{u}^l, \underline{u}^0$ and $\underline{u}^l - \underline{u}^0$ with $B_r = \Omega$, where the (S) inequality does not contain the second term in (2.11) because all the above functions have zero average over Ω.

We remark that if $\underline{u}^l \to \underline{u}^0$ weakly then, given $N > 0$:

$$\begin{aligned}
&\lim_{l\to 0}\int_0^{T_0} dt \int_\Omega |\underline{u}^l - \underline{u}^0|^2\, d\underline{x} \equiv \int_0^{T_0} dt \sum_{0<|\underline{k}|} |\underline{\gamma}^l_{\underline{k}}(t) - \underline{\gamma}^0_{\underline{k}}(t)|^2 \le \\
&\le \lim_{l\to 0} \sum_{0<|\underline{k}|<N} \int_0^{T_0} dt |\underline{\gamma}^l_{\underline{k}}(t) - \underline{\gamma}^0_{\underline{k}}(t)|^2 + \sum_{|\underline{k}|\ge N} \int_0^{T_0} dt \frac{|\underline{k}|^2}{N^2} |\underline{\gamma}^l_{\underline{k}}(t) - \underline{\gamma}^0_{\underline{k}}(t)|^2 \le \\
&\le \lim_{l\to 0} \sum_{0<|\underline{k}|<N} \int_0^{T_0} dt |\underline{\gamma}^l_{\underline{k}}(t) - \underline{\gamma}^0_{\underline{k}}(t)|^2 + \frac{1}{N^2}\int_0^{T_0} dt \int_\Omega |\underset{\sim}{\partial}(\underline{u}^l - \underline{u}^0)|^2 = \\
&= \frac{1}{N^2}\int_0^{T_0} dt \int_\Omega |\underset{\sim}{\partial}(\underline{u}^l - \underline{u}^0)|^2 \le \frac{4T_0\nu S}{N^2}
\end{aligned} \tag{B1}$$

having used also the *a priori* bounds (1.10) and the componentwise convergence of $\gamma^l_{\underline{k}}(t)$ to $\gamma^0_{\underline{k}}(t)$ (see remark 5) after theorem 3); so that $\int_0^{T_0}\int_\Omega |\underline{u}^l - \underline{u}^0|^2 \to 0$ and this proves the appropriate convergence of the first two terms and that of the last term in the r.h.s. of (1.17) to the corresponding terms of (1.18).

Remark also that by the (S) inequality (*i.e.* (2.11)) with $q = 3$, $a = \frac{3}{4}$, $\frac{q}{2} - a = \frac{3}{4}$, and by the *a priori* bounds (1.10):

$$\begin{aligned}
&\int_0^{T_0} dt \int_\Omega |\underline{u}^l - \underline{u}^0|^3\, d\underline{x} \le C \int_0^{T_0} dt\, ||\underset{\sim}{\partial}(\underline{u}^l - \underline{u}^0)||_2^{3/2}\, ||\underline{u}^l - \underline{u}^0||_2^{3/2} \le \\
&\le C\Big(\int_0^{T_0} dt\, ||\underset{\sim}{\partial}(\underline{u}^l - \underline{u}^0)||_2^2\Big)^{3/4} \Big(\int_0^{T_0} dt\, ||\underline{u}^l - \underline{u}^0||_2^6\Big)^{1/4} \le \\
&\le C(4T_0\nu S)^{3/4}(2\overline{E}_0)\int_0^{T_0} dt\, ||\underline{u}^l - \underline{u}^0||_2^2 \xrightarrow[l\to 0]{} 0
\end{aligned} \tag{B2}$$

proving the continuity of the third term in (1.17).

Finally (and similarly) if $p^l = -\Delta^{-1}\sum_{ij}\partial_i\partial_j(u_i u_j)$ and recalling the (CZ) inequality (and by (H) inequality):

$$\begin{aligned}&\int_0^{T_0} dt \int_\Omega d\underline{x} |p^l \underline{u}^l - p^0 \underline{u}^0| \le \int\int |p^l - p^0|\,|\underline{u}^l| + \int\int |p^0|\,|\underline{u}^l - \underline{u}^0| \le \\ &\Big(\int\int |p^l - p^0|^{3/2}\Big)^{2/3} \Big(\int\int |\underline{u}^l|^3\Big)^{1/3} + \Big(\int\int |p^0|^{3/2}\Big)^{2/3} \Big(\int\int |\underline{u}^l - \underline{u}^0|^3\Big)^{1/3}\end{aligned} \tag{B3}$$

where the last term tends to 0 by (B2) and, by (CZ) inequality the first is bounded by:

$$\Big(\int_0^{T_0} \int_\Omega |p^l - p^0|^{3/2}\Big) \le \Big(\int\int |\underline{u}^l - \underline{u}^0|^3\Big)^{2/3} \xrightarrow[l\to 0]{} 0 \tag{B4}$$

showing the continuity of the fourth term in the r.h.s. of (1.17).

Appendix C: Other inequalities.

As an example of bounds that can be deduced from the (S) and (CZ) inequalities we show here that $\underline{u}^l$ and p^l are respectively locally $L_{10/3}$ and $L_{5/3}$ in space time: *i.e.* (2.12). We briefly discuss also the (2.13).

One just uses the (S) inequality, (2.11), with $q = 10/3$, $a = 1$, $-a + q/2 = 3/2$:

$$\int_\Omega |\underline{u}^l|^{10/3}\, d\underline{x} \le C\Big(\int_\Omega |\underset{\sim}{\partial} \underline{u}^l|^2\, d\underline{x}\Big)\Big(\int_\Omega |\underline{u}^l|^2\, d\underline{x}\Big)^{3/2} \tag{C1}$$

so that integrating over $[0, T_0]$ and using the *a priori* estimate (1.10):

$$\int_0^{T_0} \int_\Omega |\underline{u}|^{10/3} \le C\, T_0 \nu S\, \overline{E}_0^{3/2} \tag{C2}$$

and by (CZ) inequality:

$$\int_0^{T_0} \int_\Omega |p^l|^{5/3} \le C'\, T_0 \nu S\, \overline{E}_0^{3/2} \tag{C3}$$

which shows that the uniformity of the local integrability of order 10/3 and 5/3 in space time of $\underline{u}^l$ and p^l.

The (C2) and (C3) immediately imply, by Hölder inequality, the finiteness of the quantities in (2.8). In fact the exponents 5/4, 13/4 and 3/2 in K_r could be even replaced by 5/3, 11/3 and 7/3 respectively: we shall not use such a possibility to follow [CKN] where it cannot be used because of the more general boundary conditions of that paper.

Finally the same ideas can be used to infer (2.13), (2.16). The (2.18), (3.7), (3.8) are only a little more refined and we refer to the original paper [CKN], as the simple argument given there could not be done better.

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