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Crossing the Entropy Barrier of Dynamical Zeta Functions

R. Aurich[1], J. Bolte[2], C. Matthies,

M. Sieber[1] and F. Steiner

II. Institut für Theoretische Physik
Universität Hamburg
Luruper Chaussee 149, 2000 Hamburg 50
Fed. Rep. Germany

Abstract

Dynamical zeta functions are an important tool to quantize chaotic dynamical systems. The basic quantization rules require the computation of the zeta functions on the real energy axis, where their Euler product representations running over the classical periodic orbits usually do not converge due to the existence of the so–called entropy barrier determined by the topological entropy of the classical system. We show that the convergence properties of the dynamical zeta functions rewritten as Dirichlet series are governed not only by the well–known topological and metric entropy, but depend crucially on subtle statistical properties of the Maslov indices and of the multiplicities of the periodic orbits that are measured by a new parameter for which we introduce the notion of a *third entropy*. If and only if the third entropy is nonvanishing, one can cross the entropy barrier; if it exceeds a certain value, one can even compute the zeta function in the physical region by means of a convergent Dirichlet series. A simple statistical model is presented which allows to compute the third entropy. Four examples of chaotic systems are studied in detail to test the model numerically.

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1 Introduction

During the last years much effort has been undertaken to find semiclassical quantization rules for classically chaotic Hamiltonian systems as a counterpart to the WKB– and EBK–quantization for classically integrable systems.

A major breakthrough has been achieved by Gutzwiller [1, 2], when he derived his *periodic–orbit formula*, which expresses the trace of the Green's function in a semiclassical approximation as a sum over all classical periodic orbits. One problem that goes with this trace formula is that the periodic–orbit sum considered as a function on the complex energy plane does in general not converge on the real axis, i.e. in the physical region. Gutzwiller himself [3] discovered that his trace formula is a special version of an exact mathematical identity, known as *Selberg's trace formula* [4], for a specific dynamical system –the free motion of a particle on a surface with constant negative Gaussian curvature, i.e. the motion on a surface endowed with a hyperbolic metric. The Selberg trace formula provides a continuous variety of convergent periodic–orbit sum rules [5], as it is an identity for the traces of certain functions of the Hamiltonian, which is, in suitable units, the negative of the hyperbolic Laplacian in this case. The idea of "smearing" the Hamiltonian with an appropriate test function can be carried over to the general case of Gutzwiller's trace formula to yield also then a variety of absolutely convergent periodic–orbit sum rules [6]. The price to pay for the convergence of these sums is that the smeared Green's functions do not exhibit poles at the (semiclassical) quantum energies, but only show peaks of finite widths. One such possible regularization is the Gaussian smearing that has proven useful for several systems [7]. But knowing only a finite part of the length spectrum of primitive periodic orbits permits only to resolve a finite part –the lower end– of the energy spectrum with finite accuracy.

The case of the free motion on hyperbolic surfaces shows that there exists an alternative to using the (Selberg) trace formula directly, since the regularized trace of the resolvent of the Hamiltonian can be expressed by the logarithmic derivative of a meromorphic function of a complex variable directly related to the complex momentum. It turns out that the quantum energies are exactly given by the non–trivial zeroes of this *Selberg zeta function* [4] on the critical line. The task of quantizing this system is therefore reduced to the computation of these non–trivial zeroes. It would appear that the identification of the quantum energies as zeroes of an oscillating function can be done with much higher precision than to identify them from peaks of some smeared Green's function. However, the representation of the zeta function as an Euler product, as derived from the trace formula, does in general not converge on the critical line and therefore cannot be used directly to calculate the zeta function in the region of interest. Within the semiclassical approximation, all this holds also for general dynamical systems to which Gutzwiller's trace formula applies. The role played by Selberg's zeta function is now played by so–called *dynamical zeta functions*.

The fact that the Euler product representing a given dynamical zeta function does not converge on the critical line, but rather on a half–plane not containing the physical region, is known as the problem of the *entropy barrier*, since the abscissa of absolute convergence of the Euler product is given by the topological entropy of the classical system. These considerations have led to the belief that it is impossible to find a quantization rule for chaotic systems using dynamical zeta functions [8].

To some extend one can view the Riemann zeta function as a model for a dynamical zeta function of an unknown dynamical system. Again there is an entropy barrier, since both the Dirichlet series and the Euler product representing the Riemann zeta function do not converge in the critical strip, where the famous Riemann–zeroes are located. But in this case it is known how to compute these non–trivial zeroes using the *Riemann–Siegel formula* [9]. This formula allows a controlled numerical computation on the critical line. It would therefore be highly desirable to derive an analogue of the Riemann–Siegel formula for general dynamical zeta functions. A Riemann–Siegel lookalike formula for general chaotic systems has been proposed, but attempts to prove this formula have to cope with serious convergence problems [10].

In this article we investigate the convergence properties of certain representations of dynamical zeta functions as Dirichlet series and show that the convergence is determined by a new (classical) parameter called *third entropy.* A model is presented that allows to compute the third entropy and thus the region of convergence for these Dirichlet series using a simple random walk model for the coefficients of the series.

Our paper is organized as follows. In sec. 2 we review the theory of dynamical zeta functions relevant to the semiclassical quantization of chaotic systems and describe how the zeta functions may be represented as Dirichlet series. We also introduce the notion of a third entropy. A statistical model for the third entropy is developed in sec. 3. Numerical checks of our model are presented in sec. 4 for four chaotic systems: the hyperbola billiard, the geodesic flows on two hyperbolic octagons and Artin's billiard. Finally, our results are summarized in sec. 5.

2 Dynamical Zeta Functions and the Third Entropy

To be definite, we concentrate in this article on some simple dynamical systems, which nevertheless show the typical behaviour of classically chaotic Hamiltonian systems. The systems under consideration will be either plane billiards or geodesic flows (free motion) on hyperbolic surfaces.

Chaotic plane billiards consist of domains $D \subset \mathbb{R}^2$ with piecewise smooth boundaries ∂D, such that the motion of a particle sliding freely on D and being elastically reflected on ∂D is chaotic. The quantum Hamiltonian is $H = -\Delta$ (we always choose units in which $\hbar = 1 = 2m$) and the wave functions are required to vanish on ∂D (Dirichlet boundary–value problem). We also require H to have a purely discrete spectrum, $0 < E_1 \leq E_2 \leq \ldots$, $E_n = p_n^2$.

