An Infinite Volume Expansion for Many Fermion Green's Functions

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Abstract We prove the convergence of a simplified cluster/Mayer expansion at one energy scale for three space-time dimensional many Fermion systems. The bounds are uniform in the scale. We iterate them to show that the sum of all diagrams that contain no two or four-legged subdiagrams converges. Our results are suited to a multiscale construction of the full system.

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§I Introduction

In this paper we consider many Fermion systems formally characterized by the effective potential

$$\mathcal{G}(\psi^e, \bar{\psi}^e) = \log \frac{1}{Z} \int e^{-\lambda \mathcal{V}(\psi+\psi^e, \bar{\psi}+\bar{\psi}^e)} d\mu_C(\psi, \bar{\psi})$$

for the external fields $\psi^e, \bar{\psi}^e$. Here, $d\mu_C(\psi, \bar{\psi})$ is the fermionic Gaussian measure in the Grassmann variables $\{\psi(\xi), \bar{\psi}(\xi) \mid \xi \in \mathbb{R} \times \mathbb{R}^d \times \{\uparrow, \downarrow\}\}$ with propagator

$$C(\xi, \bar{\xi}) = \delta_{\sigma, \bar{\sigma}} \int \frac{d^{d+1}p}{(2\pi)^{d+1}} \frac{e^{i<p, \xi - \bar{\xi}>_-}}{ip_0 - e(\mathbf{p})}$$

where

$$< p, \xi >_- = \mathbf{p} \cdot \mathbf{x} - p_0 \tau$$

and

$$e(\mathbf{p}) = \frac{\mathbf{p}^2}{2m} - \mu.$$

The variable $\xi = (\tau, \mathbf{x}, \sigma)$ consists of time, space and spin components and the $(d+1)$-momentum $p = (p_0, \mathbf{p})$. The interaction is given by

$$\mathcal{V} = \frac{1}{2} \int \prod_{i=1}^{4} d\xi_i \quad V(\xi_1, \xi_2, \xi_3, \xi_4)\, \bar{\psi}(\xi_1) \bar{\psi}(\xi_2) \psi(\xi_4) \psi(\xi_3)$$

where the kernel $V(\xi_1, \xi_2, \xi_3, \xi_4)$ is translation invariant with $V(0, \xi_2, \xi_3, \xi_4)$ integrable and

$$\int d\xi = \sum_{\sigma \in \{\uparrow, \downarrow\}} \int_{\mathbb{R}} d\tau \int_{\mathbb{R}^d} d\mathbf{x}.$$

The partition function

$$Z = \int e^{-\lambda \mathcal{V}(\psi, \bar{\psi})} d\mu_C(\psi, \bar{\psi})$$

so that $\mathcal{G}(0,0) = 0$.

The Euclidean Green's functions

$$G_p(\xi_1, \bar{\xi}_1, \ldots, \xi_p, \bar{\xi}_p) = \prod_{i=1}^{p} \frac{\delta^2}{\delta \psi^e(\xi_i) \delta \bar{\psi}^e(\bar{\xi}_i)} \mathcal{G}$$

generated by the effective potential are the connected Green's functions amputated by the free propagator. By definition, $\mathcal{G}$ exists when the norm

$$|||G_p||| = \max_j \sup_{\xi_j} \int \prod_{i \neq j} d\xi_i \; |G_p(\xi_1, \cdots, \xi_{2p})|$$

of each of its moments, $G_p,\ p \geq 1$, is finite. Intuitively, $|||G_p|||$ is the supremum in momentum space of G_p. In fact, the supremum in momentum space was used as the standard norm on vertices in [FT2].

