NEWTON'S METHOD AND PERIODIC SOLUTIONS OF NONLINEAR WAVE EQUATIONS *

Walter Craig

Department of Mathematics

Brown University

Providence, RI 02912

C. E. Wayne

Department of Mathematics

Pennsylvania State University

University Park, PA 16802

Abstract. We prove the existence of periodic solutions of the nonlinear wave equation

$$\partial_t^2 u = \partial_x^2 u - g(x,u) \ ,$$

satisfying either Dirichlet or periodic boundary conditions on the interval $[0, \pi]$. The coefficients of the eigenfunction expansion of this equation satisfy a nonlinear functional equation. Using a version of Newton's method, we show that this equation has solutions provided the nonlinearity $g(x,u)$ satisfies certain generic conditions of nonresonance and genuine nonlinearity.

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1. Introduction:

In this paper we prove the existence of periodic solutions of nonlinear wave equations of the form

$$\partial_t^2 u = \partial_x^2 u - g(x,u) \ , \tag{1.1}$$

for nonlinearities $g(x,u)$ which satisfy certain conditions of nonresonance and genuine nonlinearity. We assume that one has either periodic or Dirichlet boundary conditions at the ends of the interval $[0,\pi]$. This problem is called the free vibration problem for the nonlinear string and has been extensively studied. The review [B], contains over 60 references, and in the eight years since it was written the number has increased much further. In this introduction we will try to describe in very general terms the methods we use to study this problem and how these methods relate to, and differ from, previous approaches to this problem.

The first real breakthrough on this problem was due to Rabinowitz [R]. He rephrased the problem as a variational problem and was able to prove that under appropriate assumptions on the non-linearity $g(x,u)$ one had periodic solutions whenever the time period was a rational multiple of π. Many authors have used Rabinowitz's variational methods to obtain related results. None, however, were able to circumvent the restriction on the period. In addition, these variational techniques have not yet shed light on the existence of quasi-periodic solutions of (1.1). On the other hand, the variational techniques are global, and they place few restrictions on the strength of the non-linear term.

More recently, a quite different approach which uses the Kolmogorov, Arnold, Moser (KAM) theory has been developed by Kuksin [K] and Wayne [W]. This approach uses the fact that (1.1) is a hamiltonian system and modifies the classical KAM ideas to work in this infinite dimensional context. This has two advantages–first, it allows one to construct solutions whose periods are irrational multiples of π and second, it easily extends to give quasi-periodic as well as periodic solutions. A disadvantage is that since the KAM theory has an essentially perturbative character, it is restricted to equations with weak non-linearity, or equivalently, to solutions of small norm.

In this paper we propose yet a third approach. Our method is reminiscent of the Lyapunov-Schmidt method of classical bifurcation theory in that we split the problem into two pieces, one of which is finite dimensional and corresponds to the null space of the linearized operator, and the other piece infinite dimensional. In contrast to the Lyapunov-Schmidt method in our problem the linearization of the infinite dimensional piece does not have bounded inverse. Indeed, it's failure to have bounded inverse is related to the restriction in the work of Rabinowitz and others to solutions whose period is a rational multiple of π.

Our method is perturbative and begins by expanding the nonlinear term in (1.1) as $g(x,u) = g_1(x)u + g_2(x)u^2 + \ldots$. If one ignores terms of $\mathcal{O}(u^2)$ or higher, (1.1) becomes a linear equation whose solutions can be explicitly computed in terms of the eigenvectors, $\{\psi_j(x)\}$, and eigenvalues, $\{\omega_j^2\}$, of the Sturm-Liouville operator $L = (-\frac{d^2}{dx^2} + g_1(x))$. We then make the *Ansatz* that a periodic solution of (1.1) with angular frequency Ω exists and write it as

$$u(x,t) = \sum_{j,k} \hat{u}(j,k)\psi_j(x)e^{ik\Omega t} \quad . \tag{1.2}$$

Substituting (1.2) into (1.1) gives an infinite system of nonlinear algebraic equations which the coefficients $\{\hat{u}(j,k)\}$ must solve. We construct solutions of this system of equations using Newton's method taking as our initial approximation to the solution linear combinations of Kronecker δ-functions at the lattice sites $(j,k) = (1,\pm 1)$. This initial guess corresponds to a periodic solution of the linearized equation with angular frequency ω_1. We are able to prove that our interative scheme converges to a solution of the equations for $\{\hat{u}(j,k)\}$, which, when substituted into (1.2) gives a periodic solution of (1.1). This construction yields a family of periodic orbits with frequencies in a Cantor set of positive measure. One interesting point is that the linear operator which must be inverted in order to apply Newton's method is closely related to the lattice Schrödinger operators studied by Fröhlich and Spencer [FS] in their work on localization theory. From a technical point of view, this connection between localization theory and dynamical systems strikes us as one of the more interesting aspects of the present approach. Similar ideas, with applications to partial differential-difference equations have been previously developed in the work of Albanese, Fröhlich and Spencer [AFS].

Our method, like the KAM method is perturbative, and is restricted to the study of solutions of small norm. However, it differs from the KAM method in several ways. First of all, the present theory is not a transformation theory–we do not proceed by making a sequence of canonical transformations, nor do we transform the system to some normal form. Secondly, the present method makes no direct use of the hamiltonian nature of the problem. Thus, we hope it will be applicable to non-hamiltonian problems.

The existence theorem does not apply to all choices of nonlinearity g, it requires that certain conditions of linear nonresonance and genuine nonlinearity are satisfied. These are analogs of well known difficulties in the theory of dynamical systems. These conditions depend only upon the *3-jet* of g, and are finite in number. The set of nonlinearities which satisfy them is generic, indeed it is open and dense, and is described more precisely in section 6. These conditions can in principle be checked in explicit cases. For the classical examples of the nonlinear Klein Gordon and the sine Gordon equations, the nonlinearity depends upon one parameter, and we show that, for an open set of full measure of this parameter, the conditions are satisfied and the existence theorem applies.

Thinking of the analogy with the Lyapunov center theorem for finite dimensional hamiltonian systems in the neighborhood of an elliptic equilibrium point, one expects to obtain a family of periodic orbits bifurcating from the orbit of the linearized equations whenever no integer multiple of the frequency of the solution being perturbed coincides with the frequency of any other normal mode. There is a significant difference when the problem has infinitely many degrees of freedom. In the finite dimensional case there are smooth families of periodic solutions bifurcating from the solution of the linear problem. In the case of the wave equation we will construct a smooth curve bifurcating from the solution of the linear equation. However we cannot prove that all points on this curve give rise to solutions of the wave equation–only that there is a (Cantor) set of frequencies of positive measure such that for any point on the curve whose frequency lies in this Cantor set one has a solution. This difference arises from the fact that if there are only finitely many frequencies $\omega_1, \omega_2, \omega_3, \ldots, \omega_N$, and if $m\omega_1 \neq \omega_j$, for $j = 2, 3, \ldots, N$, then $m\Omega \neq \omega_j$, for all Ω in some interval surrounding ω_1. Thus, one typically obtains solutions for the nonlinear problem for an interval of frequencies. For infinitely many frequencies, however, even if $m\omega_1 \neq \omega_j$, for $j = 2, 3, \ldots$, there will in general be a dense set of Ω's for which $m\Omega = \omega_j$, for some j. It is the process of excising these resonant frequencies which gives rise to the Cantor set on which the construction of solutions is successful.

As remarked above, the KAM approach to this problem also yields the existence of quasi-periodic solutions for equations like (1.1). We believe that an extension of the present method will also yield quasi-periodic solutions, and plan further work on this problem. We remark that such results would be of interest even for finite dimensional systems since they would give a proof of the existence of invariant tori for hamiltonian systems near an elliptic equilibrium point whose dimension is less than or equal to the number of degrees of freedom of the systems, different from that of [E], or [P].

We conclude with an outline of the remainder of the paper. In the next section we state our principle results, transform the wave equation to a problem on a two-dimensional lattice, and apply our results to discuss two well known examples, the Klein-Gordon and sine-Gordon equations. In Section 3 we state the induction hypotheses which allow us to derive the results in Section 2. Section 4 contains the verification of these induction hypotheses, while Section 5 explains how to control the inverse of the linearized operator which arises in Newton's method. This analysis is connected with the theory of localization in Schrödinger operators. Finally, in Section 6 we derive some estimates we need which link the wave equation to the lattice problem, and discuss the issues of genericity of the nonlinearity.

2. Results

We present results for nonlinear wave equations, which are obtained by a reduction of the problem to one for nonlinear equations on lattices. The reduction to the lattice equations and the corresponding existence results are also given in this section. Solutions are obtained by an induction procedure based on the Nash Moser method which is described in Sections 3 and 4.

2.1. Nonlinear wave equations

The equation we study in this paper is the nonlinear wave equation on a bounded interval $0 \leq x \leq \pi$ in one space dimension.

$$\partial_t^2 u = \partial_x^2 u - g(x,u) \tag{2.1}$$

We will assume that the nonlinear term is analytic in both variables in the region $\{(x,u); |\mathrm{Im}\, x| < \overline{\sigma}\}$, periodic in x with period π, and with Taylor expansion in u,

$$g(x,u) = g_1(x)u + g_2(x)u^2 + g_3(x)u^3 + \cdots \tag{2.2}$$

We are seeking solutions which are periodic in time, with period $2\pi/\Omega$, which satisfy certain self adjoint boundary conditions on a spatial interval. If we linearize about the solution $u \equiv 0$ we obtain

$$\partial_t^2 v = \partial_x^2 v - g_1(x) v. \tag{2.3}$$

Two examples of boundary conditions that we will address are Dirichlet conditions, $u(0,t) = 0 = u(\pi,t)$, and periodic conditions $u(x+\pi,t) = u(x,t)$. Solutions of the linear equation (2.3) are given by separation of variables and an eigenfunction expansion. Let $\{\psi_j(x)\}_{j=1}^{\infty}$ be normalized eigenfunctions for the the linear differential operator

$$L(g_1)\psi = (-\frac{d^2}{dx^2} + g_1(x))\psi \tag{2.4}$$

with the proper boundary conditions, with eigenvalues $\{\omega_j^2\}_{j=1}^{\infty}$. (For periodic boundary conditions, it is more convenient to begin labeling the eigenvalues and eigenfunctions with $j = 0$–this causes no essential difference.) Solutions of (2.3) are given by

$$v(x,t) = \sum_{j=1}^{\infty} r_j \cos(\omega_j t + \xi_j)\psi_j(x) \tag{2.5}$$

which are parametrized by the amplitudes r_j and the phases ξ_j. For real ω_j each function $\cos(\omega_j t + \xi)\psi_j(x)$ is time periodic, with frequency ω_j. A more general solution to the linear equation (2.3) is time periodic only if for all nonzero amplitudes r_j there exist a full set of rational relations between the associated frequencies ω_j; that is, there exists ω and integers k_j such that for all j with $r_j \neq 0$, $\omega_j = k_j\omega$. Unless a full set of resonance conditions are satisfied the general solution is quasiperiodic or almost periodic. We seek periodic solutions to the full nonlinear problem (2.1) near the linear solutions $r\cos(\omega_j t + \xi)\psi_j(x)$, with frequency near the real linear frequency ω_j. In this process any coincidence or near coincidence of linear frequencies causes resonance and other phenomena related to small divisors in the full nonlinear problem. However the following results demonstrate that for most nonlinearities $g(x,u)$ an iterative construction can overcome these difficulties, to prove the existence of periodic solutions to (2.1) of small amplitude.

Theorem 2.1. *Consider equation (2.1) with periodic boundary conditions on the interval $0 \leq x \leq \pi$. For an open dense set of nonlinear terms $g(x,u)$ there exist time periodic real analytic solutions. More precisely there exist parameters (Ω, r, ξ) and* *solutions $u(x,t;r)$ such that*

$$
\begin{aligned}
|u(x,t;r) - r\cos(\Omega t + \xi)\psi_j(x)| &< Cr^2 \\
|\Omega - \omega_j| &< Cr^2
\end{aligned}
\tag{2.6}
$$

for a set of r of positive measure.

In section 6 the topology of the class of nonlinear equations will be discussed, and a more precise statement regarding the open dense set of nonresonant, genuinely nonlinear terms will be described. A similar statement holds for solutions of the nonlinear wave equation satisfying Dirichlet boundary conditions.

We note that not only will the set of amplitudes r of the solutions we construct have positive measure, but the set of frequencies of the periodic solutions will also have positive measure. Thus, we are assured of having solutions of irrational period.

Theorem 2.2. *Consider Dirichlet boundary conditions for equation (2.1). Additionally ask that $g(x,u) = -g(-x,-u)$. Among this class of nonlinearities there is an open dense set such that there exist time periodic solutions of (2.1). That is, there exist parameters (Ω, r, ξ) and solutions $u(x,t;r)$ satisfying*

$$
\begin{aligned}
|u(x,t;r) - r\cos(\Omega t + \xi)\psi_j(x)| &< Cr^2 \\
|\Omega - \omega_j| &< Cr^2
\end{aligned}
\tag{2.7}
$$

for a set of r of positive measure.

Remarks: The conditions on the nonlinear term $g(x,u)$ in Theorem 2.1 and Theorem 2.2 are quite explicit. They depend only upon the coefficients $g_1(x)$, $g_2(x)$, $g_3(x)$, in other words only upon the $3 - jet$ of g. Roughly, there is a condition on g_1 in order to avoid certain primary resonances in the linear equation, and a condition of genuine nonlinearity placed upon g_1, g_2 and g_3. Both are open conditions, excluding sets which are essentially of codimension 1. The precise nature of the good set will be described in more detail below. Unfortunately the case $g_1(x) = 0$ is too resonant for the present methods to handle, and is in the excluded set. On the other hand for an open set of constants m^2 of full Lebesgue measure the case $g_1(x) = m^2$ is included in the conditions of the theorems, thus the nonlinear Klein Gordon-equation and the sine-Gordon equations, and nonlinear perturbations of them, are covered by our results. It is possible to prove similar existence theorems for other boundary conditions as well; these should be self adjoint as well as satisfying other conditions on the eigenfunction expansion and the description of the nonlinearity. The precise conditions that are required are discussed in Section 6.

The inverse of the linearized operator in (2.3) plays a role in the existence results. When applied to time-periodic functions with frequency Ω the point spectrum of the operator is $\{\omega_j^2 - \Omega^2 k^2; 1 \le j < \infty, \ -\infty < k < \infty\}$. For most choices of Ω and coefficient $g_1(x)$ this is a dense set in $\mathbf{R}$. In particular, spectrum will accumulate at zero, a phenomenon which is often called the small divisor problem. It is in this case that the results Theorem 2.1 and Theorem 2.2 are most interesting. Because of the small divisors, the method of solution is of the Nash Moser type, alternating a Newton iteration with approximate inversion of the linearized operator. Both theorems above follow from a more general result in the form of a Nash-Moser type theorem for nonlinear equations posed on the lattice $\mathbf{Z}^+ \times \mathbf{Z}$. Two dimensional lattices are not special, and the theorem is easily generalized; in our situation two lattice directions suffice to index the temporal and spatial eigenfunctions used to describe the above problems in nonlinear waves.

The lattice problem arises by expanding solutions $u(x,t)$ of (2.1) in eigenfunction-Fourier series. The coefficients $U(i,j)$ in this expansion must satisfy nonlinear equations on the lattice of the form

$$W(U) + V(\Omega)U = 0 \quad . \tag{2.8}$$

Denoting lattice sites $x = (j,k) \in \mathbf{Z}^+ \times \mathbf{Z}$ and $\Omega \in \mathbf{R}$ a frequency parameter, the form of $V(\Omega)$ is a diagonal linear operator on sequences $U(x)$,

$$V(\Omega)(x,y) = (\omega_j^2 - \Omega^2 k^2)\delta(x,y). \tag{2.9}$$

where $\delta(x,y)$ is the Kronnecker delta. The sequence of frequencies $\{\omega_j\}_{j=1}^\infty$ satisfies

$$|\omega_j - j| < C_g,$$

which is the case for the eigenvalues of the linear operator (2.4). Let $\mathcal{H}_\sigma = \{U(x) \in \ell^2(\mathbf{Z}^+ \times \mathbf{Z}); \sum_{x\in\mathbf{Z}^+\times\mathbf{Z}} e^{2\sigma|x|}|U(x)|^2 < \infty\}$, a Hilbert space of sequences. For $\sigma < \overline{\sigma}$ the nonlinear term $W(U)$ is a real analytic mapping from $\mathcal{H}_\sigma \to \mathcal{H}_{\sigma-\gamma}$, for any $0 < \gamma \le \sigma$. We ask that $W(0) = 0$, $D_U W(0) = 0$, and furthermore that $W(U)$ satisfy certain natural conditions of genuine nonlinearity, best explained below. For the nonlinear wave equation the constant $\overline{\sigma}$ is determined by the analyticity properties of the term $g(x,u)$.

Again we are led to linearize the nonlinear problem (2.8) about the solution $U(x) = 0$, obtaining

$$V(\Omega)\varphi = 0.$$

Solutions of this are simply $\varphi(x) = \delta_y(x)$, with $y = (j,k)$, and $\Omega = (\omega_j/k)$. Each nonzero eigenspace of $V(\Omega)$ is at least two dimensional, from the form of $V(\Omega) = (\omega_j^2 - \Omega^2 k^2)$, which is spanned by the vectors in $\mathcal{H}_\sigma$ supported on the lattice sites $y = (j,k)$ and $\overline{y} = (j,-k)$.

The nonlinear existence theorem focuses on solutions near the linearized solution space, with frequency near the value $\Omega = \omega_j$ of the linearized problem. In fact with little loss of generality we will assume that ω_1 is real and focus on a neighborhood of $\Omega = \omega_1$, with solutions supported near $y = (1,1)$ and $\overline{y} = (1,-1)$.

Central to the construction is the inversion of the linearized operator $H(U) = V(\Omega) + DW(U)$ of , about an approximate solution U. This involves an analysis of the small eigenvalues of $H(U)$. These are connected with the geometry of the lattice points x at which is close to zero.

A major part of this paper is the analysis of the linearized operator $H(U) = V(\Omega) + DW(U)$ of (2.8). We assume that $DW(u)$ is selfadjoint, which will be the case for the lattice problems which come from the nonlinear wave equation. By an analogy with quantum mechanics we call $H(U)$ a **Hamiltonian operator**, and the matrix of the inverse operator $G(U)(z) = (H(U) - z\mathbf{1})^{-1}$ the **Green's function**. Central to the construction is the approximate inversion of $H(U)$ about an approximate solution U. This involves an analysis of the small eigenvalues of $H(U)$, and the geometry of the lattice sites x at which $V(\Omega)(x,x)$ is close to zero. We define a **singular site** to be a lattice point $x = (j,k)$ at which $|V(\Omega)(x,x)| = |\omega_j^2 - \Omega^2 k^2| < d_s$, where d_s is a small parameter which is specified in the next section. Any connected set of singular sites will be called a singular region. We will show that by restricting the frequency Ω appropriately, singular regions for the Dirichlet problem consist only of isolated sites, while for the periodic problem singular regions will consist of no more than pairs of adjacent sites.

In order to make the first step in an existence theorem for (2.1) with periodic or Dirichlet boundary conditions we ask for certain conditions of nonresonance among the linear frequencies $\{\omega_j\}_{j=1}^{\infty}$. This does not have to be a condition among infinitely many of them, but at least a large enough number of the initial frequencies.

Definition 2.3. *A sequence $\{\omega_j\}_{j=1}^{\infty}$ is (L_0, d_0)-nonresonant with ω_1 if there exists some $\tau > 5$ such that for all $|j| + |k| \ \leq \ L_0$ the following conditions hold:*

$$|\omega_1^2 k^2 - \omega_j^2| > d_0 \quad \text{if} \quad (j,k) \ \neq \ (1, \pm 1). \tag{2.10}$$

and

$$|k\omega_1 - j| > \frac{d_0}{(|j| + |k|)^{\tau}}, \quad \text{for} \quad (j,k) \ \neq \ (0,0). \tag{2.11}$$

Proposition 2.4. *An open dense set of frequency sequences $\{\omega_j\}_{j=1}^{\infty}$ are (L_0, d_0)-nonresonant with ω_1 for some L_0, d_0 with $d_0 = o(L_0^{-1/2})$.*

This condition is on the equation linearized about $U = 0$, thus depends only upon the coefficient $g_1(x)$. We defer to Section 6 the discussion of the precise topology in which the above set of frequency sequences $\{\omega_j\}_{j=1}^{\infty}$ is dense, and the proof of this proposition; however the set of coefficients g_1 satisfying (2.11) is open and dense in $L^2(0, \pi)$, and has open intersection with the π periodic, analytic potentials.

2.2. Symmetries of the equation

The wave equation (2.1) has certain elementary properties of symmetry, relevant to this paper, that are reflected in the nonlinear lattice systems (2.8). The sequences $u \in \mathcal{H}_\sigma$ among which we construct solutions are complex, however they will correspond to real solutions of the wave equation. Denote the involution on the lattice

$$x = (j,k) \to \overline{x} = (j, -k) \tag{2.12}$$

and the complex conjugate of U by $\overline{U}$, then the reality condition on sequences is that $\overline{U(x)} = U(\overline{x})$. We will require that the lattice equation (2.8) is covariant with respect to this symmetry,

$$\begin{aligned} \overline{V(\Omega)U(x)} &= V(\Omega)\overline{U(x)} = V(\Omega)U(\overline{x}) \\ \overline{W(U(x))} &= W(\overline{U(x)}) = W(U(\overline{x})). \end{aligned} \tag{2.13}$$

The wave equation respects an additional translational symmetry; $t \to t+T$, $T \in \mathbf{R}$. That is, time translation leaves the equation and the boundary conditions invariant. We will consider lattice systems which also possess a continuous symmetry of this form. The translation operator on sequences in $\ell^2(\mathbf{Z}^+ \times \mathbf{Z})$ is the diagonal operator

$$T_\xi U(x) = e^{ik\xi} U(x),$$

where $x = (j,k)$. T_ξ is a unitary operator on $\ell^2(\mathbf{Z})$ and on all the $\mathcal{H}_\sigma$ spaces, and it preserves the reality condition. From the nature of the diagonal operator notice that $T_\xi V(\Omega) U = V(\Omega) T_\xi U$. We will further require that the nonlinearity W satisfy

$$T_\xi W(U) = W(T_\xi U). \tag{2.14}$$

The nonlinear terms in lattice problems arising from the nonlinear wave equation satisfy (2.14), because the system is autonomous. Our construction will be of families of solutions invariant with respect to this translation. The interpretation is that this is the construction of embedded invariant circles of solutions of (2.1) in the space $\mathcal{H}_\sigma$.

The final requirement on the lattice problem is that the linearized operator $H(U) = V(\Omega) + DW(U)$ is selfadjoint. Since $V(\Omega)$ is real and diagonal, this is the condition on the nonlinear term $W(U)$ that

$$DW(U)(x,y) = \overline{DW(U)}(y,x) \quad . \tag{2.15}$$

Again, this condition holds for problems stemming from the nonlinear wave equation.

We expect that similar constructions can be obtained for invariant tori of higher dimension, giving rise to solutions of the nonlinear wave equation (2.1) which are quasiperiodic in time. In a second publication we plan to address this and other problems from our point of view.

2.3. Results for lattice problems

The main existence theorem for periodic solutions for nonlinear lattice problems (2.8) can now be stated. We fix the exponent $1/2 < \eta < 1$.

Theorem 2.5. *Consider equations (2.8) which satisfy the reality and translation invariance conditions (2.13), (2.14), (2.15). Suppose further that the nonlinear term in (2.8) satisfies hypotheses* **H1-H3** *of Section 6. There is a constant L_* such that if $\{\omega_j\}_{j=1}^{\infty}$ is (L_0, d_0)-nonresonant with ω_1 for $d_0 > L_0^{-\eta}$, for some $L_0 > L_*$, then there is an open set of nonlinearities W such that there exist uncountably many solutions $U(x) \in \mathcal{H}_{\overline{\sigma}/2}$ of (2.8). More precisely there exist (r, θ, Ω) such that these solutions satisfy*

$$\begin{aligned} &\|U - rT_\theta(\delta_y + \delta_{\overline{y}})\|_{\overline{\sigma}/2} \leq Cr^2 \\ &|\Omega - \omega_1| \leq Cr^2 \end{aligned} \tag{2.16}$$

These sequences remain solutions when acted upon by the translation T_ξ; they are embedded circles in the space $\mathcal{H}_{\overline{\sigma}/2}$.

Both Theorem 2.1 and Theorem 2.2 follow from this theorem. Indeed, consider solutions to the nonlinear problem (2.1) described in terms of the eigenfunctions, or normal modes, of the linearized equations. One expands a function $u(x,t)$ which is $2\pi/\Omega$ periodic in time, satisfying the correct spatial boundary conditions (Dirichlet on $[0,\pi]$, or periodic), in terms of the eigenfunction expansion. Let $\xi = \Omega t$, then

$$u(x,\xi) = \sum_{(j,k)\in \mathbf{Z}^+\times\mathbf{Z}} U(j,k)e^{ik\xi}\psi_j(x). \tag{2.17}$$

Square integrable time periodic solutions u to the wave equation (2.1) correspond to sequences $U(j,k) \in \ell^2(\mathbf{Z}^+ \times \mathbf{Z})$ which solve the lattice equation (2.8). The nonlinear term in the wave equation is $g(x,u) - g_1(x)u$. This corresponds to the nonlinearity for the lattice system

$$W(U)(j,k) = \int_0^{2\pi}\int_0^{\pi} \psi_j(x)e^{-ik\xi}(g(x,u) - g_1(x)u)dxd\xi. \tag{2.18}$$

We call equation (2.8) the **mode interaction equation** for the nonlinear wave equation (2.1). If $U(x) \in \mathcal{H}_\sigma$ then the solution $u(x,\xi)$ given by (2.17) is analytic. If

$U(\overline{x}) = \overline{U(x)}$, then $u(x, \xi)$ is real, and vice versa. Furthermore, with g analytic in the region $\{(x, u); |\text{Im } x| < \overline{\sigma}\}$, and with our choices of boundary conditions, the lattice nonlinearity $W \in C^\omega(\mathcal{H}_\sigma; \mathcal{H}_{\sigma-\gamma})$ for all $0 < \gamma \leq \sigma < \overline{\sigma}$. Other boundary conditions will also result in analytic nonlinearities on the spaces $\mathcal{H}_\sigma$, this is discussed in detail in section 6. For these, theorems similar to Theorem 2.2 and Theorem 2.1 also hold.

Conversely, starting from a solution $U(x) \in \mathcal{H}_\sigma$ of the lattice equation (2.8), consider the function $u(x, \xi) = \sum_{(j,k) \in \mathbf{Z}^+ \times \mathbf{Z}} U(j, k) \psi_j(x) e^{ik\xi}$ and its translates by T_θ. Since $U \in \mathcal{H}_\sigma$ they form an analytic family, in fact an embedded circle. Setting $\xi = \Omega t$, a real analytic solution of the nonlinear wave equation (2.1) is obtained.

We can now give a more precise description of the existence result. The proof of Theorem 2.5 is by a Nash Moser iteration scheme, proving a result which is in spirit very close to the theorem of Kolmogorov, Arnold and Moser. In fact the conclusions are reminiscent of these results; we construct families of solutions of (2.8) invariant under translation T_ξ, parametrized by $r \in C$ a Cantor set. That is, the periodic solutions that we find occur not in smooth curves but in totally disconnected families, Cantor sets foliated by invariant circles. There is a set of positive measure of amplitudes r for which there are solutions. This feature of totally disconnected families of solutions is familiar in the study of invariant tori of quasiperiodic orbits for Hamiltonian systems near elliptic stationary points. The fundamental reason behind this phenomenon is that Hamiltonian systems possessing infinitely many degrees of freedom have the possibility for the generation of a dense set of linear resonances even for periodic orbits.

In order to describe the detailed existence result we need to define the condition of genuine nonlinearity. Let $B_0 = \{x \in \mathbf{Z}^+ \times \mathbf{Z} : |j| + |k| \leq L_0\}$ be a bounded subdomain of the lattice $\mathbf{Z}^+ \times \mathbf{Z}$, and define Π_0 to be the orthogonal projection onto $\ell^2(B_0)$. This projection commutes with T_ξ, and furthermore Π_0 commutes with the lattice involution (2.12) and thus preserves the reality condition. Consider an approximate problem to (2.8) for a sequence $U_0 \in \ell^2(B_0)$

$$\Pi_0\big(W(U_0) + V(\Omega)U_0\big) = 0. \tag{2.19}$$

If the sequence $\{\omega_j\}_{j=1}^\infty$ is (L_0, d_0) nonresonant then $\Pi_0 V(\Omega)$ restricted to $\ell^2(B_0)$ has only a double eigenvalue at $\Omega = \omega_1$, with eigenvectors supported on the lattice sites $N = \{(1, \pm 1)\}$. We denote the orthogonal projection onto $\ell^2(N)$ by Q, and set $P = (\mathbf{1} - Q)$. Parametrize a neighborhood of zero in the null space of $\Pi_0 V(\omega_1)$ by $\mathbf{p} =$

$(p_1, p_2) \to \varphi(\mathbf{p})$, $\|\mathbf{p}\| < r_0$, with $\varphi(\mathbf{p}) = (p_1 + ip_2)\delta_{(1,1)}(x) + (p_1 - ip_2)\delta_{(1,-1)}(x)$. Equation (2.19) possesses a branch of nontrivial solutions $(U(\mathbf{p}), \Omega_0(\mathbf{p}))$ bifurcating from $(\mathbf{p}, \Omega) = (0, \omega_1)$. Denoting rotations by angle ξ in the plane $\mathbf{p} \in \mathbf{R}^2$ also by T_ξ, then $T_\xi \varphi(\mathbf{p}) = \varphi(T_\xi \mathbf{p})$, and the above branch of solutions is T_ξ invariant. The condition of genuine nonlinearity is that for this branch of approximate solutions the frequency parameter Ω_0 is nondegenerate in $\mathbf{p}$.

Definition 2.6. *The problem (2.8) is said to satisfy a twist condition if the bifurcation surface of the approximate problem $\mathcal{C}_0 = (\mathbf{p}, \Omega_0(\mathbf{p}))$ satisfies*

$$\begin{aligned} &\Omega_0(0) = \omega_1 \\ &\det \partial^2_{\mathbf{p}} \Omega_0(0) \neq 0 \end{aligned} \tag{2.20}$$

Since the branch of solutions is invariant under T_ξ then $\partial_{\mathbf{p}} \Omega_0(0) = 0$ automatically .

Recall that η is a small positive constant fixed above.

Theorem 2.7. *Let the sequence $\{\omega_j\}_{j=1}^\infty$ be (L_0, d_0) nonresonant with ω_1 for some $L_0 > L_*$ with $d_0 > L_0^{-\eta}$. If the nonlinear term in (2.8) satisfies hypotheses* **H1-H3***, there is a neighborhood $\mathcal{N}_0 = \{(\mathbf{p}, \Omega); \|\mathbf{p}\| < r_0, |\Omega - \omega_1| < r_0^2\}$ in the parameter space and a function $u(x; \mathbf{p}, \Omega) \in \mathcal{H}_{\sigma/2}$, $Qu = 0$ which is C^∞ in the parameters $(\mathbf{p}, \Omega) \in \mathcal{N}_0$ satisfying: (i) there is a Cantor set $\mathcal{N} \subseteq \mathcal{N}_0$ in parameter space, invariant under T_ξ, such that for $(\mathbf{p}, \Omega) \in \mathcal{N}$, $u(x; \mathbf{p}, \Omega)$ is a solution of the first bifurcation equation*

$$P\big(W(\varphi(\mathbf{p}) + u) + V(\Omega)u\big) = 0. \tag{2.21}$$

(ii) Set the exponent $0 < \nu < 1 - \eta$. If additionally the approximate problem (2.19) on $\ell^2(B_0)$ satisfies a twist condition

$$\det \partial^2_{\mathbf{p}} \Omega_0(0) \geq \mathcal{K}_0^2 > 0 \tag{2.22}$$

with $\mathcal{K}_0^2 > L_0^{-\nu}$, then there exists a C^∞ surface $\mathcal{C} = (\mathbf{p}, \Omega(\mathbf{p}))$ invariant under T_ξ satisfying the second bifurcation equation

$$Q\big(W(\varphi(\mathbf{p}) + u(\mathbf{p})) + V(\Omega(\mathbf{p}))u(\mathbf{p})\big) = 0. \tag{2.23}$$

where the intersection $C = \{\mathbf{p}; \mathcal{C}(\mathbf{p}) \cap \mathcal{N} \neq \emptyset\}$ has positive measure.

The intersection of $\mathcal{C}$ with $\mathcal{N}$ is of course the solution set for equation (2.8), consisting typically of a Cantor set foliated by invariant circles. The measure of the set $C = \{\mathbf{p}; \mathcal{C}(\mathbf{p}) \cap \mathcal{N} \neq \emptyset\}$ is relatively large, on the order of πr_0^2.

We note that even if the non-degeneracy condition fails to hold, one may have periodic solutions. Such situations are explored in [CW].

It is possible that the actual bifurcation point $(0, \omega_1) \notin \mathcal{N}$ due to an exact or near resonance of ω_1 and ω_j, with j such that $(j,k) \notin B_0$. However if none of these occur then there is a result on the density of the periodic orbits within radii $0 < \|\mathbf{p}\| < r_0$. Fix exponents $0 < \overline{\tau}$ and $0 < \overline{\alpha}$ such that $\overline{\alpha} + 1 > \overline{\tau}$.