A hyperbolic surface may be represented as a fundamental domain of some discrete subgroup Γ (a Fuchsian group) of $PSL(2,\mathbb{R})$ in the complex upper half–plane $\mathcal{H} = \{z = x + iy \,|\, y > 0\}$ with Poincaré metric $ds^2 = y^{-2}(dx^2 + dy^2)$. $PSL(2,\mathbb{R})$ operates on $\mathcal{H}$ via fractional linear transformations. The wave functions are required to be periodic with respect to Γ–transformations (Γ–automorphic functions) and again the spectrum should be discrete, $0 = E_0 < E_1 \leq E_2 \leq \ldots$, $E_n = p_n^2 + \frac{1}{4}$.

The smeared, absolutely convergent version [6] of Gutzwiller's semiclassical trace formula reads

$$\sum_{n=0}^{\infty} h(p_n) \sim 2\int_0^\infty dp\, p\, h(p) \; < d(p) > \; + \; \sum_\gamma \sum_{k=1}^{\infty} \frac{l_\gamma \chi_\gamma^k g(k l_\gamma)}{e^{k\lambda_\gamma l_\gamma/2} - \sigma_\gamma^k e^{-k\lambda_\gamma l_\gamma/2}} \; . \tag{1}$$

Here $h(p)$ is an even function, holomorphic in the strip $|Im\, p| \leq \tau - \frac{\overline{\lambda}}{2} + \varepsilon$, $\varepsilon > 0$, that decreases faster than $|p|^{-2}$ for $|p| \to \infty$; $g(x) = \int_{-\infty}^{+\infty} \frac{dp}{2\pi} e^{ipx} h(p)$ is its Fourier–transform. $< d(p) >$ is the mean energy density, expressed as a function of momentum p. The sum on the r.h.s. of (1) runs over all primitive periodic orbits γ with lengths l_γ. $\chi_\gamma \in \{\pm 1\}$ is a *character* attached to γ, where it is assumed that the *Maslov index* of γ is even, and σ_γ is the sign of the trace of the monodromy matrix. λ_γ is the (scaled) Lyapunov exponent of γ and $\overline{\lambda}$ is the asymptotic average of all these exponents, which is also called the *metric entropy*, because it measures the mean rate at which phase space gets distorted in the neighbourhood of a periodic orbit [2]. The *topological entropy* $\tau > 0$ measures the exponential proliferation of periodic orbits,

$$N(l) := \#\{\gamma | l_\gamma \leq l\} \sim \frac{e^{\tau l}}{\tau l} \ , \quad l \to \infty \ . \tag{2}$$

To treat the systems considered here on the same footing, we introduce the complex variable $s := \frac{\overline{\lambda}}{2} - ip$. For the motion on hyperbolic surfaces one knows that both the metric as well as the topological entropy have the value one, since all Lyapunov exponents are equal to one. In the case of plane billiards the two entropies have to be calculated numerically. Notice that for billiards with area$(D) < \infty$ one expects under rather general assumptions Pesin's theorem [11] $\tau = \overline{\lambda}$ to hold, whereas for non–compact systems like the hyperbola billiard and some scattering systems one finds $\overline{\lambda} > \tau$ [2].

¿From the trace formula (1) one can derive [12] the trace of the regularized resolvent in the following form

$$\sum_{n=1}^{\infty} \left[\frac{1}{E_n - E(s)} - \frac{1}{E_n} \right] \sim B - \phi(s) + \frac{1}{2s - \overline{\lambda}} \frac{Z'(s)}{Z(s)} \ . \tag{3}$$

(B is a constant, which is irrelevant for our discussion in this paper.) Notice that (1) and (3) become exact relations in the case of the free motion on a hyperbolic surface. $\phi(s)$ is a function with known analytic properties, $Im\, \phi(\frac{\overline{\lambda}}{2} \pm ip) = \pm\pi < d(p) >$, $p \in I\!R$. In (3) the *dynamical zeta function* $Z(s)$ appears, which is defined by the Euler product

$$Z(s) := \prod_\gamma \prod_{m=0}^{\infty} \left(1 - \chi_\gamma \sigma_\gamma^m e^{-(s + m\lambda_\gamma + \frac{1}{2}(\lambda_\gamma - \overline{\lambda}))l_\gamma} \right) \ , \ Re\, s > \tau \ . \tag{4}$$

The *critical line*, on which in the semiclassical limit the non–trivial zeroes of $Z(s)$ are located, is the line $s = \frac{\overline{\lambda}}{2} - ip$, $p \in I\!R$, i.e. $Re\, s = \frac{\overline{\lambda}}{2}$.

For the following discussion it is convenient to deal with a simpler function, which contains, however, the same information on the non–trivial zeroes as $Z(s)$. This is the *Ruelle–type zeta function*

$$R(s) := \prod_\gamma \left(1 - \hat{\chi}_\gamma e^{-s l_\gamma} \right) \ , \ Re\, s > \tau \ , \tag{5}$$

where $\hat{\chi}_\gamma := \chi_\gamma \, e^{-\frac{1}{2}(\lambda_\gamma - \overline{\lambda})l_\gamma}$.

Expanding the product over the primitive periodic orbits in (5) transforms the Euler product into a generalized *Dirichlet series*

$$R(s) = \sum_\rho A_\rho e^{-sL_\rho} \ , \ Re\, s > \tau \ . \tag{6}$$

Here the sum runs over all *Dirichlet–orbits*, or briefly *D–orbits*, defined by $\rho = \gamma_1 \oplus \ldots \oplus \gamma_n$, i.e. over all formal combinations of primitive periodic orbits with *Dirichlet–lengths*, or briefly *D–lengths*, $L_\rho := l_{\gamma_1} + \ldots + l_{\gamma_n}$. The coefficients in the series are determined by the quantities $\hat{\chi}_\gamma$ attached to the primitive orbits that constitute a D–orbit, $A_\rho := \prod_{i=1}^{n}(-\hat{\chi}_{\gamma_i})$. Since the Euler product (5) converges absolutely for $Re\, s > \tau$, this also holds for the Dirichlet series (6). Dirichlet series such as (6) converge in right half–planes $Re\, s > \sigma_c$ and converge absolutely in right half–planes $Re\, s > \sigma_a$, $\sigma_a \geq \sigma_c$. Therefore, the series (6) converges in the strip

$\sigma_c < Re\, s \leq \sigma_a$ only conditionally. The *abscissae of convergence* can be computed from the formulae

$$\begin{aligned} \sigma_a &= \limsup_{N\to\infty} \frac{1}{L_N} \log \sum_{n=1}^{N} |A_n| \ , \\ \sigma_c &= \limsup_{N\to\infty} \frac{1}{L_N} \log |\sum_{n=1}^{N} A_n| \ , \end{aligned} \tag{7}$$

once the D–orbits have been ordered according to their lengths, $L_1 \leq L_2 \leq L_3 \leq \ldots$.