Our long term goal is to give a rigorous proof that the standard model for an interacting system of electrons and phonons has a superconducting ground state at sufficiently low temperature. Perturbation theory and, in particular, the renormalization of the two point function was controlled in [FT1]. (See [BG] for related results.) A renormalization group flow for the four point function was defined and analyzed in [FT2]. Two additional ingredients are required to complete this program. First an infinite volume expansion that combines power counting at fixed energy with the exclusion principle and second, control of the Goldstone boson. This paper provides the first ingredient, in three space-time dimensions. With the exception of Lemma 3 all components of the expansion apply in all dimensions. We restrict to $d = 2$ only when it is necessary to do so.

As in [FT1,2], the model is sliced into energy regimes by decomposing momentum space into shells around the Fermi surface. The j^{th} slice has covariance

$$C^{(j)}(\xi, \bar{\xi}) = \delta_{\sigma,\bar{\sigma}} \int \frac{d^{d+1}p}{(2\pi)^{d+1}} \frac{e^{i<p,\xi-\bar{\xi}>_-}}{ip_0 - e(\mathbf{p})} f_j(p),$$

where

$$f_j(p) = f\left(M^{-2j}\left(p_0^2 + e(\mathbf{p})^2\right)\right)$$

effectively forces $|ip_0 - e(\mathbf{p})| \sim M^j$. The function $f \in C_0^\infty([1, M^4])$. The parameter M is strictly bigger than one so that the scales near the Fermi surface have j near $-\infty$. The model is defined in finite volume and at fixed scale by the following lemma. However, the radius of convergence depends on volume and scale in a patently unsatisfactory way.

$$\int d\xi = \sum_{\sigma \in \{\uparrow,\downarrow\}} \int_{\mathbb{R}} d\tau \int_{\mathbb{R}^d} d\mathbf{x}.$$

Lemma 1

$$\mathcal{G}_\Lambda^{(j)}(\psi^e,\bar\psi^e)=\log\,\frac{1}{Z_\Lambda^{(j)}}\int e^{-\lambda\mathcal{V}_\Lambda(\psi+\psi^e,\bar\psi+\bar\psi^e)}d\mu_{C^{(j)}}(\psi,\bar\psi)$$

where

$$\mathcal{V}_\Lambda=\frac{1}{2}\int_{\Lambda^4}\prod_{i=1}^4 d\xi_i\quad V(\xi_1,\xi_2,\xi_3,\xi_4)\,\bar\psi(\xi_1)\bar\psi(\xi_2)\psi(\xi_4)\psi(\xi_3)$$

and

$$Z_\Lambda^{(j)}=\int e^{-\lambda\mathcal{V}_\Lambda(\psi,\bar\psi)}d\mu_{C^{(j)}}(\psi,\bar\psi)$$

is analytic in λ in a neighborhood of the origin that includes at least the disk of radius const $\left(M^{2j}|\Lambda|\right)^{-1}$.

Proof : Expand the exponential

$$\begin{aligned}
Z_\Lambda^{(j)}&=\int e^{-\lambda\mathcal{V}_\Lambda(\psi,\bar\psi)}d\mu_{C^{(j)}}(\psi,\bar\psi)\\
&=\sum_{n=0}^\infty\frac{(-\lambda)^n}{n!}\int\mathcal{V}_\Lambda(\psi,\bar\psi)^n d\mu_{C^{(j)}}(\psi,\bar\psi)\\
&=\sum_{n=0}^\infty\frac{(-\lambda)^n}{2^n n!}\int_{\Lambda^{4n}}\left(\prod_{j=1}^n\prod_{i=1}^2 d\xi_{j,i}d\bar\xi_{j,i}\right)\left(\prod_{j=1}^n V(\bar\xi_{j,1},\bar\xi_{j,2},\xi_{j,1},\xi_{j,2})\right)\det\left[C^{(j)}(\xi_k,\bar\xi_\ell)\right]
\end{aligned}$$

where the indices k and ℓ run over $\{(j,i)\,|\,1\le j\le n\,,\ 1\le i\le 2\}$.