Theorem 2.8. *An (L_0, d_0)-nonresonant sequence $\{\omega_j\}_{j=1}^{\infty}$ is fully nonresonant with ω_1 if there exist positive constants c_1 and c_2 such that for all $(j,k) \in \mathbf{Z}^+ \times \mathbf{Z}$, $(j,k) \neq (0,0)$,*

$$|k\omega_1 - j| > \frac{c_1}{(|j| + |k|)^{\overline{\tau}}},$$

and if for $(j,k) \neq (1, \pm 1)$,

$$|k^2\omega_1^2 - \omega_j^2| > \frac{c_2}{(|j| + |k|)^{\overline{\alpha}}} \quad .$$

In this case the Cantor set of Theorem 2.7 has $\mathbf{p} = 0$ as an accumulation point. Furthermore there is an estimate of the density of periodic orbits near $\mathbf{p} = 0$. There are constants $\mu > 0$, C_g such that for all $0 < r_1 < r_0$,

$$\mathrm{meas}\{r \in (0, r_1); \|\mathbf{p}\| = r, \ (\mathbf{p}, \Omega(\mathbf{p})) \in \mathcal{N}\} \geq r_1(1 - C_g r_1^{\mu}).$$

Because it depends upon the details of the induction process, the proof of this density result is deferred to section 6. In the proof estimates of the size of the exponent μ will be given.

2.4. The Nonlinear Klein-Gordon and sine-Gordon equations

Principal examples of problems of the form (2.1) are the nonlinear Klein Gordon equation

$$\partial_t^2 u = \partial_x^2 u - m^2 u + (m^2/3) u^3 \tag{2.24}$$

and the related sine-Gordon equation

$$\partial_t^2 u = \partial_x^2 u - m^2 \sin(u). \tag{2.25}$$

Both equations (2.24) and (2.25) have frequency sequences

$$\{\omega_j\}_{j=1}^{\infty} = \{\sqrt{j^2 + m^2}\}_{j=1}^{\infty}$$

for the Dirichlet problem, and

$$\{\sqrt{4[(j+1)/2]^2 + m^2}\}_{j=0}^{\infty}$$

for the periodic problem. (The term $[(j+1)/2]$ inside the square root means the integer part of $(j+1)/2$.)

When posed with periodic boundary conditions on the interval $[0, 2\pi]$, (2.25) is a completely integrable Hamiltonian system. This is not the case for (2.24), or for (2.25) with Dirichlet conditions posed at $x = 0, \pi$. Traveling wave solutions for (2.24) and (2.25) satisfying periodic boundary conditions on the interval $[0, \pi]$ are easily described using phase plane analysis for functions $u(x - ct)$. Thus, we will concentrate on the case of Dirichlet boundary conditions. On a formal level this problem was discussed by J.B. Keller & L. Ting [KT], and the curvature of the solution surfaces $\partial_{\mathbf{p}}^2 \Omega(0)$ was derived. These solutions are related to a class called 'breather solutions' in the literature, which are spatially localized, time periodic solutions to nonlinear wave equations posed on all of $x \in \mathbf{R}$. Theorem 2.2 implies the existence of small amplitude time periodic solutions of both (2.24) and (2.25) for a set of values of the parameter m^2 of full measure. We have only to check that the hypotheses of Theorem 2.7 are satisfied for the associated lattice problem. Either of these equations could be perturbed by an additional nonlinear term, $h(x, u)$, and as long as the total nonlinearity satisfied the twist condition, the results discussed below would still hold.

Theorem 2.9. *Consider a sequence of constants d_0, L_0 such that $d_0 \log(L_0) \to 0$. There is an open set $\mathcal{M}$ of full Lebesgue measure such that if $m^2 \in \mathcal{M}$, then the* *frequency sequences* $\{\omega_j\}_{j=1}^{\infty}$ *for the equations (2.24) and (2.25) are* (d_0, L_0) *nonresonant with* ω_1 *for some* (d_0, L_0).

Proof. Fix d_0, L_0 and consider an arbitrary interval $[a, b]$ of parameters m^2. For the Dirichlet problem, the first (d_0, L_0)-nonresonance condition is violated for those m^2 such that $|k^2(1 + m^2) - (j^2 + m^2)| \leq d_0$. That is

$$|(\frac{j^2 - k^2}{k^2 - 1}) - m^2| \leq \frac{d_0}{|k^2 - 1|},$$

hence by excising a closed interval of length $2d_0/|k^2-1|$ about every point $(j^2-k^2)/(k^2-1)$, $|j| + |k| \leq L_0$, $(j, k) \neq (1, \pm 1)$ which falls within the interval $[a, b]$, the remaining values of m^2 satisfy the first condition of Definition 2.3. Note that this imposes no condition for $k = \pm 1$, and that $m^2 = 0$ is excised. The diophantine condition of Definition 2.3 is violated for those values of m^2 such that $|k\sqrt{m^2 + 1} - j| \leq d_0/(|j|+|k|)^\tau$. Excising a closed interval of length $2d_0/(|j| + |k|)^\tau$ about every point $m^2 = (j/k)^2 - 1$ as well, the remaining parameters satisfy both conditions in Definition 2.3. Call this open set $\mathcal{M}(d_0, L_0)$. The only (j, k) that need to be considered are those for which $(1 + a)k^2 - C_0 \leq j^2 \leq (1 + b)k^2 + C_0$. The total measure of the excised intervals is estimated by

$$\sum_{\substack{|j|+|k|\leq L_0 \\ k\neq\pm 1 \\ (j^2/k^2)\in(a-C_1, b+C_1)}} \frac{2d_0}{|k^2 - 1|} + \frac{2d_0}{(|j| + |k|)^\tau} \leq C d_0 \log(L_0),$$

as long as $\tau \geq 2$. The set $\mathcal{M} \cap [a, b] = \cup_{d_0, L_0} \mathcal{M}(d_0, L_0)$, thus if $d_0 \log(L_0) \to 0$, the set $\mathcal{M}$ is of full measure. The proof in the case of periodic boundary conditions is similar. $\square$

Computing the curvature of the approximate bifurcation branches for the equation (2.11) is a straightforward exercise, and it is non-zero and independent of the choice of approximate domain B_0 as long as $L_0 \geq 6$. Theorem 2.5 implies the following existence result

Theorem 2.10. *Consider the equations (2.24) and (2.25), satisfying Dirichlet boundary conditions on the interval* $[0, \pi]$. *The curvature of the branch bifurcating from the frequency* ω_1 *is*

$$\partial_{\mathbf{p}}^2 \Omega(0) = \frac{-3m^2}{16\sqrt{1 + m^2}} \mathbf{1}. \tag{2.26}$$

Thus the nonlinear problem satisfies the twist condition, and for $m^2 \in \mathcal{M}$ and $d_0 > L_0^{-\eta}$, there exist small periodic solutions with frequency close to ω_1.

If the sign of the nonlinear term in the equation (2.24) is reversed, that is, if $g(x,u) = m^2(u + (1/3)u^3)$ then the curvature (2.26) reverses sign, but still retains a twist and the existence theorem holds.

3. The Induction Argument

3.1. Notation

In this subsection we introduce notation relating to the function spaces in which we solve (2.8) . We wish to show that the solutions of these lattice problems decay exponentially; to measure this decay we introduce a family of Hilbert spaces. When specifying points in $\mathbf{Z}^+ \times \mathbf{Z}$ by a single symbol, we will use x, y, etc. When we wish to specify their components we will use (j,k), (l,m), etc. For points $x = (j,k) \in \mathbf{Z}^+ \times \mathbf{Z}$, we always use the ℓ^1 norm, $|x| = |j| + |k|$.

Define a family of Hilbert spaces $\mathcal{H}_\sigma \subset \ell^2(\mathbf{Z}^+ \times \mathbf{Z})$, which consist of elements of $\ell^2(\mathbf{Z}^+ \times \mathbf{Z})$ for which the norm

$$\|u\|_\sigma^2 = \sum_{x \in \mathbf{Z}^+ \times \mathbf{Z}} |u(x)|^2 e^{2\sigma|x|}$$

is finite. We denote the inner product in these Hilbert spaces by $\langle \cdot, \cdot \rangle_\sigma$. If $\sigma = 0$, $\mathcal{H}_\sigma = \ell^2$. We will denote the inner product in ℓ^2 by either $\langle \cdot, \cdot \rangle_0$ or $\langle \cdot, \cdot \rangle$, depending on the circumstances. Similarly, $\| \cdot \|$ will mean $\| \cdot \|_0$–*i.e.* the ℓ^2 norm. This is similar to sequence spaces used in [P2].

If $S : \mathcal{H}_\sigma \to \mathcal{H}_\sigma$, let $\|S\|_\sigma$ be the usual Hilbert space operator norm. The following proposition shows that these norms respect the exponential weights.

Proposition 3.1. *The norm is a Banach algebra norm;*

$$\| ST \|_\sigma \leq \| S \|_\sigma \| T \|_\sigma \, .$$

If $\| S \|_\sigma \leq C_s$ *there is a sup norm estimate of the matrix elements of* S*;*

$$|S(x,y)| \leq C_s e^{-\sigma|x-y|}.$$

If for some σ *the matrix elements of an operator* S *are bounded by*

$$|S(x,y)| \leq C e^{-\sigma|x-y|},$$

then for all $0 \leq \gamma < \sigma$

$$\| S \|_{\sigma-\gamma} \leq C/\gamma^2.$$

An additional simple but important property of these norms is embodied in the following lemma.

Lemma 3.2. *Let* $B_L = \{(j,k) \in \mathbf{Z}^+ \times \mathbf{Z}; |j|+|k| \leq L\}$*. Let* Π_L *be orthogonal projection onto* $\ell^2(B_L)$*, and let* $\mathbf{1} - \Pi_L$ *be its orthogonal complement. If* $0 < \gamma \leq \sigma$ *then*

$$\|\Pi_L f\|_\sigma \leq e^{\gamma L} \|f\|_{\sigma-\gamma}$$
$$\|(\mathbf{1} - \Pi_L) f\|_{\sigma-\gamma} \leq e^{-\gamma L} \|f\|_\sigma.$$

Proof. *From the definition of the norm,*

$$\begin{aligned} \|\Pi_L f\|_\sigma \leq & \sup_{|j|+|k|\leq L} e^{\gamma(|j|+|k|)} \|f\|_{\sigma-\gamma} \\ \leq & e^{\gamma L} \|f\|_{\sigma-\gamma} \\ \|(\mathbf{1} - \Pi_L) f\|_{\sigma-\gamma} \leq & \sup_{|j|+|k|>L} e^{-\gamma(|j|+|k|)} \|f\|_\sigma \\ \leq & e^{-\gamma L} \|f\|_\sigma. \end{aligned}$$

□

We will also use the analyticity of these solutions as functions of parameters. For this purpose, if $\mathcal{N} \subset \mathbf{R}^2 \times \mathbf{R}$, we define complex domains $D(\mathcal{N};\rho) = \{(\mathbf{z},\Omega) \in \mathbf{C}^2 \times \mathbf{C}; \sqrt{\|\mathbf{z}-\mathbf{p}\|^2 + |\Omega_1 - \Omega|^2} < \rho$ for some $(\mathbf{p},\Omega_1) \in \mathcal{N}\}$.

3.2. The bifurcation problem on B_0

The induction procedure of this paper is started by solving an approximate bifurcation problem (2.19), consisting of the full equation (2.8), restricted to the lattice subdomain B_0. For $u \in \ell^2(B_0)$, we write $V_0(\Omega)u = \Pi_0 V(\Omega)u$ and $W_0(u) = \Pi_0 W(u)$. Linearizing (2.19) around $u \equiv 0$, we obtain $V_0(\Omega)\phi = 0$. If the sequence $\{\omega_j\}_{j=1}^{\infty}$ is (d_0, L_0)-nonresonant with ω_1, then $V_0(\omega_1)$ has a two-dimensional null space $\ell^2(N)$ parameterized by $\phi(\mathbf{p})$, $\mathbf{p} \in \mathbf{R}^2$. As we have indicated above, these will be solved using a Lyapunov-Schmidt decomposition;

$$P(W_0(\phi(\mathbf{p}) + u) + V_0(\Omega)u) = 0 \quad , \tag{3.1}$$

$$Q(W_0(\phi(\mathbf{p}) + u) + V_0(\Omega)u) = 0 \quad . \tag{3.2}$$

This pair of equations is equivalent to (2.19).

With these definitions in hand, we can construct the first approximation to the solution of (2.8). For any subset $B \subset \mathbf{Z}^2$, we write $\overline{B} = B \backslash N$, where N is the set of lattice points supporting the null space of $V_0(\omega_1)$. Define $\mathcal{N}_0 = \{(\mathbf{p}, \Omega) \in \mathbf{R}^2 \times \mathbf{R} \mid ||\mathbf{p}|| < r_0, |\Omega - \omega_1| < r_0^2\}$, a neigborhood of the point $(0, \omega_1)$ at which bifurcation branches are to be constructed.

Lemma 3.3. *Suppose that the sequence $\{\omega_j\}_{j=1}^{\infty}$ is (L_0, d_0) nonresonant with ω_1. For any $0 < r_0 < (1/6C_W)d_0/L_0^2$ and $\rho_0 < d_0/(3CL_0^2)$ there exists a solution $u_0(x; \mathbf{p}, \Omega) \in \ell^2(B_0)$ of (3.1) which is analytic in $D(\mathcal{N}_0, \rho_0)$. Furthermore,*

$$T_\xi u_0(x; \mathbf{p}, \Omega) = u_0(x; T_\xi \mathbf{p}, \Omega),$$

and u_0 satisfies the estimate

$$\|u_0\|_\sigma < \frac{3C_W L_0}{d_0} \|\mathbf{p}\|^2 \tag{3.3}$$

for $\sigma \leq \sigma_* - (1/L_0)$.

For $(\mathbf{p}, \Omega)$ in the complex subdomain $D(\mathcal{N}_0, \rho_0/2)$ the Cauchy estimates implies that

$$\begin{aligned}
\|\partial_\Omega^\beta u_0\|_\sigma \leq& \frac{3C_W L_0}{d_0} \frac{\|\mathbf{p}\|^2}{(\rho_0/2)^\beta} \\
\|\partial_{\mathbf{p}} \partial_\Omega^\beta u_0\|_\sigma \leq& \frac{3C_W L_0}{d_0} \left(\frac{\|\mathbf{p}\|}{(\rho_0/2)^\beta} \right) \left(1 + \frac{r_0}{(\rho_0/2)} \right),
\end{aligned} \tag{3.4}$$

and in general for $|\alpha| \geq 2$,

$$\|\partial_{\mathbf{p}}^{\alpha}\partial_{\Omega}^{\beta}u_0\|_{\sigma} \leq \frac{3C_W L_0}{d_0}\frac{1}{(\rho_0/2)^{|\alpha|+\beta-2}}\big(1+\frac{r_0^2}{(\rho_0/2)^2}\big).$$

Proof. Linearizing the function on the left hand side of (3.1) about $u_0 = 0$ and setting $(\mathbf{p}, \Omega) = (0, \omega_1)$ gives

$$PV_0(\omega_1) = H_{\overline{B_0}}(\omega_1).$$

By hypothesis this is invertible, with inverse $G_{\overline{B_0}}(\omega_1)$ bounded by C/d_0. Equation (3.1) is finite dimensional, solvable by the implicit function theorem to give a solution analytic in a complex neighborhood of $(\mathbf{p}, \Omega) = (0, \omega_1)$. The point of this lemma is to estimate the size of this neighborhood. Solutions of (3.1) are fixed points of the mapping

$$u_1 = -G_{\overline{B_0}}(\omega_1)\big(W_0(\phi(\mathbf{p}) + u_0) + (H_{\overline{B_0}}(\Omega) - H_{\overline{B_0}}(\omega_1))u_0\big). \tag{3.5}$$

The symmetry of the equation with respect to T_ξ assures the covariance property of the solution.

Unique analytic solutions are assured for parameter values such that the mapping in (3.5) preserves a neighborhood of the origin, on which it is a contraction. To find a neighborhood that is mapped to itself, we use hypothesis **H1** to estimate

$$\begin{aligned}\|u_1\|_\sigma \leq & \frac{C}{d_0}\|W_0(\phi(\mathbf{p}) + u_0)\|_\sigma + \|G_{\overline{B_0}}(\omega_1)\big(H_{\overline{B_0}}(\Omega) - H_{\overline{B_0}}(\omega_1)u_0\big)\|_\sigma \\ \leq & \frac{C_W}{d_0\gamma}(\|\mathbf{p}\|^2 + \frac{1}{\gamma^2}\|u_0\|^2_{\sigma+\gamma}) + \sup_{(j,k)\in\overline{B_0}}\left|\frac{\omega_j^2 - \Omega^2 k^2}{\omega_j^2 - \omega_1^2 k^2} - 1\right|\|u_0\|_\sigma\end{aligned}$$

Using Lemma 3.2,

$$\begin{aligned}\|u_1\|_\sigma \leq & \frac{C_W}{d_0\gamma}(\|\mathbf{p}\|^2 + \frac{e^{2\gamma L_0}}{\gamma^2}\|u_0\|^2_\sigma) \\ & + \frac{CL_0^2}{d_0}|\Omega - \omega_1|\|u_0\|_\sigma,\end{aligned}$$

and by optimizing over γ such that $\sigma + \gamma \leq \sigma_*$ we find that $\gamma = 1/L_0$ and

$$\|u_1\|_\sigma \leq C_W\frac{L_0}{d_0}(\|\mathbf{p}\|^2 + L_0^2\|u_0\|^2_\sigma) + C\frac{L_0^2}{d_0}|\Omega - \omega_1|\|u_0\|_\sigma \quad . \tag{3.6}$$

We require $r_0 = \rho_0 < d_0/(3(C_W + C)L_0^2)$, and define $C_u = 3C_W L_0/d_0$. For $(\mathbf{p}, \Omega) \in D(\mathcal{N}_0; \rho_0)$, the complex neighborhood of $\mathcal{N}_0$, if $\|u_0\|_\sigma \leq C_u\|\mathbf{p}\|^2$ then $\|u_1\|_\sigma < C_u\|\mathbf{p}\|^2$ also. Thus the fixed point of the map will lie in this neighborhood.

To obtain a contraction estimate, we use hypothesis **H2** of section 6.

$$
\begin{aligned}
&\|G_{\overline{B_0}}(\omega_1)(W_0(\phi(\mathbf{p})+u_2)-W_0(\phi(\mathbf{p})+u_1)\|_\sigma \\
&\qquad \le \frac{C}{d_0}\| \int_0^1 DW_0((\phi(\mathbf{p})+tu_2+(1-t)u_1)\ dt\ (u_2-u_1)\|_\sigma \\
&\qquad \le \frac{C_W L_0^2}{d_0}(\|\mathbf{p}\|+L_0\|u_1\|_\sigma+L_0\|u_2\|_\sigma)\|u_2-u_1\|_\sigma
\end{aligned}
$$

In order that this gives a contraction mapping when $\|\mathbf{p}\| < r_0+\rho_0$ and $\|u_1\|_\sigma, \|u_2\|_\sigma < C_u\|\mathbf{p}\|^2$, we ask that $\rho_0 = r_0 < (1/9C_W)(d_0/L_0^2)$.

The second term is easier,

$$
\|G_{\overline{B_0}}(\omega_1)(H_{\overline{B_0}}(\Omega)-H_{\overline{B_0}}(\omega_1))(u_2-u_1)\|_\sigma \le \frac{CL_0^2}{d_0}|\Omega-\omega_1|\|u_2-u_1\|_\sigma
$$

which gives a contracting estimate if $|\Omega-\omega_1| < r_0^2+\rho_0$, and $2C(r_0^2+\rho_0) < d_0/L_0^2$.

Estimate (3.6) leads to an a priori estimate on the fixed point.

$$
\|u_0\|_\sigma \le C_u\|\mathbf{p}\|^2 = \frac{3C_W L_0}{d_0}\|\mathbf{p}\|^2.
$$

This is the upper bound stated in (3.3).

□

Over the neighborhood $\mathcal{N}_0$ the problem (2.19) is reduced to finding the zero set of the equation (3.2). A trivial solution branch is $\{(\mathbf{p},\Omega) : \mathbf{p}=0, -r_0^2 < (\omega_1-\Omega) < r_0^2\}$, and we seek an additional family of solutions parametrized by $\{\mathbf{p} : \|\mathbf{p}\| < r_0\}$. As the equations (3.1) and (3.2) and the solution $u_0(x;\mathbf{p},\Omega)$ behave covariantly with respect to the translation T_ξ, the zero set of (3.2) is invariant under T_ξ. Properties of this set are given in the next result.

Lemma 3.4. *The mapping* $(\mathbf{p},\Omega) \to G_0(\mathbf{p},\Omega) = Q\,(W_0(\phi(\mathbf{p})+u_0(\mathbf{p},\Omega)) + V_0(\Omega)\phi(\mathbf{p}))$ *is analytic on* $\mathcal{N}_0$*, zero for* $\{(\mathbf{p},\Omega) : \mathbf{p}=0, -r_0^2 < (\omega_1-\Omega) < r_0^2\}$*, and has a Taylor expansion at* $\mathbf{p}=0$*,* $\Omega=\omega_1$ *with the following Taylor coefficients:*

$$
\begin{aligned}
\partial_{\mathbf{p}} G_0(0,\omega_1) &= 0 \\
\partial_\Omega G_0(0,\omega_1) &= 0,
\end{aligned}
\tag{3.7}
$$

and

$$\begin{aligned}\partial_{\mathbf{p}}^2 G_0(0,\omega_1) &= 0\\ \partial_{\Omega}^2 G_0(0,\omega_1) &= 0\\ \partial_{\mathbf{p}}\partial_{\Omega} G_0(0,\omega_1) &= Q\left(\partial_\Omega V_0(\omega_1))\partial_{\mathbf{p}}\phi(0)\right)\\ &= -2\omega_1\partial_{\mathbf{p}}\phi(0)\quad .\end{aligned}\tag{3.8}$$

If $\omega_1 \neq 0$, *the last term is nonvanishing.*

Thus the zero set of $G_0(\mathbf{p},\Omega)$ in a neighborhood of $(\mathbf{p},\Omega) = (0,\omega_1)$ consists of the Ω axis union a surface $(\mathbf{p},\Omega_0(\mathbf{p}))$, given as a graph over a neighborhood of zero in the $\mathbf{p}$-plane.

Proof. The analyticity follows immediately since the implicit function theorem guarantees that $u_0(\mathbf{p},\Omega)$ is analytic, $\phi(\mathbf{p})$ is analytic by construction, and W and V are analytic by assumption. We next compute the Taylor expansion of the function G_0 at $(\mathbf{p},\Omega) = (0,\omega_1)$. Clearly $Q\left(W_0(\phi(\mathbf{p}) + u_0(\mathbf{p},\Omega)) + V_0(\Omega)\phi(\mathbf{p})\right)|_{(\mathbf{p},\Omega)=(0,\omega_1)} = 0$, and the first derivatives are

$$\begin{aligned}\partial_{p_j} G_0(0,\omega_1) =& Q\left(V_0(\omega_1)\partial_{p_j}\phi(0)\right) = 0 \ \ ; \ j = 1,2\ ,\\ \partial_\Omega G_0(0,\omega_1) =& Q\left(\partial_\Omega V_0(\omega_1)\phi(0)\right) = 0\quad ,\end{aligned}\tag{3.9}$$

where we use that $\phi(0) = 0$ and $\partial_{p_j}\phi(0)$ is in the null space of $V_0(\omega_1)$. To compute the second derivatives we use that $D_u W_0(0) = 0$, $\|u_0\|_\sigma \leq C_u\|\mathbf{p}\|^2$, and the fact that (3.2) is covariant with respect to T_ξ. Indeed,

$$\begin{aligned}&T_\pi Q\left(W_0(\phi(\mathbf{p}) + u_0(\mathbf{p},\Omega)) + V_0(\Omega)\phi(\mathbf{p})\right)\\ &\quad = Q\left(W_0(\phi(-\mathbf{p}) + u_0(-\mathbf{p},\Omega)) + V_0(\Omega)\phi(-\mathbf{p})\right)\\ &\quad = -Q\left(W_0(\phi(\mathbf{p}) + u_0(\mathbf{p},\Omega)) + V_0(\Omega)\phi(\mathbf{p})\right)\quad .\end{aligned}$$

Thus, G_0 is odd in $\mathbf{p}$, and both $\partial_{\mathbf{p}}^2 G_0(0,\omega_1) = 0$ and $\partial_\Omega^2 G_0(0,\omega_1) = 0$. The mixed partial is nonzero, however, as long as $\omega_1 \neq 0$.

$$\begin{aligned}\partial_\Omega\partial_{p_j} G_0(0,\omega_1) =& Q\big(D^2 W_0(0)(\partial_{p_j}\phi + \partial_{p_j}u_0)(\partial_\Omega u_0)\\ &+ DW_0(0)(\partial_\Omega\partial_{p_j}u_0) + \partial_\Omega V_0(\Omega)\partial_{p_j}\phi\big)|_{(\mathbf{p},\Omega)=(0,\omega_1)}\\ =& Q\left(\partial_\Omega V_0(\omega_1)\partial_{p_j}\phi(0)\right)\\ =& -2\omega_1\partial_{p_j}\phi(0)\quad .\end{aligned}\tag{3.10}$$

□

The T_ξ invariance of the zero set of $G_0(\mathbf{p}, \Omega)$ also implies that the surface $(\mathbf{p}, \Omega_0(\mathbf{p}))$ obeys $\partial_{p_j}\Omega_0(0) = 0$, that is, it is tangent to the plane $\{(\mathbf{p}, \Omega); \Omega = \omega_1\}$. By a further Taylor expansion we will compute $\partial^2_{p_j p_k}\Omega_0(0)$, which pertains to the twist condition of the nonlinear problem.

Lemma 3.5. *If* $\sqrt{r_0^2 + \rho_0^2}(3C_W L_0/d_0) < C$, *and* $(r_0^2 + \rho_0^2)C_W L_0/(d_0\rho_0) << 1$, *the solution surface* $(\mathbf{p}, \Omega_0(\mathbf{p}))$ *is defined over the full neighborhood* $\{\mathbf{p}; \|\mathrm{Re}\ \mathbf{p}\| < r_0, \|\mathrm{Im}\ \mathbf{p}\| < \rho_0\}$. *The surface satisfies*

$$\begin{aligned} \Omega_0(0) &= \omega_1 \\ \partial_{p_j}\Omega_0(0) &= 0 \\ \partial^2_{p_j p_k}\Omega_0(0) &= \mathcal{K}_0\delta_{jk}, \end{aligned} \tag{3.11}$$

where

$$\begin{aligned} \mathcal{K}_0 =& \frac{1}{2\omega_1}\Big(\frac{1}{6}\langle \partial_{\mathbf{p}}\phi, (D^3W_0(0)(\partial_{\mathbf{p}}\phi)^3)\rangle \\ & - \frac{1}{2}\langle (D^2W_0(0)(\partial_{\mathbf{p}}\phi)^2), G_{\overline{B_0}}(\omega_1)(D^2W_0(0)(\partial_{\mathbf{p}}\phi)^2)\rangle\Big). \end{aligned} \tag{3.12}$$

Proof. The expression for the curvature is derived from the Taylor expansion of the mapping $G_0(\mathbf{p}, \Omega)$ at $(0, \omega_1)$. Differentiating (3.1) with respect to $\mathbf{p}$, we easily find that

$$\begin{aligned} u_0(0, \omega_1) &= 0 \\ \partial_{p_j}u_0(0, \omega_1) &= \partial_\Omega u_0(0, \omega_1) = 0 \end{aligned} \tag{3.13}$$

and $\partial^2_{p_j p_k}u_0(0, \omega_1)$ satisfies

$$P(D^2W_0(0)[\partial_{p_j}\phi, \partial_{p_k}\phi] + V_0(\omega_1)\partial^2_{p_j p_k}u_0) = 0.$$

Since $PV_0(\omega_1)$ has inverse $G_{\overline{B_0}}$,

$$\partial^2_{p_j p_k}u_0(0, \omega_1) = -G_{\overline{B_0}}(\omega_1)(D^2W_0(0)[\partial_{p_j}\phi, \partial_{p_k}\phi]).$$

In order to compute the curvature of the surface, the relevant third order term in the Taylor expansion of the mapping $G_0(\mathbf{p}, \Omega)$ is $\partial^3_{\mathbf{p}}G_0(0, \omega_1)$.

$$\partial^3_{\mathbf{p}}G_0(0, \omega_1) = Q((D^3W_0(0)(\partial_{\mathbf{p}}\phi)^3) + 3(D^2W_0(0)[\partial_{\mathbf{p}}\phi, \partial^2_{\mathbf{p}}u_0])) \quad . \tag{3.14}$$

Using the expression for $\partial_{\mathbf{p}}^2 u_0$ and (3.13) this gives (3.12) for the curvature of the nontrivial zero set $(\mathbf{p}, \Omega_0(\mathbf{p}))$ of the mapping G_0 at $\mathbf{p} = 0$. Incidentally we may easily deduce that $\partial_\Omega \partial_{\mathbf{p}}^2 G_0(0, \omega_1) = 0$, $\partial_\Omega^3 G_0(0, \omega_1) = 0$ since the mapping is odd.

We also establish that the surface $(\mathbf{p}, \Omega_0(\mathbf{p}))$ is defined and analytic throughout the complex neighborhood $\{\mathbf{p}; \|\mathbf{p} - \mathbf{p}_1\| < \rho_0$ for some $\mathbf{p}_1 \in \mathbf{R}$ with $\|\mathbf{p}_1\| < r_0\}$. Since the branch is simple it suffices to show that $\partial_{\mathbf{p}} \Omega_0(\mathbf{p})$ is bounded. Using that $G_0(\mathbf{p}, \Omega_0(\mathbf{p})) = 0$ and differentiating with respect to $\mathbf{p}$, we find

$$Q\big(V_0(\Omega)\partial_{p_j}\phi + \partial_\Omega V_0(\Omega)\phi\partial_{p_j}\Omega + DW_0(\phi(\mathbf{p}) + u_0)(\partial_{p_j}\phi + \partial_{p_j}u_0 + \partial_\Omega u_0 \partial_{p_j}\Omega)\big) = 0.$$

Thus

$$Q\big(\partial_\Omega V_0(\Omega)\phi + (DW_0)\partial_\Omega u_0)\big)\partial_{p_j}\Omega = -Q\big(V_0(\Omega)\partial_{p_j}\phi + DW_0\partial_{p_j}(\phi + u_0)\big).$$

Quite clearly $\|Q(V_0(\Omega)\partial_{p_j}\phi + DW_0\partial_{p_j}(\phi + u_0))\|_0 < C$, so that the only possible singularities occur when $Q(\partial_\Omega V_0(\Omega)\phi + DW_0\partial_\Omega u_0)$ vanishes. The first term is explicit – $Q\partial_\Omega V_0\phi(\mathbf{p}) = 2\Omega\phi(\mathbf{p})$, which is bounded below by $c\|\mathbf{p}\|$ for Ω bounded away from zero. Using Lemma 3.3, and a Cauchy estimate for the second term,

$$\begin{aligned}\|QDW_0\partial_\Omega u_0\|_0 &\le C_W(\|\mathbf{p}\| + \|u_0\|_\sigma)\|\partial_\Omega u_0\|_\sigma \\ &\le C_W(\|\mathbf{p}\| + \frac{3C_W L_0}{d_0}\|\mathbf{p}\|^2)\frac{6C_W L_0}{\rho_0 d_0}\|\mathbf{p}\|^2 \\ &\le \frac{c}{2}\|\mathbf{p}\| \quad ,\end{aligned}$$

as long as constants are chosen as in the hypotheses of the lemma.

□

3.3. The Induction Hypotheses

Starting from the approximate solution of the previous subsection we will inductively construct better approximate solutions in ever larger regions of the lattice and prove that in the limit they converge to a solution of the original problem (2.8). In this subsection, we will state the inductive hypotheses, and in the next, we verify that they suffice to prove Theorem 2.7.

We begin the statement of the induction scheme with the definition of the constants which appear in the induction and a description of their role which we hope will help the readers orient themselves.

Inductive Constants:

(1) At the $n^{\rm th}$ step in the iteration we solve a finite dimensional problem in a lattice box $B_n = \{(j,k) : |j| + |k| \le L_n\}$, whose size is given by

$$L_n = 2^n L_0 \qquad n \ge 1 \ .$$

(2) The amount by which our $n^{\rm th}$ approximate solution fails to be a true solution is measured by

$$\epsilon_n = \epsilon_0^{\kappa^n} \ \ , \text{ with } \ \kappa > 1 \text{ and } \ n \ge 1 \ \ .$$

The constants ϵ_0 and κ are fixed in section 3.5.

(3) As the iteration proceeds we encounter worse and worse small denominators whose effects are estimated by

$$\delta_n = \frac{1}{L_n^\alpha} \ \ ,$$

where α is a fixed constant.

(4) There is a length scale ℓ_n which estimates the distance over which the effects of a resonance δ_n can be felt. Set

$$\ell_n = L_n^\beta \ \ ,$$

with β is a small constant. We will need to estimate the spectra of the local Hamiltonian operators, $H_{C_{\ell_n}(S)} = (V(\Omega) + D_u W)|_{C_{\ell_n}(S)}$, which are defined on disks $C_{\ell_n}(S) \equiv \{x \in \mathbf{Z}^+ \times \mathbf{Z} : \text{dist}(x,S) < \ell_n\}$, Hence we require that $\ell_n << L_n$.

(5) The approximate solutions will decay exponentially in the size of the indices $|(j,k)|$. The exponential decay rate will change during the iteration. This rate is determined by

$$\sigma_{n+1} = \sigma_n - 6\gamma_n \ \ , \ \gamma_n = \frac{\sigma_0}{64(n+2)^2} \ \ , \ n \ge 0 \ \ .$$

Here, $\sigma_0 < \sigma_*$ is defined in Section 6, and is related to $\overline{\sigma}$, the width of the strip in which the coefficients of (2.1) are analytic. Note that for all $n \ge 0$, $\sigma_n > \overline{\sigma}/2$.

(6) The solutions will depend on parameters (such as the frequency Ω and $\mathbf{p}$, the amplitude and phase of the periodic solution). We require them to be analytic in a complex neighborhood, whose size is governed by

$$\rho_n = \frac{\rho_0 \delta_n}{L_n^2} \ \ , \ n \ge 1 \ ,$$

where ρ_0 is the size of our original analyticity domain.