Going from the Euler product (5) to the Dirichlet series (6) did not change the absolute convergence, therefore one concludes that $\sigma_a = \tau$. If one forms the so–called *pseudo–orbits* [10] according to $\tilde{\rho} := m_1\gamma_1 \oplus \ldots \oplus m_n\gamma_n$ with *pseudo–lengths* $\tilde{L}_\rho := m_1 l_{\gamma_1} + \ldots + m_n l_{\gamma_n}$, $m_k \in I\!N$, then it can be rigorously shown [13] for compact hyperbolic surfaces that these proliferate according to

$$\widetilde{N}(L) := \#\{\tilde{\rho}|\tilde{L}_\rho \leq L\} \sim a\, e^{\tau L} \ , \quad L \to \infty \ , \tag{8}$$

with $a = \frac{Z(2)}{Z'(1)}$ and $\tau = 1$. The same conclusion can be drawn from (2), however not rigorously, for all the systems under consideration, with a being an unknown parameter then. Since $N(L) \leq N_D(L) \leq \widetilde{N}(L)$ for all L, where $N_D(L) := \{\rho|L_\rho \leq L\}$ counts the number of D–orbits with D–lengths smaller than or equal to L, one concludes that $L_N \sim \frac{1}{\tau}\log N$, $N \to \infty$.

As long as it is excluded that all characters χ_γ are negative, the coefficients A_ρ have different signs. Inspecting (7) then shows that there is a possibility for σ_c to become smaller than σ_a. We thus conclude that an important and novel parameter is provided by the difference $\sigma_a - \sigma_c$, which we call the *third entropy* δ:

$$\delta := \sigma_a - \sigma_c \ . \tag{9}$$

δ satisfies the general bounds $0 \leq \delta \leq \tau$, where the upper bound follows from

$$\delta \leq \limsup_{N\to\infty} \frac{\log N}{L_N} = \tau \ .$$

$\sigma_a = \tau$ thus yields the lower bound $\sigma_c \geq 0$.

¿From eqs. (9) and (7) and the definition of the coefficients A_n one sees that the parameter δ is a measure of the statistical properties of the Maslov indices and of the multiplicities of the periodic orbits, as will be discussed in more detail below. δ contains information beyond the topological and metric entropy, and thus the name “third entropy” seems to be appropriate. The need for a third entropy has clearly been foreseen by Gutzwiller [2], but nothing has been done as yet to develop this idea and to give a precise definition of it.

Obviously, the third entropy determines whether the entropy barrier at $Re\, s = \sigma_a = \tau$ can be crossed using the Dirichlet series representation. Four different cases for the value of the third entropy have to be distinguished:

1. $\delta = 0$: the entropy barrier is impenetrable;
2. $0 < \delta < \tau - \frac{\overline{\lambda}}{2}$: the entropy barrier is transparent, but the critical line cannot be reached;
3. $\delta = \tau - \frac{\overline{\lambda}}{2}$: the entropy barrier is transparent, but it is not known in general whether (6) converges on the critical line;
4. $\tau \geq \delta > \tau - \frac{\overline{\lambda}}{2}$: the entropy barrier is transparent and the Dirichlet series (6) converges conditionally on the critical line.

The fourth case is the most desirable one, but a priori either case can occur. The first case is realized for the Riemann zeta function, which therefore cannot be viewed as generic in this respect.

We want to present in the following a rather simple random walk model for the coefficients of the Dirichlet series, that allows to predict the third entropy from a few input parameters. For a given chaotic dynamical system it should then be possible to decide from the input parameters, whether and how far the entropy barrier can be crossed. For some systems σ_c has been previously calculated directly from (7), and the Dirichlet series has been evaluated numerically on the critical line [14] rather successfully. For a different method to calculate the zeroes on the critical line, see [15]. As already mentioned, the advantage of this method is that the quantum energies can be read of as zeroes of an oscillating function, which is a much more accurate method than identifying them as peaks of finite widths of some smearing function. Using the same input, i.e. the same set of primitive lengths, therefore allows to resolve considerably more quantum energies, when one calculates the zeta function on the critical line.

3 A Statistical Model for the Third Entropy

Let us first consider an idealized situation where the Ruelle–type zeta function is given by the generalized Dirichlet series ($B_0 = 1$)

$$R_M(s) := \sum_{n=0}^{\infty} B_n e^{-sL_n} \ , \ Re\, s > \tau_M \ , \tag{10}$$

and where the D–lengths $0 = L_0 < L_1 < L_2 < \dots$ are supposed to be non–degenerate and to grow asymptotically like $L_N \sim \frac{1}{\tau_M} \log N$, $N \to \infty$. Furthermore we assume that the coefficients B_n $(n > 0)$ are randomly distributed such that the value of B_k is independent of the value of B_n for $k \neq n$. The coefficients B_n should all be distributed according to the same probability density $p(B)$ with mean $< B >= 0$ and variance σ_B^2. Then, according to the central limit theorem for the distribution of sums of independent random variables, the (partial) sums

$$S_N := \sum_{n=1}^{N} B_n \tag{11}$$

obey, in the limit $N \to \infty$, a normal distribution with mean $< S_N >= N < B >= 0$ and variance $\sigma_S^2 = N\sigma_B^2$.

Under these assumptions we can evaluate approximately formula (7) for the abscissa σ_c of conditional convergence for $R_M(s)$ by replacing $|S_N|$ by $< S_N^2 >^{1/2} = \sqrt{\sigma_S^2} = \sqrt{N\sigma_B^2}$:

$$\begin{aligned} \sigma_c = \limsup_{N\to\infty} \frac{\log|S_N|}{L_N} &\cong \tau_M \lim_{N\to\infty} \frac{\log\sqrt{N\sigma_B^2}}{\log N} \\ &= \frac{1}{2}\tau_M \ . \end{aligned} \tag{12}$$

This simple statistical model thus provides us with a prediction for the location of the abscissa of conditional convergence of the zeta function as well as for the third entropy, i.e. $\delta \simeq \frac{1}{2}\tau_M$.