The (k,ℓ) matrix element

$$\begin{aligned}
C^{(j)}(\xi_k,\bar\xi_\ell)&=\delta_{\sigma_k,\bar\sigma_\ell}\int\frac{d^{d+1}p}{(2\pi)^{d+1}}\frac{e^{i<p,\xi_k-\bar\xi_\ell>_-}}{ip_0-e(\mathbf{p})}f_j(p)\\
&=\left\langle A_k,\bar A_\ell\right\rangle
\end{aligned}$$

where

$$A_k(p,\alpha):=\delta_{\alpha,\sigma_k}\frac{e^{i<p,\xi_k>_-}}{[ip_0-e(\mathbf{p})]^{\frac{1}{2}}}f_j(p)^{1/2}$$

and

$$\bar A_\ell(p,\bar\alpha):=\delta_{\bar\alpha,\bar\sigma_\ell}\left\{\frac{e^{-i<p,\xi_\ell>_-}}{[ip_0-e(\mathbf{p})]^{\frac{1}{2}}}\right\}^*f_j(p)^{1/2}$$

are in $L^2\left(\mathbb{R}^{d+1}\times\{\uparrow,\downarrow\}\right)$. Here, any branch of the square root will do, since $ip_0-e(\mathbf{p})$ does not vanish on the domain of integration.

Consequently, by Gram's inequality,

$$\begin{aligned}\left|\det\left[C^{(j)}(\xi_k,\bar{\xi}_\ell)\right]\right| &\le \prod_k \|A_k\|_2 \ \|\bar{A}_k\|_2 \\ &\le \text{const}^n \ M^{2jn}\end{aligned}$$

since

$$\begin{aligned}\|\bar{A}_k\|_2^2 = \|A_k\|_2^2 &= \int \frac{d^{d+1}\mathbf{p}}{(2\pi)^{d+1}} \left|ip_0 - e(\mathbf{p})\right|^{-1} f_j(p) \\ &\le \text{const } M^j.\end{aligned}$$

Substituting,

$$\begin{aligned}\left|Z_\Lambda^{(j)}\right| &\le \sum_{n=0}^{\infty} \frac{|\lambda|^n}{2^n n!} \int_{\Lambda^{4n}} \left(\prod_{j=1}^{n}\prod_{i=1}^{2} d\xi_{j,i} d\bar{\xi}_{j,i}\right) \left|\prod_{j=1}^{n} V(\bar{\xi}_{j,1},\bar{\xi}_{j,2},\xi_{j,1},\xi_{j,2})\right| \left|\det\left[C^{(j)}(\xi_k,\bar{\xi}_\ell)\right]\right| \\ &\le \sum_{n=0}^{\infty} \frac{|\text{const } \lambda|^n}{n!} \int_{\Lambda^{4n}} \left(\prod_{j=1}^{n}\prod_{i=1}^{2} d\xi_{j,i} d\bar{\xi}_{j,i}\right) \left|\prod_{j=1}^{n} V(\bar{\xi}_{j,1},\bar{\xi}_{j,2},\xi_{j,1},\xi_{j,2})\right| M^{2jn} \\ &\le \sum_{n=0}^{\infty} \frac{|\text{const } \lambda|^n}{n!} \left(|\Lambda| M^{2j} \|\!|V|\!\|\right)^n \\ &= \exp\left(\text{const } |\lambda||\Lambda| M^{2j} \|\!|V|\!\|\right).\end{aligned}$$

It follows that the partition function is an entire function of λ.

Each Taylor coefficient of the expansion of the numerator

$$\int e^{-\lambda \mathcal{V}_\Lambda(\psi+\psi^e,\bar{\psi}+\bar{\psi}^e)} d\mu_{C^{(j)}}(\psi,\bar{\psi})$$

in powers of the external fields is of the form

$$\int P(\psi,\bar{\psi}) e^{-\lambda \mathcal{V}_\Lambda(\psi,\bar{\psi})} d\mu_{C^{(j)}}(\psi,\bar{\psi})$$

for some polynomial P. It is estimated in exactly the same way and is also an entire function of λ.