We are now in a position to state the induction hypotheses. The equation (2.8) is equivalent to two equations obtained by the Lyapunov Schmidt decomposition of the full lattice problem

$$F(\mathbf{p},\Omega,u) \equiv P\big(W(\phi(\mathbf{p})+u)+V(\Omega)u\big)=0 \tag{3.15}$$

$$G(\mathbf{p},\Omega,u) \equiv Q\big(W(\phi(\mathbf{p})+u)+V(\Omega)\phi(\mathbf{p})\big)=0 \quad . \tag{3.16}$$

There exists a constant C_G and a positive exponent μ such that, provided the conditions stated in section 3.5 are satisfied, the following induction statements are true.

(n.1) There is a function $u_n(\mathbf{p},\Omega;x) = u_{n-1}(\mathbf{p},\Omega;x) + v_{n-1}(\mathbf{p},\Omega;x) = u_0(\mathbf{p},\Omega;x) + \sum_{j=0}^{n-1} v_j(\mathbf{p},\Omega;x)$, defined on $\mathcal{N}_0 \times B_n$, and analytic on $D(\mathcal{N}_n,\rho_n)$ satisfying:

(i) For any $(\mathbf{p},\Omega) \in D(\mathcal{N}_j,\rho_j)$,

$$\|v_j(\mathbf{p},\Omega;\cdot)\|_{\sigma_j-\gamma_j} \le \|\mathbf{p}\|^2 \epsilon_j C_G^{j+1}/\delta_{j+1}\gamma_j^{12} \quad , \quad j=0,\ldots,n-1 \quad ,$$

and on $\mathcal{N}_0$, one has

$$\|\partial_\Omega^\beta v_j(\mathbf{p},\Omega;\cdot)\|_{\sigma_j-\gamma_j} \le \frac{C_G^{j+1}\epsilon_j}{\delta_{j+1}\gamma_j^{12}}(\beta+3)!\frac{\|\mathbf{p}\|^2}{(\rho_{j+1}/2)^\beta}$$

$$\|\partial_\mathbf{p}\partial_\Omega^\beta v_j(\mathbf{p},\Omega;\cdot)\|_{\sigma_j-\gamma_j} \le \frac{C_G^{j+1}\epsilon_j}{\delta_{j+1}\gamma_j^{12}}(\beta+4)!\frac{\|\mathbf{p}\|}{(\rho_{j+1}/2)^\beta}\big(1+\frac{r_0}{\rho_0}\big).$$

For higher $\mathbf{p}$ derivatives there is a general estimate

$$\|\partial_\mathbf{p}^\alpha\partial_\Omega^\beta v_j(\mathbf{p},\Omega;\cdot)\|_{\sigma_j-\gamma_j} \le \frac{C_G^{j+1}\epsilon_j}{\delta_{j+1}\gamma_j^{12}}\frac{(|\alpha|+\beta+3)!}{\rho_{j+1}^{|\alpha|+\beta-2}} \quad .$$

This estimate quantifies our control of the C^k norms of v_j.

(ii) The function $u_n(x,\mathbf{p},\Omega)$ is an approximate solution of the first bifurcation equation (3.15). For $(\mathbf{p},\Omega) \in D(\mathcal{N}_n,\rho_n/2)$,

$$\|F(u_n)\|_{\sigma_n} \le \|\mathbf{p}\|^2\epsilon_n \quad .$$

(n.2) There exists a closed set of parameters, $(\mathbf{p},\Omega) \in \mathcal{N}_{n+1} \subset \mathcal{N}_n \subset \ldots \subset \mathcal{N}_0$, with the following properties:

(i) If $(\mathbf{p},\Omega) \in \mathcal{N}_{n+1}$, and if x_i and x_j are any two singular sites in B_n^c, which are not in the same singular region, then the distance between x_i and x_j is greater than $2\ell_{n+1}$.

(ii) If S is a singular region in $B_{n+1}\backslash B_n$, then for any $(\mathbf{p},\Omega) \in D(\mathcal{N}_{n+1}, \rho_{n+1})$,

$$\mathrm{dist}(\mathrm{spec}(H_S(\mathbf{p},\Omega;u_n)),0) > \delta_{n+1}$$

$$\mathrm{dist}(\mathrm{spec}(H_{C_{\ell_{n+1}}(S)}(\mathbf{p},\Omega;u_n)),0) > \delta_{n+1}$$

(iii) For any C^∞ curve, $\Omega(\mathbf{p}) = \mathcal{K}\|\mathbf{p}\|^2(1+C(\|\mathbf{p}\|)) + \omega_1$, with $|\mathcal{K}| \geq L_0^{-\nu}$ and $|C(\cdot)|_{C^1} < 1/2$, then

$$\mathrm{meas}\left\{\|\mathbf{p}\| \in (-r_0, r_0); (\mathbf{p},\Omega(\mathbf{p})) \in \mathcal{N}_{n+1}\right\} \geq r_0(1 - Cr_0^\mu) \quad .$$

(Recall that the exponent ν was defined in Theorem 2.5.)

The induction hypotheses $(n.1)$ and $(n.2)$ suffice to prove Theorem 2.7. We will verify this in the next subsection. The inductive estimates imply in particular that throughout the parameter domain $\mathcal{N}_0$ there is a bound on the derivatives of u_n.

$$\|\partial_{\mathbf{p}}^\alpha \partial_\Omega^\beta u_n\|_{\sigma_n - \gamma_n} \;\leq\; \|\partial_{\mathbf{p}}^\alpha \partial_\Omega^\beta u_0\|_{\sigma_n-\gamma_n} + \|\sum_{j=0}^{n} \partial_{\mathbf{p}}^\alpha \partial_\Omega^\beta v_j\|_{\sigma_n-\gamma_n}.$$

Thus for $\alpha = 0$, we estimate

$$\|\partial_\Omega^\beta u_n\|_{\sigma_n-\gamma_n} \;\leq\; \frac{CL_0}{d_0}\bigl(1 + \frac{C\epsilon_0^{1/\kappa}}{\delta_0}(\beta!)^b\bigr)\frac{\|\mathbf{p}\|^2}{(\rho_0/2)^\beta} \quad .$$

For $|\alpha| = 1$,

$$\|\partial_{\mathbf{p}}\partial_\Omega^\beta u_n\|_{\sigma_n-\gamma_m} \;\leq\; \frac{CL_0}{d_0}\bigl(1 + \frac{C\epsilon_0^{1/\kappa}}{\delta_0}((\beta+1)!)^b\bigr)\frac{\|\mathbf{p}\|}{(\rho_0/2)^\beta}\bigl(1 + \frac{r_0}{(\rho_0/2)}\bigr) \quad .$$

and in general

$$\|\partial_{\mathbf{p}}^\alpha \partial_\Omega^\beta u_n\|_{\sigma_n-\gamma_m} \;\leq\; \frac{CL_0}{d_0}\frac{1}{(\rho_0/2)^{|\alpha|+\beta-2}}\bigl(1 + \frac{C\epsilon_0^{1/\kappa}}{\delta_0}((|\alpha|+\beta)!)^b(1 + \frac{r_0^2}{(\rho_0/2)^2})\bigr) \quad . \tag{3.17}$$

These are obtained using elementary bounds and the above choices of inductive constants. The exponent $b = 1/\log\kappa + 1$. The approximate solution u_n is analytic in

$D(\mathcal{N}_n, \rho_n/2)$, and an estimate similar to (3.17) holds over the complex neighborhood $D(\mathcal{N}_n, \rho_n/4)$, where in fact the exponent $b = 1/\kappa$.

The induction hypotheses also allow us to prove the following important result about the inverse of the linearized operator which appears in Newton's method.

Theorem 3.6. *Suppose that the induction hypotheses $(j.1)$ and $(j.2)$ hold for $j = 0, 1, \ldots, n$. For any non-singular lattice region A and $E_{n+1} \subset \overline{B_{n+1}} \cup A$, the Green's function is analytic on $D(\mathcal{N}_{n+1}, \rho_{n+1})$, and satisfies*

$$\|G_{E_{n+1}}(\mathbf{p}, \Omega, u_n)\|_{\sigma_n - \gamma_n} \leq \frac{C_G^{n+1}}{\delta_{n+1}\gamma_n^{12}} \quad . \tag{3.18}$$

Under perturbations of u_n a similar estimate holds. If

$$\|u - u_n\|_{\sigma_n - \gamma_n} \leq \|\mathbf{p}\|^2 \epsilon_n C_G^{n+1}/\delta_{n+1}\gamma_n^{12} \quad ,$$

the estimate holds,

$$\|G_{E_{n+1}}(\mathbf{p}, \Omega, u)\|_{\sigma_n - 2\gamma_n} \leq \frac{2C_G^{n+1}}{\delta_{n+1}\gamma_n^{12}} \quad . \tag{3.19}$$

We delay the proof of this theorem until section 5, where a detailed analysis of the linearized operators is presented.

3.4. Proof of Theorem 2.7.

In this section we demonstrate that the induction hypotheses lead to a proof of Theorem 2.7.

Set $\mathcal{N} = \cap_{n \geq 0} \mathcal{N}_n$, the parameter domain for which we obtain a solution of the first bifurcation equation (3.15) . The parameters γ_n governing loss of exponential decay in the induction process satisfy $\sigma_n \geq \overline{\sigma}/2$, for all n. Thus the induction hypothesis $(n.1)(i)$ implies that $u_n(\mathbf{p}, \Omega; x)$ converges to a sequence $u(\mathbf{p}, \Omega; x) \in \mathcal{H}_{\overline{\sigma}/2}$,

$$u(\mathbf{p}, \Omega; x) \equiv u_0(\mathbf{p}, \Omega; x) + \sum_{j=0}^{\infty} v_j(\mathbf{p}, \Omega; x),$$

which is C^∞ as a function of $(\mathbf{p},\Omega)$ on the parameter domain $\mathcal{N}_0$. Induction hypothesis $(n.1)(ii)$ implies that for $(\mathbf{p},\Omega)\in\mathcal{N}\subseteq\mathcal{N}_0$ the sequence u satisfies the first bifurcation equation (3.15).

To obtain a solution to the full nonlinear lattice problem, it remains to solve the second bifurcation equation (2.23). This is a finite dimensional problem, recovering the zero set of the mapping G. Not surprisingly, we will show that the solution is close to the approximate solution that was discussed in Section 3.2.

Lemma 3.7. *The mapping* $(\mathbf{p},\Omega)\to G(\mathbf{p},\Omega)=Q(W(\phi(\mathbf{p})+u(\mathbf{p},\Omega))+V(\Omega)\phi(\mathbf{p}))$ *is* C^∞ *on* $\mathcal{N}_0$*, zero for* $\{(\mathbf{p},\Omega);\mathbf{p}=0,\ -r_0^2<(\omega_1-\Omega)<r_0^2\}$ *and has a Taylor expansion at* $\mathbf{p}=0$*,* $\Omega=\omega_1$ *with the following Taylor coefficients:*

$$\partial_{\mathbf{p}}G(0,\omega_1)=0\quad,\quad\partial_\Omega G(0,\omega_1)=0\quad,$$

and

$$\partial_{\mathbf{p}}^2G(0,\omega_1)=0\quad,\quad\partial_\Omega^2G(0,\omega_1)=0\quad,$$
$$\partial_{\mathbf{p}}\partial_\Omega G(0,\omega_1)=Q(\partial_\Omega V(\omega_1))\partial_{\mathbf{p}}\phi(0)=-2\omega_1\partial_{\mathbf{p}}\phi(0)\quad.$$

Proof. The fact that G is C^∞ follows from the fact that u is C^∞ on $\mathcal{N}_0$, which follows from the uniform bounds in (3.17). The Taylor coefficients are computed as in the proof of Lemma 3.4.

□

The zero set of the mapping $G(\mathbf{p},\Omega)$ consists of the Ω-axis union a surface $(\mathbf{p},\Omega(\mathbf{p}))$ which is invariant under the translations T_ξ, and is given as a graph over the neighborhood $\{\mathbf{p}\in\mathbf{R}^2;\|\mathbf{p}\|<r_0\}$. This is the surface of solutions of the second bifurcation equation (2.23). It is close to the approximate solution surface $(\mathbf{p},\Omega_0(\mathbf{p}))$. Intersections of $(\mathbf{p},\Omega(\mathbf{p}))$ with $\mathcal{N}$ correspond to solutions of the first bifurcation equation as well, thus are solutions of the full problem (2.8). Such intersections are guaranteed if the surface $(\mathbf{p},\Omega(\mathbf{p}))$ has nonzero curvature at zero.

Lemma 3.8. *The solution surface* $(\mathbf{p},\Omega(\mathbf{p}))$ *of (2.23) is defined in* $\{\mathbf{p}\in\mathbf{R}^2;\|\mathbf{p}\|<r_0\}$*. The surface satisfies*

$$\Omega(0)=\omega_1\quad,\quad\partial_{p_j}\Omega(0)=0\quad,\partial_{p_jp_k}\Omega(0)=\mathcal{K}\delta_{jk}\quad,$$

where

$$\mathcal{K} \;=\; \frac{1}{6\omega_1}\langle \partial_{\mathbf{p}}\phi, (D_u^3 W(0)(\partial_{\mathbf{p}}\phi)^3)\rangle + \frac{1}{2\omega_1}\langle \partial_{\mathbf{p}}\phi, D_u^2 W(0)[\partial_{\mathbf{p}}\phi, (\partial_{\mathbf{p}}^2 u)]\rangle.$$

The difference $|\mathcal{K}-\mathcal{K}_0| < C\big(e^{-(\overline{\sigma}L_0)/2} + \epsilon_0^{1/\kappa} L_0/d_0\big)/(\delta_0\rho_0^2)$.

Proof. The proof is essentially the same as that for Lemma 3.5, using the solution u in place of the approximate solution u_0. The estimates (3.4) on u_0 from the bifurcation theory on B_0 are replaced by (3.17) from the induction argument. The expression for the curvature can be rewritten as;

$$\begin{aligned}\mathcal{K} \;&= \mathcal{K}_0 + \frac{1}{6\omega_1}\langle \partial_{\mathbf{p}}\phi, (\mathbf{1}-\Pi_0)D_u^3 W(0)(\partial_{\mathbf{p}}\phi)^3\rangle \\ &+ \;\frac{1}{2\omega_1}\langle \partial_{\mathbf{p}}\phi, (\mathbf{1}-\Pi_0)D_u^2 W(0)[\partial_{\mathbf{p}}\phi, \partial_{\mathbf{p}}^2 u]\rangle \\ &+ \frac{1}{2\omega_1}\langle \partial_{\mathbf{p}}\phi, D_u^2 W_0(0)[\partial_{\mathbf{p}}\phi, (\partial_{\mathbf{p}}^2 u - \partial_{\mathbf{p}}^2 u_0)]\rangle\end{aligned}$$

The difference is estimated using (3.17).

$$\begin{aligned}|\mathcal{K}-\mathcal{K}_0| \;&\le C\big(\|(\mathbf{1}-\Pi_0)D_u^3 W(0)(\partial_{\mathbf{p}}\phi)^3\|_0 + \|(\mathbf{1}-\Pi_0)D_u^2 W(0)[\partial_{\mathbf{p}}\phi, \partial_{\mathbf{p}}^2 u]\|_0 \\ &\quad + \;\|D_u^2 W_0(0)[\partial_{\mathbf{p}}\phi, (\partial_{\mathbf{p}}^2 u - \partial_{\mathbf{p}}^2 u_0)]\|_0\big) \\ &\le Ce^{-(\overline{\sigma}L_0)/2}(1 + \|\partial_{\mathbf{p}}^2 u\|_{\overline{\sigma}/2}) + \frac{C_W}{(\overline{\sigma}/2)^2}\|\partial_{\mathbf{p}}^2(u-u_0)\|_0 \\ &\le Ce^{-(\overline{\sigma}L_0)/2}\big(1 + \frac{1}{\delta_0\rho_0^2}\big) + \frac{C_W}{(\overline{\sigma}/2)^2}\,\frac{L_0\epsilon_0^{1/\kappa}}{d_0\delta_0}\left(1+\frac{r_0^2}{\rho_0^2}\right).\end{aligned} \tag{3.20}$$

Assuming that $|\mathcal{K}_0| > L_0^{-\nu}$, if

$$\big(e^{-(\overline{\sigma}L_0/2)} + \epsilon_0^{1/\kappa} L_0/d_0\big)/(\delta_0\rho_0^2) << L_0^{-\nu},$$

the curvature of the surface $(\mathbf{p}, \Omega(\mathbf{p}))$ satisfies $|\mathcal{K}| > cL_0^{-\nu}$, and thus by the induction hypothesis $(n.2)(iii)$ the set $(\mathbf{p}, \Omega(\mathbf{p}))\cap\mathcal{N}$ is nonempty. This inequality will be verified in Section 3.5 which follows

$\square$

The Klein-Gordon case $G(x,u) = m^2(u-u^3)$ is somewhat special in these considerations of curvature of the bifurcation branches. Firstly $D_u^2 W(0) = 0$, since the nonlinearity appears only at cubic order. Additionally for $L_0 \ge 6$, $(\mathbf{1}-\Pi_0)D_u^3 W(0)(\partial_{\mathbf{p}}\phi)^3 = 0$, thus the expression (3.20) gives that $\mathcal{K} = \mathcal{K}_0$, and the twist condition is satisfied automatically.

3.5. Final Reckoning

The proof of convergence of this induction is based on a proper choice of constants that appear in the analysis throughout the paper. There are roughly three requirements to be satisfied. First the inductive constants α, β, τ and κ must be chosen so that the Newton iteration method is convergent. This involves estimates of the truncation in approximating the nonlinearity, the excision process in the parameter domain, and the iterative construction of the Green's function. Secondly, the initial approximate bifurcation problem and parameter neighborhood must be sufficiently large so that, even after the excisions of the induction steps, the remaining solution set has large measure. Finally, we must be able to carry out the analysis of convergence for an open dense set of nonlinearities $g(x,u)$ for the wave equation, thus requirements of Proposition 2.4 and Theorems 2.8 and 2.9 must be satisfied. In this subsection we show that all the requirements of the paper can be simultaneously satisfied. We accordingly make choices of the inductive parameters α, β, τ and κ, and show that the principal requirements on the parameters r_0, ρ_0 and d, d_0 and d_s reduce to a condition that L_0, the initial radius of an approximating lattice domain, be sufficiently large. As technical aspects of the induction verification appear in sections 4 and 5, this subsection occasionally refers ahead to to formulae in these sections.

Before the induction starts, a bifurcation analysis is performed on the initial domain B_0 of radius L_0. The apriori estimates on the initial domain require that the induction starts with $u_0 \in \mathcal{H}_{\sigma_0}$, $\sigma_0 + \gamma_0 < \sigma_* - 1/L_0$. Setting $d_0 = L_0^{-\eta}$ for some $1/2 < \eta < 1$, and $r_0 = \rho_0 = cL_0^{-(2+\eta)}$, for c some small positive number, we are able to satisfy the hypotheses of Lemma 3.3 and Lemma 3.5,

$$\begin{aligned} r_0 <& C\frac{d_0}{L_0^2} \\ \rho_0 <& C\frac{d_0}{L_0^2} \\ r_0^2\frac{L_0}{d_0\rho_0} <&< 1 \end{aligned} \tag{3.21}$$

We next turn to the choice of the inductive constants and the question of convergence. The sequence $u_0 \in \ell^2(B_0) \cap \mathcal{H}_{\sigma_0}$ is an exact solution of the restricted equation

$\Pi_0 F(u_0) = 0$, thus there exists $c_0 > 0$ such that

$$\begin{aligned} \|F(u_0(\mathbf{p},\Omega))\|_{\sigma_0} &= \|(\mathbf{1} - \Pi_0) W(\phi(\mathbf{p}) + u_0)\|_{\sigma_0} \\ &\le \frac{C_W}{\gamma_0} e^{-\gamma_0 L_0} \big(\|\mathbf{p}\|^2 + \frac{1}{\gamma_0^2}\|u_0\|^2_{\sigma_0+\gamma_0}\big) \\ &\le e^{-c_0 L_0}\|\mathbf{p}\|^2 \end{aligned}$$

as long as $L_0 \ge L_*$, is chosen sufficiently large so that $\sigma_* - 1/L_0 - \gamma_0 \ge \sigma_0 > \overline{\sigma}/2$. We have used (3.21) and **H1** to estimate the nonlinear term. Make the choice

$$\epsilon_0 = e^{-c_0 L_0}.$$

For $n > 0$ the iteration will have a supergeometric rate of convergence as long as the hypotheses of Proposition 4.1 are satisfied;

$$\epsilon_{n-1} \frac{C C_G^{n+1}}{\delta_n \gamma_{n-1}^{14}} \left(e^{-\gamma_{n-1} L_n} + \frac{r_0^2 \epsilon_{n-1} C_G^{n+1}}{\delta_n \gamma_{n-1}^{13}} \right) \le \epsilon_n. \tag{3.22}$$

Using the definition of the inductive constants, (3.22) will follow from the pair of inequalities

$$C_G^{n+1} L_0^{\alpha} e^{\alpha \log(2) n} \big(\frac{c}{|n|^2}\big)^{14} e^{-L_0 c 2^n/|n|^2} \le \frac{1}{2}\epsilon_0^{\kappa^{n-1}(\kappa-1)} \tag{3.23}$$

and

$$L_0^{-2(2+\eta)} C_G^{2(n+1)} L_0^{2\alpha} e^{2\alpha\log(2)n} \big(\frac{c}{|n|^2}\big)^{27} \le \frac{1}{2}\epsilon_0^{\kappa^{n-1}(\kappa-2)} \tag{3.24}$$

The governing factor of the left hand side of (3.23) is $e^{-cL_0 2^n/|n|^2}$, while from the definition of ϵ_0 that of the right hand side is $e^{-c_0 L_0(1-(1/\kappa))\kappa^n}$. If necessary decrease c_0 so that $c_0(1-(1/\kappa)) < 2c$. For L_* sufficiently large the estimate (3.23) will hold for all $n > 0$, uniformly in $L_0 > L_*$.

A similar discussion verifies that (3.24) will hold for all $n > 0$ if $L_0 > L_*$ is sufficiently large. The right hand side is governed by the factor $e^{(\gamma_0 L_0 (2-\kappa)\kappa^{n-1})}$, while the left hand exponent grows at most linearly in n and logarithmically in L_0. For $L_0 > L_*$, L_* chosen sufficiently large, it too will hold for all $n > 0$.

In order to make the excision process work, relations (4.4)–(4.7) of Proposition 4.6 will have to hold. Set

$$2d = 2d_s = d_0 = L_0^{-\eta}$$

If we use the definitions above, we see that if $(1-\tau\beta) > 0$, $\beta(1+c_0\log(2)) < 1$ and $L_0 > L_*$ is chosen large, then the first two relations of (4.4)-(4.7) are satisfied. The following three relations will have already been satisfied in (3.21). Without yet discussing the measure of the remaining parameter region, we will verify the rest of the requirements of section 4. In order to separate the localizing neighborhoods $C_{\ell_n}(S)$, the hypothesis of Lemma 4.8 is that

$$d > L_0^{\tau\beta}(L_0^{-1} + r_0^2 L_0^{\beta}) = L_0^{\tau\beta-1} + L_0^{\beta(\tau+1)-2(2+\eta)} \quad ,$$

which will hold for $\eta < (1-\tau\beta)$ and $\beta(\tau+1) < \eta + 4$, which can be satisfied by taking β small. The extension of these estimates to the complex domain will go through if (4.17) holds. By definition $L_n\rho_n < \delta_n/2$, and one checks readily that $(1 + L_0/d_0(1 + (r_0/\rho_0)))\rho_{n+1} = (1 + 2L_0^{(1+\eta)})L_0^{-(2+\eta)}\delta_{n+1}/L_{n+1}^2 \leq \delta_{n+1}$.

In order to construct the Green's function using the procedure of section 5, define $2d_s = d_0 = L_0^{-\eta}$, and ask that $L_0 > L_*$ is sufficiently large so that $C_W r_0/(\overline{\sigma}^2 d_s) \leq C\sigma_*$. This permits the construction of the Green's function on non-singular domains. If we choose

$$\sigma_0 + 5\gamma \equiv (\frac{35}{32}) < \sigma_* - 2\log(1 + 4\sqrt{C_W r_0/(\overline{\sigma}^2 d_s)}) \quad ,$$

then the restrictions on σ_0 in both Theorem 5.1 and Proposition 5.3 can be satisfied.

This is the second of two conditions to be satisfied in defining σ_0, the starting decay rate of the induction. The requirements of the extension lemmas 5.3 and 5.4 are that $r_0 < Cd_0$ and $r_0 < \gamma_0^4 d_s$, which are both satisfied as in (3.21) by an increase in L_* if necessary. The more intricate patching techniques of Theorem 5.6 require additionally that

$$\frac{Cr_0 e^{-\gamma_{n-1}\ell_{n+1}/2}}{\delta_{n+1}^2\gamma_{n-1}^{34}} \leq 1. \tag{3.25}$$

Using the definitions above, (3.25) reads

$$CL_0^{-(2+\eta)}L_0^{2\alpha}e^{\alpha\log(2)(n+1)} \times C|n|^{68} \times e^{-(c/2|n|^2)L_0^{\beta}}e^{\beta\log(2)(n+1)} < 1 \quad .$$

As before, for any choices of α, β there is an L_* such that this will hold for all $n \geq 1$, uniformly in $L_0 > L_*$. Using (3.25), the hypotheses of Lemmas 5.9 and 5.10 follow accordingly. The second type of extension of the Green's function is addressed in Theorem 5.12, where it is asked that

$$r_0^2\epsilon_n \leq \frac{\delta_{n+1}^2\gamma_n^{27}}{2C_W C_G^{2n}}. \tag{3.26}$$

Using the definitions, it is clearly tantamount to

$$L_0^{-2(2+\eta)}\epsilon_0^{\kappa^n} \leq L_0^{-2\alpha}e^{-2\alpha\log(2)(n+1)}\Big(\frac{c}{|n+1|^2}\Big)^{27}\frac{1}{2C_W C_G^{2n}},$$

which can be uniformly satisfied for $L_0 > L_*$, for all $n \geq 0$ as long as L_* is sufficiently large. Corollary 5.13 follows from Theorem 5.12 if the correction term v_j is small in norm. That is, given (3.22) and (3.23), we have

$$\|v_j\|_{\sigma_j-\gamma_j} \leq \|\mathbf{p}\|^2\frac{\epsilon_j C_G^j}{\delta_{j+1}\gamma_j^{12}} \leq \frac{1}{4^{j+1}}\frac{\gamma_{j+1}\gamma_0^4 d_s}{2C_W C},$$

which is even easier to satisfy than (3.26).

The remaining requirements that are placed on the inductive constants come from Proposition 4.6, where an estimate is made of the measure of the parameter set after the n^{th} excision. This will involve the choice of α and τ. To satisfy (4.5),

$$\begin{aligned} &c\sum_{j=0}^{n}\frac{d}{L_j^\tau}(1+2r_0^2L_{j+1}) \\ &= CL_0^{-(\tau+\eta)}\sum_{j=0}^{n}e^{-\tau\log(2)j} + 4Cr_0^2L_0^{-(\tau+\eta-1)}\sum_{j=0}^{n}e^{-\tau\log(2)j} \\ &\leq C\frac{L_0^{-(\tau+\eta)}}{\tau} \leq r_0^2 \end{aligned} \tag{3.27}$$

which will hold, uniformly in n, as long as $\tau > 4+\eta$ and L_* is large. We will construct the $\mathcal{N}_{n+1}$ by first constructing two intermediate sets $\mathcal{N}_{n+1}^1$ and $\mathcal{N}_{n+1}^2$. The estimate of (3.27) bounds the measure of the excisions that are made in constructing $\mathcal{N}_{n+1}^{(1)}$. To control the size of excisions in constructing $\mathcal{N}_{n+1}^{(2)}$, we require

$$\begin{aligned} &\sum_{j=0}^{n}CL_j(1+r_0^2L_{j+1})\ell_{j+1}^2\frac{\delta_{j+1}}{\mathcal{K}L_j^2} \\ &\leq CL_0^{-(\alpha+1-2\beta-\nu)}\sum_{j=0}^{n}e^{-(\alpha+1-2\beta)\log(2)j} \\ &\quad + CL_0^{-(\alpha+2(2+\eta)-2\beta-\nu)}\sum_{j=0}^{n}e^{-(\alpha-2\beta)\log(2)j} \\ &\leq r_0^2 \quad . \end{aligned} \tag{3.28}$$

As long as we satisfy the requirements that

$$\alpha > 2\beta + \nu$$
$$\alpha + 1 - 2\beta > 2(2+\eta)$$
$$\alpha + 4 + 2\eta - 2\beta > 2(2+\eta)$$

then (3.28) will hold uniformly in $L_0 > L_*$ and $n > 0$.

Addressing (4.6), we see that the inequalities will be satisfied if the curvature $\mathcal{K}$ of the surface is restricted from being too small. Using the above choices,

$$C(\frac{1}{d_s} + \frac{L_0}{d_0}(1 + \frac{r_0}{\rho_0})) = C(L_0^{\eta} + 2L_0^{1+\eta}) \le \mathcal{K} L_0^2 2^{2n}.$$

This will hold for all $n \ge 0$, uniformly in $L_0 > L_*$ if the curvature satisfies

$$\mathcal{K} > L_0^{-\nu}, \qquad 0 \le \nu < 1 - \eta \tag{3.29}$$

We define ν in the twist condition of Theorem 2.7, for a bifurcation surface must intersect the sets $\mathcal{N}_{n+1} = \mathcal{N}_{n+1}^{(1)} \cap \mathcal{N}_{n+1}^{(2)}$ to exhibit a solution of the equation. To estimate the measure of this intersection, we show that for α and τ reasonably large, (4.7) is satisfied for some exponent μ. The first inequality is

$$\begin{aligned}
&\sum_{j=0}^{n} \sqrt{\frac{C}{\mathcal{K}} \frac{d}{L_j^{\tau}} (1 + r_0^2 L_0^{j+1})} \\
&\qquad \le \sum_{j=0}^{n} \sqrt{\frac{C}{\mathcal{K}}} \big(L_0^{-(\eta+\tau)/2} e^{-\tau \log(2) j/2} \\
&\qquad\qquad + L_0^{-(\eta+\tau-1)/2} L_0^{-(2+\eta)} e^{-(\tau-1)\log(2)(j+1)/2} \big) \\
&\qquad \le C \sqrt{\frac{1}{\mathcal{K}}} \big(L_0^{-(\tau+\eta)/2} + L_0^{-(\tau+\eta+3)/2} \big)
\end{aligned} \tag{3.30}$$

where the exponential sums are bounded uniformly in n as long as $\tau > 1$. This sum is dominated by $r_0^{(1+\mu)} = L_0^{-(2+\eta)(1+\mu)}$ if $\mathcal{K} > L_0^{-\nu}$ and $0 < \mu < (\tau - (\eta + \nu + 4))/2$. Considerations of the second inequality are similar.

$$\begin{aligned}
&\sum_{j=0}^{n} \sqrt{C L_j (1 + r_0^2 L_0^{j+1}) \times \ell_{j+1}^2 \times \frac{1}{\mathcal{K}} \times \frac{\delta_{j+1}}{L_j^2}} \\
&\qquad \le \sum_{j=0}^{n} \sqrt{\frac{C}{\mathcal{K}}} \big(L_0^{-(\alpha+1-2\beta)/2} e^{-(\alpha+1-2\beta)\log(2) j/2} \\
&\qquad\qquad + L_0^{-(\alpha+2(2+\eta)-2\beta)/2} e^{-(\alpha-2\beta)\log(2) j/2} \big) \\
&\qquad \le \sqrt{\frac{C}{\mathcal{K}}} \big(L_0^{-(\alpha+1-2\beta)/2} + L_0^{-(\alpha+2(2+\eta)-2\beta)/2} \big),
\end{aligned} \tag{3.31}$$

where we must require that $\alpha > 2\beta$. Allowing $\mathcal{K} > L_0^{-\nu}$, this expression will be dominated by $r_0^{(1+\mu)}$ for

$$0 < \mu < \frac{\alpha - (\nu + 2(\beta + \eta) + 3)}{2(2 + \eta)}.$$

Clearly a choice of α and τ large, and β small is available so that these inequalities will hold throughout the induction.

The last relations that should be verified pertain to the the density of the nonlinearities to which the results apply. We require that $d_0 = o(L_0^{-1/2})$ in order to satisfy the hypotheses of Proposition 2.4. This gives a genericity result for the coefficients g_1. We thus restrict $1/2 < \eta < 1$ in order to do this. Secondly, we ask for bounds on the curvature in the twist condition. Setting $|\mathcal{K}| \geq L_0^{-\nu}$ with $\nu > 0$ gives a bound which is decreasing in L_0, and again there is an open dense set of nonlinearities possessing a sufficiently large twist for Theorem 2.7 to apply.

4. Verification Of The Induction Hypotheses

In this section we begin the verification of the induction hypotheses. There are two main subsections; in the first, we show that if $(j.1)$ and $(j.2)$ hold for $j = 0, 1, \ldots, n-1$ then we can prove $(n.1)$, and in the second we construct sets $\mathcal{N}_{n+1}$ which satisfy $(n.2)$.

4.1. Verification Of $(n.1)$

Assume that $(j.1)$ and $(j.2)$ hold $j = 0, 1, \ldots, n-1$. Then $(n.1)$ is a consequence of the following estimate on the Newton iteration.