In the following we want to argue in favour of a slight variation of this model applicable to real dynamical systems.

We will call a chaotic dynamical system of the type introduced in the preceding section *ideal*, if the length spectrum of primitive periodic orbits is not degenerate. Assuming, furthermore, an irregular distribution of primitive lengths then guarantees that the length spectrum of D–orbits is not degenerate either.

The signs of the coefficients A_ρ in (6) crucially depend on whether the number of primitive orbits γ constituting the D–orbit ρ with $\chi_\gamma = +1$ is even or odd. Arranging the D–lengths in ascending order, as in (10), means that a change in sign, when going from A_n to A_{n+1}, depends on whether the numbers of primitive orbits with positive characters in L_n and L_{n+1} differ in parity. The absence of multiplicities and the irregularity of the primitive length spectrum makes this change, at least in the limit of long D–orbits we are interested in, random. Therefore the signs of the coefficients A_n can be considered as a random walk process. The probabilities for A_n to be positive or negative are the same, since there is an equal number of D–orbits with an even number of primitive orbits with positive character, as there are D–orbits with an odd number of such primitive orbits. For the hyperbolic surfaces we have $\hat{\chi}_\gamma = \chi_\gamma \in \{\pm 1\}$ and hence $A_n \in \{\pm 1\}$. In the case of a plane billiard, however, $\hat{\chi}_\gamma = C_\gamma \chi_\gamma$, where the C_γ's are distributed about one. Therefore the A_n's are distributed about ± 1 with zero mean. In both cases one has a distribution of the coefficients with mean $< A >= 0$ and variance σ_A. This is exactly a distribution of the type discussed above.

For an ideal chaotic system with topological entropy $\tau > 0$ we therefore predict for the abscissa of conditional convergence $\sigma_c = \frac{1}{2}\tau$ and for the third entropy $\delta = \frac{1}{2}\tau$. Notice that in this case the third entropy is determined solely by the topological entropy and therefore is not an independent quantity. This fact, however, is due to the "idealness" of the system and is not typical for generic systems. Whether the metric entropy $\overline{\lambda}$ is smaller, equal, or larger than the topological entropy determines in this ideal situation, whether case 2,3, or 4 in the classification of δ in the preceding section is realized. Thus the Dirichlet series is conditionally convergent on the critical line if $\overline{\lambda} > \tau$. In the case $\overline{\lambda} = \tau$ [11] one may have conditional convergence or not.

In reality most systems will not be of the ideal type and there will indeed occur multiplicities of primitive lengths and therefore of D–lengths, too. But knowing the asymptotic behaviour of the multiplicities of D–lengths allows to modify the statistical model so as to be applicable to realistic systems. In this process a new parameter is introduced that makes the third entropy an independent quantity, which is determined by the topological entropy and this new parameter.

In most cases as illustrated by the models that will be discussed in the next section, the mean multiplicities $< g(l) >$ of primitive lengths either asymptotically approach a constant $\bar{g}$ for $l \to \infty$ or proliferate exponentially, $< g(l) >\sim \frac{r}{l} e^{l/2}$, $r = const.$, $l \to \infty$. The multiplicity $g_D(L)$ of a D–length $L = l_1 + \ldots + l_n$ then reads $g_D(L) = \prod_{i=1}^{n} g(l_i)$ for the vast majority of D–orbits composed of primitive periodic orbits of different lengths. One can now argue, under a few reasonable assumptions, that the mean multiplicity $< g_D(L) >$ of a D–length L behaves asymptotically like

$$< g_D(L) >\sim d\, e^{\alpha L} \ , \quad L \to \infty \ , \tag{13}$$

for some positive constants α and d. In the examples, for which we shall test our model, we have checked (13) numerically and found good agreement, see fig. 1.

In the systems under consideration, primitive orbits γ can only be degenerate with respect to their lengths, if they share the same character χ_γ and Lyapunov exponent λ_γ. Therefore the coefficients A_ρ of D–orbits ρ with the same D–length L are all equal. In the Dirichlet series (6) the sum over all D–orbits ρ can thus be replaced by a sum over distinct D–lengths L_n. We denote the common coefficient of these degenerate D–orbits by A_n ($A_0 = 1 = g_D(0)$). Thus

$$\begin{aligned} R(s) &= \sum_{n=0}^{\infty} A_n g_D(L_n)\, e^{-sL_n} \\ &= \sum_{n=0}^{\infty} A_n d\, e^{-(s-\alpha)L_n} \left[\frac{g_D(L_n)}{de^{\alpha L_n}}\right] \ , \quad Re\, s > \tau \ . \end{aligned} \tag{14}$$

The location of the abscissa of convergence is determined by the coefficients of the Dirichlet series for large L_n. In this regime we can omit the factor in the bracket, as it approaches one in the mean due to (13). Let us introduce

$$\hat{R}(t) := \sum_{n=0}^{\infty} A_n d\, e^{-tL_n} \ , \quad Re\, t > \tau - \alpha \ , \tag{15}$$

which is of the same type as $R_M(s)$ in eq. (10) with $B_n := A_n d$, $< B >= 0$, $\sigma_B^2 = d^2\sigma_A^2$, $\tau_M = \tau - \alpha$. The abscissa of conditional convergence $\hat{\sigma}_c$ of $\hat{R}(t)$ thus is according to eq. (12) $\hat{\sigma}_c = \frac{1}{2}\tau_M = \frac{1}{2}(\tau - \alpha)$. $\hat{R}(t)$ differs from $R(s)$ by a constant shift in the argument, $s = t + \alpha$, and by the omission of the factor $\frac{1}{d}e^{-\alpha L_n} g_D(L_n)$ in each term, which is always positive and bounded as $L_n \to \infty$. Furthermore, this factor approaches one in the mean. One therefore expects that its omission does not influence the value of σ_c. We thus conclude $\sigma_c = \hat{\sigma}_c + \alpha$, i.e.

$$\sigma_c = \frac{1}{2}(\tau + \alpha) \ . \tag{16}$$

Our model then yields for the third entropy the value

$$\delta = \frac{1}{2}(\tau - \alpha) \ . \tag{17}$$

Since $0 < \alpha < \tau$, the third entropy is bounded by $0 < \delta < \frac{1}{2}\tau$. The question whether the Dirichlet series representing $R(s)$ can be evaluated on the critical line can now be answered once one knows the three entropies of the system, i.e. the values of the three quantities τ, $\overline{\lambda}$ and α. The condition to be fulfilled is $\delta > \tau - \frac{\overline{\lambda}}{2}$, i. e.

$$\alpha < \overline{\lambda} - \tau \ .$$

This condition means in particular that $\overline{\lambda} > \tau$ has to be realized for the system. If $\alpha < \overline{\lambda} - \tau$ holds, the Dirichlet series (6) will converge on the critical line $s = \frac{\overline{\lambda}}{2} - ip$, $p \in I\!R$, and it can be calculated numerically with the available part of the length spectrum as an input. Such a computation has been carried out rather successfully for some systems in [14].