The Green's functions of $\mathcal{G}$ are finite sums of finite products of such numerators divided by a power of $Z_\Lambda^{(j)}$. Thus they are meromorphic with poles at the zeroes of the partition function. Therefore the radius of convergence is at least the absolute value of the smallest root of $Z_\Lambda^{(j)}$.

When $\lambda = 0$, the partition function is one. As above

$$\left|\frac{d}{d\lambda} Z_\Lambda^{(j)}\right| \le \text{const } |\Lambda| M^{2j} \|\!|V|\!\| \exp\left(\text{const } |\lambda||\Lambda| M^{2j} \|\!|V|\!\|\right)$$

so that

$$\left| Z_\Lambda^{(j)}(\lambda) - 1 \right| \le \text{const } |\lambda||\Lambda| M^{2j} \|V\| \exp\left(\text{const } |\lambda||\Lambda| M^{2j} \|V\|\right).$$

Finally, for $|\lambda||\Lambda|M^{2j}\|V\| \le \text{const}$,

$$\text{const } |\lambda||\Lambda| M^{2j} \|V\| \exp\left(\text{const } |\lambda||\Lambda| M^{2j} \|V\|\right) \; < \; 1 \; .$$

■

Let $G_p^{(j,\Lambda)}$ be the p-point Green's function generated by $\mathcal{G}_\Lambda^{(j)}$. By the Lemma above the Taylor series

$$G_p^{(j,\Lambda)} = \sum_{n=0}^{\infty} g_n(p,j,\Lambda)\lambda^n$$

has a strictly positive, though possibly j and Λ dependent, radius of convergence. The main result of this paper is

Theorem 1

Let $d = 2$. There exists a const *, independent of j and Λ, such that*

$$\|g_n(p,j,\Lambda)\| \le \text{const}^{\,n+p} M^{(2-5p/2)j} \|V\|^n$$

where

$$\|g_n(p,j,\Lambda)\| = \max_k \sup_{\xi_k} \int \prod_{i \ne k} d\xi_i \; |g_n(p,j,\Lambda)(\xi_1,\cdots,\xi_{2p})|.$$

Furthermore the limits

$$g_n(p,j) = \lim_{\Lambda \to \mathbb{R}^3} g_n(p,j,\Lambda)$$

exist and the infinite volume Green's functions at scale j

$$G_p^{(j)} = \sum_{n=0}^{\infty} g_n(p,j)\lambda^n$$

are analytic in $|\lambda| < R = (\text{const}\,\|V\|)^{-1}$.

The reader might have expected the perturbative power counting factor $M^{\frac{1}{2}(4-2p)j}$ as in [FT2, Lemma III.1a, (III.16a)] rather than $M^{(2-5p/2)j}$. However in this theorem we do not try to optimize the analysis with respect to external legs, and we state the theorem at the level of ordinary Green's functions, rather than amputated functions, which appear in a multiscale analysis. A finer, more powerful though more complicated bound, which is operationally equivalent to the perturbative one is given in Theorem 2 of Section III. By operationally equivalent we mean, for example, that the bound of Theorem 2 is adapted to a multiscale analysis and can be iterated in a renormalization group flow. In fact Theorem 2 of Section III states that the sum over all "completely convergent" graphs and all scale assignments to the lines of the graphs is absolutely convergent and analytic for all coupling constants λ in a fixed disk around zero. Recall that "completely convergent" [Ri] means that there are neither two nor four point subgraphs with internal scales higher than those of the external legs. Theorem 2 shows that our single slice analysis is the correct building block for a multiscale analysis.