Proposition 4.1. *If the inductive parameters satisfy*

$$\epsilon_{n-1} \frac{CC_G^{n+1}}{\delta_n \gamma_{n-1}^{14}} \left[e^{-\gamma_{n-1} L_n} + \frac{r_0^2 \epsilon_{n-1} C_G^{n+1}}{\delta_n \gamma_{n-1}^{13}} \right] \leq \epsilon_n \ , \tag{4.1}$$

then there is a function $u_n(\mathbf{p}, \Omega; z) = u_0(\mathbf{p}, \Omega; z) + \sum_{j=0}^{n-1} v_j(\mathbf{p}, \Omega; z)$, defined on $\mathcal{N}_0 \times B_n$, which obeys the following estimates:

(1) For any $j = 0, \ldots, n-1$, v_j is analytic on $D(\mathcal{N}_j; \rho_j/2)$ and for any $(\mathbf{p}, \Omega)$ in this set it satisfies

$$\|v_j(\mathbf{p}, \Omega; \cdot)\|_{\sigma_j - \gamma_j} \leq \epsilon_j \|\mathbf{p}\|^2 C_G^{j+1} / \delta_{j+1} \gamma_j^{12} \ .$$

Furthermore, there exists a positive constant C_3 such that v_j is a C^∞ function of $(\mathbf{p},\Omega)$ on all of $\mathcal{N}_0$, satisfying the following estimates.

$$\begin{aligned}\|\partial_\Omega^\beta v_j(\mathbf{p},\Omega;\cdot)\|_{\sigma_j-\gamma_j} \leq& C_3\epsilon_j C_G^{j+1}\frac{(\beta+3)!}{(\delta_{j+1}\gamma_j^{12}\rho_{j+1}^\beta)}\|\mathbf{p}\|^2 \quad ,\\ \|\partial_\mathbf{p}\partial_\Omega^\beta v_j(\mathbf{p},\Omega;\cdot)\|_{\sigma_j-\gamma_j} \leq& C_3\epsilon_j C_G^{j+1}\frac{(\beta+4)!}{(\delta_{j+1}\gamma_j^{12}\rho_{j+1}^\beta)}(1+\frac{r_0}{\rho_0})\|\mathbf{p}\| \quad .\end{aligned}$$

and for $|\alpha|\geq 2$,

$$\|\partial_\mathbf{p}^\alpha\partial_\Omega^\beta v_j(\mathbf{p},\Omega;\cdot)\|_{\sigma_j-\gamma_j} \leq C_3\epsilon_j C_G^{j+1}\frac{(|\alpha|+\beta+3)!}{(\delta_{j+1}\gamma_j^{12}\rho_{j+1}^{|\alpha|+\beta-2})} \quad .$$

The usual multi-index notation for derivatives is used in this estimate.

(2) For every $(\mathbf{p},\Omega)\in D(\mathcal{N}_n;\rho_n/2)$, the function u_n satisfies

$$\|F(u_n(\mathbf{p},\Omega))\|_{\sigma_n} \leq \|\mathbf{p}\|^2\epsilon_n \quad .$$

Proof. The idea of this proposition is to construct v_{n-1} using Newton's method. We expect the solution to be small far from the origin, so rather than trying to solve $F(u_{n-1}+v_{n-1})=0$, we study the approximation $\Pi_n F(u_{n-1}+v_{n-1})=0$, where Π_n is the orthogonal projection onto $\ell^2(B_n)$. Since

$$\Pi_n F(u_{n-1}+v_{n-1}) \approx \Pi_n\left[F(u_{n-1})+DF(u_{n-1})v_{n-1}\right] \quad ,$$

a better approximation of a solution to the problem is given by performing an iteration step based on Newton's method. This entails inverting the linearized operator $\Pi_n(DF(u_{n-1}))^{-1}\Pi_n, = G_{\overline{B_n}}(u_{n-1})$. This Green's function is not defined on all of $\mathcal{N}_0$, but on the subset $D(\mathcal{N}_n,\rho_n)$ it is boundedly invertible. On this set we define the correction to u_{n-1},

$$\tilde{v}_{n-1} = -\Pi_n(DF(u_{n-1}))^{-1}\Pi_n F(u_{n-1}).$$

From the estimates of Theorem 3.6, $G_{\overline{B_n}}$ is analytic on $D(\mathcal{N}_n;\rho_n)$, and satisfies (3.19). Furthermore by $(n-1.1)(ii)$, we know that $\|PF(u_{n-1}(\mathbf{p},\Omega))\|_{\sigma_{n-1}} \leq \|\mathbf{p}\|^2\epsilon_{n-1}$, on $D(\mathcal{N}_{n-1},\rho_{n-1}/2)$. Thus, on $D(\mathcal{N}_n;\rho_n/2)$, $G_{\overline{B_n}}$ and $\Pi_n F(u_{n-1})$ are analytic, and

$$\begin{aligned}\|\tilde{v}_{n-1}\|_{\sigma_{n-1}-\gamma_{n-1}} \leq& \|G_{\overline{B_n}}\|_{\sigma_{n-1}-\gamma_{n-1}}\|\Pi_n F(u_{n-1})\|_{\sigma_{n-1}-\gamma_{n-1}}\\ \leq& \epsilon_{n-1}\|\mathbf{p}\|^2 C_G^n/(\delta_n\gamma_{n-1}^{12}) \quad ,\end{aligned}$$

where we used (3.19) to estimate $\|G_{\overline{B_n}}\|_{\sigma_{n-1}-\gamma_{n-1}}$. One has

Lemma 4.2. *$\tilde{v}_{n-1}(\mathbf{p},\Omega;x)$ is analytic on $D(\mathcal{N}_n;\rho_n)$ and for any $(\mathbf{p},\Omega)$ in this domain it satisfies*

$$\|\tilde{v}_{n-1}(\mathbf{p},\Omega;\cdot)\|_{\sigma_{n-1}-\gamma_{n-1}} \leq \epsilon_{n-1}\|\mathbf{p}\|^2 \frac{C_G^n}{\delta_n \gamma_{n-1}^{12}} \quad . \tag{4.2}$$

We now construct v_{n-1} by smoothly extending $\tilde{v}_{n-1}(\mathbf{p},\Omega;x)$ to all of $\mathcal{N}_0$, doing this essentially by setting $\tilde{v}_{n-1} = 0$ on $\mathcal{N}_0 \backslash \mathcal{N}_n$, however using some care in order to obtain in the limit a C^∞ function. We use the following from the appendix of [C].

Lemma 4.3. *For every $R > 0$ and for every compact set $\Delta \subset \mathbf{C}^d$, there exists a $\chi \in C^0(\mathbf{C}^d) \cap C^\infty(\mathbf{R}^d) : \mathbf{C}^d \to [0,1]$. χ has support in $Y_R(\Delta) \equiv \cup_{\eta_0 \in \Delta}\{\eta \in \mathbf{C}^d; \|\eta - \eta_0\| \leq R\}$ and $\chi(\eta) = 1$ for every $\eta \in Y_{R/2}(\Delta)$. Finally, for every positive integer k,*

$$\sup_{\mathbf{R}^d} \|\partial_\eta^k \chi\| \leq \frac{|k|(|k|+2)!}{R^{|k|}} \quad .$$

Define χ_{n-1} to be a function satisfying the hypotheses of the lemma with $d = 3$, $\Delta = \overline{\mathcal{N}_n}$, and $R = \rho_n/2$, and set $v_{n-1} = \chi_{n-1}\tilde{v}_{n-1}$.

Lemma 4.4. *Within the domain of analyticity $D(\mathcal{N}_n;\rho_n/2)$ the Cauchy estimate applied to (4.2) controls all derivatives $\partial_\Omega^\beta \partial_{\mathbf{p}}^\alpha v_{n-1}(\mathbf{p},\Omega;\cdot)$. Using Lemma 4.3, the estimate on all of $\mathcal{N}_0$ is,*

$$\begin{aligned}
\|\partial_\Omega^\beta v_{n-1}(\mathbf{p},\Omega;\cdot)\|_{\sigma_{n-1}-\gamma_{n-1}} \leq & C_3 \epsilon_{n-1} C_G^n \frac{(\beta+3)!}{(\delta_n \gamma_{n-1}^{12} \rho_{n+1}^\beta)} \|\mathbf{p}\|^2 \quad , \\
\|\partial_{\mathbf{p}} \partial_\Omega^\beta v_{n-1}(\mathbf{p},\Omega;\cdot)\|_{\sigma_{n-1}-\gamma_{n-1}} \leq & C_3 \epsilon_{n-1} C_G^n \frac{(\beta+4)!}{(\delta_n \gamma_{n-1}^{12} \rho_{n+1}^\beta)} \big(1 + \frac{r_0}{\rho_0}\big) \|\mathbf{p}\| \quad .
\end{aligned}$$

and for $|\alpha| \geq 2$,

$$\|\partial_{\mathbf{p}}^\alpha \partial_\Omega^\beta v_{n-1}(\mathbf{p},\Omega;\cdot)\|_{\sigma_{n-1}-\gamma_{n-1}} \leq C_3 \epsilon_{n-1} C_G^n \frac{(|\alpha|+\beta+3)!}{(\delta_n \gamma_{n-1}^{12} \rho_{n+1}^{|\alpha|+\beta-2})} \quad .$$

Lemma 4.2 and Lemma 4.4, together with the remark that for $\mathrm{dist}((\mathbf{p},\Omega),\mathcal{N}_n) \leq \frac{1}{2}\rho_n$, $v_{n-1} = \tilde{v}_{n-1}$, imply the first assertion of Proposition 4.1.

We complete the proof of Proposition 4.1 by verifying that $u_n \equiv u_{n-1} + v_{n-1}$ is an approximate solution of the first bifurcation equation $F(u) = 0$. Note that

$$\begin{aligned} F(u_n) =& F(u_{n-1} + v_{n-1}) = \{(\mathbf{1} - \Pi_n)F(u_n) \\ &+ \Pi_n \left(F(u_{n-1} + v_{n-1}) - [F(u_{n-1}) + DF(u_{n-1})v_{n-1}]\right)\} \quad , \end{aligned} \tag{4.3}$$

where we used the fact that v_{n-1} was constructed so that
$\Pi_n[F(u_{n-1}) + DF(u_{n-1})v_{n-1}] = 0$, (on $D(\mathcal{N}_n; \rho_n)$.)

We can bound the various terms in (4.3) by using the following observations:

(i) From hypotheses **H2** and **H3** of Section 6, there exist positive constants, C_W and σ, (with $\sigma < \sigma_0$) such that the operator norm satisfies $\|D_u W(u)\|_{\sigma-\gamma} \le C_W/\gamma^2$ and $\|D_u^2 W(u)[v, w]\|_{\sigma-\gamma} \le C_W/\gamma^3 \|v\|_\sigma \|w\|_{\sigma-\gamma}$, for all u with $\|u\|_\sigma \le 1$.

(ii) Using the explicit form of $V(\Omega)$, we see that $|V(\Omega)(j,k)| \le (\Omega_m^2 + 1)(j^2 + k^2)$, thus

$$\|V(\Omega)v\|_{\sigma-\gamma} \le (\Omega_m^2 + 1) \sup_{j,k} (j^2 + k^2) e^{-\gamma(|j|+|k|)} \|v\|_\sigma \le \frac{4(\Omega_m^2 + 1)}{\gamma^2} \|v\|_\sigma \ .$$

where $\Omega_m =$ the maximum value of $\Omega = \omega_1 + \sqrt{r_0^2 + \rho_0^2}$.

(iii) By Lemma 3.2 $\|(\mathbf{1} - \Pi_n)w\|_{\sigma-\gamma} \le e^{-\gamma L_n} \|w\|_\sigma$.

Point (iii) implies that

$$\|(\mathbf{1} - \Pi_n)F(u_n)\|_{\sigma_n} \le e^{-\gamma_{n-1} L_n} \|F(u_n)\|_{\sigma_{n-1} - 2\gamma_{n-1}} \ .$$

The fundamental theorem of calculus implies that

$$\begin{aligned} \|F(u_n)\|_{\sigma_{n-1}-2\gamma_{n-1}} \le& ||F(u_{n-1})||_{\sigma_{n-1}-2\gamma_{n-1}} \\ &+ \| \int_0^1 (DF(u_{n-1} + t v_{n-1}) v_{n-1}) dt \|_{\sigma_{n-1}-2\gamma_{n-1}} \end{aligned}$$

By $(n - 1.1)$ the first term is bounded by $\epsilon_{n-1} \|\mathbf{p}\|^2$. Points (i) and (ii) imply that

$$\| \int_0^1 (DF(u_{n-1} + t v_{n-1}) v_{n-1}) dt \, \|_{\sigma_{n-1}-2\gamma_{n-1}} \le \left[\frac{C_W + 4(\Omega_m^2 + 1)}{\gamma_{n-1}^2} \right] \|v_{n-1}\|_{\sigma_{n-1}-\gamma_{n-1}} \quad ,$$

completing the estimate of the truncation error.

We now estimate $[F(u_{n-1}+v_{n-1}) - F(u_{n-1}) - DF(u_{n-1})v_{n-1}]$, using the fundamental theorem of calculus to rewrite it as as $\int_0^1 \int_0^t D^2F(u_{n-1}+sv_{n-1})v_{n-1}v_{n-1}dsdt$. This is quadratic in v_{n-1}. From the first of the observations above,

$$\| \int_0^1 \int_0^t D^2F(u_{n-1}+sv_{n-1})v_{n-1}v_{n-1}dsdt\|_{\sigma_{n-1}-2\gamma_{n-1}} \le \frac{C_W}{\gamma_{n-1}^3} \|v_{n-1}\|^2_{\sigma_{n-1}-\gamma_{n-1}} \quad .$$

In deriving this estimate we used the fact that $D_u^2F(u_{n-1}) = D_u^2W(u_{n-1})$. Combining these three estimates, we obtain the lemma:

Lemma 4.5. *The exists a constant C, depending on C_W and Ω_m, such that if*

$$\epsilon_{n-1}\frac{CC_G^{n+1}}{\delta_n\gamma_{n-1}^{14}}\left[e^{-\gamma_{n-1}L_n} + \frac{r_0^2\epsilon_{n-1}C_G^{n+1}}{\delta_n\gamma_{n-1}^{13}}\right] \le \epsilon_n$$

then

$$\|F(u_n)\|_{\sigma_n} \le \|\mathbf{p}\|^2\epsilon_n \quad ,$$

for $(\rho,\Omega) \in D(\mathcal{N}_n, \rho_n/2)$.

Proof. The estimates above imply that the truncation error is

$$\begin{aligned}
\|(\mathbf{1}-\Pi_n)F(u_n)\|_{\sigma_n} &\le e^{-\gamma_{n-1}L_n}\|F(u_n)\|_{\sigma_{n-1}-2\gamma_{n-1}} \\
&\le \|\mathbf{p}\|^2\epsilon_{n-1}e^{-\gamma_{n-1}L_n}\left[1+\frac{(C_W+4(\Omega_m^2+1)C_G^n}{\delta_n\gamma_{n-1}^{14}}\right] \\
&\le \|\mathbf{p}\|^2\epsilon_{n-1}e^{-\gamma_{n-1}L_n}\frac{CC_G^n}{\delta_n\gamma_{n-1}^{14}} \quad .
\end{aligned}$$

The contribution from the quadratic error is bounded by

$$\|\Pi_n\left\{F(u_{n-1}+v_{n-1}) - [F(u_{n-1}) + DF(u_{n-1})v_{n-1}]\right\}\|_{\sigma_n} \le \frac{\epsilon_{n-1}^2\|\mathbf{p}\|^4C_WC_G^{2n}}{\delta_n^2\gamma_{n-1}^{27}} \quad .$$

The proposition follows if the sum of these two terms to be less than ϵ_n. Combining these two estimates and using the fact that $\|\mathbf{p}\|^2 \le r_0^2$, this follows from the hypothesis of Lemma 4.5.

4.2. Verification of $(n.2)$

We continue now by showing how one constructs the set $\mathcal{N}_{n+1}$, described in the induction hypothesis $(n.2)$. We will construct $\mathcal{N}_{n+1}$ using the properties of u_n given by $(n.1)$. Recall that we constructed $\mathcal{N}_0$ in Section 2.3. We denote by C a constant independent of n and of the inductive constants. The following proposition not only implies $(n.2)$, but includes some other useful information as well.

Proposition 4.6. *Assume that the following relationships hold between the inductive constants.*

$$\begin{aligned} &\frac{Cd}{2^{\tau+1}} < L_n^{(1-\tau\beta)}, \quad \tau\beta < 1 \\ &\beta(1+c_0\log(2)) < 1 \\ &\frac{C_W L_0 r_0^2}{\overline{\sigma}^3 d_0 \rho_0} < 1 < \Omega_{min} L_n^2 \\ &\frac{16 C_W}{\overline{\sigma}^2} r_0 < d_s \\ &(1+\frac{L_0}{d_0} r_0) < C \end{aligned} \tag{4.4}$$

Then there exists a closed set $\mathcal{N}_{n+1} \subseteq \mathcal{N}_n$ *such that:*

(a) If $(\mathbf{p},\Omega) \in \mathcal{N}_{n+1}$*, then any two singular sites in* B_n^c*, which are not in the same singular region, are separated by a distance of at least* $2\ell_{n+1}$.

(b) If S *is a singular region in* $B_{n+1}\backslash B_n$ *and* $(\mathbf{p},\Omega) \in D(\mathcal{N}_{n+1},\rho_{n+1})$ *then*

$$\operatorname{dist}(\operatorname{spec}(H_{C_{\ell_{n+1}}(S)}(\mathbf{p},\Omega;u_n)),0) > \delta_{n+1}$$

and

$$\operatorname{dist}(\operatorname{spec}(H_S(\mathbf{p},\Omega;u_n)),0) > \delta_{n+1}$$

If the inductive constants satisfy

$$\begin{aligned} &\sum_{j=0}^{n} C\frac{d}{L_j^\tau}(1+2r_0^2 L_{j+1}) < r_0^2 \\ &\sum_{j=0}^{n} CL_j(1+r_0^2 L_{j+1}) \times \ell_{j+1}^2 \times \frac{\delta_{j+1}}{L_j^2} < r_0^2 \end{aligned} \tag{4.5}$$

then $\mathcal{N}_{n+1}$ *has positive measure. If additionally,*

$$C(\frac{1}{d_s} + \frac{L_0}{d_0}(1+\frac{r_0}{\rho_0})) < \frac{\mathcal{K} L_n^2}{4} \tag{4.6}$$

and for some $\mu > 0$,

$$\begin{aligned} \sum_{j=0}^{n} \sqrt{\frac{Cd}{\mathcal{K} L_j^{\tau}}(1 + r_0^2 L_{j+1})} &< r_0^{1+\mu} \\ \sum_{j=0}^{n} \sqrt{C L_j (1 + r_0^2 L_{j+1}) \times \ell_{j+1}^2 \times \frac{\delta_{j+1}}{\mathcal{K} L_j^2}} &< r_0^{1+\mu} \end{aligned} \tag{4.7}$$

then the set $\mathcal{N}_{n+1}$ satisfies the following intersection condition:

(c) Let $\Omega(\mathbf{p}) = \omega_1 + \mathcal{K}\|\mathbf{p}\|^2(1 + C(\|\mathbf{p}\|))$, be a surface with non-degenerate quadratic contact at $\mathbf{p} = 0$. If $|\mathcal{K}| > L_0^{-\nu}$, and $|C(\cdot)|_{C^1} \leq 1/2$, then the set $\{0 \leq r < r_0;\ r = \|\mathbf{p}\|,\ (\mathbf{p}, \Omega(\mathbf{p})) \in \mathcal{N}_{n+1}\}$ has measure greater than or equal to $r_0(1 - C r_0^{\mu})$.

Remark. In fact the set $\mathcal{N}_{n+1}$ is invariant under the rotations $\mathbf{p} \to T_\xi \mathbf{p}$. This invariance is a consequence of the invariance of the spectrum of the linearized operator $H_{B_{n+1}}(u)$, under the rotations $u \to T_\xi u$.

Lemma 4.7. *The spectrum of the operator $H_{E_{n+1}}(T_\xi u)$ is independent of ξ.*

Proof. The properties of the nonlinear function in (2.8) are that $T_\xi F(u) = F(T_\xi u)$, and furthermore T_ξ commutes with orthogonal projection onto $\ell^2(E)$ for any $E \subset \mathbf{Z}^+ \times \mathbf{Z}$. Differentiating with respect to u we find $T_\xi DF(u)v = DF(T_\xi u)T_\xi v$, thus $DF(u)$ and $DF(T_\xi u)$ are unitarily equivalent, and the result follows.

□

Part (a) of Proposition 4.6 can be proven without restricting $\mathbf{p}$–we need only place conditions on Ω. From the diophantine condition on ω_1 there is an integer N_0 such that for the initial induction steps $0 \leq n \leq N_0$, the separation condition (a) is satisfied for all $(\mathbf{p}, \Omega) \in \mathcal{N}_0$. This is the result of the next lemma, whose hypotheses form a subset of those of Proposition 4.6.

Lemma 4.8. *Assume that $\tau\beta < 1$, $\beta(1 + c_0 \log(2)) < 1$, and that $d > L_0^{\tau\beta}(1/L_0 + r_0^2 L_0^{\beta})$. There exists a constant $c_0 = c_0(\tau, \beta)$ such that, if L_0 is sufficiently large and $N_0 = c_0 \log(L_0)$, then for all $n \leq N_0$ and $(\mathbf{p}, \Omega) \in \mathcal{N}_n$, if $x = (k, j)$, $x' = (k', j')$ are a pair of singular sites in B_n^c which are not in the same singular region, they are well separated;*

i.e.

$$\mathrm{dist}(x, x') \geq 2\ell_{n+1} \quad .$$

Proof. Assume that $|x - x'| < 2\ell_{n+1}$. We will derive a contradiction. If x is a singular site then $|j^2\Omega^2 - \omega_k^2| < d_s$. Factoring the left hand side of this inequality and assuming that j and k are non-negative this implies $|j\Omega - \omega_k| < d_s/|j\Omega + \omega_k| \leq C(\Omega_m)d_s/(|j|+|k|)$, for some constant $C(\Omega_m)$. Similar estimates hold for $|j'\Omega - \omega_{k'}|$, and for the cases when j and j' are negative. We now use the asymptotics of the Sturm-Liouville operator. For the case of Dirichlet boundary conditions there exists a constant C_{g_1} such that $|\omega_k - k| < C_{g_1}/k$, for all $k \geq 1$. (For a review of the properties of these eigenvalues see, for example [PT].) For periodic boundary conditions, if $k = 2m$ or $k = 2m - 1$, $m \geq 1$, then $|\omega_k - 2m| < C_{g_1}/m$. Note that since x and x' are in B_n^c, there exists a constant $C(\Omega_m)$ such that $\min(|k|, |k'|) \geq C(\Omega_m)L_n$. In the case of Dirichlet boundary conditions we have

$$\begin{aligned} |(j - j')\Omega - (k - k')| &= |(j - j')\Omega - (\omega_k - \omega_{k'}) + (\omega_k - \omega_{k'}) - (k - k')| \\ &\leq \frac{2C(\Omega_m)d_s}{L_n} + \frac{2C(\Omega_m)C_{g_u}}{L_n} \leq \frac{C(g_1, \Omega)}{L_n} \quad . \end{aligned} \tag{4.8}$$

Since $x \in B_n^c$ we have $|j|+|k| \geq L_n$, with a similar estimate for x'. We are assuming that $\mathrm{dist}(x, x') = |j - j'| + |k - k'| \leq 2\ell_{n+1} = 2L_{n+1}^\beta = 2(2^{n+1}L_0)^\beta$. But $n \leq N_0 = c_0 \log(L_0)$ so $\mathrm{dist}(x, x') \leq 2(2^{c_0 \log(L_0)}L_0)^\beta = 2(L_0^{\beta(1+c_0 \log 2)}) < L_0$, if $\beta(1 + c_0 \log 2) < 1$ and L_0 is sufficiently large. The principal frequency ω_1 is (d_0, L_0)-nonresonant, thus it satisfies a finite diophantine condition over the lattice points within B_0. Hence as long as $2\ell_{n+1} < L_0$ and $|\Omega - \omega_1| < r_0^2$, $(x - x') \in B_0$ and the components satisfy

$$\begin{aligned} |(j - j')\Omega - (k - k')| &\geq |(j - j')\omega_1 - (k - k')| - |(j - j')(\Omega - \omega_1)| \\ &\geq \frac{d}{(2\ell_{n+1})^\tau} - 2\ell_{n+1}r_0^2 \quad . \end{aligned} \tag{4.9}$$

By the hypotheses of the lemma, $4\ell_{n+1}r_0^2 < \frac{1}{2}\frac{d}{(2\ell_{n+1})^\tau}$, so we find $|(j - j')\Omega - (k - k')| \geq \frac{1}{2}\frac{d}{(2\ell_{n+1})^\tau}$. Combining this with (4.8) gives

$$\frac{1}{2}\frac{d}{(2\ell_{n+1})^\tau} \leq \frac{C(g_1, \Omega_m)}{L_n}$$

or, using $\ell_{n+1} = L_{n+1}^\beta$, $L_{n+1}^{(1-\tau\beta)} \leq 2^{\tau(1+\beta)}C(g_1, \Omega)/d$. If $\tau\beta < 1$, and L_0 is sufficiently large, this inequality cannot be true and the lemma follows.

$\square$

In the case of periodic boundary conditions, the argument needs to be modified only slightly to accound for the different asymptotics of the eigenvalues. The estimate in (4.8) is replaced by

$$|(j-j')\Omega - 2([k/2]-[k'/2])| \leq \frac{C(g_1,\Omega_m)}{L_n} \quad , \tag{4.10}$$

where $[\cdot]$ denotes the integer part. Combining this estimate with the lower bound coming from the fact that ω_1 is (d_0, L_0) nonresonant, we find that $\mathrm{dist}(x,x')$ cannot be less than $2\ell_{n+1}$ except in one special case. This special case occurs when $k = 2m$ and $k' = 2m-1$ or vice-versa. Then (4.10) becomes $|(j-j')\Omega| \leq C(g_1,\Omega_m)/L_n$. But if, in addition, $j = j'$ we cannot apply the diophantine estimate to obtain the contradiction. Thus, if (j,k) and (j',k') are any pair of singular sites with either $j \neq j'$ or $\{k,k'\} \neq \{2m, 2m-1\}$, then they are separated by a distance of at least $2\ell_{n+1}$.

Remark. As a corollary of the proof of this lemma, we see that in the case of Dirichlet boundary conditions, the singular regions consist of isolated singular sites, while in the case of periodic boundary conditions the singular regions consist either of isolated sites, or pairs of adjacent sites, with $j = j'$, and $\{k,k'\} = \{2m, 2m+1\}$.

Singular sites are not necessarily well separated for all $(\mathbf{p},\Omega) \in \mathcal{N}_0$, for induction steps $n > N_0$. In order to insure that part (a) of Proposition 4.6 will hold when $n > N_0$, we delete from $(\omega_1 - r_0^2, \omega_1 + r_0^2)$ those values of Ω for which it would fail to be true. For this purpose we use the following elementary result in Diophantine approximation.

Lemma 4.9. *For any bounded interval of the real line, I, of length $|I|$,*

$$\begin{aligned}\mathrm{meas}\{\Omega \in I; |j\Omega - k| < d(|j|+|k|)^{-\tau}, \quad \text{for} \quad L_n \leq |j|+|k| \leq 4L_{n+1}\} \\ \leq \frac{Cd}{L_n^\tau}(1+|I|L_n) \quad .\end{aligned} \tag{4.11}$$

For $n > N_0$ define $\mathcal{N}_{n+1}^{(1)}$ inductively by deleting from $\mathcal{N}_n$ all points $(\mathbf{p},\Omega)$ such that $|j\Omega - k| < d(|j|+|k|)^{-\tau}$, for some (j,k) with $L_n \leq |j|+|k| \leq 4L_{n+1}$. This will delete finitely many open disks from $\mathcal{N}_n^{(1)}$. We assert that for $\Omega \in \mathcal{N}_{n+1}^{(1)}$, then any two singular sites $x, x' \in B_n^c$ which are not in the same singular region satisfy

$\mathrm{dist}(x, x') \geq 2\ell_{n+1}$. Indeed the argument used in Lemma 4.8 can be modified to prove this degree of separation. The one change is that in the estimate leading to (4.9), if $\Omega \in \mathcal{N}^{(1)}_{n+1}$, then $|j\Omega - k| \geq d(|j| + |k|)^{-\tau}$ for all $0 < |j| + |k| \leq 4L_{n+1}$. Thus, (4.9) is replaced by

$$|(j - j')\Omega - (k - k')| \geq \frac{d}{(2\ell_n)^\tau} \quad . \tag{4.12}$$

Thus we have obtained the following result.

Lemma 4.10. *For any $n \geq 0$, $(\mathbf{p}, \Omega) \in \mathcal{N}^{(1)}_{n+1}$, and any pair of singular sites in B^c_n, which are not in the same singular region,*

$$\mathrm{dist}(x, x') \geq 2\ell_{n+1} \quad .$$

Since we will define $\mathcal{N}_{n+1}$ to be a subset of $\mathcal{N}^{(1)}_{n+1}$, this lemma implies part (a) of Proposition 4.6.

Next consider part (b) of Proposition 4.6, where we must address the dependence of the local hamiltonians and their spectra upon the parameters $(\mathbf{p}, \Omega)$. In order to belong to a singular region S, a lattice point $x = (j, k)$ must satisfy $(\omega_1 - r_0^2)|k| - C_0 \leq |j| \leq (\omega_1 + r_0^2)|k| + C_0$. Furthermore, in either of the above problems a singular region consists of either one site alone, or a pair of adjacent sites of the form $x_1 = (j, 2m - 1)$, $x_2 = (j, 2m)$. Suppose $S \subseteq B_{n+1} \backslash B_n$ is a set of this type so it has the possibility for being a singular region for some $(\mathbf{p}, \Omega) \in \mathcal{N}_0$. The local hamiltonian for the surrounding neighborhood $C_{\ell_{n+1}}(S)$ is

$$H_{C_{\ell_{n+1}}(S)}(\mathbf{p}, \Omega; u_n) = \big(D_u W(\phi(\mathbf{p}) + u_n) + V(\Omega)\big)|_{C_{\ell_{n+1}}(S)}.$$

This matrix has at most $4\ell^2_{n+1}$ many eigenvalues, which we list in order; $e_i(\mathbf{p}, \Omega)$, for $i = 1, 2, \ldots$. In this subsection we will study the behavior of the eigenvalues of H_S and $H_{C_{\ell_{n+1}}(S)}$ with respect to parameters. The goal is to control the subsets of $\mathcal{N}_0$ on which either H_S or $H_{C_{\ell_{n+1}}(S)}$ has a small eigenvalue, of size $|e_i(\mathbf{p}, \Omega)| < \delta_{n+1}$. (To save space, unless the meaning is unclear we will write $H_{C_{\ell_{n+1}}(S)}$ as $H_{C(S)}$.)

Lemma 4.11. *For S fixed, each eigenvalue $e_i(\mathbf{p}, \Omega)$ of $H_{C(S)}$ is continuous in $(\mathbf{p}, \Omega) \in \mathcal{N}_0$, differentiable almost everywhere, and is strictly monotone decreasing in Ω if $S \subseteq$*

$\{(j,k) \in \mathbf{Z}^+ \times \mathbf{Z}, k > 0\}$. *(They are strictly increasing in the opposite case* $S \subseteq \{(j,k) \in \mathbf{Z}^+ \times \mathbf{Z}, k < 0\}$*.)*

Proof: For $S \subseteq \{(j,k) \in \mathbf{Z}^+ \times \mathbf{Z}, k > 0\} \cap B_n^c$, then as long as $\ell_{n+1} < L_n/(2\omega_1) + C_0$, every lattice site $x = (j,k) \in C(S)$ satisfies $k > L_n/2$, that is, all of $C(S)$ is contained within the positive k quadrant. The local hamiltonian $H_{C(S)}$ is monotone decreasing in Ω. Indeed,

$$\partial_\Omega H_{C(S)}(\mathbf{p}, \Omega; u_n) = \big(\partial_\Omega V(\Omega) + D_u^2 W(\phi(\mathbf{p}) + u_n)(\partial_\Omega u_n)\big)_{C(S)},$$

and since for every $x = (j,k) \in C(S),\ |k| > L_n/2$,

$$\big(\partial_\Omega V(\Omega)\big)_{C(S)}(x,y) = -2\Omega(k^2 \delta(x,y)) < -\Omega_{min} L_n^2/2(\delta(x,y)).$$

The second term is bounded by

$$\begin{aligned} \|D_u^2 W(\phi(\mathbf{p}) + u_n)[\partial_\Omega u_n, w]\|_0 &\le \frac{C_W}{(\sigma_n - \gamma_n)^3} \|\partial_\Omega u_n\|_{\sigma_n - \gamma_n} \|w\|_0, \\ &\le \frac{8 C_W L_0 r_0^2}{\overline{\sigma}^3 d_0 \rho_0} \|w\|_0 \end{aligned} \tag{4.13}$$

For $(C_W L_0 r_0^2/(\overline{\sigma}^3 d_0 \rho_0)) \le 1 \le \Omega_{min} L_n^2$, the quadratic form $\langle \psi, H_{C(S)} \psi \rangle$ is monotone decreasing in Ω, and thus for fixed $\mathbf{p}$ the eigenvalues $e_i(\mathbf{p}, \Omega)$ are monotone decreasing in Ω.

Note that if there are no quadratic terms in the nonlinearity W, then

$$\|D_u^2 W[\partial_\Omega u_n, w]\|_0 \le \frac{8 C_W}{\overline{\sigma}^3} \frac{L_0 r_0^3}{d_0 \rho_0} \|w\|, \tag{4.14}$$

a better estimate than (4.13).

The eigenvalues vary continuously with respect to $(\mathbf{p}, \Omega) \in \mathcal{N}_0$, however they are not necessarily smooth in $\mathcal{N}_0$. Examples of Rellich, [R], show that even a two by two matrix depending analytically on several parameters may not have differentiable eigenvalues. This lack of smoothness occurs at points of collision between eigenvalues. However on sets where either all eigenvalues are distinct, or the multiplicity of all eigenvalues does not change, it is possible to choose eigenvalues and eigenvectors to be smooth functions of $(\mathbf{p}, \Omega)$, [K]. These sets of constancy for eigenvalue multiplicity of $H_{C(S)}$ are semianalytic sets, thus $e_i(\mathbf{p}, \Omega)$ are differentiable almost everywhere on $\mathcal{N}_0$, and the eigenvectors can be chosen differentiably almost everywhere as well. $\square$

Let us fix a set S which has the possibility of being singular for some $(\mathbf{p}, \Omega) \in \mathcal{N}_0$, and assume that for $x \in C(S), k > 0$. We are concerned with the location within $\mathcal{N}_0$ of the zero sets $Z_i \equiv \{(\mathbf{p}, \Omega) \in \mathcal{N}_0; e_i(\mathbf{p}, \Omega) = 0\}$. For $\mathbf{p} = 0, e_i(0, \Omega) = -(\Omega^2 m^2 - \omega_\ell^2)$ for some $(\ell, m) \in C(S)$, so that it vanishes only for $\Omega = \omega_\ell / m$. For each $\mathbf{p} \in \{\|\mathbf{p}\| < r_0\}$, $e_i(\mathbf{p}, \Omega)$ is monotone decreasing, thus $Z_i \cap \{\mathbf{p} = \text{const}\}$ consists of at most one point. By the continuity and strict monotonicity of $e_i(\mathbf{p}, \Omega)$, Z_i is given by a surface $(\mathbf{p}, \Omega_i(\mathbf{p}))$, where $\Omega_i(\mathbf{p})$ is continuous but not necessarily smooth, and $\Omega_i(0) = \omega_\ell / m$ for some $(\ell, m) \in C(S)$. Using Lemma 4.7, the sets Z_i are invariant under T_ξ.