Finally, we want to remark on the conditionality of the convergence for $\sigma_c < Re\, s \le \sigma_a$. For $Re\, s > \sigma_a$ the Dirichlet series may be summed in any order of its terms; the value of the sum does not depend on this order. But for $\sigma_c < Re\, s \le \sigma_a$ this is no longer the case. For any s there is an ordering of the terms, such that the conditionally convergent sum may take any desired value. Our point, however, is the following: Define $R(s)$ for $Re\, s > \sigma_a$ (by the Euler product for instance). Then form the Dirichlet series (6) and arrange the terms in ascending order of the D–lengths. For this ordering we concluded that $\sigma_c = \frac{1}{2}(\tau + \alpha)$. Keeping the ordering fixed one can continue the Dirichlet series beyond the entropy barrier up to $Re\, s > \sigma_c$. The ordering played an essential role in our statistical model for the coefficients. We argued that the choice of the signs going from A_n to A_{n+1} would be random, since the parities of the numbers of primitive orbits with positive characters in L_n and L_{n+1} are random. If one introduced any regularity in the ordering of terms, such as e.g. first summing all the terms with positive A_n's and then the negative ones, the random walk hypothesis would break down. Therefore, the value of σ_c, and thus also the validity of our model, strongly depends on the chosen order of terms in the Dirichlet series. The conditionality of the convergence does not, however, touch the analytic properties of this representation of the zeta function, as the Dirichlet series still converges uniformly. Hence the holomorphy of the zeta function is not destroyed for $Re\, s > \sigma_c$, when it is being represented by an "only" conditionally convergent series.

4 Application to Four Chaotic Systems

In this section we shall present a test of our statistical model for the third entropy, i.e. for the location of the abscissa of convergence, by investigating four specific chaotic dynamical systems that have already previously been studied in quite some detail: the hyperbola billiard, two different hyperbolic octagons (Hadamard–Gutzwiller model), and Artin's billiard. All these systems are of the type described in sec. 2 and will be explained in more detail below. For each system we shall plot the sequence $L_N^{-1} \log |\sum_{n=1}^N A_n|$ obtained from the numerical data against L_N, the lim sup of which yields σ_c. We then compare these sequences with our theoretical value $\frac{1}{2}(\tau + \alpha)$.

4.1 The Hyperbola Billiard

Our first example will be a plane billiard whose domain $D_0 \subset I\!R^2$ is given by $D_0 = \{(x,y) \in I\!R^2 | x \geq 0,\ y \geq 0,\ x \cdot y \leq 1\}$. Although the area of D_0, measured with the usual Euclidean metric on $I\!R^2$, is infinite, the spectrum of the Laplacian $\Delta = \partial_x^2 + \partial_y^2$ is discrete. This billiard has been previously studied in [16, 17]. The primitive length spectrum has been completely determined up to $l = 25$ and the quantum energies have been computed up to $E = 1,500$. ¿From the calculated length spectrum the metric and topological entropies have been determined to be $\overline{\lambda} = 0.705\ldots$ and $\tau = 0.592\ldots$. This system possesses a reflection symmetry across the $(x = y)$–axis in D_0. Dividing out this symmetry and thus considering only a desymmetrized system results in studying a billiard on the domain $D = \{(x,y) \in D_0 |\ x \geq y\}$. The stationary Schrödinger equation then is the eigenvalue equation for $-\Delta$ with Dirichlet boundary conditions on ∂D. In the following we will always deal with this desymmetrized hyperbola billiard. It is found that the metric and the topological entropies remain unchanged after desymmetrization. Notice that $\overline{\lambda} > \tau$.

¿From Gutzwiller's trace formula (1) applied to the full billiard domain D_0 one can derive the trace formula for the desymmetrized system, for which (1) does not apply directly, since on D there exists the primitive periodic orbit γ_0 running along the $(x = y)$–axis, which has to be treated separately. One then finds [17] that the Euler product for the dynamical zeta function reads for $Re\, s > \tau$

$$\begin{aligned} Z(s) = \quad & \prod_{n=0}^{\infty} \left(1 - \chi_{\gamma_0} \sigma_{\gamma_0}^{2n+1}\, e^{-(s+(2n+1)\lambda_{\gamma_0} + \frac{1}{2}(\lambda_{\gamma_0} - \overline{\lambda}))l_{\gamma_0}}\right) \\ & \cdot \prod_{\gamma \neq \gamma_0} \prod_{m=0}^{\infty} \left(1 - \chi_\gamma \sigma_\gamma^m\, e^{-(s+m\lambda_\gamma + \frac{1}{2}(\lambda_\gamma - \overline{\lambda}))l_\gamma}\right) \quad . \end{aligned} \tag{18}$$

($l_{\gamma_0} = 2\sqrt{2}$, $\sigma_{\gamma_0} = \chi_{\gamma_0} = -1$, $\lambda_{\gamma_0} = \frac{1}{2\sqrt{2}} \log(3 + 2\sqrt{2})$). The character for a primitive orbit γ is $\sigma_\gamma = \chi_\gamma = (-1)^{n_\gamma}$, where n_γ denotes the number of reflections from ∂D when traversing the orbit once. The Ruelle–type zeta function has the Euler product representation

$$R(s) = \prod_{\gamma} \left(1 - \hat{\chi}_\gamma\, e^{-sl_\gamma}\right) \ , \qquad Re\, s > \tau \ , \tag{19}$$

where $\hat{\chi}_\gamma = \chi_\gamma\, e^{-\frac{1}{2}(\lambda_\gamma - \overline{\lambda})l_\gamma}$ for $\gamma \neq \gamma_0$ and $\hat{\chi}_{\gamma_0} = 0$. This function has the Dirichlet series representation

$$R(s) = \sum_{\rho} A_\rho\, e^{-sL_\rho} \ , \tag{20}$$

which converges absolutely for $Re\, s > \tau$. Here the coefficients read $A_\rho = \prod_{\gamma\in\rho}(-\hat{\chi}_\gamma)$.