A theorem of this kind is usually proven with a standard cluster/Mayer expansion [GJS,Br,Ri]. Space-time, $\mathbb{R}^{d+1}$, is paved by cubes Δ of side M^{-j} dual to the decay rate M^j of the propagator. The decay rate is primarily determined by the thickness of the shell in momentum space. Then one expands in coupling constants that control the interaction between boxes. One essential prerequisite for the convergence of such expansions is the j independent estimate $|Z(\Delta)| \le \text{const}$. For our models, one can see in perturbation theory that this estimate fails.

The logarithm of the partition function is given perturbatively by the sum of all connected vacuum graphs. In evaluating a connected vacuum graph at scale j, each propagator contributes $(ip_0 - e(\mathbf{p}))^{-1} \sim M^{-j}$ and the volume of integration for each momentum loop is $\sim M^{2j}$. Hence the value of a vacuum graph, of order n, with the position of one vertex held fixed is $\sim M^{-j(2n)} M^{2j(n+1)} \sim M^{2j}$. Integrating the fixed vertex over Δ gives $|\Delta| M^{2j} = M^{-(d-1)j}$. This result is a reflection of the Pauli exclusion principle. The shell in momentum space about the Fermi surface has volume M^{2j}, while the position space volume of the "dual" box Δ is $M^{-(d+1)j}$. The Pauli exclusion principle now permits $M^{-(d-1)j}$ electrons to be located in Δ with momentum restricted to the shell. For $d = 1$ there is no true Fermi surface and consequently one electron in Δ. As d grows the Pauli exclusion principle becomes progressively weaker and the estimate on the partition function in Δ becomes more and more j dependent.

There are three naive ways to force the volume in phase space to be independent of j. One either makes the box Δ smaller, or decomposes the shell into sufficiently small sectors, or both. In each case, the number of electrons in such a constrained region would be of order one, achieving duality in the sense of the exclusion principle. The first alternative, however, violates duality in the sense of decay of the propagator.

Let us decompose the shell into $M^{-(d-1)j}$ sectors of side M^j, by constructing a smooth partition of unity

$$1 = \sum_{m=1}^{M^{-(d-1)j}} \eta_m(\mathbf{p}), \qquad \eta_m(\mathbf{p}) = \eta_m\left(\frac{\mathbf{p}}{|\mathbf{p}|}k_F\right)$$

of the Fermi surface, where η_m is supported on the union of the m^{th} sector, S_m, and its neighbors, whose number is at most $3^{d-1}-1$. There is a corresponding decomposition of the covariance

$$C^{(j)} = \sum_{m=1}^{M^{-(d-1)j}} C^{(j,m)}$$

where

$$C^{(j,m)}(\xi,\bar{\xi}) = \delta_{\sigma,\bar{\sigma}} \int \frac{d^{d+1}p}{(2\pi)^{d+1}} \frac{e^{i<p,\xi-\bar{\xi}>_-}}{ip_0 - e(\mathbf{p})} f_j(p)\, \eta_m(\mathbf{p})$$

and of the fields

$$\psi = \sum_{m=1}^{M^{-(d-1)j}} \psi^{(m)}, \qquad \bar{\psi} = \sum_{m=1}^{M^{-(d-1)j}} \bar{\psi}^{(m)}.$$

The standard power counting bound for an individual graph is still easy to prove when there are sectors. First, one selects a spanning tree for the graph. To each line not in the tree there is a corresponding momentum loop, obtained by joining its ends through a path in the tree. This construction produces a complete set of independent loops. Ignoring unimportant constants, each propagator is bounded by its supremum M^{-j}. The volume of integration for each loop is now $M^{(d+1)j}$. A priori, there is one sector sum with $M^{-(d-1)j}$ terms for each line. But, by conservation of momentum, there is only one sector sum per loop. Thus, if there are n vertices and E external lines, the supremum in momentum space of the graph is bounded by

$$\begin{aligned}\prod_{\text{lines}} M^{-j} \prod_{\text{loops}} M^{(d+1)j} M^{-(d-1)j} &= M^{-j(4n-E)/2} M^{2j[(4n-E)/2-(n-1)]} \\ &= M^{\frac{1}{2}(4-E)j}.\end{aligned}$$