Small neighborhoods of these sets Z_i contain all the parameter values in $\mathcal{N}_0$ for which the local hamiltonian $H_{C(S)}$ has a small eigenvalue.

Lemma 4.12. *Let $e_i(\mathbf{p}, \Omega)$ be an eigenvalue of $H_{C(S)}$ such that $|e_i(\mathbf{p}, \Omega)| < \delta_{n+1}$. Then there exists Ω_1 with $|\Omega - \Omega_1| < C\delta_{n+1}/L_n^2$ such that $e_i(\mathbf{p}, \Omega_1) = 0$.*

Thus by excising all $(\mathbf{p}, \Omega) \in \mathcal{N}_0$ with $\text{dist}(\Omega, \Omega_1) < C\delta_{n+1}/L_n^2$, for any $(\mathbf{p}, \Omega_1) \in Z_i$, $i = 1, 2, \ldots$, the remaining parameters avoid small eigenvalues for $H_{C(S)}$. If this excision is performed for all regions $S \subseteq B_{n+1} \backslash B_n$ which are potentially singular regions, the remaining set of parameters will satisfy Proposition 4.6, (b). Our goal will be to estimate the measure, and the geometry of the remaining parameter region.

Proof. For fixed $\mathbf{p}$, $e_i(\mathbf{p}, \Omega)$ is a decreasing function of Ω, thus it is differentiable almost everywhere, and by Fatou's lemma

$$e_i(\mathbf{p}, \Omega_1) - e_i(\mathbf{p}, \Omega_2) \geq \int_{\Omega_1}^{\Omega_2} -\partial_\Omega e_i(\mathbf{p}, \omega)\, d\omega$$

for any $\Omega_1 \leq \Omega_2$. Given $e_i(\mathbf{p}, \Omega)$ and associated eigenvector ψ_i,

$$e_i = \langle \psi_i, H_{C(S)} \psi_i \rangle,$$

thus where e_i and ψ_i are differentiable,

$$\begin{aligned} \partial_\Omega e_i =& \langle \psi_i, \partial_\Omega H_{C(S)} \psi_i \rangle + 2\langle \partial_\Omega \psi_i, H_{C(S)} \psi_i \rangle \\ =& \langle \psi_i, \partial_\Omega H_{C(S)} \psi_i \rangle + 2e_i \langle \partial_\Omega \psi_i, \psi_i \rangle \end{aligned} .$$

The eigenvectors are normalized, so that

$$\begin{aligned}1 =&|\psi_i|^2 = \langle \psi_i, \psi_i\rangle \\ 0 =&2\langle \partial_\Omega \psi_i, \psi_i\rangle,\end{aligned}$$

hence

$$\partial_\Omega e_i = \langle \psi_i, \partial_\Omega H_{C(S)}\psi_i\rangle. \tag{4.15}$$

This is known as the Feynman-Hellman formula. We can estimate the derivative,

$$\begin{aligned}-\partial_\Omega e_i &\geq 2\Omega_{min} L_n^2 - \|D_u^2 W[\partial_\Omega u_n, \psi_i]\|_0 \\ &\geq \Omega_{min} L_n^2,\end{aligned} \tag{4.16}$$

as long as $8C_W L_0 r_0^2/d_0\rho_0\overline{\sigma}^3 < \Omega_{min}L_n^2$. Now suppose that $0 \leq e_i(\mathbf{p}, \Omega_1) \leq \delta_{n+1}$. Then

$$\begin{aligned}(\Omega - \Omega_1)(\Omega_{min}/2)L_n^2 &\leq \int_{\Omega_1}^{\Omega} -\partial_\Omega e_i(\mathbf{p}, \omega)\ d\omega, \\ &\leq e_i(\mathbf{p}, \Omega_1) - e_i(\mathbf{p}, \Omega) \ \leq\ \delta_{n+1} - e_i(\mathbf{p}, \Omega),\end{aligned}$$

thus e_i must vanish for a nearby value $\Omega = \Omega_2$, and furthermore $|\Omega_2 - \Omega_1| < C\delta_{n+1}/L_n^2$. □

To obtain the subset $\mathcal{N}_{n+1}^{(2)} \subseteq \mathcal{N}_n \subseteq \mathcal{N}_0$ on which all local hamiltonians avoid small eigenvalues, delete from $\mathcal{N}_n^{(2)}$ all points $(\mathbf{p}, \Omega)$ such that there is some $S \subseteq \{(j,k) \in B_{n+1}\backslash B_n;\ (\omega_1 - r_0^2)|k| - C_0 < j < (\omega_1 + r_0^2)|k| + C_0\}$, and some $(\mathbf{p}, \Omega_1) \in Z_i$ a zero set for an eigenvalue of the operators H_S or $H_{C_{\ell_{n+1}}(S)}$ such that $|\Omega_1 - \Omega| < C\delta_{n+1}/L_n^2$. The remaining parameters in $\mathcal{N}_{n+1}^{(2)}$ will now satisfy

$$|e_i(\mathbf{p}, \Omega)| > 2\delta_{n+1}.$$

These lower bounds on the size of eigenvalues are now extended to the complex domain $D(\mathcal{N}_{n+1}, \rho_{n+1})$. For $(\mathbf{p}, \Omega) \in D(\mathcal{N}_{n+1}, \rho_{n+1})$, there are real $(\mathbf{p}_1, \Omega_1) \in \mathcal{N}_{n+1}$ with $\sqrt{\|\mathbf{p} - \mathbf{p}_1\|^2 + |\Omega - \Omega_1|^2} < \rho_{n+1}$. Compute the operator norm of the difference of the local hamiltonians, $(H_{C(S)}(\mathbf{p}, \Omega; u_n) - H_{C(S)}(\mathbf{p}_1, \Omega_1; u_n))$;

$$\begin{aligned}&\|H_{C(S)}(\mathbf{p}, \Omega; u_n) - H_{C(S)}(\mathbf{p}_1, \Omega_1; u_n)\|_0 \\ &\quad \leq \|(V(\Omega) - V(\Omega_1))C(S)\|_0 + \|(D_u W(\phi(\mathbf{p}) + u_n) - D_u W(\phi(\mathbf{p}_1) + u_n))_{C(S)}\|_0 \\ &\quad \leq CL_{n+1}^2\rho_{n+1} + \frac{C_W}{\overline{\sigma}^3}\big(\|\partial_{\mathbf{p}}(\phi + u_n)\|_{\sigma_n - \gamma_n}\|\mathbf{p} - \mathbf{p}_1\| + \|\partial_\Omega u_n\|_{\sigma_n - \gamma_n}|\Omega - \Omega_1|\big) \\ &\quad \leq CL_{n+1}^2\rho_{n+1} + \frac{C_W}{\overline{\sigma}^3}\big(1 + \frac{L_0}{d_0}(1 + \frac{r_0}{\rho_0}) + \frac{L_0 r_0^2}{d_0\rho_0}\big)\rho_{n+1} \\ &\quad \leq \delta_{n+1}\end{aligned}$$

as long as both

$$\begin{aligned} CL_{n+1}\rho_{n+1} &< \delta_{n+1}/2, \\ C(1+(L_0/d_0)(1+(r_0/\rho_0))\rho_{n+1} &< \delta_{n+1}. \end{aligned} \tag{4.17}$$

The remaining parameters in $D(\mathcal{N}_{n+1}, \rho_{n+1})$ will now satisfy (b) of Proposition 4.6. We can estimate the total measure of the region $\mathcal{N}_0$ that is excised by this process. The number of possible singular regions S is bounded by $CL_n(1+r_0^2 L_{n+1})$. For each region S and surrounding neighborhood $C_{\ell_{n+1}}(S)$ we excise $C\delta_{n+1}/L_n^2$ of zero sets Z_i. There are one or two small eigenvalues for S and at most $4\ell_{n+1}^2$ for $C_{\ell_{n+1}}(S)$. Each of these neighborhoods is invariant under T_ξ, thus the total measure of $\mathcal{N}_0$ that is excised is bounded by

$$(CL_n(1+r_0^2 L_{n+1})\ell_{n+1}^2)(\delta_{n+1}/L_n^2)\pi r_0^2. \tag{4.18}$$

Define $\mathcal{N}_{n+1} \equiv \mathcal{N}_{n+1}^{(1)} \cap \mathcal{N}_{n+1}^{(2)}$, the good parameter region for the $(n+1)^{st}$ induction step. Conditions (4.5) insure that the sum over n of the measure of these excisions will be small when compared to meas $\mathcal{N}_0 = 2\pi r_0^4$.

This is still not enough to prove statement (c) of Proposition 4.6, which is the guarantee that a smooth surface $(\mathbf{p}, \Omega(\mathbf{p}))$ with nonzero curvature will intersect $\mathcal{N}_{n+1}$ for a set of $\mathbf{p}$ of large measure. This will rely on more detailed knowledge of the geometry of the sets Z_i.

Lemma 4.13. *If $L_0 r_0^2/d_0\rho_0 < c_0$, then there is a constant C_1 such that the set Z_i lies between the cones*

$$\frac{\omega_\ell}{m} \pm \frac{C_1}{L_n^2}\|\mathbf{p}\|$$

for some $(\ell, m) \in C_{\ell_{n+1}}(S)$.

Proof. We have chosen $e_i(\mathbf{p}, \Omega)$ so that $e_i(0, \omega_\ell/m) = 0$ for some $x \in (\ell, m) \in C(S)$. The operator $H_{C(S)}$ is monotone decreasing along any line $(\mathbf{p}(s), \Omega(s)) = (s\mathbf{p}_0, \omega_\ell/m + sC_1/L_n^2) \in \mathcal{N}_0$. Indeed, using the arguments of Lemma 4.12, we have

$$\frac{d}{ds}H_{C(S)}(\mathbf{p}(s), \Omega(s); u_n) = \frac{C_1}{L_n^2}\partial_\Omega H_{C(S)} + \mathbf{p}_0 \cdot \partial_{\mathbf{p}} H_{C(S)}.$$

The $\mathbf{p}$ derivative is bounded by

$$\begin{aligned}
\|\mathbf{p}_0 \cdot \partial_{\mathbf{p}} H_{C(S)}\|_0 \leq & \|D_u^2 W(\phi(\mathbf{p}) + u_n)(\partial_{\mathbf{p}}(\phi + u_n))\|_0 \|\mathbf{p}_0\| \\
\leq & \frac{C_W}{(\sigma_n - \gamma_n)^3} \|\mathbf{p}\| \|\partial_{\mathbf{p}}(\phi + u_n)\|_{(\sigma_n - \gamma_n)} \\
\leq & \frac{C_W}{\sigma^3} \|\mathbf{p}_0\| (1 + \frac{C L_0}{d_0}(1 + \frac{r_0}{\rho_0})).
\end{aligned}$$

Under the hypotheses of the lemma there is a constant C_1 such that the monotonicity of $\partial_\Omega H_{C(S)}$ dominates the variations given by $\mathbf{p} \cdot \partial_{\mathbf{p}} H_{C(S)}$, thus the operator is monotone decreasing along $(\mathbf{p}(s), \Omega(s))$, thus so are its eigenvalues.

To show that Z_i lies to the right of the cone $(\omega_\ell/m) + (C_1/L_n^2)\|\mathbf{p}\|$, consider

$$\begin{aligned}
e_i(\mathbf{p}(s), \Omega(s)) &= e_i(\mathbf{p}(s), \Omega(s)) - e_i(0, \omega_\ell/m) \\
&\leq \int_0^1 \frac{d}{ds} e_i(\mathbf{p}(t), \Omega(t)) \; dt \leq 0.
\end{aligned}$$

The cone $(\omega_\ell/m) - (C_1/L_n^2)\|\mathbf{p}\|$ is shown to be a lower bound with a similar argument. $\square$

Remark. In the case where the nonlinear term W contains no quadratic terms, there is a better estimate of both $\partial_\Omega W$ from (4.14), and $\partial_{\mathbf{p}} W$;

$$\|D_u^2 W(\phi + u_n)(\partial_{\mathbf{p}}(\phi + u_n)\|_0 \leq \frac{C_W}{\sigma^3} \left(1 + \frac{L_0 r_0^2}{d_0 \rho_0}\right) \|\mathbf{p}\|. \tag{4.19}$$

The sets Z_i can then be shown to lie between the paraboloids

$$\frac{\omega_\ell}{m} \pm \frac{C_1}{L_n^2} \|\mathbf{p}\|^2.$$

The more subtle control over the behavior of the sets Z_i that we require is that within the region $\mathcal{N}_{n+1}^{(1)}$, there is a form of parabolic estimate even for the general nonlinearity.

Lemma 4.14. *Choose $C_1 = \hat{C}(1/d_s + (L_0/d_0)(1 + (r_0/\rho_0))$. If $(\mathbf{p}_1, \Omega_1), (\mathbf{p}_2, \Omega_2) \in Z_i$ also lie within the same component of $\mathcal{N}_{n+1}^{(1)}$, then*

$$|\Omega_2 - \Omega_1| < \frac{C_1}{L_n^2} \big| \|\mathbf{p}_2\|^2 - \|\mathbf{p}_1\|^2 \big|. \tag{4.20}$$

That is, the sets Z_i are restricted by paraboloids rather than cones, whenever they intersect $\mathcal{N}^{(1)}_{n+1}$. The proof of this lemma depends upon several facts. First, for $(\mathbf{p}, \Omega) \in \mathcal{N}^{(1)}_{n+1}$ there can be at most two eigenvalues of the local hamiltonians $H_S(\mathbf{p}, \Omega; u_n)$ and $H_{C(S)}(\mathbf{p}, \Omega; u_n)$ which are smaller than $3d_s/4$. Certainly both domains S and $C(S)$ have only one singular region, so they contain at most two singular sites at which $|V(\Omega)(x)| < d_s$. Then the operator norm of $D_uW(\phi(\mathbf{p}) + u_n)$ is bounded;

$$\begin{aligned}\|D_uW(\phi(\mathbf{p}) + u_n)\|_0 <& \frac{C_W}{(\sigma_n - \gamma_n)^2}\Big(\|\mathbf{p}\| + \frac{CL_0}{d_0}\|\mathbf{p}\|^2\Big) \\ \leq& \frac{4C_W}{\overline{\sigma}^2}\|\mathbf{p}\| \quad .\end{aligned}$$

Thus if $16C_W r_0/\overline{\sigma}^2 < d_s$, the operator norm of D_uW is bounded by $d_s/4$, and the eigenvalues of $V(\Omega)$ can be perturbed by at most this much.

Secondly, the eigenvectors associated with the small eigenvalues of $H_{C(S)}$ have special structure. They are supported principally on the singular region S, with estimates of their extent over the other sites of $C(S)$. Denote as usual P_S the orthogonal projection onto $\ell^2(S)$. Let (ψ_i, e_i) be eigenvector-eigenvalue pairs for the eigenvalues such that $|e_i| < 3d_s/4$.

Proposition 4.15. *Under the hypotheses of Proposition 4.6, if* $(\mathbf{p}, \Omega) \in \mathcal{N}^{(1)}_{n+1}$, *then*

$$\|(\mathbf{1} - P_S)\psi_i\|_0 \leq \frac{C\|\mathbf{p}\|}{d_s\overline{\sigma}^{10}}$$

Proof. This result is based upon Corollary 5.5 on the structure of the Green's function for a region of the character of $C(S)$. There is a contour $\mathcal{C}$ surrounding all eigenvalues of $H_{C(S)}$ with $|e_i| \leq d_s/2$ such that $\mathrm{dist}(\mathcal{C}, \mathrm{spec} H_{C(S)}) > d_s/8$. For $C_W\|\mathbf{p}\|/\overline{\sigma}^2 < Cd_s$, whenever $\zeta \in \mathcal{C}$ the hypotheses of Corollary 5.5 are satisfied, and can represent $G_{C(S)} = G_D \oplus G_S + R$, where $D = C(S)\backslash S$, G_D is analytic over $|\zeta| < 3d_s/4$, and $R(x, y)$ satisfies the estimate

$$|R(x, y)| \leq \frac{C\|\mathbf{p}\|}{d_s^2\overline{\sigma}^{10}} e^{-(\sigma_n - 3\gamma_n)[\mathrm{dist}(x,S) + \mathrm{dist}(S,y)]} .$$

In particular $\|R\|_0 \leq C\|\mathbf{p}\|/(d_s^2\overline{\sigma}^{10})$.

Define the orthogonal projection P_E related to the small eigenvalues

$$P_E = \frac{1}{2\pi i}\oint_{\mathcal{C}} \mathbf{0} \oplus G_S(\zeta)\ d\zeta$$

which has range within $\ell^2(S)$. Since G_D is analytic in a $3d_s/4$ neighborhood of the origin,

$$\begin{aligned}P_E &= \frac{1}{2\pi i}\oint_{\mathcal{C}} G_D(\zeta) \oplus G_S(\zeta)\, d\zeta \\ &= \frac{1}{2\pi i}\oint_{\mathcal{C}} G_{C(S)}(\zeta)\, d\zeta - \frac{1}{2\pi i}\oint_{\mathcal{C}} R\, d\zeta\end{aligned}$$

When applied to an eigenvector ψ_i of a small eigenvalue e_i lying within the contour $\mathcal{C}$,

$$\begin{aligned}(\mathbf{1} - P_E)\psi_i &= \Big(\psi_i - \frac{1}{2\pi i}\oint_{\mathcal{C}} G_{C(S)}\psi_i\, d\zeta\Big) - \frac{1}{2\pi i}\oint_{\mathcal{C}} R\psi_i\, d\zeta \\ &= \Big(\psi_i - \frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{1}{\zeta - e_i}\psi_i d\zeta\Big) - \frac{1}{2\pi i}\oint_{\mathcal{C}} R\psi_i\, d\zeta \\ &= -\frac{1}{2\pi i}\oint_{\mathcal{C}} R\psi_i\, d\zeta\end{aligned}$$

Estimating the norm of the remainder,

$$\begin{aligned}\|(\mathbf{1} - P_S)\psi_i\|_0 &\le \|(\mathbf{1} - P_E)\psi_i\|_0 \\ &\le \frac{1}{2\pi}\|\int_{\mathcal{C}} R\psi_i\, d\zeta\|_0 \le \frac{C\|\mathbf{p}\|}{d_s\sigma^8},\end{aligned}$$

where we have used that the contour has circumference proportional to d_s.

□

For $(\mathbf{p}, \Omega) \in \mathcal{N}_{n+1}^{(1)}$ we are now prepared to derive better estimates on the behavior of the sets Z_i. Keeping in mind the Feynman Hellman formula (4.15) for perturbation of the eigenvalues $e_i(\mathbf{p}, \Omega)$, we compute $\langle \psi_i, \partial_{\mathbf{p}} H_{C(S)}(\mathbf{p}, \Omega; u_n)\psi_i\rangle$ and estimate its size.

Proposition 4.16. *Let $C_2 = C(1/d_s + L_0/d_0(1 + r_0/\rho_0))$. For $(\mathbf{p}, \Omega) \in \mathcal{N}_{n+1}^{(1)}$, there is an improved estimate*

$$|\langle \psi_i, \partial_{\mathbf{p}} H_{C(S)}(\mathbf{p}, \Omega; u_n)\psi_i\rangle| = |\langle \psi_i, D_u^2 W(\phi(\mathbf{p}) + u_n)[\partial_{\mathbf{p}}(\phi + u_n), \psi_i]\rangle| \le C_2\|\mathbf{p}\|. \quad (4.21)$$

Proof. The gradient with respect to $\mathbf{p}$ of $H_{C(S)}$ is $D_u^2 W(\phi(\mathbf{p}) + u_n)(\partial_{\mathbf{p}}(\phi + u_n))$. Using hypothesis **H3** this is described $(D_u^2 W(\phi + u_n))_{C(S)} = (C + \Delta)_{C(S)}$. Using this along with the structure result for the eigenfunctions of $H_{C(S)}$, $\langle \psi_i, D_u^2 W(\phi(\mathbf{p}) + u_n)[\partial_{\mathbf{p}}(\phi + u_n), \psi_i]\rangle$ will be shown to be proportional to $\|\mathbf{p}\|$.

$$\begin{aligned}&\langle \psi_i, D_u^2 W(\phi(\mathbf{p}) + u_n)[\partial_{\mathbf{p}}(\phi + u_n), \psi_i]\rangle = \langle P_S\psi_i, C[\partial_{\mathbf{p}}\phi, P_S\psi_i]\rangle \\ &+ 2\langle (\mathbf{1} - P_S)\psi_i, C[\partial_{\mathbf{p}}\phi, P_S\psi_i]\rangle + \langle (\mathbf{1} - P_S)\psi_i, C[\partial_{\mathbf{p}}\phi, (\mathbf{1} - P_S)\psi_i]\rangle \\ &+ \langle \psi_i, \Delta[\partial_{\mathbf{p}}\phi, \psi_i]\rangle + \langle \psi_i, (C + \Delta)[\partial_{\mathbf{p}} u_n, \psi_i]\rangle.\end{aligned} \quad (4.22)$$

Referring to **H3** (see (6.5)), $\langle P_S\psi_i, C[\partial_{\mathbf{p}}\phi, P_S\psi_i]\rangle = 0$, and we estimate the remaining terms. The second term of (4.22) is

$$|\langle(\mathbf{1}-P_S)\psi_i, C[\partial_{\mathbf{p}}\phi, P_S\psi_i]\rangle| \leq \|\mathbf{p}\|\frac{C}{d_s}\frac{C_W}{(\overline{\sigma}/2)^4}$$

and the third term is even better behaved;

$$|\langle(\mathbf{1}-P_S)\psi_i, C[\partial_{\mathbf{p}}\phi, (\mathbf{1}-P_S)\psi_i]\rangle| < \|\mathbf{p}\|^2\frac{c^2C_W}{(d_s^2\overline{\sigma}^4)}.$$

Next,

$$|\langle\psi_i, \Delta[\partial_{\mathbf{p}}\phi, \psi_i]\rangle| \leq \frac{C_W}{(\overline{\sigma}/2)^4}\|\mathbf{p}\|,$$

and finally, the most significant term is

$$\begin{aligned}|\langle\psi_i, (C+\Delta)[\partial_{\mathbf{p}}u_n, \psi_i]\rangle| &\leq \frac{C_W}{(\overline{\sigma}/2)^4}\|\partial_{\mathbf{p}}u_n\|_{\sigma_n-\gamma_n}\\ &\leq \frac{C_W}{(\overline{\sigma}/2)^4}C\frac{L_0}{d_0}(1+\frac{r_0}{\rho_0})\|\mathbf{p}\|\end{aligned}$$

where we have used the induction hypothesis $(n.1)$ to control $\|\partial_{\mathbf{p}}u_n\|_{\sigma_n-\gamma_n}$. Assuming that $\|\mathbf{p}\| \leq r_0 < d_s$ and collecting terms, the result follows.

□

This is enough information to prove the parabolic estimates on the sets $Z_i \cap \mathcal{N}^{(1)}_{n+1}$. Let $(\mathbf{p}_1, \Omega_1) \in Z_i \cap \mathcal{N}^{(1)}_{n+1}$, and consider a path $\big(\mathbf{p}(s), \Omega_\pm(s)\big) = \big(\mathbf{p}_1 + s(\mathbf{p}_1/\|\mathbf{p}_1\|), \Omega_1 \pm C_1(\|\mathbf{p}(s)\|^2 - \|\mathbf{p}_1\|^2)/2L_n^2\big)$. At points on this path at which $e_i(\mathbf{p}, \Omega)$ is differentiable,

$$\begin{aligned}\frac{d}{ds}e_i =&\ \partial_\Omega e_i\frac{d\Omega_\pm}{ds} + \partial_{\mathbf{p}}e_i\frac{d\mathbf{p}}{ds}\\ =&\ \pm\langle\psi_i, \partial_\Omega H_{C(S)}\psi_i\rangle\frac{C_1\|\mathbf{p}(s)\|}{L_n^2} + \langle\psi_i, \partial_{\mathbf{p}}H_{C(S)}\psi_i\rangle\cdot\frac{\mathbf{p}(s)}{\|\mathbf{p}(s)\|}\end{aligned}$$

Choosing the plus sign to illustrate one case,

$$\frac{d}{ds}e_i \leq -\Omega_{min}\frac{C_1}{L_n^2}\times L_n^2\|\mathbf{p}(s)\| + C_2\|\mathbf{p}(s)\|,$$

where (4.15), (4.16), and (4.21) are used to estimate the inner products. By choosing the constant $\hat{C}$ in Lemma 4.14 sufficiently large, we can insure tha $C_1\Omega_{min} > C_2$ which implies that $de_i/ds \leq 0$ at points of differentiability along the curve $(\mathbf{p}(s), \Omega_+(s))$. Hence

$$e_i(\mathbf{p}(s), \Omega_+(s)) - e_i(\mathbf{p}_1, \Omega_1) \leq \int_0^s \frac{d}{ds} e_i(\mathbf{p}(t), \Omega_+(t))\, dt \leq 0,$$

and Z_i must remain to the left of $(\mathbf{p}(s), \Omega_+(s))$ as long as this curve remains within $\mathcal{N}_{n+1}^{(1)}$. A similar argument shows that Z_i lies to the right of the curve $(\mathbf{p}(s), \Omega_-(s))$. With this knowledge we can prove Lemma 4.14. Let $(\mathbf{p}_1, \Omega_1)$ and $(\mathbf{p}_2, \Omega_2) \in Z_i$, lying within the same component of $\mathcal{N}_{n+1}^{(1)}$, with $\|\mathbf{p}_1\| \leq \|\mathbf{p}_2\|$. Since Z_i is invariant under T_ξ we may also assume that $\mathbf{p}_1$ and $\mathbf{p}_2$ are parallel. Consider the curves $(\mathbf{p}(s), \Omega_\pm(s))$, with $\mathbf{p}(t) = \mathbf{p}_2$ for some t. If $(\mathbf{p}(t), \Omega_\pm(t)) \in \mathcal{N}_{n+1}^{(1)}$ then by our analysis

$$\Omega_-(t) \leq \Omega_2 \leq \Omega_+(t) \tag{4.23}$$

from which we conclude that

$$|\Omega_2 - \Omega_1| \leq \frac{C_1}{L_n^2} \big| \|\mathbf{p}_2\|^2 - \|\mathbf{p}_1\|^2 \big|$$

Recall that $\mathcal{N}_{n+1}^{(1)}$ consists of open disks that are constant in the $\mathbf{p}$ direction, thus if either of $(\mathbf{p}(t), \Omega_\pm(t))$ is not within $\mathcal{N}_{n+1}^{(1)}$, then automatically (4.23) holds. This finishes the proof of Lemma 4.14.

We now are in position to prove part (c) of Proposition 4.6. Consider an arbitrary C^∞ surface $\mathcal{C} = (\mathbf{p}, \Omega(\mathbf{p}))$ which is invariant under T_ξ; $\Omega(\mathbf{p}) = \omega_1 + \mathcal{K}\|\mathbf{p}\|^2(1 + \|\mathbf{p}\|C(\|\mathbf{p}\|))$, where $\mathcal{K} \neq 0$. We will see that this surface intersects the good set of parameters $\mathcal{N}_{n+1} = \mathcal{N}_{n+1}^{(1)} \cap \mathcal{N}_{n+1}^{(2)}$ in a large set. In practice these surfaces are solution sets of the second bifurcation equation (3.14)(b), and intersections with $\cup_{n=0}^{\infty} \mathcal{N}_n$ correspond to solutions of the full equations (2.8).

The total measure of Ω within $\{\omega_1 - r_0^2 < \Omega < \omega_1 + r_0^2\}$ which has been excised in order to construct $\mathcal{N}_{n+1}^{(1)}$ from $\mathcal{N}_n^{(1)}$ is estimated in Lemma 4.9 by $Cd/L_n^\tau(1 + r_0^2 L_n)$. From the parabolic nature of $\mathcal{C}$ an estimate of the new excisions is given by,

$$\text{meas}\{0 \leq r < r_0; (\mathbf{p}, \Omega(\mathbf{p})) \in \mathcal{N}_n^{(1)} \backslash (\mathcal{N}_{n+1}^{(1)}) \text{ with } \|\mathbf{p}\| = r\} \leq \sqrt{\frac{Cd}{|\mathcal{K}| L_n^\tau}(1 + r_0^2 L_{n+1})} \tag{4.24}$$

The parabolic estimates Lemma 4.14 are used to control the effect of the excisions in order to satisfy condition (b) on the surface $\mathcal{C}$. Suppose that $(\mathbf{p}_1, \Omega(\mathbf{p}_1)) \in \mathcal{C} \cap Z_i \cap \mathcal{N}_{n+1}^{(1)}$, and let $(\mathbf{p}, \Omega) \in \mathcal{C}$ be another point in $\mathcal{N}_{n+1}^{(1)}$, with $|\Omega - \Omega_1| < \delta_{n+1}/L_n^2$. We have an estimate on the quantity $|\|\mathbf{p}\|^2 - \|\mathbf{p}_1\|^2|$. Without loss of generality consider $\mathbf{p}$ parallel to $\mathbf{p}_1$. On $Z_i \cap \mathcal{N}_{n+1}^{(1)}$, $|\Omega_i(\mathbf{p}) - \Omega_i(\mathbf{p}_1)| \leq (C_1/L_n^2)|\|\mathbf{p}\|^2 - \|\mathbf{p}_1\|^2|$, while on $\mathcal{C}$, $|\Omega(\mathbf{p}) - \Omega(\mathbf{p}_1)| > |\mathcal{K}/2||\|\mathbf{p}\|^2 - \|\mathbf{p}_1\|^2|$. Thus in forming $\mathcal{N}_{n+1}^{(2)}$ by deletion of δ_{n+1}/L_n^2 neighborhoods of each Z_i, any other points of $\mathcal{N}_0$ that are excised satisfy

$$\begin{aligned}\frac{\delta_{n+1}}{L_n^2} \geq &|\Omega(\mathbf{p}) - \Omega_i(\mathbf{p})| \geq |\Omega(\mathbf{p}) - \Omega(\mathbf{p}_1)| - |\Omega_i(\mathbf{p}_1) - \Omega_i(\mathbf{p})| \\ \geq &(\frac{|\mathcal{K}|}{2} - \frac{C_1}{L_n^2})|\|\mathbf{p}\|^2 - \|\mathbf{p}_1\|^2|.\end{aligned}$$

As long as $C_1/L_n^2 < |\mathcal{K}|/4$, we have $|\|\mathbf{p}\|^2 - \|\mathbf{p}_1\|^2| < (4/|\mathcal{K}|)\delta_{n+1}/L_n^2$. Eliminating such $\mathbf{p}$ for each possible singular site and each resulting eigenvalue set Z_i, the total measure of the parameters $\mathbf{p}$ excluded in constructing $\mathcal{N}_{n+1}^{(2)}$ from $\mathcal{N}_n^{(2)}$ is controlled by

$$\begin{aligned}&\mathrm{meas}\{0 \leq r < r_0; \|\mathbf{p}\| = r,\ (\mathbf{p}, \Omega(\mathbf{p})) \in \mathcal{N}_{n+1}^{(1)} \cap (\mathcal{N}_n^{(2)} \backslash \mathcal{N}_{n+1}^{(2)})\} \\ &\leq \sqrt{C L_n (1 + r_0^2 L_{n+1}) \times \ell_{n+1}^2 \times (1/|\mathcal{K}|)(\delta_{n+1}/L_n^2)}\end{aligned} \tag{4.25}$$

With this we have finished the proof of Proposition 4.6.

□

5. The Green's Function

The principal goals of this section are to prove estimates on solutions to the linearized first bifurcation equation (2.21), where the linearization is about an approximate solution obtained in the induction. Actually, solutions are obtained for the linearized equations restricted to finite subdomains $\ell^2(B_n)$ of $\ell^2(\mathbf{Z}^+ \times \mathbf{Z})$; this restriction is both the analog of the smoothing procedure of the classical Nash-Moser method, and it allows us to consider only finitely many possibly singular sites at each induction step. In the induction the domains $B_n \subseteq B_{n+1}$ on which we approximately solve (2.21) grow. In addition, the tolerance for small denominators, or small spectra, increases, quantified by the sequence δ_n. The full nonlinear equations (2.8) also have small or zero eigenvalues, stemming from the null space of the operator $H_{B_n}(0, \omega_1)$; however the Lyapounov Schmidt decomposition allows us to work independently of these small eigenvalues. In the induction the linearized operators H_{B_n} and the Green's functions G_{B_n} depend upon the parameters in the problem, the frequency Ω and the parametrization $\mathbf{p}$ of the null space of $H_{B_0}(0, \omega_1)$,. The first step of the induction defines $G_{B_0}(\mathbf{p}, \Omega)$ for $(\mathbf{p}, \Omega) \in \mathcal{N}_0$ a full neighborhood of the bifurcation point $(0, \omega_1)$. In subsequent induction steps we only obtain control of $G_{B_n}(\mathbf{p}, \Omega)$ in subsets $\mathcal{N}_n \subseteq \mathcal{N}_{n-1} \subseteq \cdots \mathcal{N}_0$, on which the tolerance of the n^{th} linearized operator $H_{B_n}(\mathbf{p}, \Omega, u_n)$ for small spectra are satisfied.