Since the system possesses a time reversal invariance, most primitive lengths are twofold degenerate. The only exceptions are those orbits that are reflected into themselves from the boundary ∂D. Their lengths occur without multiplicities. But as their number becomes negligible when going to higher and higher lengths, the mean multiplicity $< g(l) >$ of primitive lengths approaches two for $l \to \infty$. This leads to an exponential growth of the mean multiplicity $< g_D(L) >$ of D–lengths. In fig. 1.a we show a plot of $< g_D(L) >$ as calculated from the known length spectrum. We use all 195,113 primitive orbits up to $l = 25$, among which we find 101,265 different lengths. Out of these 806,028 D–orbits of 459,204 different D–lengths with $L \leq 25$ can be formed. A fit with the exponential expression (13) yields $d = 0.958\ldots$ and $\alpha = 0.026\ldots$. From these parameters we predict according to our model for the abscissa of conditional convergence $\sigma_c = \frac{1}{2}(\tau + \alpha) = 0.309\ldots$. In fig. 2.a this prediction is shown as the dotted line and is compared with the numerical approximation to the exact definition of σ_c according to eq. (7). It is seen that our model is completely consistent with the true values, at least in the limited L–range available to us.

¿From our model we can also derive the third entropy for the hyperbola billiard. Its value is $\delta = \sigma_a - \sigma_c = \frac{1}{2}(\tau - \alpha) = 0.283\ldots$. Since $\tau - \frac{\overline{\lambda}}{2} = 0.239\ldots$, one sees that $\delta > \tau - \frac{\overline{\lambda}}{2}$, which implies according to the classification of sec. 2 that the Dirichlet series converges conditionally on the critical line. Thus (20) may be used to compute the zeta function on $Re\, s = \frac{\overline{\lambda}}{2}$ and to find thereby its non–trivial zeroes there, which determine the quantum mechanical energies. In the first reference of [14] such a calculation has been carried out for the hyperbola billiard.

4.2 Two Hyperbolic Octagons

The two dynamical systems to be discussed next will be provided by two different hyperbolic octagons. These correspond to compact Riemann surfaces M of genus $g = 2$, realized as fundamental domains of Fuchsian groups Γ on the Poincaré upper half–plane $\mathcal{H} = \{z = x + iy|\, y > 0\}$ with hyperbolic metric $ds^2 = y^{-2}(dx^2 + dy^2)$. The surfaces M are represented as $M = \Gamma\backslash\mathcal{H}$, where Γ is a discrete, torsion–free subgroup of $PSL(2, I\!R)$ isomorphic to the fundamental group $\pi_1(M)$. Γ operates on $\mathcal{H}$ via fractional linear transformations,

$$z \in \mathcal{H}\ , \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \ : \qquad \gamma z = \frac{az + b}{cz + d} \ . \tag{21}$$

The fundamental domain $\mathcal{F}$ of Γ may be realized as a domain in $\mathcal{H}$ bounded by a $4g$–gon (i.e. an octagon for $g = 2$). The first of the octagons considered in this section is the so–called *regular octagon* [18, 19], which represents the most symmetric Riemann surface of genus two. The second one is some arbitrarily chosen *asymmetric octagon.*

The Hamiltonian for the free motion of a particle on M is $H = -\Delta$, where $\Delta = y^2(\partial_x^2 + \partial_y^2)$ is the hyperbolic Laplacian on $\mathcal{H}$. The eigenfunctions of H are realized as functions on $\mathcal{H}$ which are invariant under the operation of Γ, so–called Γ–automorphic functions: $\psi(\gamma z) = \psi(z)$, for all $\gamma \in \Gamma$. Then H has a discrete spectrum $0 = E_0 < E_1 \leq E_2 \leq \ldots$, $E_n = p_n^2 + \frac{1}{4}$. Gutzwiller's (smeared) trace formula is exact for this system, since it is identical to Selberg's trace formula [4] ($g \geq 2$),

$$\sum_{n=0}^{\infty} h(p_n) = 2(g-1) \int_0^{\infty} dp\, p\, h(p) \tanh(\pi p) + \sum_{\gamma} \sum_{k=1}^{\infty} \frac{l_\gamma g(k l_\gamma)}{2 \sinh(k l_\gamma / 2)} \; . \tag{22}$$

The smearing function $h(p)$ has to fulfill the requirements stated in sec. 1. Comparing the Selberg trace formula with (1) shows that all the primitive periodic orbits in this system have positive characters $\chi_\gamma = +1$. The topological and metric entropies are $\tau = 1$ and $\overline{\lambda} = 1$, respectively. The dynamical zeta function (Selberg's zeta function) is [4]

$$Z(s) = \prod_{\gamma} \prod_{n=0}^{\infty} \left(1 - e^{-(s+n) l_\gamma}\right) \; , \quad Re\, s > 1 \; . \tag{23}$$

¿From this one infers that the Dirichlet series for the corresponding Ruelle–type zeta function is given by (compare (6))

$$R(s) = \sum_{\rho} (-1)^{|L_\rho|} e^{-s L_\rho} \; , \quad Re\, s > 1 \; , \tag{24}$$

where $|L_\rho|$ denotes the number of primitive periodic orbits constituting the D–orbit ρ.

The two octagons we are looking at in this section differ in one point: the regular octagon has a huge symmetry group (of order 96), whereas the asymmetric octagon possesses only one symmetry operation. The latter symmetry is always present for Riemann surfaces of genus two, since these are all hyperelliptic and thus have the hyperelliptic involution as a symmetry. The classical dynamics on the regular octagon excels by a large degeneracy of the length spectrum of primitive periodic orbits; the mean multiplicity grows exponentially, $< g(l) > \sim \frac{8\sqrt{2}}{l} e^{l/2}$, $l \to \infty$, see [18, 19]. On the other hand, in the case of the asymmetric octagon the mean multiplicity approaches the constant value of four. This is due to the time reversal invariance and to the hyperelliptic symmetry. In any case the mean multiplicity of D–lengths grows exponentially, $< g_D(L) > \sim d\, e^{\alpha L}$, $L \to \infty$. One sees that a higher symmetry of the system leads to a faster growth of the degeneracy in the length spectrum, i.e. to a larger value of α. This in turn results in a higher value for the abscissa of conditional convergence as can be seen from our formula $\sigma_c = \frac{1}{2}(\tau + \alpha)$.