In the course of a non-perturbative construction, estimates cannot be made graph by graph because there are too many of them. Rather, the perturbation series must be blocked and the blocks estimated as units. The blocks are estimated using the exclusion principle to implement strong cancellations between the roughly $n!^2$ graphs of order n. However, once the series is blocked, momentum loops can't be defined and the argument leading to the estimate above cannot be made. Conservation of momentum has to be implemented at each vertex, rather than through loops. Even though the volume cutoff Λ breaks exact conservation of momentum, many of the $M^{-2\ell(d-1)j}$ terms in the sector sums for a general 2ℓ-legged vertex must be zero.

Lemma 2

Fix $m \in \mathbf{Z}^{d+1}$ *and* $\ell \geq 2$. *Then, the number of* 2ℓ*-tuples*

$$\{S_1, \ \cdots \ S_{2\ell}\}$$

of sectors for which there exist $\mathbf{k}_i \in \mathrm{I\!R}^d$, $i = 1, \ \cdots, \ 2\ell$ *satisfying*

$$\mathbf{k}_i' \in S_i, \quad |\mathbf{k}_i - \mathbf{k}_i'| \leq \text{const } M^j \ , \quad i = 1, \ \cdots, \ 2\ell$$

and

$$|\mathbf{k}_1 + \ \cdots \ + \mathbf{k}_{2\ell}| \leq \text{const}\,(1 + |m|)\, M^j$$

is bounded by

$$\text{const}^{\,\ell}(1 + |m|)^d M^{-(2\ell-1)(d-1)j} M^j \,\{1 + |j|\delta_{d,2}\delta_{\ell,2}\}\,.$$

In particular, for a four legged vertex, the number of 4*-tuples is at most*

$$\text{const}\,(1 + |n|)^d M^{(-3d+4)j}\,\{1 + |j|\delta_{d,2}\}\,.$$

Here, $\mathbf{k}' = \frac{\mathbf{k}}{|\mathbf{k}|}$ *denotes the projection of* $\mathbf{k}$ *onto the Fermi surface.*

Lemma 2 is proved in the next section. Specializing to two space dimensions, the number of active sector 4-tuples at a vertex is of order $|j|M^{-2j}$. Four planar vectors of equal length whose sum is zero form a parallelogram. The factor M^{-2j} is natural since a parallelogram is determined by two of its sides. The logarithmic factor $|j|$ is not an artifact of our bounds. It arises from the degenerate situation in which all four vectors are roughly collinear. One of the main technical difficulties of the paper is to overcome the logarithm. Note that in three dimensions, the parallelogram is hinged and the logarithm $|j|$ jumps to the power $M^{-j/2}$.

To circumvent the logarithm in two dimensions we divide the Fermi circle into sectors of length $M^{j/2}$ rather than M^j through a smooth partition of unity

$$1 = \sum_{\ell=1}^{M^{-j/2}} \zeta_\ell(\mathbf{p}), \qquad \zeta_\ell(\mathbf{p}) = \zeta_\ell\left(\frac{\mathbf{p}}{|\mathbf{p}|}k_F\right)$$

where ζ_ℓ is supported on the ℓ^{th} sector, Σ_ℓ, and its 2 neighbors. We denote by r_ℓ the center of Σ_ℓ. The new sectors are long and skinny since they are still M^j thick. We shall show in a moment that the sector propagator

$$C^{(j,\ell)}(\xi,\bar\xi) = \delta_{\sigma,\bar\sigma}\int \frac{d^3p}{(2\pi)^{d+1}}\frac{e^{i<p,\xi-\bar\xi>_-}}{ip_0 - e(\mathbf{p})} f_j(p)\,\zeta_\ell(\mathbf{p})$$

decays anisotropically. To accommodate the anisotropy, we will introduce a different lattice of dual boxes for each sector. The boxes will be short in the direction perpendicular to r_ℓ and long in the direction of r_ℓ.