5.1. The Green's function on nonsingular domains

The first bounds on the Green's function concern the nonresonant case. We denote $N = \{n_j\}$ the lattice sites which support the null vectors of $H_{B_0}(0, \omega_1)$ and $\overline{B}_0 = B_0 - N$. For $(\mathbf{p}, \Omega)$ in a neighborhood $\tilde{\mathcal{N}}$ of $(0, \omega_1)$ the diagonal elements $|V(\Omega)(n_j, n_j)|$ can be very small. However, since we assumed that our initial frequencies were (d_0, L_0) nonrenonant, all of the sites of $\overline{B}_0$ have $|V(\omega_1)(x, x)| > d_0$. On the full lattice $\mathbf{Z}^+ \times \mathbf{Z}$ we call a site x **singular**, if $|V(\Omega)(x, x)| < d_s$, where d_s is a parameter we defined in Section 3.5. When restricted to nonsingular sites, there is a σ_0 such that the operator $H(\mathbf{p}, \Omega)$ is invertible on $\mathcal{H}_\sigma$ for all $0 \leq \sigma < \sigma_0$. This is the conclusion of the following.

Theorem 5.1. *Let $H_A = V(\Omega) + DW$ be a linear operator defined on $\ell^2(A)$, for any domain $A \subseteq \mathbf{Z}^+ \times \mathbf{Z}$ consisting entirely of sites where $|V(x)| > \delta$. Suppose that $DW(x, y)$ satisfies an estimate of the form*

$$|DW(x, y)| \leq \epsilon e^{-\sigma|x-y|} \quad . \tag{5.1}$$

Then for

$$\frac{\epsilon}{\delta} < \min\left[\frac{\sigma^2}{64}, \frac{\sigma}{8}\right] \quad ,$$

and for all $|z| \leq \frac{1}{2}\delta$ the Green's function $G_A(z) = (H_A - z\mathbf{1})^{-1}$ exists and decays exponentially off diagonally

$$|G_A(z)(x, y)| \leq \frac{32}{\delta} e^{-\sigma'|x-y|} \tag{5.2}$$

where $\sigma' = \sigma - 4\log(1 + 4\sqrt{(\epsilon/\delta)})$. Furthermore if H_A depends analytically on parameters, then so does $G_A(z)$. In operator norms, for any $0 < \gamma \leq \sigma'$ there is a constant C_0 such that

$$\| G_A(z) \|_{\sigma'-\gamma} \leq \frac{C_0}{\delta\gamma^2} \quad . \tag{5.3}$$

Proof. This proof is neatest via path expansion of the resolvent $(H_A - z\mathbf{1})^{-1}$. First subtract the diagonal from DW,

$$H_A = V(\Omega) + DW = V_1 + DW_1$$

where $V_1 = V + \text{diag } DW$. For $\epsilon \leq (1/4)\delta$ the elements of the diagonal matrix $|V_1(x,x)| \geq \frac{3}{4}\delta$, while diag $DW_1 = 0$, and otherwise $DW_1(x,y) \leq \epsilon e^{-\sigma|x-y|}$. Certainly $|V_1^{-1}(x,x)| \leq 4\,\delta^{-1}$. Consider the Neumann series for the Green's function $G_A(z)(x,y) = (H_A - z\mathbf{1})^{-1}(x,y) = (V_1 - z\mathbf{1} + DW_1)^{-1}(x,y)$, considering DW_1 as the perturbation,

$$(H_A - z\mathbf{1})^{-1} = (V_1 - z\mathbf{1} + DW_1)^{-1} = (V_1 - z\mathbf{1})^{-1}(1 + DW_1(V_1 - z\mathbf{1})^{-1})^{-1}.$$

Set $T = -DW_1(V_1 - z\mathbf{1})^{-1}$, which also vanishes on the diagonal, so that at least formally,

$$G_A(z) = (H_A - z\mathbf{1})^{-1} = (V_1 - z\mathbf{1})^{-1} \sum_{n=0}^{\infty} T^n. \tag{5.4}$$

We of course have $|T(x,y)| \leq 4\epsilon\delta^{-1}e^{-\sigma|x-y|}$. We will sum (5.4) over paths, thereby preserving an exponential decay estimate for the Green's function. Consider a path $\beta = (x, \beta_1, \beta_2, \cdots \beta_{n-1}, y)$ from x to y in the domain A, with the number of steps taken $\#\beta = n$, with total length $l(\beta) = \sum_j |\beta_j - \beta_{j-1}|$. Then

$$T^n(x,y) = \sum_{\substack{\text{paths}\,\beta: x\to y \\ \#\beta = n}} \prod_{j=1}^{n} T(\beta_{j-1}, \beta_j) \quad , \tag{5.5}$$

where we make the identification $\beta_0 = 0$ and $\beta_n = y$. Inserting the estimate on $|T(\beta_{j-1}, \beta_j)|$ from above we find,

$$\begin{aligned} |T^n(x,y)| &\leq \sum_{\substack{\text{paths}\,\beta: x\to y \\ \#\beta = n}} (4\epsilon/\delta)^n \prod_{j=1}^{n} e^{-\sigma|\beta_j - \beta_{j-1}|} \qquad (5.6) \\ &= \sum_{\substack{\text{paths}\,\beta: x\to y \\ \#\beta = n}} (4\epsilon/\delta)^n e^{-\sigma l(\beta)}. \end{aligned}$$

The full Neumann series is now considered.

$$|\sum_{n=0}^{\infty} T^n(x,y)| \leq \sum_{n=0}^{\infty} \sum_{L=|x-y|}^{\infty} \sum_{\substack{\text{paths}\,\beta: x\to y \\ \ell(\beta)=L,\, \#\beta = n}} (4\epsilon\delta^{-1})^n e^{-\sigma L}. \tag{5.7}$$

This gives an estimate of the Green's function in terms of the number of paths from x to y of length L with n steps. For our two dimensional lattice a sufficiently good upper bound on the number of paths from x to y is $\#\{\beta : l(\beta) = L, \#\beta = n\} \leq 4^n \binom{L-1}{n-1}^2$. The resulting estimate of (5.7) is

$$\begin{aligned}
\delta(x,y) + |\sum_{n=1}^{\infty} T^n(x,y)| &\leq \delta(x,y) + \sum_{L=|x-y|}^{\infty} \sum_{n=1}^{L} (4^2\epsilon\delta^{-1})^n \binom{L-1}{n-1}^2 e^{-\sigma l} \\
&\leq \delta(x,y) + \sum_{L=|x-y|}^{\infty} \left(\sum_{n=1}^{L} (4^2\epsilon\delta^{-1})^{n/2} \binom{L-1}{n-1} \right)^2 e^{-\sigma L} \\
&\leq \delta(x,y) + \sum_{L=|x-y|}^{\infty} (4^2\epsilon\delta^{-1})(1 + 4\sqrt{(\epsilon/\delta)})^{2(L-1)} e^{-\sigma L}
\end{aligned}$$

This sum converges if $\frac{\epsilon}{\delta} < \frac{\sigma^2}{64}$ giving the estimate

$$\frac{32\epsilon}{\sigma\delta} e^{-(\sigma - 2\log(1+4\sqrt{(\epsilon/\delta)}))|x-y|} \quad .$$

Combining this estimate with (5.4) yields (5.2). Estimate (5.3) then follows from (5.2) by way of Proposition 3.1. These estimates are uniform in parameters, as long as the hypotheses of the theorem remain valid, therefore analyticity properties of H_A are preserved in $G_A(z)$.

□

Corollary 5.2. *In fact one sees from the proof that a stronger estimate is true*

$$|G_A(x,y;z)| \leq \frac{4}{\delta} \left\{ \delta(x,y) + \frac{32\epsilon}{\sigma\delta} e^{-\sigma'|x-y|} \right\} \quad . \tag{5.8}$$

Let A be any non-singular domain and let $E_0 \subset \overline{B_0} \cup A$. Consider the operator $H_{E_0} = V(\Omega) + DW(u_0)$. If $(r_0^2 + \rho_0)L_0^2 \leq d_0/2$, then for every $\Omega \in D(\mathcal{N}_0, \rho_0)$, $|V(\Omega)(x,x)| > d_0/2$, for every point $x \in \overline{B_0}$. This follows from the form of $V(\Omega)$, and the fact that $|V(\omega_1)(x,x)| > d_0$ by the (d_0, L_0) non-resonance hypothesis. If

$$d_s \leq d_0/2 \quad , \tag{5.9}$$

all sites of E_0 are nonsingular. The operator $DW(u_0)$ is bounded from $\mathcal{H}_{\sigma+\gamma_0}$ to $\mathcal{H}_\sigma$ provided $\sigma < \sigma_* - L_0^{-1}$ as in Lemma 3.3. Indeed hypothesis **H2** of Section 6 implies that

$$|(DW(u_0))(x,y)| \leq ((C_W\|\mathbf{p}\|)/\gamma_0^3)e^{-\sigma_0|x-y|} \quad .$$

In order to make the labeling of the indices consistent, define $\sigma_{-1} \equiv \sigma_0 + 6\gamma_{-1}$, with $\gamma_{-1} \equiv \sigma_0/64$. Note that both of these definitions are consistent with the original inductive definitions in Section 3.3. Thus, we can apply Theorem 5.1 and we obtain the following estimate.

Proposition 5.3. *Let A be any non-resonant region and let B_0 satisfy the hypotheses of (2.21). If in addition*

$$\frac{C_W r_0}{d_s\gamma_0^3} < (\frac{\sigma_0}{8})^2 \quad , \tag{5.10}$$

$$(r_0^2+\rho_0)L_0^2 \leq d_0/2 \quad , \quad d_s \leq d_0/2 \quad , \tag{5.11}$$

$$\sigma_{-1} \equiv (\frac{35}{32})\sigma_0 < \sigma_* - 2\log(1+4\sqrt{\epsilon/\delta}) \quad ,$$

and if $E_0 \subset \overline{B_0} \cup A$ then

$$\| G_{E_0} \|_{\sigma_{-1}-\gamma_{-1}} \leq \frac{C_0}{d_s\gamma_0^3} \quad .$$

Remark. If the nonlinearity in (2.8) has no quadratic term, the estimate on $DW(u_0)$ can be improved to

$$|(DW(u_0))(x,y)| \leq \frac{C_W\|\mathbf{p}\|^2}{\gamma_0^3}e^{-\sigma_0|x-y|} \quad .$$

This allows us to replace hypothesis (5.10) by

$$\frac{C_W r_0^2}{d_s\gamma_0^3} < (\frac{\sigma_0}{8})^2 \quad , \tag{5.12}$$

and still obtain the same estimate on the Green's function.

Remark. This proves (3.18) when $n = 0$.

Note that the hypothesis is independent of the domains E_0. This emphasizes the fact that it is small values on the diagonal $V(\Omega)$, and the associated singular sites which play a major role in determining the decay of the Green's functions for $H_{B_n}(\mathbf{p},\Omega,u_n)$.

At each level of the induction there are two steps to perform. The domain under consideration is extended from B_n to $B_{n+1} \subseteq \mathbf{Z}^+ \times \mathbf{Z}$, and the Green's function $G_{\overline{B_{n+1}}}$ on the larger domain must be constructed and estimated. Secondly, a Newton step is performed to construct the next correction $v_n(x, \mathbf{p}, \Omega)$ to the approximate solution $u_n(x; \mathbf{p}, \Omega)$. These are the analog of the smoothing step and the Newton iteration of the classical Nash-Moser method. In the first of these steps we construct $G_{\overline{B_{n+1}}}(\mathbf{p}, \Omega, u_n; z)$ from $G_{\overline{B_n}}$ for $(\mathbf{p}, \Omega) \in \mathcal{N}_{n+1}$. This is the most detailed construction as there may be many resonant sites in the region $B_{n+1} \backslash B_n$. The second step also requires us to estimate the inverse of the linearized operator $H_{\overline{B_{n+1}}}(\mathbf{p}, \Omega; u)$, for $u = u_n + v_n$. Due to the rapid convergence of Newton's method the adjustment $v_n(x; \mathbf{p}, \Omega)$ is very small and once one has control of $G_{\overline{B_{n+1}}}(\mathbf{p}, \Omega, u_n; z)$ the construction of $G_{\overline{B_{n+1}}}(\mathbf{p}, \Omega, u_n + v_n; z)$ is relatively painless.

5.2. Domain Extension:

The induction hypotheses produce a set of parameters $(\mathbf{p}, \Omega) \in \mathcal{N}_n$ such that for any $E_n \subset \overline{B_n} \cup A$, where A is a nonsingular region, and for any $|z| < \delta_n/2$ the Green's function has the estimate

$$\| G_{E_n} \|_{\sigma_{n-1} - \gamma_{n-1}} \leq \frac{C_G^n}{\delta_n \gamma_{n-1}^{12}}. \tag{5.13}$$

Take L_0 large enough that $\delta_0 < d_s/C_0$. Proposition 5.3 implies that this estimate also holds for $n = 0$. In this subsection we will extend this induction estimate to the domains $E_{n+1} \subset \overline{B_{n+1}} \cup A$, with tolerance δ_{n+1} as long as A remains nonsingular, and all sites within $B_{n+1} \backslash B_n \equiv A_{n+1}$ obey the induction hypotheses $(j.1)$ and $(j.2)$ $j = 1, \ldots, n$. In this process we will see that the geometry of the resonant sites is important, as well as the severity of the resonance. In fact this process of domain extension can be carried out in any situation in which both the geometry and severity of the resonances are under control.

Let $\{S_k\}$ be the set of singular regions of $H_{B_{n+1}}(\mathbf{p}, \Omega, u_n)$ within A_{n+1}, and define $C_{\ell_{n+1}}(S_k)$ to be balls of radius ℓ_{n+1} about S_k, isolating neighborhoods of the singular regions from the rest of B_{n+1}. By assumption $(n.2)$, for all parameter values $(\mathbf{p}, \Omega) \in \mathcal{N}_{n+1}$ these regions are well separated, and $C(S_k)$ are disjoint. The regions S_k, of course, vary with Ω. The inductive estimate on the Green's function will be proved by decomposing the domain B_{n+1} into subdomains, constructing Green's functions on the subdomains, and then patching the full Green's function together using a resolvent expansion. Related estimates and resolvent expansions have appeared in work of [FS]. First we consider the Green's function on the neighborhoods $C_{\ell_{n+1}}(S_k)$.

Lemma 5.4. *Suppose that $S \subset A_{n+1}$ is a singular region, and $(\mathbf{p}, \Omega) \in \mathcal{N}_{n+1}$ so that $(j.1)$ and $(j.2)$ $j = 1, 2, \ldots$ hold, then for $|z| < \frac{1}{2}\delta_{n+1}$, and*

$$r_0 < \delta_0 \gamma_0^3 \quad , \tag{5.14}$$

then

$$|G_{C_{\ell_{n+1}}(S)}(x, y; z)| \leq \frac{C}{\delta_{n+1}\gamma_{n-1}^{10}} e^{-(\sigma_{n-1} - 3\gamma_{n-1})|x-y|} \quad . \tag{5.15}$$

This implies

$$\|G_{C_{\ell_{n+1}}(S)}\|_{\sigma_{n-1} - 4\gamma_{n-1}} \leq \frac{C}{\delta_{n+1}\gamma_{n-1}^{12}} \quad .$$

Remark. In the case where the nonlinearity in (2.8) contains no quadratic term, condition (5.14) can be replaced by

$$r_0^2 < \delta_0 \gamma_0^3 \quad . \tag{5.16}$$

Proof. The proof uses resolvent identities to decompose the Green's function into several parts, allowing matrix elements to be estimated. Let S be the singular region itself, and $D = C_{\ell_{n+1}}(S) - S$ the annulus surrounding this region. Induction hypothesis $(n.2)$ implies that the singular regons are so far apart that the annulus D contains no other singular sites. The linear operator $H_{C_{\ell_{n+1}}(S)}$ is decomposed into block diagonal and off diagonal parts,

$$H_{C_{\ell_{n+1}}(S)} = H_D \oplus H_S + \Gamma_{DS}. \tag{5.17}$$

The operators H_D and H_S are the restrictions of $H_{C_{\ell_{n+1}}(S)}$ to the subdomains D and S, and the term Γ_{DS} accounts for the interactions between these domains. There is accordingly a decomposition of the resolvent, giving the first resolvent identity,

$$G_{C_{\ell_{n+1}}(S)} = G_D \oplus G_S + G_D \oplus G_S \Gamma_{DS} G_{C(S)} \tag{5.18}$$

(Again, to save space, we will write $G_{C(S)}$ rather than $G_{C_{\ell_{n+1}}(S)}$.) Iterating this and its adjoint we obtain the second resolvent identity

$$\begin{aligned} G_{C_{\ell_{n+1}}(S)} = G_D \oplus G_S + G_D \oplus G_S \Gamma_{DS} G_D \oplus G_S \\ + G_D \oplus G_S \Gamma_{DS} G_{C(S)} \Gamma_{DS} G_D \oplus G_S \end{aligned} \tag{5.19}$$

The Hamiltonian H_D has diagonal elements which are nonsingular, and satisfies the hypotheses of Corollary 5.2. Thus

$$|G_D(x,y;z)| \leq \frac{2C_0}{\delta_0 \gamma_0^3} e^{-(\sigma_{n-1}-2\gamma_{n-1})|x-y|} \quad .$$

The set S consists of only one or two sites. The interaction term has zero diagonal, and decays exponentially off diagonal;

$$|\Gamma_{DS}(x,y)| \leq \frac{C_W \|\mathbf{p}\|}{\gamma_{n-1}^3} e^{-(\sigma_n-2\gamma_{n-1})|x-y|} \quad .$$

We will treat estimate (5.15) in three cases.

If both $x, y \in S$, then $|G_{C_{\ell_{n+1}}(S)}(x,y;z)| \leq 2/\delta_{n+1}$, since by the induction hypothesis $\mathrm{dist}(\mathrm{spec}(H_{C_{\ell_{n+1}}(S)}), 0) > \delta_{n+1}$ The estimate (5.15) follows since $|x-y| \leq 1$. Secondly, if $x \in D$ and $y \in S$ (or vice versa), the first resolvent identity is used

$$\begin{aligned} |G_{C(S)}(x,y;z)| &\leq |\sum_{\substack{p \in D \\ q \in S}} G_D(x,p) \Gamma_{DS}(p,q) G_{C(S)}(q,y)| \\ &\leq \sum_{p \in D} \frac{2C_0}{\delta_0 \gamma_0^3} e^{-(\sigma_{n-1}-2\gamma_{n-1})|x-p|} \frac{C_W \|\mathbf{p}\|}{\gamma_{n-1}^3} e^{-(\sigma_n-2\gamma_{n-1})|p-q|} \frac{2}{\delta_{n+1}} \\ &\leq \frac{C\|\mathbf{p}\|}{\delta_0 \gamma_0^3 \delta_{n+1} \gamma_{n-1}^5} e^{-(\sigma_{n-1}-3\gamma_{n-1})|x-y|}. \end{aligned} \tag{5.20}$$

Note that $G_D \oplus G_S(x,y) = 0$ in this case and we have used the fact that $|q-y| \leq 1$..

Finally, consider the case when both $x, y \in D$. Using the second resolvent identity,

$$
\begin{aligned}
|G_{C_{\ell_{n+1}}(S)}(x,y)| \le |G_D(x,y)| + | \sum_{\substack{p,t\in D\\ r,q,s\in S}} & G_D(x,p)\Gamma_{DS}(p,q)\times \\
& \times G_{C_{\ell_{n+1}}(S)}(q,r)\Gamma_{DS}(r,t)G_D(t,y)| \\
\le \frac{2C_0}{\delta_0\gamma_0^3} e^{-(\sigma_{n-1}-2\gamma_{n-1})|x-y|} & \\
+ \sum_{\substack{p,t\in D\\ q,r\in S}} \frac{2C_0}{\delta_0\gamma_0^3} e^{-(\sigma_{n-1}-2\gamma_{n-1})|x-p|} & \frac{C_W\|\mathbf{p}\|}{\gamma_{n-1}^3} e^{-(\sigma_{n-1}-2\gamma_{n-1})|p-q|} \frac{2}{\delta_{n+1}} \times \\
& \times \frac{C_W\|\mathbf{p}\|}{\gamma_{n-1}^3} e^{-(\sigma_{n-1}-2\gamma_{n-1})|r-t|} \frac{2C_0}{\delta_0\gamma_0^3} e^{-(\sigma_{n-1}-2\gamma_{n-1})|t-y|} \quad .
\end{aligned}
\tag{5.21}
$$

We have used the fact that $\Gamma_{DS}(p,q) = 0$ if both of its arguments are in D to eliminate the term linear in Γ from (5.21). We estimate this exponential sum with a loss of decay γ_{n-1}. This results in a bound on (5.21) of the form

$$
\left\{ \left(\frac{2C_0}{\delta_0\gamma_0^3}\right) + \left(\frac{2C_0}{\delta_0\gamma_0^3}\right)^2 \left(\frac{C_W\|\mathbf{p}\|}{\gamma_{n-1}^3}\right)^2 \frac{2}{\delta_{n+1}} \frac{C}{\gamma_{n-1}^4} \right\} e^{-(\sigma_{n-1}-3\gamma_{n-1})|x-y|} \quad . \tag{5.22}
$$

Thus inequality (5.15) follows from (5.20) and (5.21) using $\|\mathbf{p}\| < r_0 < \delta_0\gamma_0^3$.

□

In fact a useful corollary of the proof describes in more detail the structure of the local Green's function in neighborhoods containing only one singular region.

Corollary 5.5. *Assume that $C_W r_0/d_s < \overline{\sigma}^2/16$. Suppose that $S \subseteq A_{n+1}$ is a singular region such that $C_{\ell_{n+1}}(S)\backslash S \equiv D$ is nonsingular. Then*

$$
G_{C_{\ell_{n+1}}(S)}(x,y;\zeta) = G_S \oplus G_D + R \tag{5.23}
$$

where $G_D(x,y;\zeta)$ is analytic for $|\zeta| < 3d_s/4$. For any ζ, $|\zeta| < 3d_s/4$ such that $\mathrm{dist}(\mathrm{spec}(H_{C_{\ell_{n+1}}(S)}),\zeta) > d_s/8$, *then*

$$
\begin{aligned}
\|R\|_0 &\le \frac{C\|\mathbf{p}\|}{d_s^2} \quad , \\
|R(x,y)| &\le \frac{C\|\mathbf{p}\|}{d_s^2\gamma_{n-1}^{10}} e^{-(\sigma_{n-1}-3\gamma_{n-1})[\mathrm{dist}(x,S)+\mathrm{dist}(S,y)]} \quad , \\
\|R\|_{\sigma_{n-1}-4\gamma_{n-1}} &\le \frac{C\|\mathbf{p}\|}{d_s^2\gamma_{n-1}^{12}}
\end{aligned}
\tag{5.24}
$$

Continuing the proof of the induction step, the neighborhoods $C_{\ell_{n+1}}(S_j)$ are patched back into the region B_{n+1}. This process is performed using a Neumann series based on the resolvent identities. In fact this expansion has superior convergence properties if two sets of domains are used in alternation: the domains $B_{n+1} - C_{\ell_{n+1}}(S_k)$, and $C_{\ell_{n+1}}(S_k)$, and the same sort of domains made from B_{n+1} by excising only the resonant regions $\{S_k\}$,rather than the disks $C(S_k)$. Of course the Green's functions $G_{\{S_k\}}$ also satisfy (5.15), by virtue of induction hypothesis $(n.2)(ii)$.

Theorem 5.6. *Suppose that the induction hypotheses* $(j.1)$ *and* $(j.2)$, $j = 1, \ldots, n$, *are satisfied. Let* A *be any additional nonresonant domain. There exist constants* C *and* C_G *independent of* n *such that if*

$$\frac{r_0}{\delta_0 \gamma_0^3} < 1 \quad , \quad \frac{r_0 C e^{-\gamma_{n-1}\ell_{n+1}/2}}{\delta_{n+1}\gamma_{n-1}^{15}} < 1 \quad ,$$

and

$$\frac{r_0 C C_G^n e^{-\gamma_{n-1}\ell_{n+1}/2}}{\delta_{n+1}^2 \gamma_{n-1}^{34}} < 1 \quad ,$$

and if $E_{n+1} \subset \overline{B_{n+1}} \cup A$, *then the Green's function* $G_{E_{n+1}}(x, y; z)$ *satisfies*

$$\| G_{E_{n+1}} \|_{\sigma_{n-1}-5\gamma_{n-1}} \leq \frac{C_G^{n+1}}{\delta_{n+1}\gamma_{n-1}^{12}}. \tag{5.25}$$

A central tool in the proof of this theorem is an estimate of the many exponential sums that arise in resolvent expansions. In particular there arise many sums over classes of paths which, by the arrangement of the decomposition of domains, are forced to contain a "long step". In the induction the size of this step will typically be $\ell_{n+1}/2$. The following elementary lemma concerns such an estimate.

Lemma 5.7. *Let* $\{\beta\}_{n,\ell}$ *be the collection of all paths from* x *to* y *in a domain* $B \subseteq \mathbf{Z}^+ \times \mathbf{Z}$, *with a fixed number of steps,* $\#\beta = n$, *and a length* $\ell(\beta)$ *of at least* ℓ. *Then for any* $0 < \gamma < \sigma$,

$$\sum_{\{\beta\}_{n,\ell}} e^{-\sigma\ell(\beta)} \leq \frac{C^{2n}}{\gamma^{2n}} e^{-(\sigma-\gamma)\ell}$$

Proof.

$$\sum_{\{\beta\}_{n,\ell}} e^{-\sigma\ell(\beta)} \leq e^{-(\sigma-\gamma)\ell} \sum_{\{\beta\}_{n,\ell}} e^{-\gamma\ell(\beta)} \quad .$$

Since $\ell(\beta) = |x-\beta_1| + |\beta_1-\beta_2| + \ldots + |\beta_{n-1}-y|$, we interpret $((x-\beta_1),(\beta_1-\beta_2),\ldots,(\beta_{n-1}-y))$ as a vector in $\mathbf{Z}^{2n}$. Then,

$$\sum_{\{\beta\}_{n,\ell}} e^{-\gamma\ell(\beta)} \le \sum_{\substack{v\in\mathbf{Z}^{2n}\\|v|\ge\ell}} e^{-\gamma|v|} \le \frac{C^{2n}}{\gamma^{2n}} \quad ,$$

by an elementary exponential estimate, and the lemma follows.

□

Proof. (of Theorem 5.6). We estimate $G_{E_{n+1}}$, for an arbitrary subset $E_{n+1} \subset \overline{B_{n+1}} \cup A$ by a path expansion. The paths appearing in the expansion of $G_{E_{n+1}}$ are a subset of those appearing in the expansion of $G_{\overline{B_{n+1}}\cup A}$, so the estimate on $G_{E_{n+1}}$ follows from that on $G_{\overline{B_{n+1}}\cup A}$. The procedure is to isolate each singular region $S_k \subset A_{n+1}$ with neighborhoods $C_{\ell_{n+1}}(S_k)$. Since the radius of $C_{\ell_{n+1}}(S_n)$ is ℓ_{n+1} the resulting alternating expansions will be composed of elements whose path expansions consist of long paths of length $\mathcal{O}(\ell_{n+1})$. The result is a convergent Neumann expansion and good estimates of exponential decay.

The resolvent expansion for $G_{\overline{B_{n+1}}\cup A}(x,y;z)$ is the Neumann series constructed from resolvent identities where two sets of domains are used alternately. Let $B = E \cup_j D_j = \overline{E} \cup_j \overline{D}_j$, where E, $\{D_j\}$ and $\overline{E}$, $\{\overline{D}_j\}$ are collections of disjoint domains of the lattice. Then we can express

$$H_B = H_E \oplus_j H_{D_j} + \Gamma_{ED} = H_{\overline{E}} \oplus_j H_{\overline{D}_j} + \Gamma_{\overline{ED}}. \tag{5.26}$$

Applying the first resolvent identity to both of these relations

$$\begin{aligned} G_B &= G_E \oplus_j G_{D_j} + G_E \oplus_j G_{D_j}\Gamma_{ED}G_B \\ &= G_{\overline{E}} \oplus_j G_{\overline{D}_j} + G_{\overline{E}} \oplus_j G_{\overline{D}_j}\Gamma_{\overline{ED}}G_B. \end{aligned} \tag{5.27}$$

Alternately using the first and second of these formulas, a formal expansion of the Green's function is obtained.

$$\begin{aligned} G_B = G_E \oplus_j G_{D_j} &+ G_E \oplus_j G_{D_j}\Gamma_{ED}G_{\overline{E}} \oplus_j G_{\overline{D_j}} \\ &+ \sum_{\ell=1}^{\infty}(G_E \oplus_j G_{D_j}\Gamma_{ED}G_{\overline{E}} \oplus_j G_{\overline{D}_j}\Gamma_{\overline{ED}})^{\ell}\times \\ &\times \left(G_E \oplus_j G_{D_j} + G_E \oplus_j G_{D_j}\Gamma_{ED}G_{\overline{E}} \oplus_j G_{\overline{D_j}}\right). \end{aligned} \tag{5.28}$$

We will use the expansion in this paper with the collections $\{D_j\}$ and $\{\overline{D_j}\}$ the set of singular regions, S_k, or their neighborhoods $C_{\ell_{n+1}}(S_k)$.

For situations in which all operators in this expression have exponential off diagonal decay, and where there is a large distance between domains E and $\overline{D}_j$, (5.28) results in an expansion with superior convergence properties. The proof of the estimate of $G_{\overline{B_{n+1}} \cup A}$ consists in making choices of the sets $E, \{D_j\}$ and $\overline{E}, \{\overline{D}_j\}$, and estimating the formal sum (5.28). Cases are chosen as follows:

Case 1: If $\mathrm{dist}(x, S_k) > \frac{1}{2}\ell_{n+1}$, for all singular regions $S_k \subset A_{n+1}$, we set $D_j = \{S_j\}$, where the index runs over all singular regions in A_{n+1}. Then define $E = \overline{B}_{n+1} \cup A - (\cup_j D_j)$. Also let $\overline{D}_j = C_{\ell_{n+1}}(S_j)$, and $\overline{E} = \overline{B}_{n+1} \cup A - (\cup_j \overline{D}_j)$.

Case 2: If $\mathrm{dist}(x, S_k) \le \frac{1}{2}\ell_{n+1}$, for some singular region $S_k \subset A_{n+1}$, chose $D_j = C_{\ell_{n+1}}(S_j)$, for each of the singular regions $S_j \subset A_{n+1}$, and $E = \overline{B}_{n+1} \cup A - (\cup_j D_j)$. Similarly, take $\overline{D}_j = \{S_j\}$, and $\overline{E} = \overline{B}_{n+1} \cup A - (\cup_j \overline{D}_j)$.

Address first the block diagonal term $G_E \oplus_j G_{D_j}$ of (5.28). In both cases, the domain $E \subset \overline{B_n} \cup A$, for some set A consisting entirely of non-singular sites. Thus, by the Theorem 3.6, $\| G_E \|_{\sigma_{n-1}-2\gamma_{n-1}} \le 2C_G^n/(\delta_n \gamma_{n-1}^{12})$. On the other hand, G_{D_ℓ} is either of the form $G_{\{S_\ell\}}(x, y; z)$ or $G_{C_{\ell_{n+1}}(S_\ell)}(x, y; z)$, so Lemma 5.4 implies that $\|G_{D_\ell}\|_{\sigma_{n-1}-4\gamma_{n-1}} \le C_0/(\delta_{n+1}\gamma_{n-1}^{12})$ for all singular regions of A_{n+1}. Thus, we have proved

Lemma 5.8. *If the hypotheses of Lemma 5.4 hold then there exists a constant $C > 0$ such that*

$$\| G_E \oplus_j G_{D_j} \|_{\sigma_{n-1}-4\gamma_{n-1}} \le \frac{CC_G^n}{\delta_{n+1}\gamma_{n-1}^{12}} \quad . \tag{5.29}$$

The term in (5.28) linear in the coupling Γ involves the choices of decomposition more directly. In Case 1,

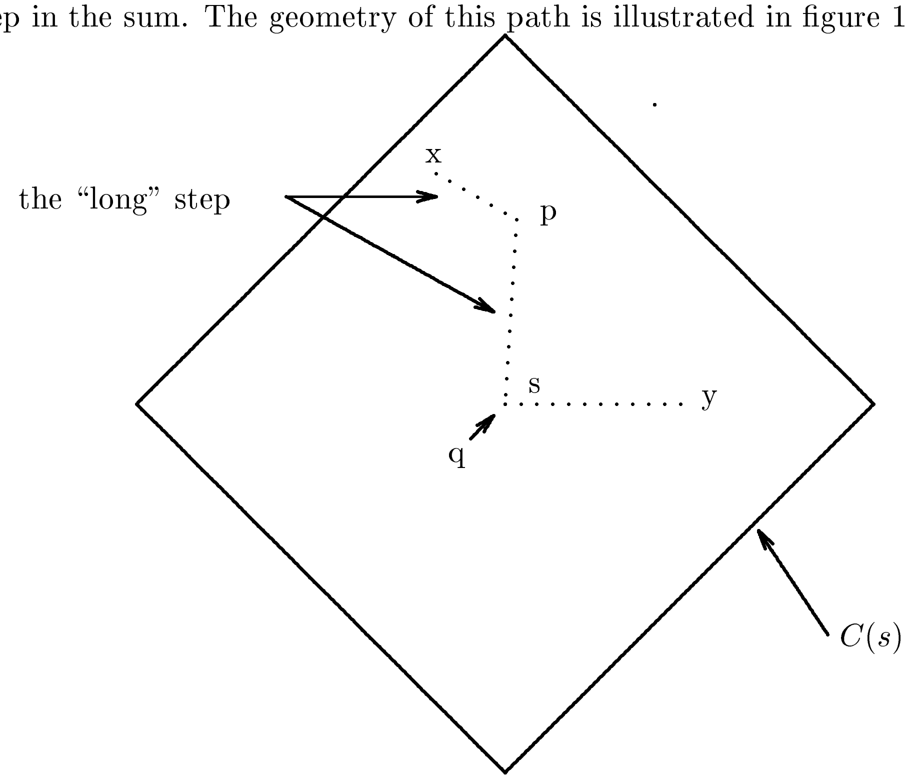
$$(G_E \oplus_j G_{D_j})\Gamma_{ED}(G_{\overline{E}} \oplus_j G_{\overline{D}_j})(x, y) = \sum_{\substack{p \in E \\ q \in D_j}} G_E(x, p)\Gamma_{ED}(p, q)G_{\overline{D}_j}(q, y) \quad . \tag{5.30}$$

In deriving this expression we used the fact that $x \in E$. Because G_E has block diagonal form, the only nonzero contributions to the sum are for $p \in E$. This implies that

$\Gamma_{ED}(p,q) = 0$ unless $q \in D_j$ for some index j. The estimates of exponential off diagonal decay of operators are now used; $|G_E(x,p)| \le (2C_G^n/(\delta_n \gamma_{n-1}^{12}))e^{-(\sigma_{n-1}-2\gamma_{n-1})|x-p|}$, by Theorem 3.6, $|\Gamma_{ED}(p,q)| \le (C_W \|\mathbf{p}\|/\gamma_{n-1}^3)e^{-(\sigma_{n-1}-2\gamma_{n-1})|p-q|}$, and

$$|G_{\overline{D}_j}(q,y)| \le (C/(\delta_{n+1}\gamma_{n-1}^{10}))e^{-(\sigma_{n-1}-3\gamma_{n-1})|q-y|},$$

from Lemma 5.4. Since $\mathrm{dist}(x, S_\ell) > \ell_{n+1}/2$ and in the sum the only nonzero coupling terms are for $q \in D_j = \{S_\ell\}$; $|x-p| + |p-q| > \ell_{n+1}/2$. We refer to this as the long step in the sum. The geometry of this path is illustrated in figure 1.