For the regular octagon the complete primitive length spectrum up to $l_{max} = 18.092025\ldots$ has been determined in [19]. It consists of 4,232,092 orbits of 1,500 different lengths. The D–length spectrum up to $L = l_{max}$ then consists of 26,469,856 D–orbits with 2,336 different D–lengths. In fig. 1.b we show a fit of $d\, e^{\alpha L}$ to $< g_D(L) >$. The fit parameters are $d = 4.5631\ldots$ and $\alpha = 0.4658\ldots$. Thus our prediction is $\sigma_c = \frac{1}{2}(1 + \alpha) = 0.7329\ldots$. We compare this value in fig. 2.b with a numerical evaluation of (7). Reasonably good agreement is found.

In the case of the asymmetric octagon we used the generator method described in [18] to determine the primitive length spectrum. We formed words in the group generators of lengths up to 12 and truncated the length spectrum at $l = 13$. Due to this method the spectrum is, however, not complete, but there are some orbits missing of lengths larger than about $l = 12$. We got 36,336 primitive orbits with 9,758 different lengths and generated 173,775 D–orbits with 29,062 different D–lengths up to $L = 13$. In fig. 1.c we show a fit to $< g_D(L) >$, from which we obtain the parameters $d = 1.5311\ldots$ and $\alpha = 0.1148\ldots$. Hence we predict $\sigma_c = \frac{1}{2}(1 + \alpha) = 0.5574\ldots$ which is compared in fig. 2.c with the numerical evaluation of (7). Again we find reasonably good agreement.

4.3 Artin's Billiard

Our final example of a chaotic dynamical system will be a billiard system on the Poincaré upper half–plane $\mathcal{H}$ constructed as follows. Let $\mathcal{F}_0$ be a fundamental domain on $\mathcal{H}$ for the modular group $\Gamma = PSL(2, \mathbb{Z})$. It may be chosen as $\mathcal{F}_0 = \{z \in \mathcal{H} \,|\, |z| > 1$ for $-\frac{1}{2} < x < 0$ and $|z| \geq 1$ for $0 \leq x \leq \frac{1}{2}\,\}$. $\mathcal{F}_0 \simeq \Gamma \backslash \mathcal{H}$ is a Riemann surface of genus zero with one cusp, at which $\mathcal{F}_0$ extends to infinity. The hyperbolic area of $\mathcal{F}_0$ is finite, area$(\mathcal{F}_0) = \frac{\pi}{3}$. The Hamiltonian for the free motion on $\mathcal{F}_0$ is again minus the hyperbolic Laplacian and the eigenfunctions are required to be Γ–automorphic. The spectrum of this Hamiltonian is both continuous and discrete, where the non–zero eigenvalues are embedded in the continuous spectrum $[\frac{1}{4}, \infty)$. One finds that the scattering waves, corresponding to the continuous spectrum, are even under the symmetry operation $z \mapsto -\bar{z}$. Desymmetrizing the system with respect to this symmetry by considering only the odd wavefunctions leads to a system defined on the half–domain $\mathcal{F} = \{\, z \in \mathcal{F}_0 \mid x \geq 0\,\}$. The eigenfunctions of the Laplacian then have to obey Dirichlet boundary conditions on $\partial\mathcal{F}$. One therefore obtains a billiard system in a non–compact hyperbolic triangle extending to infinity. The desymmetrization procedure has projected out the continuous spectrum completely due to the even symmetry of the scattering waves under the reflection $z \mapsto -\bar{z}$. The remaining, discrete spectrum is known to satisfy $\frac{1}{4} < E_1 \leq E_2 \leq \ldots$. Numerically the lowest state has energy $E_1 = 91.14134\ldots$, see [20].

The resulting system has been called [21] *Artin's billiard*, since it was first considered by Artin in 1924 [22]. It was the first dynamical system that could be proven to be ergodic by Artin [22].

Venkov [23] refined Selberg's trace formula for the modular group to the case under consideration. In the notation introduced above it reads

$$\begin{aligned}\sum_{n=1}^{\infty} h(p_n) \;=\; & \frac{1}{24}\int_{-\infty}^{+\infty} dp\, p\, h(p)\, \tanh(\pi p) \;+\; \frac{1}{4}\int_{-\infty}^{+\infty} dp \left(\frac{1}{4} + \frac{2}{3\sqrt{3}}\cosh\left(\frac{\pi p}{3}\right)\right)\frac{h(p)}{\cosh(\pi p)} \\ & + \sum_{\gamma}\sum_{k=1}^{\infty} \frac{\chi_\gamma^k l_\gamma g(k l_\gamma)}{e^{k l_\gamma/2} - \chi_\gamma^k e^{-k l_\gamma/2}} \;-\; \frac{3}{4} g(0)\log 2 \\ & - \frac{1}{4}\int_{-\infty}^{+\infty} dp\, h(p)\, \psi\left(\frac{1}{2} + ip\right) \,.\end{aligned} \tag{25}$$

Here $\psi(z)$ denotes $\psi(z) = \frac{d}{dz}\log\Gamma(z)$. The sum over primitive periodic orbits runs over two distinct classes of orbits. These either come from orbits γ_s on $\mathcal{F}_0$ that are symmetric under $z \mapsto -\bar{z}$ or from ones without symmetry, denoted by γ_u. The latter orbits have characters $\chi_{\gamma_u} = +1$, whereas the former ones have characters $\chi_{\gamma_s} = -1$ [21].

The Selberg zeta function for Artin's billiard has the Euler product representation [21]

$$Z(s) = \prod_{\gamma}\prod_{n=0}^{\infty}\left(1 - \chi_\gamma^{n+1}\, e^{-(s+n)l_\gamma}\right) \,, \quad Re\, s > 1 \,. \tag{26}$$

As in the previous examples, the Ruelle–type zeta function can be derived from $Z(s)$ and a Dirichlet series can be found for it

$$R(s) = \sum_{\rho} A_\rho\, e^{-sL_\rho} \,, \quad Re\, s > 1 \,. \tag{27}$$

A D–orbit ρ now consists of n_s primitive periodic orbits of the symmetric type with negative characters and of n_u orbits of the other type with positive characters. Therefore the coefficients of the Dirichlet series are $A_\rho = \prod_{\gamma\in\rho}(-\chi_\gamma) = (-1)^{n_u}$. Thus A_ρ only depends on the parity of n_u.