The reason that approximately collinear configurations generate a logarithm for sectors of length M^j, but not for sectors of length $M^{j/2}$, may be seen in the proof (§II) of

Lemma 3

Let $d = 2$ and divide the Fermi circle into sectors Σ_ℓ, $\ell = 1,\cdots,M^{-j/2}$ of width $M^{j/2}$. Fix $m \in \mathbf{Z}^3$ and any sector Σ_{ℓ_1}. The number of sector quadruples $\{\Sigma_{\ell_1},\Sigma_{\ell_2},\Sigma_{\ell_3},\Sigma_{\ell_4}\}$ for which there exist $\mathbf{k}_i \in \mathrm{I\!R}^2$, $i = 1, \cdots, 4$ satisfying

$$\mathbf{k}_i' \in \Sigma_{\ell_i}, \quad |\mathbf{k}_i - \mathbf{k}_i'| \le \text{const}\, M^j\ , \quad i = 1, \cdots, 4$$

and

$$|\mathbf{k}_1 + \cdots + \mathbf{k}_4| \le \text{const}\ (1+|m|)\, M^j$$

is bounded by

$$\text{const}\ (1+|m|^2)M^{-j/2}.$$

The anisotropic decay of the covariance is given in

Lemma 4

(a) Let

$$\rho^{(j,\ell)}(\xi,\bar{\xi}) \stackrel{\rm def}{=} 1+M^j|\xi_0-\bar{\xi}_0|+M^j|\xi_\parallel-\bar{\xi}_\parallel|+M^{j/2}|\xi_\perp-\bar{\xi}_\perp|$$

where $\xi_\parallel$ is the component of ξ parallel to r_ℓ and $\xi_\perp$ is the component orthogonal to r_ℓ. The same notation is used for $\bar{\xi}$. Then, for any $\gamma>0$,

$$|C^{(j,\ell)}(\xi,\bar{\xi})| \le {\rm const}\;\; M^{3j/2}\rho^{(j,\ell)}(\xi,\bar{\xi})^{-\gamma}.$$

(b)

$$\left|D_0^{n_0}D_\parallel^{n_1}D_\perp^{n_2}\left(e^{-i<\mathbf{r}_\ell,\xi-\bar{\xi}>_-}C^{(j,\ell)}(\xi,\bar{\xi})\right)\right| \le {\rm const}\;\; M^{(\frac{3}{2}+n_0+n_1+\frac{n_2}{2})j}\rho^{(j,\ell)}(\xi,\bar{\xi})^{-\gamma}.$$

Here $D_0=\frac{\partial}{\partial\xi_0}$, $D_\parallel=\frac{\mathbf{r}_\ell}{|\mathbf{r}_\ell|}\cdot\nabla_\xi$ and $D_\perp=\hat{\pi}_\ell\cdot\nabla_\xi$ where $\hat{\pi}_\ell$ is any unit vector perpendicular to $\mathbf{r}_\ell$.

Proof : (a) A pointwise bound on

$$C^{(j,\ell)}(\xi,\bar{\xi}) = \delta_{\sigma,\bar{\sigma}}\int \frac{d^3p}{(2\pi)^3}\frac{e^{i<p,\xi-\bar{\xi}>_-}}{ip_0-e(\mathbf{p})}f_j(p)\,\zeta_\ell(\mathbf{p})$$

is obtained by observing that the integrand is bounded by M^{-j} and that the volume of integration is $M^{\frac{5}{2}j}$. Multiplying $C^{(j,\ell)}$ by $\rho^{(j,\ell)}(\xi,\bar{\xi})^\gamma$ is converted, by integration by parts, into p-derivatives acting on