A typical term in the expansion with $p \in E$, and $q \in D$

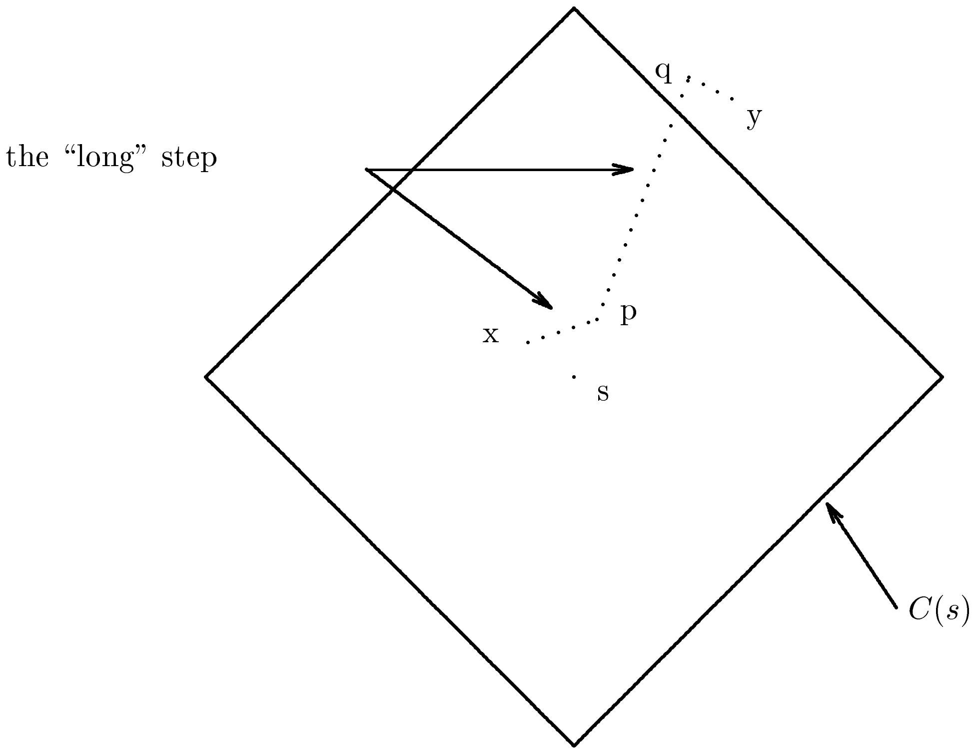
figure 1

This is just the situation addressed by Lemma 5.7, thus the sum (5.30) is bounded by

$$\begin{aligned}
&|\sum_{\substack{p\in E\\ q\in D_j}} G_E(x,p)\Gamma_{ED}(p,q)G_{\overline{D}_j}(q,y)| \\
&\qquad \le \sum_{\substack{p\in E\\ q\in D_j}} \frac{CC_G^n\|\mathbf{p}\|}{\delta_n\delta_{n+1}\gamma_{n-1}^{25}} e^{-(\sigma_{n-1}-3\gamma_{n-1})(|x-p|+|p-q|+|q-y|)} \qquad (5.31) \\
&\qquad \le \frac{CC_G^n\|\mathbf{p}\|}{\delta_{n+1}^2\gamma_{n-1}^{27}} e^{-\gamma_{n-1}\ell_{n+1}/2} e^{-(\sigma_{n-1}-4\gamma_{n-1})|x-y|}
\end{aligned}$$

In case 2, the sums corresponding to (5.30) are

$$
\begin{aligned}
&\left|(G_E \oplus_j G_{D_j})\Gamma_{ED}(G_{\overline{E}} \oplus_j G_{\overline{D}_j})(x,y)\right| \\
&\qquad = \Big| \sum_{\substack{p\in D_\ell \\ q\in E\cup_{k\neq\ell} D_k}} G_{D_\ell}(x,p)\Gamma_{ED}(p,q)(G_{\overline{E}}(q,y) \oplus_k G_{\overline{D_k}}(q,y))\Big| \\
&\qquad \leq \sum_{\substack{p\in D_\ell \\ q\in E\cup_{k\neq\ell} D_k}} \left(\frac{C}{\delta_{n+1}\gamma_{n-1}^{10}}\right)\left(\frac{C_W\|\mathbf{p}\|}{\gamma_{n-1}^{3}}\right)\left(\frac{2}{\delta_{n+1}}+\frac{C_G^n}{\delta_n\gamma_{n-1}^{12}}\right)\times \qquad (5.32) \\
&\qquad\qquad \times e^{-(\sigma_{n-1}-3\gamma_{n-1})(|x-p|+|p-q|+|q-y|)} \\
&\qquad \leq \frac{CC_G^n\|\mathbf{p}\|}{\delta_{n+1}^2\gamma_{n-1}^{27}} e^{-\gamma_{n-1}\ell_{n+1}/2} e^{-(\sigma_{n-1}-4\gamma_{n-1})|x-y|}
\end{aligned}
$$



A typical term in the expansion with $p \in D$, and $q \in E$

figure 2

In this case the "long step" results from the fact that $q \in E \cup_{k\neq\ell} D_k$ while $\mathrm{dist}(x, s_\ell) \leq \ell_{n+1}/2$. Thus $|x-p| + |p-q| > \ell_{n+1}/2$, and again Lemma 5.7 can be applied to the exponential sum. The geometry of this case is illustrated in figure 2. Combining estimates (5.31) and (5.32) we have the following decay result.

Lemma 5.9. *There exists a constant $C > 0$, independent of n such that if*

$$\frac{r_0 C e^{-\gamma_{n-1}\ell_{n+1}/2}}{\delta_{n+1}\gamma_{n-1}^{15}} < 1 \quad , \tag{5.33}$$

then

$$\| \, (G_E \oplus_j G_{D_j})\Gamma_{ED}(G_{\overline{E}} \oplus_j G_{\overline{D}_j}) \, \|_{\sigma_{n-1}-5\gamma_{n-1}} \leq \frac{1}{4}\frac{C_G^n}{\delta_{n+1}\gamma_{n-1}^{12}} \quad . \tag{5.34}$$

Remark. In the case where the nonlinearity contains no quadratic terms, (5.33) can be replaced by

$$\frac{r_0^2 C e^{-\gamma_{n-1}\ell_{n+1}/2}}{\delta_{n+1}\gamma_{n-1}^{15}} < 1 \quad . \tag{5.35}$$

Finally, we bound the infinite series in (5.28). The estimates are much like those immediately preceeding. We will prove

Lemma 5.10. *There exists a constant $C > 0$, independent of n, such that if*

$$\frac{r_0^2 C G_G^n e^{-\gamma_{n-1}\ell_{n+1}/2}}{\delta_{n+1}^2\gamma_{n-1}^{34}} < \frac{1}{4} \quad , \tag{5.36}$$

then

$$\| \, G_E \oplus_j G_{D_j}\Gamma_{ED} G_{\overline{E}} \oplus_j G_{\overline{D}_j}\Gamma_{\overline{ED}} \, \|_{\sigma_{n-1}-5\gamma_{n-1}} \leq \frac{1}{4} \quad . \tag{5.37}$$

Proof. Consider first case 1. Then,

$$|G_E \oplus_j G_{D_j}\Gamma_{ED} G_{\overline{E}} \oplus_j G_{\overline{D}_j}\Gamma_{\overline{ED}}(x,y)| = \Big| \sum_{\substack{p\in E \\ q\in D_j \\ r\in\overline{D}_j}} G_E(x,p)\Gamma_{ED}(p,q)G_{\overline{D}_j}(q,r)\Gamma_{\overline{ED}}(r,y)\Big| \tag{5.38}$$

Here the restrictions on the location of p and q are the same as in (5.30), while $r \in \overline{D_j}$ because $q \in D_j \subseteq \overline{D}_j$. In fact the sum vanishes unless $y \in \overline{E} \cup_{k\neq j} \overline{D}_j$, for only then is $\Gamma_{\overline{ED}}(r,y) \neq 0$. These paths also have a long step between x and q, whence $|x-p| + |p-q| > \ell_{n+1}/2$. Using additionally the estimate $|\Gamma_{\overline{ED}}(r,y)| \leq$

$(\|\mathbf{p}\|C_W/\gamma_{n-1}^3)e^{-(\gamma_{n-1}-2\gamma_{n-1}|r-y|)}$, (5.38) is bounded by an exponential sum;

$$
\begin{aligned}
&|G_E \oplus_j G_{D_j}\Gamma_{ED}G_{\overline{E}} \oplus_j G_{\overline{D}_j}\Gamma_{\overline{ED}}(x,y)| \\
&\leq \sum_{\substack{p\in E\\ q\in D_j\\ r\in\overline{D}_j}} \left(\frac{CC_G^n}{\delta_n\delta_{n+1}\gamma_{n-1}^{22}}\right)\left(\frac{C_W\|\mathbf{p}\|}{\gamma_{n-1}^4}\right)^2 e^{-(\sigma_{n-1}-3\gamma_{n-1})(|x-p|+|p-q|+|q-r|+|r-y|)} \\
&\leq \frac{CC_G^n\|\mathbf{p}\|^2}{\delta_{n+1}^2\gamma_{n-1}^{32}}e^{-\gamma_{n-1}\ell_{n+1}/2}e^{-(\sigma_{n-1}-4\gamma_{n-1})|x-y|)} \quad .
\end{aligned}
$$

Lemma 5.7 is used above in estimating the exponential sums over paths with large steps. In Case 2, a very similar computation, using the estimates of (5.32) leads to an identical conclusion–we omit the details.

If we now apply Proposition 3.1 we find

$$
\| G_E \oplus_j G_{D_j}\Gamma_{ED}G_{\overline{E}} \oplus_j G_{\overline{D}_j}\Gamma_{\overline{ED}} \|_{\sigma_{n-1}-5\gamma_{n-1}} \leq \frac{CC_G^n\|\mathbf{p}\|^2}{\delta_{n+1}^2\gamma_{n-1}^{34}}e^{-\gamma_{n-1}\ell_{n+1}/2} \quad .
$$

Thus, if

$$
\frac{CC_G^n\|\mathbf{p}\|^2}{\delta_{n+1}^2\gamma_{n-1}^{34}}e^{-\gamma_{n-1}\ell_{n+1}/2} < 1/4,
$$

the operator norm $\| G_E \oplus_j G_{D_j}\Gamma_{ED}G_{\overline{E}} \oplus_j G_{\overline{D}_j}\Gamma_{\overline{ED}} \|_{\sigma_{n-1}-5\gamma_{n-1}} \leq 1/4$, and the Lemma 5.10 follows.

$\square$

Remark. If the nonlinearity in (2.8) contains no quadratic term, estimate (5.36) may be replaced by

$$
\frac{r_0^4 CG_G^n e^{-\gamma_{n-1}\ell_{n+1}/2}}{\delta_{n+1}^2\gamma_{n-1}^{34}} < \frac{1}{4} \quad , \tag{5.39}
$$

The full resolvent expansion (5.28) is now easily estimated in operator norm. Using estimates (5.29), (5.34) and (5.26),

$$
\begin{aligned}
\| G_B \|_{\sigma_{n-1}-5\gamma_{n-1}} \leq & \| G_E \oplus_j G_{D_j} \|_{\sigma_{n-1}-5\gamma_{n-1}} \\
& + \| G_E \oplus_j G_{D_j} \Gamma_{ED} G_{\overline{E}} \oplus_j G_{\overline{D}_j} \|_{\sigma_{n-1}-5\gamma_{n-1}} \\
& + \sum_{\ell=1}^{\infty} \| (G_E \oplus_j G_{D_j}) \Gamma_{ED} (G_{\overline{E}} \oplus_j G_{\overline{D}_j}) \Gamma_{\overline{ED}} \|^{\ell}_{\sigma_{n-1}-5\gamma_{n-1}} \times \\
& \quad \big(\| G_E \oplus_j G_{D_j} \|_{\sigma_{n-1}-5\gamma_{n-1}} \\
& \qquad + \| (G_E \oplus_j G_{D_j}) \Gamma_{ED} (G_{\overline{E}} \oplus_j G_{\overline{D}_j}) \|_{\sigma_{n-1}-5\gamma_{n-1}} \big) \\
\leq & \Big(\frac{C C_G^n}{\delta_{n+1}\gamma_{n-1}^{12}} + \frac{1}{4} \frac{C C_G^n}{\delta_{n+1}\gamma_{n-1}^{12}} \Big) \sum_{\ell=0}^{+\infty} \frac{1}{4^{\ell}} \\
\leq & \frac{G_G^{n+1}}{\delta_{n+1}\gamma_{n-1}^{12}} \quad .
\end{aligned}
\tag{5.40}
$$

The requirements on the choice of γ_{n-1} will be easily satisfied in the induction, as the radii ℓ_{n+1} of isolating neighborhoods of resonant sites grow exponentially, while the tolerance δ_{n+1} for small divisors behaves similarly, but does not appear in the exponent of constants. This finishes the proof of Theorem 5.6.

□

5.3. The Newton step:

The second step in the induction is to perform a correction to the approximation given by Newton's method. That is, the new approximate solution is $u_{n+1} = u_n + v_n$, where v_n is obtained as a solution of a linear equation on the domain $\overline{B_{n+1}}$, using the Green's function $G_{\overline{B_{n+1}}}(\mathbf{p}, \Omega, u_n)$. Linearizing the nonlinear equation about this new approximation changes it; $H_{\overline{B_{n+1}}}(\mathbf{p}, \Omega, u_{n+1})$ and its inverse must be estimated. Due to the rapid convergence of the Newton scheme these corrections are small and the estimates are not difficult. From section 3, we know that

$$
\| v_n \|_{\sigma_n - \gamma_n} \leq \frac{\|\mathbf{p}\|^2 \epsilon_n C_G^{n+1}}{\delta_{n+1}\gamma_n^{12}}. \tag{5.41}
$$

The following estimates are designed to correct the operator $H_{B_{n+1}}(u_{n+1})$ and the Green's function $G_{\overline{B_{n+1}}}$ due to these adjustments – in fact they are valid for any sequence $v(x)$ satisfying inequality (5.41).

Lemma 5.11. *If one adds to the function $u_n(x)$ any $v(x)$ satisfying the estimate (5.41), the modified Hamiltonian has the form*

$$H_{E_{n+1}}(u_n + v) = H_{E_{n+1}}(u_n) + R_{n+1}(u_n, v) \quad , \tag{5.42}$$

for any $E_{n+1} \subset \overline{B_{n+1} \cup A}$. Then R_{n+1} satisfies the estimate

$$\| R_{n+1}(u_n; v) \|_{\sigma_n - 2\gamma_n} \leq \frac{\|\mathbf{p}\|^2 \epsilon_n C_W C_G^{n+1}}{\delta_{n+1}\gamma_n^{15}} \tag{5.43}$$

Proof. The linear operator $H_B(u) = V_B(\Omega) + DW_B(\mathbf{p}, u)$ consists of a linear diagonal part, which is independent of $u_n + v_n$, and an off diagonal piece $DW(\phi(\mathbf{p}) + u)$. From estimate **H2** of Section 6, $D_u^2 W$ has operator norm bounded by $C_W \|\mathbf{p}\| \gamma^{-3}$ on $\mathcal{H}_{\sigma_n - \gamma}$. The estimate on R_{n+1} follows from Taylor's theorem.

□

When $(\mathbf{p}, \Omega) \in \mathcal{N}_{n+1}$ the spectrum of $H_{E_{n+1}}(u_n)$ is bounded by $\delta_{n+1}/2$ away from zero. The induction provides for perturbations R_{n+1} satisfying (5.43), which are substantially smaller than this. The Green's function $G_{E_{n+1}}(u_{n+1})$ can be constructed from $G_{E_{n+1}}(u_n)$ and R_{n+1} via Neumann series for z in the set $\{z; |z| \leq \delta_{n+1}/4\}$. This gives the following result for the construction.

Theorem 5.12. *Under perturbations of size (5.43), the Green's function remains a bounded operator on $\mathcal{H}_{\sigma_n - 2\gamma_n}$ for all $|z| \leq \delta_{n+1}/4$, as long as*

$$r_0^2 \epsilon_n \leq \frac{\delta_{n+1}^2 \gamma_n^{27}}{2 C_W C_G^{2(n+1)}} \quad ,$$

and one has the estimate that

$$\| G_{E_{n+1}}(u_{n+1}) \|_{\sigma_n - 2\gamma_n} \leq \frac{2 C_G^{(n+1)}}{\delta_{n+1} \gamma_n^{12}} . \tag{5.44}$$

Proof. Simple Neumann series gives that

$$G_{E_{n+1}}(u_n + v) = G_{E_{n+1}}(u_n) \left(1 + R_{n+1} G_{E_{n+1}}(u_n)\right)^{-1} ,$$

so that it suffices to bound $\| R_{n+1} G_{E_{n+1}}(u_n) \|_{\sigma_n - 2\gamma_n} \leq 1/2$. The stated condition on the constants assures that this will be the case.

□

Note that Theorem 5.6 and Theorem 5.12 imply Theorem 3.6 which we used repeatedly in section 4.

The same method allows us to carry through an estimate on the Green's function on nonsingular domains. One has the

Corollary 5.13. *Let $A \subset \mathbf{Z}^+ \times \mathbf{Z}$ be a nonsingular domain and $u_n = u_0 + \sum_{j=0}^{n-1} v_j$, for $n \geq 1$. If*

$$\|v_j\|_{\sigma_j - \gamma_j} \leq \frac{1}{4^{j+1}} \frac{\gamma_{j+1}^6 \gamma_0^3 \delta_0}{2C_W C_0}$$

and if $|z| < \delta_{n+1}/2$, then

$$\|G_A(u_n; z)\|_{\sigma_{n-1} - 2\gamma_{n-1}} \leq \frac{2C_0}{d_s \gamma_0^3} \quad .$$

The proof is again a simple application of Neumann series–we do not repeat it.

6. Analysis of the nonlinear operator

For the nonlinear wave equation (2.1) it is natural to assume that the nonlinear term $g(x, s)$ is periodic in x with period π, and analytic in the region $\{(x, s) \in \mathbf{C}^2; |\mathrm{Im}\, x| \leq \overline{\sigma}\}$. These conditions translate into conditions for the nonlinear operator W applied to sequences defined over the lattice $\mathbf{Z}^+ \times \mathbf{Z}$. In stating these estimates, we will fix an index $\sigma_* < \overline{\sigma}$, and then give estimates that will hold for all $\sigma < \sigma_*$. The constants will depend upon $\overline{\sigma}$ and σ_*, but will be uniform in $\sigma < \sigma_*$.

This section also addresses several of the properties of the nonlinear wave equation that have been stated in Section 2, including the properties of genericity of (d_0, L_0)-nonresonant frequency sequences $\{\omega_j\}_{j=1}^{\infty}$. This will finish the proofs of the main theorems, Theorem 2.1 and Theorem 2.2. Finally, there is a proof of Theorem 2.8 on the accumulation of periodic orbits at zero in the fully non-resonant case.

6.1. Analyticity properties of W

The basic property of the nonlinear operator W in the nonlinear lattice equation (2.8) is that it is an analytic mapping $W : \mathcal{H}_\sigma \to \mathcal{H}_{\sigma-\gamma}$, for any $\sigma < \sigma_*$. This works well with the techniques of estimating exponentially decaying matrices in Section 5. However we will use slightly stronger hypotheses on the nonlinearity. These hold in the case of the nonlinear wave equation of the original problem, as we will prove below. Since we are interested in solutions of the nonlinear problem which are small perturbations of the normal modes of the linear problem, these properties are stated in terms of perturbations from a given linear mode $\phi(\mathbf{p})$. The additional assumptions are estimates on perturbations of $W(\phi(\mathbf{p}))$ and its first and second derivatives. Let $0 < \sigma < \sigma_*$.

H1 If $u \in \mathcal{H}_\sigma$, with $Qu = 0$ and $\|u\|_\sigma \le r_0$, then for any $0 < \gamma \le \sigma$, $W(\phi(\mathbf{p}) + u) \in \mathcal{H}_{\sigma-\gamma}$, and

$$\|W(\phi(\mathbf{p}) + u)\|_{\sigma-\gamma} \le \frac{C_W}{\gamma}(\|\mathbf{p}\|^2 + \frac{1}{\gamma^2}\|u\|_\sigma^2). \tag{6.1}$$

H2 If $u \in \mathcal{H}_\sigma$, with $Qu = 0$ and $\|u\|_\sigma \le r_0$, then for any $0 < \gamma \le \sigma$, $D_u W(\phi(\mathbf{p}) + u)$ is a bounded operator on $\mathcal{H}_{\sigma-\gamma}$, with operator norm given by the estimate

$$\|D_u W(\phi(\mathbf{p}) + u)\|_{\sigma-\gamma} \le \frac{C_W}{\gamma^2}(\|\mathbf{p}\| + \frac{1}{\gamma}\|u\|_\sigma) \tag{6.2}$$

H3 If $u \in \mathcal{H}_\sigma$, with $Qu = 0$ and $\|u\|_\sigma \le r_0$, then for any $0 < \gamma \le \sigma$, The second derivative $D_u^2 W(\phi(\mathbf{p}) + u)$ satisfies the estimates

$$\|D_u^2 W(\phi(\mathbf{p}) + u)[v, w]\|_{\sigma-\gamma} \le \frac{C_W}{\gamma^3}\|v\|_\sigma \|w\|_{\sigma-\gamma}. \tag{6.3}$$

Furthermore this bilinear operator can be written in the form $D_u^2 W = C + \Delta$, where Δ satisfies

$$\|\Delta[v, w]\|_{\sigma-\gamma} \le \frac{C_W}{\gamma^3}(\|\mathbf{p}\| + \frac{1}{\gamma}\|u\|_\sigma)\|v\|_\sigma \|w\|_{\sigma-\gamma}. \tag{6.4}$$

The operator $C = D_u^2 W(0)$ satisfies the property

$$\langle \delta(z), C[\phi(\mathbf{p}), \delta(z')]\rangle_0 = 0 \tag{6.5}$$

for any z, z' with $(z - z') = (0, 0)$ or $(\pm 1, 0)$.

The hypotheses (6.4) and (6.5) are natural, and always satisfied by the nonlinear wave equation, where C depends only upon the coefficient $g_2(x)$. In fact if $D_u^2 W(0) = 0$, that is the nonlinear term is of cubic or higher order at zero, then $C = 0$. The role of hypotheses (6.4) and (6.5) is to avoid overly strong nonlinear coupling between resonances associated with lattice sites within the same singular region. They are used in section 4 in the analysis of the linearized operators $D_u W$ and their dependence on parameters. In the above estimates, the constant C_W is of course independent of γ.

6.2. Circumstances in which $H1$-$H3$ are satisfied.

We consider two cases of boundary conditions for equation (2.1) for which hypotheses $H1$-$H3$ are satisfied by the associated nonlinear operator on the lattice. The first case is for periodic boundary conditions on the interval $[0, \pi]$, and the second is Dirichlet boundary conditions imposed on the endpoints $0, \pi$. In this second case there is an additional restriction on the non-linearity.

Periodic Boundary Conditions: Assign periodic boundary conditions to the Sturm-Liouville operator $L_{g_1} = -\frac{d^2}{dx^2} + g_1(x)$ on $[0, \pi]$. The potential g_1 is assumed to be periodic, of period π, and analytic in the strip $\{x;\ |\mathrm{Im}\, x| < \overline{\sigma}\}$. The spectral asymptotics for Sturm-Liouville operators guarantees that there are at most finitely many negative eigenvalues. Label the eigenvalues

$$\omega_0^2 < \omega_1^2 \leq \omega_2^2 < \omega_3^2 \leq, \ldots \quad .$$

and let $\{\psi_n(x)\}_{n=0}^{\infty}$ be the corresponding sequence of eigenfunctions. The main technical result of this section is that these eigenfunctions have uniform properties of analyticity in $\{x;\ |\mathrm{Im}\, x| < \overline{\sigma}\}$.

Proposition 6.1. *Expand* $\psi_n(x) = \sum_m \hat{\psi}_n(m) e^{imx}$. *Then for* $\sigma_* < \overline{\sigma}$, *there are constants* $C_1(g_1, \sigma_*)$ *and* $C_2(g_1, \sigma_*)$, *(independent of* n*) such that for* $m > 0$ *one has*

$$|\hat{\psi}_n(m)| \leq C_1(g_1, \sigma_*) e^{-\sigma_* |n - |m||} \quad . \tag{6.6}$$

Conversely, if one expands $e^{inx} = \sum_m \hat{e}_n(m) \psi_m(x)$ *then*

$$|\hat{e}_n(m)| \leq C_2(g_1, \sigma_*) e^{-\sigma_* ||n| - m|} \quad . \tag{6.7}$$

Remark. The eigenfunctions $\psi_n(x)$ for the periodic problem are analytic in the strip $\{x; |\mathrm{Im}\ x| < \overline{\sigma}\}$, thus this result "almost" follows from analyticity of the eigenfunctions. The one thing that must be checked is that the constants $C_{1,2}(g_1, \sigma)$ can be chosen uniformly in n. The proof of this proposition will be given below, following a discussion of some of its consequences.

From this proposition we obtain the following useful corollary. Let $\mathcal{D}_\sigma = \{(x,\xi) \in \mathbf{C}^2; |\mathrm{Im}\ x| < \sigma, |\mathrm{Im}\ \xi| < \sigma\}$. If $\tilde{f}(x,\xi)$ is analytic on $\mathcal{D}_\sigma$, and periodic with period π in x and 2π in ξ, define $|||\tilde{f}|||_\sigma = \sup_{(x,\xi)\in\mathcal{D}_\sigma} |\tilde{f}(x,\xi)|$. Let f be the coefficients in the eigenfunction expansion of $\tilde{f}$.

Corollary 6.2. *There exist constants $C_1(g_1, \sigma_*)$ and $C_2(g_1, \sigma_*)$ such that for all $0 < \gamma \le \sigma < \sigma_*$, $|||\tilde{f}|||_{\sigma-\gamma} < (C_1/\gamma)||f||_\sigma$ and $||f||_{\sigma-\gamma} < (C_2/\gamma)|||\tilde{f}|||_{\sigma-\gamma}$.*

Proof. One just writes out $\tilde{f}(x,\xi)$ in terms of its eigenfunction expansion, substitutes for the factors $\psi_n(x)$ their expansion in terms of exponentials, and then uses the estimates of Proposition 6.1 to bound the resulting sums. To estimate f one writes down the definition of this quantity, and again bounds it using the estimates of Proposition 6.1.

□

With the proposition in hand, verifying the hypotheses on W is straightforward. We begin by supposing that $u \in \mathcal{B}_\sigma$ with $\|u\|_\sigma \le C_u \|\mathbf{p}\|^2$ and that $g(x,u)$ is analytic for $(x,u) \in \{|\mathrm{Im}\ x| \le \overline{\sigma}\}$, for some $\overline{\sigma} > 0$. Then $\tilde{W}(\phi(\mathbf{p}) + u) = (g - g_1)(x, \tilde{\phi}(\mathbf{p}) + \tilde{u})$. By Corollary 6.2, for any $0 < \gamma \le \sigma$, $|||\tilde{\phi}(\mathbf{p}) + \tilde{u}|||_{\sigma-\gamma/2} < (C/\gamma)(\|\mathbf{p}\| + C_u\|\mathbf{p}\|^2))$. Then, $(g - g_1)(x, \tilde{\phi}(\mathbf{p})(x,\xi) + \tilde{u}(x,\xi))$ is analytic for $(x,\xi) \in \mathcal{D}_{\sigma-\gamma/2}$. Furthermore,

$$\|W(\phi + u)\|_{\sigma-\gamma} \le \frac{C_2}{\gamma} |||(g - g_1)(x, \tilde{\phi} + \tilde{u}(x,\xi))|||_{\sigma-\gamma/2} \quad .$$

Since $(g - g_1)$ is of at least quadratic order in $\tilde{u}$ for $|||\tilde{u}||| \le 1$, $|||(g-g_1)(x, \tilde{\phi}+\tilde{u})|||_{\sigma-\gamma/2} \le C_3 |||\tilde{\phi}(\mathbf{p}) + \tilde{u}|||^2_{\sigma-\gamma/2}$ as long as $|||\tilde{\phi}(\mathbf{p}) + \tilde{u}|||_{\sigma-\gamma/2} \le 1$. Thus,

$$\|W(\tilde{\phi} + \tilde{u})\|_{\sigma-\gamma} \le \frac{C_W}{\gamma} (\|\mathbf{p}\|^2 + \frac{1}{\gamma^2} \|u\|^2_{\sigma-\gamma})$$

This verifies **H1**.

Remark. In the case when the nonlinearity contains no quadratic term in its Taylor series, one can improve this estimate to

$$\|W(\phi(\mathbf{p})+u)\|_{\sigma-\gamma} \leq \frac{C_W}{\gamma}(\|\mathbf{p}\|^3 + \frac{1}{\gamma^3}\|u\|^3_{\sigma-\gamma}). \quad ,$$

with similar improvements reflected in the estimates of the first and second derivatives of W.

We now prove that the derivative of W satisfies **H2**. Let $x=(j,k)$ and $x'=(j',k')$ be two points in $\mathbf{Z}^+ \times \mathbf{Z}$. Then

$$\begin{aligned} D_u W(\phi(\mathbf{p})+u)(x,x') &= \langle \delta(x), D_u W(\phi(\mathbf{p})+u)\delta(x')\rangle_0 \\ &= \int\int \overline{\psi}_j(x)\psi_{j'}(x)e^{i\xi(k'-k)} D_u(g-g_1)(\tilde{\phi}(x,\xi)+\tilde{u}(x,\xi))dxd\xi \end{aligned}$$

If we write $\overline{\psi}_j(x) = \sum_n \overline{\hat{\psi}}_j(n)e^{-inx}$, and a similar expression for $\psi_{j'}(x)$, we have

$$D_u W(x,x') = \sum_{n,n'} \overline{\hat{\psi}}_j(n)\hat{\psi}_{j'}(n') \int\int e^{ix(n-n')}e^{i\xi(k-k')}D_u(g-g_1)(\tilde{\phi}(\mathbf{p})+\tilde{u})\ dxd\xi$$

Since $D_u g(\tilde{\phi}(x,\xi)+\tilde{u}(x,\xi))$ is analytic for $(x,\xi) \in \mathcal{D}_{\sigma-\gamma/2}$, the double integral is bounded in absolute value by

$$e^{-(\sigma-\gamma/2)(|k-k'|+|n-n'|)}|||D_u(g-g_1)(\tilde{\phi}(\mathbf{p})+\tilde{u})|||_{\sigma-\gamma/2} \quad .$$

Thus by Proposition 6.1

$$\begin{aligned} |D_u W(x,x')| \leq & C|||D_u(g-g_1)(\tilde{\phi}+\tilde{u})|||_{\sigma-\gamma/2} \\ & \times \sum_{n,n'} e^{-(\sigma-\gamma/2)(|k-k'|+|n-n'|)}e^{-\sigma_*(|j-|n||+|j'-|n'||)} \\ \leq & C|||D_u(g-g_1)(x,\tilde{\phi}+\tilde{u})|||_{\sigma-\gamma/2}e^{-(\sigma-\gamma/2)(|k-k'|+|j-j'|)} \end{aligned}$$

with constant $C = C(\sigma_* - \sigma)$. Then Proposition 3.1 implies

$$\|D_u W\|_{\sigma-\gamma} \leq \frac{C}{\gamma^2}|||D_u(g-g_1)(\tilde{\phi}+\tilde{u})|||_{\sigma-\gamma/2} \quad .$$

Note that since $(g-g_1)(x,s)$ is at least quadratic in s, $|D_u(g-g_1)(x,s)| \leq C|s|$, so if $\|u\|_\sigma \leq C_u\|\mathbf{p}\|^2$, then $|||D_u(g-g_1)(\tilde{\phi}+\tilde{u})|||_{\sigma-\gamma/2} \leq C(\|\mathbf{p}\| + |||\tilde{u}|||_{\sigma-\gamma/2})$. This, when combined with the previous estimates of $|||\tilde{u}|||_{\sigma/2}$ completes the verification of **H2**.

Finally we turn to the verification of **H3**. The boundedness of the second derivative $D_u^2 W$ is almost the same as the estimate for **H2**, so we will not include the details here. The decomposition $D_u^2 W = C + \Delta$ for the nonlinear wave equation arises from the Taylor expansion of the nonlinear term.

$$\begin{aligned} D_{\tilde{u}}^2 g(x, \tilde{\phi} + \tilde{u}) &= 2g_2(x) + 6g_3(x)(\tilde{\phi} + \tilde{u}) + \cdots \\ &\equiv \tilde{C} + \tilde{\Delta} \end{aligned}$$

Let $z = (j, k)$, $z' = (j', k)$ with $(j - j') = 0, \pm 1$ and $p_1 + ip_2 = re^{i\theta k}$. The behavior of the bilinear operator C claimed in (6.5) is verified by the integral

$$\langle \delta(z), C[\phi(\mathbf{p}), \delta(z')] \rangle_0 =$$
$$\int_0^{2\pi} \int_0^{\pi} \overline{\psi}_j(x) e^{-i\xi k} \; g_2(x) r \psi_1(x) \cos(\xi + \theta) e^{i\xi} \psi_{j'}(x) e^{i\xi k} \; dx d\xi = 0.$$

Thus all contributions to the second piece of the bilinear form Δ are at least linear in $\phi(\mathbf{p}) + u$. The hypothesis is that $u \in \mathcal{H}_\sigma$ so that the use of the estimates in Corollary 6.2 gives the statement (6.4).