¿From the length spectrum of primitive periodic orbits, which has been calculated in [24, 21], one knows that $< g(l) >\sim \frac{1}{l}e^{l/2}$, $l \to \infty$. Again the mean multiplicity of the D–lengths grows like $< g_D(L) >\sim d\, e^{\alpha L}$, $L \to \infty$. For our numerical computations we take 166,319 primitive orbits with 3,000 different lengths into account. These make up the full length spectrum up to $l = 14.6$. To determine the spectrum of D–orbits up to $L = 14.6$ completely, we have to form 722,226 D–orbits with 50,587 different D–lengths. In fig. 1.d we show a numerical fit of the exponential law to $< g_D(L) >$ with parameters $d = 0.336\ldots$ and $\alpha = 0.279\ldots$. This leads to the prediction $\sigma_c = \frac{1}{2}(1+\alpha) = 0.639\ldots$ for the abscissa of conditional convergence of the Dirichlet series (27). However, a comparison with the numerical value for $\sigma_c = \limsup_{N\to\infty} \frac{1}{L_N} \log|\sum_{n=1}^{N} A_n|$ in fig. 2.d shows a mismatch between the predicted and the actual value. Therefore our statistical model presented in sec. 3 appears not to be applicable to Artin's billiard.

This specific system is known, however, to be an exception among chaotic dynamical systems in another respect as well. It is a general belief that the energy–level spacings of classically integrable systems obey Poisson statistics, whereas those of classically chaotic systems are distributed like the eigenvalues of hermitian matrices in a Gaussian orthogonal or unitary ensemble (GOE or GUE statistics). For Artin's billiard a numerical computation of the quantum energies shows [25] that the level spacing distribution is Poissonian, in contrast to what is expected. This fact hints towards unexpected statistical properties of the energy spectrum. Since by the trace formula (25) the energy and length spectra are closely related, one would therefore expect possible correlations in the length spectrum.

To check this hypothesis we have investigated numerically the random walk hypothesis on which our statistical model is built upon. In fig. 3 we plot the probability for the coefficient A_{N+1} in the Dirichlet series (14) for Artin's billiard to have the same sign as the preceding coefficient A_N. According to our statistical hypothesis this probability is expected to be one half. This indeed is the case in the first three examples we considered above. For Artin's billiard we find, however, a numerical value of about 0.41. This shows that there are correlations present in the length spectrum which lead to the fact that the signs of the coefficients in the Dirichlet series are not randomly distributed, in contrast to what is expected for a generic chaotic system. We have thus found another indication that Artin's billiard is exceptional among chaotic systems.

Notice that from fig. 2d) one reads off a value of $\sigma_c \approx 0.45$, and thus the Dirichlet series (27) for Artin's billiard converges on the critical line. In ref. [21] this result has been used to compute the quantal energies in the low energy region.

5 Summary

In this article we studied the convergence properties of dynamical zeta functions for a class of classically chaotic dynamical systems. In order to use the zeta function for the quantization of a chaotic dynamical system, it is mandatory to know in which part of the complex energy plane a given representation of the zeta function converges. If it happens that a representation is available that converges on the real energy axis (the critical line), then one can use it to find the zeroes of the zeta function on this critical line. These zeroes in turn give the semiclassical energies of the system.

The derivation of the zeta function from Gutzwiller's trace formula yields the zeta function as an Euler product. Rewriting this Euler product as a Dirichlet series does not alter the region of absolute convergence. But in contrast to the case of Euler products there do exist definite statements on the regions of conditional convergence of Dirichlet series. These may be –and in all the examples considered by us they are– larger than the region of absolute convergence.

Our aim was to present a model that describes to what extent the conditional convergence of the Dirichlet series is better than the absolute convergence. Knowing the abscissa of conditional convergence one also knows whether the Dirichlet series converges on the critical line and thus whether it can be used to compute the zeta function there. It turned out that a central role is played by a new parameter called the third entropy, which measures statistical properties of a given dynamical system beyond its topological and metric entropy. Making an assumption on the randomness of the length spectrum of primitive periodic orbits of a dynamical system allowed us to set up a statistical model for the third entropy and thus for the location of the abscissa of conditional convergence. We found that the shift of this abscissa away from the abscissa of absolute convergence is given by the topological entropy that describes the strength of the exponential proliferation of primitive periodic orbits, and by the exponent α of the exponential increase of the multiplicity of D–lengths: the less these multiplicities grow the larger is the third entropy and the more the conditional convergence gets improved. We hence conclude that this single new parameter α or, equivalently the third entropy δ, has to be calculated from the length spectrum in addition to the topological and metric entropy in order to get full information about the convergence properties of the dynamical zeta function.

In sec. 4 we have tested our model in four specific chaotic systems: the hyperbola billiard, the geodesic flows on two different hyperbolic octagons, and Artin's billiard. In the first three examples we found good agreement between the prediction obtained from our model and the directly calculated value for the abscissa of conditional convergence. The model, however, fails to explain the situation in Artin's billiard. We argued that this failure is due to correlations in the length spectrum that violate the randomness hypothesis of our model. We claim that these correlations are related to the observed unexpected statistical properties of the energy spectrum of this system. In this respect we view Artin's billiard as an exceptional case and do not consider it as generic.

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Figurecaptions

Figure 1: The mean mulitiplicity $< g_D(L) >$ is shown together with the fit curve $d\, e^{\alpha L}$ for a) the hyperbola billiard, b) the regular octagon, c) the asymmetric octagon and d) Artin's billiard

Figure 2: A numerical calculation of the sequences occuring in eq. (7) is shown as a function of the D–length L_N for a) the hyperbola billiard, b) the regular octagon, c) the asymmetric octagon and d) Artin's billiard. The upper curves correspond to σ_a, the lower ones to σ_c. The full horizontal lines indicate the critical lines, whereas the dotted lines show σ_c as derived from our model for the third entropy. The dashed line in a) corresponds to $\sigma_a = \tau$; $\sigma_a = \tau = 1$ in b)–d).

Figure 3: The probability for the coefficient A_{N+1} to have the same sign as A_N in the Dirichlet series (14) for Artin's billiard is shown