Combining these estimates, we have shown that the nonlinear wave equation with periodic boundary conditions gives rise to a nonlinear lattice problem which satisfies the hypotheses **H1**,**H2** and **H3**.

Proposition 6.3. *If we consider periodic boundary conditions in (2.8), and if the nonlinearity $g(x, u)$ is π periodic in x and analytic in (x, u) for $x \in \{|\mathrm{Im}\ x| < \overline{\sigma}\}$, for some $\overline{\sigma} > 0$, then for $\sigma_* < \overline{\sigma}$, for all $0 < \sigma < \sigma_*$ the requirements* **H1** – **H3** *are satisfied for the operator W and its derivatives.*

Dirichlet Boundary Conditions: The second problem that is considered is for Dirichlet boundary conditions imposed on the interval $[0, \pi]$. We assume that the nonlinear term $g(x, u)$ is odd with respect to reflection through the origin – *i.e.* $g(-x, -u) = -g(x, u)$. Eigenfunction expansions are thus considered with respect to the Sturm-Liouville operator $L_{g_1} = -\frac{d^2}{dx^2} + g_1(x)$, with Dirichlet boundary conditions imposed on $[0, \pi]$. Thus in addition to assuming that $g_1(x)$ is analytic and periodic in a strip of width $\overline{\sigma}$ about the real axis, it is even with respect to 0 – *i.e.* $g_1(-x) = g_1(x)$. We label the eigenvalues of L_{g_1} in order,

$$\omega_1^2 < \omega_2^2 < \omega_3^2 <, \dots \; .$$

Denote $\{\psi_n(x)\}_{n=1}^{\infty}$ the eigenfunctions corresponding to ω_n. Since g_1 is even, the eigenfunctions $\psi_n(x)$ can be extended as **odd** 2π periodic analytic functions, and hence their Fourier series expansions are of the form

$$\psi_n(x) = \sum_{m=1}^{\infty} \hat{\psi}_n(m) \sin(mx) \quad .$$

The result analogous to Proposition 6.1 is

Proposition 6.4. *Expand* $\psi_n(x) = \sum_{m>0} \hat{\psi}_n(m) \sin(nx)$. *Then for* $\sigma_* < \overline{\sigma}$, *there are constants* $C_1(g_1, \sigma_*)$ *and* $C_2(g_1, \sigma_*)$, *(independent of n) such that for* $m > 0$ *one has*

$$|\hat{\psi}_n(m)| \leq C_1(g_1, \sigma_*) e^{-\sigma_* |n-m|} \quad . \tag{6.8}$$

Conversely, if one expands $\sin(mx) = \sum_n \hat{s}_n(m) \psi_m(x)$ *then*

$$|\hat{s}_n(m)| \leq C_2(g_1, \sigma_*) e^{-\sigma_* |n-m|} \quad . \tag{6.9}$$

Let $\tilde{f}(x, \xi)$ be an analytic in $\mathcal{D}_\sigma$, 2π periodic in x, ξ and an odd function of x. The analog of Corollary 6.2 is the following;

Corollary 6.5. *There exist constants* $C_1(g_1, \sigma_*)$ *and* $C_2(g_1, \sigma_*)$ *such that for* $\sigma < \sigma_*$,

$$|||\tilde{f}|||_{\sigma-\gamma} < \frac{C_1}{\gamma} \|f\|_{\sigma-\gamma},$$

and

$$\|f\|_{\sigma-\gamma} < \frac{C_2}{\gamma} |||\tilde{f}|||_{\sigma-\gamma}.$$

Using this corollary, the verification of **H1-H3** for the nonlinear term W proceeds along similar lines. We assume that $g(x, \tilde{u})$ is odd under reflection through the origin so that the nonlinear term preserves the class of 2π periodic functions which are odd in x. That is, $g(-x, \tilde{u}(-x, \xi)) = -g(x, \tilde{u}(x, \xi))$ provided $\tilde{u}(-x, \xi) = -\tilde{u}(x, \xi)$.

6.3. Proof of Propositions 6.1 and 6.4:

We now turn to the proof of the technical results Proposition 6.1 and Proposition 6.4. We will give the details for the case of periodic boundary conditions; of course the case of Dirichlet boundary conditions is similar. Let the function $g_1(x)$ be periodic with period π and analytic in the strip $\{x \in \mathbf{C};\ |\mathrm{Im}\ x| < \overline{\sigma}\}$. Consider the Sturm-Liouville problem

$$(-\frac{d^2}{dx^2} + g_1(x))\psi_n(x) = \omega_n^2 \psi_n(x) \quad , \tag{6.10}$$

with periodic boundary conditions on $[0, \pi]$. The eigenfunctions $\{\psi_n\}_{n=0}^{\infty}$ are analytic in the same strip, thus their Fourier coefficients decay exponentially. The point of the proposition is to obtain a uniform bound on this decay. We present here a proof of this fact that is in the spirit of this paper.

Taking inner products of both sides of (6.10) with the exponential functions $e_k(x) = e^{ikx}$, gives

$$k^2 \hat{\psi}_n(k) + (e_k, g_1 \psi_n) = \omega_n^2 \hat{\psi}_n(k) \quad , \tag{6.11}$$

where $\hat{\psi}_n(k) = (e_k, \psi_n)$. If we now use the fact that $\{e^{ikx}\}_{k \in \mathbf{Z}}$ is a basis for $L^2(0, 2\pi)$, we can rewrite (6.11) as

$$\sum_j (e_k, g_1 e_j) \hat{\psi}_n(j) + (k^2 - \omega_n^2) \hat{\psi}_n(k) = 0 \quad .$$

This equation can be rewritten as

$$(V_n + D)\hat{\psi}_n = 0 \quad , \tag{6.12}$$

where $\hat{\psi}_n \in \ell^2(\mathbf{Z})$ with components $\hat{\psi}_n(k)$. V_n is a diagonal operator on $\ell^2(\mathbf{Z})$ with matrix elements $V_n(k,k) = (k^2 - \omega_n^2)$, and D is an operator with matrix elements satisfying an estimate

$$\begin{aligned} |D(j,k)| \leq & \left|\int_0^{2\pi} e^{ix(j-k)} g_1(x) dx\right| \\ \leq & C_{g_1} e^{-\overline{\sigma}|j-k|} . \end{aligned} \tag{6.13}$$

Assume for convenience that n is even. The case of n odd follows from that below with only minor changes in the notation. From the asymptotics of the eigenvalues of (6.10) we know that $\omega_n \approx n$. Since $k^2 - \omega_n^2 = (k - \omega_n)(k + \omega_n)$, we see that for $k \neq \pm n$, and n sufficiently large, there exists a constant $c_{g_1} > 0$ such that

$$\left|\frac{1}{n} V_n(k,k)\right| = \left|\frac{1}{n}(k - \omega_n)(k + \omega_n)\right| \geq c_{g_1} \quad ,$$

while $|V_n(\pm n, \pm n)/n| \leq c_{g_1}/8$. On the other hand, the operator norm of D/n on $\ell^2(\mathbf{Z})$ is bounded by C_{g_1}/n, for C_{g_1} independent of n. Thus for n sufficiently large so that $C_{g_1}/n < c_{g_1}/8$, the operator $H_n \equiv (V_n + D)/n$ will have a pair of eigenvalues whose distance from the origin is less than $c_{g_1}/4$, and the remainder of the spectrum will be a distance of at least $3c_{g_1}/4$ from the origin. We decompose the lattice $\mathbf{Z}$ into two regions relative to these two classes of spectra, $S \equiv \{n, -n\}$, and $NS \equiv \mathbf{Z} - S$. The projection

$$P \equiv \frac{1}{2\pi i} \oint_{\Delta} \frac{d\zeta}{(H_n - \zeta)} \quad , \tag{6.14}$$

with Δ the circle of radius $c_{g_1}/2$ centered at the origin, will project onto the two dimensional subspace corresponding to the two 'small' eigenvalues of H_n; this eigenspace is quite close to $\ell^2(S)$. The vector $\hat{\psi}_n \in \ell^2(\mathbf{Z})$ is an element of this subspace, since $H_n \hat{\psi}_n = 0$. We estimate $\hat{\psi}_n(j)$ by

$$|\hat{\psi}_n(j)| = |(e_j, \psi_n)| \leq |(e_j, Pe_j)|^{1/2} \quad . \tag{6.15}$$

The right hand side of this inequality is bounded with the aid of the following result.

Lemma 6.6. *Let $\sigma_* < \overline{\sigma}$. There exists $N_0 > 0$ and a constant $C_{N_0} > 0$ such that for $n \geq N_0$,*

$$|(e_j, Pe_j)| \leq C_{N_0} e^{-2\sigma_*(||j|-n|)} \quad .$$

Define a decomposition of the operator H_n relative to the subspaces $\ell^2(S) \oplus \ell^2(NS) = \ell^2(\mathbf{Z})$,

$$\begin{aligned} H_0 =& \frac{1}{n} V_n|_{NS} \\ H_S =& \frac{1}{n}(V_n + D)|_S \\ H_{NS} =& \frac{1}{n}(V_n + D)|_{NS}. \end{aligned} \tag{6.16}$$

The Green's functions associated with these reduced operators are

$$\begin{aligned} G_0(\zeta) =& (H_0 - \zeta)|_{NS}^{-1} \\ G_S(\zeta) =& (H_S - \zeta)_S^{-1} \\ G_{NS}(\zeta =& (H_{NS} - \zeta)|_{NS}^{-1} \end{aligned} \tag{6.17}$$

The coupling term in this decomposition is defined by the relation

$$H_n = G_{NS} \oplus G_S + \Gamma. \tag{6.18}$$

Hilbert spaces analogous to those used in sections 3 and 5 are defined by the norm $\|v\|_{\sigma}^2 = \sum_{k\in\mathbf{Z}} |v(k)|^2 e^{2\sigma|k|}$, and $\|G\|_{\sigma}$ refers to the operator norm.

Lemma 6.7. *For $\zeta \in \Delta$, $|G(j,k;\zeta)| \leq \frac{4}{c_{g_1}}$.*

Proof. This follows immediately from our estimates on the location of the spectrum of H_n.

$\square$

Lemma 6.8. *Given $0 < \overline{\gamma} \leq \overline{\sigma}$ there exists $N_0 > 0$ such for $n \geq N_0$, and for $\zeta \in \Delta$,*

$$|G_{NS}(j,k;\zeta)| \leq \frac{8}{c_{g_1}} e^{-(\overline{\sigma}-\overline{\gamma})|j-k|} \quad .$$

Proof. This restricted Green's function is constructed via the Neumann series, with $(H_{NS} - \zeta) = (H_0 - \zeta) + (D/n)$. This decomposition gives the series

$$G_{NS}(\zeta) = G_0(\zeta) \sum_{j=0}^{\infty} (-1)^j ((D/N) G_0(\zeta))^j .$$

For $\zeta \in \Delta$ and n sufficiently large we will show that the series converges. The first point is that $\|G_0(\zeta)\|_{\overline{\sigma}} \leq 4/c_{g_1}$, and, using Proposition 3.1, $\|D/n\|_{\overline{\sigma}-\overline{\gamma}} \leq |||g_1|||_{\overline{\sigma}}/(\overline{\gamma} n)$. Thus for N_0 chosen sufficiently large, and for $n \geq N_0$, we have

$$\frac{1}{n} \|DG_0(\zeta)\|_{\overline{\sigma}-\overline{\gamma}} \leq \frac{C}{N_0} < \frac{1}{2}.$$

Hence $\|G_{NS}(\zeta)\|_{\overline{\sigma}-\overline{\gamma}} \leq 8/c_{g_1}$.

$\square$

To prove Lemma 6.6 we combine these lemmas with the resolvent identity:

$$G(\zeta) = G_S(\zeta) \oplus G_{NS}(\zeta) + (G_S(\zeta) \oplus G_{NS}(\zeta))\, \Gamma G(\zeta) \tag{6.19}$$

For $k = \pm n$, the estimate of Proposition 6.1 is simply the requirement of boundedness, so we assume that $k \neq \pm n$.

Iterating the resolvent expansion, we find that (formally)

$$\begin{aligned} G(\zeta) = G_S(\zeta) \oplus G_{NS}(\zeta) + (G_S(\zeta) \oplus G_{NS}(\zeta))\, \Gamma\, (G_S(\zeta) \oplus G_{NS}(\zeta)) \\ + \sum_{\ell=2}^{\infty} ((G_S(\zeta) \oplus G_{NS}(\zeta))\, \Gamma)^k\, (G_S(\zeta) \oplus G_{NS}(\zeta)) \end{aligned} \tag{6.20}$$

Because G_S and G_{NS} vanish unless both of their arguments lie in either the singular or non-singular region of the lattice respectively, and since Γ vanishes unless one of its arguments lies in S and the other in NS, the second term in (6.20) vanishes when evaluated at the arguments (k, k), with $k \in NS$. Furthermore, since dist.$(\mathrm{spec} H|_{NS}, 0) > 3c_{g_1}/4$, we have

$$\frac{1}{2\pi i} \oint_{\Delta} G_{NS}(k, k; \zeta) d\zeta = 0 \quad ,$$

so the first term in (6.20) also makes no contribution to the projection operator in(6.14).

We now turn to evaluate a typical term in the infinite sum. We will use

Lemma 6.9. *If $m \in S$, and $\zeta \in \Delta$, then there exists a constant $C_{g_1} > 0$ (depending on $\overline{\sigma}$ and $\overline{\gamma}$), such that*

$$\begin{aligned} |\left((G_S(\zeta) \oplus G_{NS}(\zeta))\, \Gamma \right) (m, p)| \le \frac{C_{g_1}}{N} e^{-(\overline{\sigma} - \overline{\gamma})||p|-|n||} \\ |\left((G_S(\zeta) \oplus G_{NS}(\zeta))\, \Gamma \right) (p, m)| \le \frac{C_{g_1}}{N} e^{-(\overline{\sigma} - \overline{\gamma})||p|-|n||} \quad . \end{aligned}$$

Proof. We verify the first inequality–the second follows in like fashion. Note that

$$\left((G_S(\zeta) \oplus G_{NS}(\zeta))\, \Gamma \right) (p, m) = \sum_{q \in NS} G_{NS}(p, q; \zeta) \Gamma(q, m) \quad .$$

By (6.13), $|\Gamma(q, m)| \le (C_{g_1}/N) e^{\overline{\sigma}|q-m|}$. From Lemma 6.8 we have $|G_{NS}(p, q; \zeta)| \le \frac{8}{c_{g_1}} e^{-(\overline{\sigma}-\overline{\gamma})|p-q|}$. Summing over q, and using the fact that $m = \pm n$, completes the proof.

□

As a consequence of this lemma we have

Corollary 6.10. *If $m \in S$, ℓ is a non-negative integer and $\zeta \in \Delta$, then there exists a constant $C_{g_1} > 0$ (depending on $\overline{\sigma}$ and $\overline{\gamma}$), such that*

$$|\left((G_S(\zeta) \oplus G_{NS}(\zeta))\, \Gamma \right)^{\ell} |(m, p) \le \frac{C_{g_1}}{N}^{\ell - 2} \quad .$$

Proof. The proof is an easy induction argument, the details of which we omit.

Now consider the term in the infinite sum,

$$\Big\{((G_S(\zeta)\oplus G_{NS}(\zeta))\,\Gamma)^{\ell}\,(G_S(\zeta)\oplus G_{NS}(\zeta))\Big\}(k,k)=$$
$$\sum_{\substack{m,p,r\in S\\ q,j\in NS}} G_{NS}(k,j,\zeta)\Gamma(j,m)$$
$$((G_S(\zeta)\oplus G_{NS}(\zeta))\,\Gamma)^{\ell-2}\,(m,p)G_S(p,r;\zeta)\Gamma(r,q)G_{NS}(q,k;\zeta)\quad .$$

The various factors in this sum can be bounded by (6.13), Lemma 6.7, Lemma 6.8, and Corollary 6.10. Inserting these bounds, we find that the right hand side of this equality is bounded in magnitude by

$$\sum_{\substack{m,p,r\in S\\ q,j\in NS}} \frac{8}{c_{g_1}}e^{-(\overline{\sigma}-\overline{\gamma})|k-j|}\frac{C_{g_1}}{N}e^{-\overline{\sigma}|j-m|}\left(\frac{C_{g_1}}{N}\right)^{\ell-2}\frac{8}{c_{g_1}}\frac{C_{g_1}}{N}e^{-\overline{\sigma}|r-q|}\frac{8}{c_{g_1}}e^{-(\overline{\sigma}-\overline{\gamma})|k-q|}\quad .$$

If we now sum over q and j, and use the fact that m, p, and r are all either $\pm n$, we find that this sum is bounded by

$$C\left(\frac{C_{g_1}}{N}\right)^{\ell-2}e^{-2\sigma_*||k|-n|}\quad ,$$

where we can choose σ_* to be any constant smaller than $\overline{\sigma}-\overline{\gamma}$. If N is large, this quantity can be summed over $\ell\geq 2$, and the estimate of Lemma 6.6 follows.

□

The proof of Proposition 6.1 follows from this result. For analytic potential functions g_1 the eigenfunctions $\psi_n(x)$ are π periodic, and analytic in the strip $\{x;|\mathrm{Im}\,x|<\overline{\sigma}\}$, thus the Fourier coefficients decay exponentially,

$$|\hat{\psi}_n(m)|\leq C(n)e^{-\overline{\sigma}|n-|m||}.$$

The fact that the constant can be chosen independently of n follows from estimate (6.15) and Lemma 6.6. Statement (6.7) and the results of Proposition 6.4 follow from similar arguments.

6.4. (d_0, L_0) **nonresonant spectra:**

This section is concerned with the result Proposition 2.4, and the statements that appear in Theorem 2.1 and Theorem 2.2 concerning the nature of the set of nonlinearities to which the existence theorems apply. We will first make more precise sense of the statements in these theorems concerning the topology in which the admissible coefficients are dense. Indeed we will show that the coefficients $g_1(x)$ which give rise to spectra $\{\omega_j\}$ which are (d_0, L_0)-nonresonant with ω_1 form an open dense set in $L^2(0,\pi)$, and furthermore the intersection of this set with the class $\mathcal{A}_{\overline{\sigma}}$ of coefficients g_1 which are bounded and analytic in the strip $\{x; |\text{Im } x| \leq \overline{\sigma}\}$ is open in all reasonable topologies.

The Dirichlet spectra of the Sturm-Liouville operators

$$L_{g_1}\psi = \big(-\frac{d^2}{dx^2} + g_1(x)\big)$$

are characterized by the following representation for their spectrum

$$\omega_n^2 = n^2 + \overline{g}_1 + d(n), \tag{6.21}$$

where $\overline{g}_1 = \pi^{-1} \int_0^\pi g_1(x)\, dx$, and $d(n)$ are terms of a sequence in $\ell^2(\mathbf{Z}^+)$. By contrast, the set of potentials which are analytic in the strip $\{x; |\text{Im } x| \leq \overline{\sigma}\}$ are less easily classified by the properties of the sequence $\{d(n)\}$. Without yet imposing conditions of analyticity for the potential g_1, we will ask for all spectra which satisfy the (d_0, L_0)-nonresonance conditions. That is, for (j,k) such that $|j| + |k| \leq L_0$,

$$\begin{aligned} |k^2\omega_1^2 - \omega_j^2| &> d_0, \quad \text{if} \quad (j,k) \neq (1, \pm 1) \quad \text{and} \quad , \\ |k\omega_1 - j| &> d_0(|j| + |k|)^{-\tau} \quad \text{for} (j,k) \neq (0,0) \quad . \end{aligned}$$

Denote NR the set of $L^2(0,\pi)$ potentials with Dirichlet spectrum $\{\omega_j\}_{j=1}^\infty$ giving rise to a frequency sequence which is (d_0, L_0)-nonresonant for some L_0, for $d_0 = o(L_0^{-1/2})$.

Proposition 6.11. *The set NR is open and dense in $L^2(0,\pi)$, and has open intersection with $\mathcal{A}_{\overline{\sigma}}$.*

Proof. For d_0, L_0 fixed, denote by $NR(d_0, L_0)$ the set of frequency sequences $\{\omega_j\}_{j=1}^\infty$ which are (d_0, L_0) nonresonant. We will use the fact that there is an isomorphism between the potentials in $L^2(0,\pi)$ with mean value zero, and the sequences $\{d(j)\} \in \ell^2(\mathbf{Z}^+)$, [PT]. The sequences not in $NR(d_0, L_0)$ because they violate the first condition of Definition 2.3 are characterized in terms of their spectral representation (6.21),

$$|k^2\omega_1^2 - \omega_j^2| = |k^2\omega_1^2 - (j^2 + \overline{g_1}) - d(j)| \le d_0, \tag{6.22}$$

for some (j,k), $|j| + |k| \le L_0$. The condition (6.22) involves only finitely many eigenvalues ω_j, and leads clearly to a closed condition on both the spaces $L^2(0,\pi)$ and $\mathcal{A}_{\overline{\sigma}}$. Thus $NR(d_0, L_0)$ is open in both $L^2(0,\pi)$ and $\mathcal{A}_{\overline{\sigma}}$. If g_1 is a potential, corresponding to the sequence $\{d(j)\}_{j=1}^{\infty}$, and satisfying (6.22), the maximal distance (in $\ell^2(\mathbf{Z}^+)$) to a sequence in $NR(d_0, L_0)$ is

$$(\sum_{j=1}^{L_0} d_0^2)^{1/2} = d_0 L_0^{1/2}.$$

The second condition of Definition 2.3 (*i.e.* (2.11))is a diophantine condition on the principal frequency ω_1. If $\{\omega_j\}_{j=1}^{\infty}$ is not in $NR(d_0, L_0)$ because of violating this condition, then there is an ω_1' such that $|\omega_1 - \omega_1'| < Cd_0$, and ω_1' is diophantine. Then

$$\begin{aligned} \mathrm{dist}(\{\omega_j\}_{j=1}^{\infty}, NR(d_0, L_0)) &\le d_0 L_0^{1/2} + \mathrm{dist}(\{\omega_1'\} \cup \{\omega_j\}_{j=2}^{\infty}, NR(d_0, L_0)) \\ &\le d_0 L_0^{1/2}(1 + CL_0^{-1/2}) \end{aligned}$$

The set $NR = \cup_{L_0, d_0 = o(L_0^{1/2})} NR(d_0, L_0)$ is open, and since $d_0 = o(L_0^{1/2})$, it is dense in $L^2(0,\pi)$.

□

The periodic eigenvalue problem has different spectral asymptotics than the Dirichlet case, in particular the eigenvalues come in pairs clustered about the even indexed Dirichlet eigenvalues. The analog of (6.21) is that

$$\omega_j^2 = j^2 + \overline{g_1} + p(j)$$

if $j > 0$ is even, and

$$\omega_j^2 = (j+1)^2 + \overline{g_1} + p(j)$$

if j is odd. Furthermore we know that $\sum p(j)^2 < \infty$, and $p(2j-1) \le d(2j) \le p(2j)$ for $j \ge 1$. The regularity of the potential is more clearly reflected in the sequence $\{p(j)\}_{j+1}^{\infty}$; in fact for $g_1 \in L^2$

$$\sum_{j+1}^{\infty} |p(2j) - p(2j-1)|^2 < \infty$$

and if $g_1(x) \in \mathcal{A}_{\overline{\sigma}}$, then

$$|p(2j) - p(2j-1)| \leq C_{g_1} e^{-\overline{\rho}|j|}. \tag{6.23}$$

for some $\overline{\rho}$.

Denote NR_{per} the set of $L^2(0,\pi)$ potentials for which the periodic spectrum gives a (d_0, L_0)-nonresonant frequency sequence for some L_0, for $d_0 = o(L_0^{-1/2})$. Using the Dirichlet sequences $\{d(j)\}$ to approximate the location of the periodic spectrum, and the asymptotics (6.23) to control the deviations of $p(2j-1), p(2j)$ from $d(2j)$, the excision procedure of Proposition 6.11 leads to the following result.

Proposition 6.12. *The set NR_{per} is open and dense in $L^2(0,\pi)$, and has open intersection with $\mathcal{A}_{\overline{\sigma}}$.*

6.5. Proof of Theorem 2.8:

Without the hypothesis of full nonresonance it may certainly happen that the point $(\mathbf{p}, 0) \in \mathcal{N}_0$ is eliminated from $\mathcal{N}$ at some stage of the induction, and the Cantor set of Theorem 2.7 will not have the origin as an accumulation point. In order to prove Theorem 2.8 with the hypothesis of full nonresonance we must inspect the process of excision more carefully. The excisions are of two types; constructing sequentially $\mathcal{N}_{n+1}^{(1)}$ and $\mathcal{N}_{n+1}^{(2)}$. The former is simplest to analyze, so we address it first. Excisions to enforce a diophantine condition on Ω are taken so that for all $(j,k) \in B_{n+1} \backslash B_n$,

$$|j\Omega - k| > \frac{d}{(|j| + |k|)^{\tau}}.$$

Thus the values of Ω excised at the n^{th} step must retain a certain distance from ω_1.

$$\begin{aligned} |\omega_1 - \Omega| &> |\omega_1 - j/k| - |j/k - \Omega| \\ &\geq \frac{c_1}{(|j| + |k|)^{(\overline{\tau}+1)}} - \frac{d}{(|j| + |k|)^{(\tau+1)}}. \end{aligned}$$

As long as $d/(|j| + |k|)^{(\tau+1)} < c_1/(2(|j| + |k|)^{(\overline{\tau}+1)})$, for all $n \geq N_0$, $(j,k) \in B_{n+1} \backslash B_n$, we have

$$|\Omega - \omega_1| > \frac{c_1}{2L_{n+1}^{(\overline{\tau}+1)}},$$

and thus $(\mathbf{p}, 0)$ is not excised. This requires the choice $\overline{\tau} \leq \tau$. At the n^{th} step, $\text{meas}\{\Omega \in (\omega_1 - r_0^2, \omega_1 + r_0^2); d/(|j| + |k|)^{\tau+1} > |\Omega - j/k|\} \leq dL_n^{-\tau}(1 + r_0^2 L_{n+1})$. Given a nondegenerate surface $\mathcal{C}$ described by $\Omega(\mathbf{p}) = \omega_1 + \mathcal{K}\|\mathbf{p}\|^2(1 + C(\|\mathbf{p}\|))$, consider the measure of the set $\mathcal{S}_{n+1}^{(1)}(r_1) = \{0 < r < r_1 \leq r_0; \|\mathbf{p}\| = r,\ (\mathbf{p}, \Omega(\mathbf{p})) \in \mathcal{N}_{n+1}^{(1)}\}$. If $|\mathcal{K}|r_1^2 < c_1/L_{n+1}^{(\overline{\tau}+1)}$, then no excisions have yet occurred in $(0, r_1)$, and $\text{meas}\mathcal{S}_{n+1}^{(1)}(r_1) = r_1$. For $|\mathcal{K}|r_1^2 \geq c_1/L_{n+1}^{\overline{\tau}}$, the construction of $\mathcal{N}_{n+1}^{(1)}$ could excise a subset from $\mathcal{S}_{n+1}^{(1)}(r_1)$, of measure bounded by

$$C\frac{d}{L_n^{\tau}}(1 + r_0^2 L_{n+1}) \times \sqrt{\frac{L_n^{(\overline{\tau}+1)}}{|\mathcal{K}|}}.$$

Denote $N_1 = N_1(r_1)$ the least step such that $|\mathcal{K}|r_1^2 > c/(L_{N_1}^{(\overline{\tau}+1)})$, then

$$\begin{aligned}
&\text{meas}\big((0, r_1)\setminus \cap_{n\geq N_1} \mathcal{S}_{n+1}^{(1)}(r_1)\big) \\
&\quad\leq \sum_{n\geq N_1} \frac{d}{L_n^{\tau}}(1 + r_0^2 L_{n+1}) \times \sqrt{\frac{2}{c_1|\mathcal{K}|}} \times L_{n+1}^{(\overline{\tau}+1)/2}. \\
&\quad\leq C\frac{d}{\sqrt{|\mathcal{K}|}}\big(L_{N_1}^{-(\tau-(\overline{\tau}+1)/2)} + r_0^2 L_{N_1}^{-((\tau-1)-(\overline{\tau}+1)/2)}\big).
\end{aligned}$$

If we recall that $c/L_{N_1}^{(\overline{\tau}+1)} < |\mathcal{K}|r_1^2 \leq c/L_{N_1-1}^{(\overline{\tau}+1)}$, the result is that

$$\text{meas}\ \mathcal{S}_{n+1}^{(1)}(r_1) \geq \big(r_1 - Cd|\mathcal{K}|^{(\tau-1)/(\overline{\tau}+1)-1} r_1^{2(\tau-1)/(\overline{\tau}+1)-1}\big) \quad .$$

Thus we ask that $0 < \mu < 2(\tau-1)/(\overline{\tau}+1) - 1$. In order to make the choice of constants uniform in L_0, also ask that $d|\mathcal{K}|^{(\tau-1)/(\overline{\tau}+1)-1} < 1$. This is achieved with the present choices of induction parameters $d = L_0^{-\eta}$, $|\mathcal{K}| \geq L_0^{-\nu}$ as long as $\eta + \nu((\tau-1)/(\overline{\tau}+1)-1) > 0$.

The construction of $\mathcal{N}_{n+1}^{(2)}$ involves the excision of δ_{n+1}/L_n^2 neighborhoods of the zero sets Z_i discussed in section 4. These intersect the Ω axis at points ω_ℓ/m, $(\ell, m) \in B_{n+1}\setminus B_n$. Using the hypothesis of full nonresonance, if

$$\begin{aligned}
|\omega_1 - \omega_\ell/m| \geq& \frac{c_2}{(|\ell| + |m|)^{(\overline{\alpha}+2)}} \\
\geq& \frac{c_2}{L_{n+1}^{(\overline{\alpha}+2)}} \geq \frac{\delta_{n+1}}{L_n^2}.
\end{aligned}$$

then we will retain the point $(\mathbf{p}, 0) \in \mathcal{N}_{n+1}^{(2)}$ throughout the induction. This requires the choice $\overline{\alpha} \leq \alpha$. We also ask that $\overline{\alpha} + 1 \geq \overline{\tau}$, so that the critical excisions of this construction will always fall within $\mathcal{N}_{n+1}^{(1)}$, thus the parabolic estimates (4.20) will hold in the relevant neighborhood of ω_1.

The zero set Z_i may intersect the surface $\mathcal{C}$ closer to ω_1 than ω_ℓ/m, however these must be at points where $\omega_1 + |\mathcal{K}/2| \|\mathbf{p}\|^2 \geq |\omega_\ell/m| - C_1\|\mathbf{p}\|^2/L_n^2$. That is, the points of intersection must satisfy

$$\|\mathbf{p}\|^2 > \big|\big|\frac{\omega_\ell}{m}\big| - \omega_1\big| \frac{1}{|\mathcal{K}| + C_1/L_n^2}.$$

We ask that $C_1/L_n^2 < 1$. Define the set

$$\mathcal{S}_{n+1}^{(2)}(r_1) = \{0 < r < r_1 \leq r_0;\ \|\mathbf{p}\| = r, \text{ and } (\mathbf{p}, \Omega(\mathbf{p})) \in \mathcal{N}_{n+1}^{(2)}\} \ .$$

If $r_1^2 < (|\mathcal{K}|+1)^{-1}\big|c_2/L_{n+1}^{(\overline{\alpha}+2)}\big| - \delta_{n+1}/L_n^2$, then in constructing $\mathcal{N}_{n+1}^{(2)}$ there is no excision within $\mathcal{S}_{n+1}^{(2)}$. Thus we will select $\overline{\alpha}$ so that $\delta_{n+1}/L_n^2 \leq c_2/(2L_{n+1}^{(\overline{\alpha}+2)})$. In particular we require that $\overline{\alpha} \leq \alpha$. Let $N_2 = N_2(r_1)$ be the first induction step at which excisions may occur within $(0, r_1)$. We now estimate the measure of the set $(0, r_1)\setminus \cap_{n\geq N_2} \mathcal{S}_{n+1}^{(2)}$ as in the first case.

$$\begin{aligned} &\text{meas } \big((0, r_1)\setminus \cap_{n\geq N_2} \mathcal{S}_{n+1}^{(2)}(r_1)\big) \\ &\quad \leq \sum_{n\geq N_2} L_n(1 + r_0^2 L_{n+1}) \times 2\ell_{n+1}^2 \times \frac{\delta_{n+1}}{L_n^2} \times \sqrt{\frac{L_n^{\overline{\alpha}+2}}{|\mathcal{K}| c_2}} \\ &\quad \leq \frac{C}{\sqrt{|\mathcal{K}|}} \big(L_{N_2}^{-(\alpha-(2\beta+\overline{\alpha}))} + r_0^2 L_{N_2}^{-(\alpha-(2\beta+\overline{\alpha}+1))}\big) \end{aligned}$$

Using that $C/L_{N_2+1}^{(\overline{\alpha}+2)} \leq r_1^2 \leq C/L_{N_2}^{(\overline{\alpha}+2)}$, the result is that

$$\text{meas } \mathcal{S}_{n+1}^{(2)}(r_1) \geq \big(r_1 - \frac{C}{\sqrt{|\mathcal{K}|}} r_1^{2((\alpha-2\beta-1)/(\overline{\alpha}+2)-1)}\big).$$

Thus we also require that

$$0 < \mu < 2((\alpha - 2\beta - 1)/(\overline{\alpha} + 2) - 1).$$

This finishes the proof of the theorem.

$\square$

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