$$\frac{1}{ip_0-e(\mathbf{p})}f_j(p)\,\zeta_\ell(\mathbf{p}) \quad .$$

A derivative acting on $\zeta^{(\ell)}$ produces an $M^{-\frac{1}{2}j}$ while one acting on f produces an M^{-j}. However, the directional derivative $\hat{\pi}_\ell\cdot\nabla_{\mathbf{p}}$ acting on $f_j(p)=f\left(M^{-2j}\left(p_0^2+e(\mathbf{p})^2\right)\right)$ produces

$$M^{-2j}2e(\mathbf{p})\frac{1}{2m}2\mathbf{p}\cdot\hat{\pi}_\ell$$

which is bounded by const $M^{-2j}M^jM^{j/2}|\hat{\pi}_\ell| = $ const $M^{-j/2}$ on the support of the integrand.

(b) Each derivative, with respect to ξ or $\bar{\xi}$, of

$$e^{-i<\mathbf{r}_\ell,\xi-\bar{\xi}>_-}C^{(j,\ell)}(\xi,\bar{\xi}) = \delta_{\sigma,\bar{\sigma}} \int \frac{d^3p}{(2\pi)^3} \frac{e^{i<p-\mathbf{r}_\ell,\xi-\bar{\xi}>_-}}{ip_0 - e(\mathbf{p})} f_j(p)\, \zeta_\ell(\mathbf{p})$$

brings down a factor of $p - \mathbf{r}_\ell$. On the domain of integration, the components of this vector

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in the time, $\mathbf{r}_\ell$ and $\hat{\pi}_\ell$ directions are bounded by M^j, M^j and $M^{j/2}$ respectively. This is obvious except for the $\mathbf{r}_\ell$ component. For it we have

$$\begin{aligned} \left|(p-\mathbf{r}_\ell)_{\|}\right| &\le k_F - \big(k_F - O(M^j)\big)\cos\theta \\ &\le k_F - \big(k_F - O(M^j)\big)\big(1 - O(M^{j/2})^2\big) \\ &\le O(M^j) \end{aligned}$$

since $\theta \le O(M^{j/2})$. ■

We now give a rough description of the expansion. In perturbation theory, $g_n(p,j,\Lambda)$ is written as $\frac{1}{n!}$ times the sum of approximately $n!^2$ connected Feynman diagrams of order n. Every connected diagram is spanned by a connected tree. By Cayley's Theorem there are n^{n-2} labeled trees of order n. The number of trees can be compensated for by the $\frac{1}{n!}$. The expansion starts with trees and inductively builds graphs from them by joining the remaining $4n - 2p - 2(n-1)$ legs. However, for each fixed tree, these legs may be joined in $\sim n!$ ways to form connected graphs. As in the standard cluster expansion, part of this $n!$ will be cancelled using the decay of the propagator. We now review this procedure.

Pretend that the propagator decays like $(1 + M^j|\xi - \bar{\xi}|)^{-\gamma}$. Introduce a lattice, K_j, of cubes of side M^{-j} that paves space-time. If we are in the midst of the inductive process, some, but not all, legs of the tree have been joined to form lines. At the next step we select any leg, say in $\Delta \in K_j$, that has not yet been contracted and sum over all possible ways of connecting it to another uncontracted leg τ. Block the sum

$$\sum_{\text{target legs } \tau} = \sum_{\Delta' \in K_j} \sum_{\substack{\text{target legs} \\ \text{in } \Delta'}} .$$

Ultimately we must estimate such blocked sums. This is done by

$$\left| \sum_{\Delta' \in K_j} \sum_{\substack{\text{target legs} \\ \text{in } \Delta'}} F(\tau) \right| \le \left[\sum_{\Delta' \in K_j} \big(1 + M^j \mathrm{dist}(\Delta, \Delta')\big)^{-d-2} \right] \times \sup_{\Delta' \in K_j} \left| \sum_{\substack{\text{target legs} \\ \text{in } \Delta'}} \big(1 + M^j \mathrm{dist}(\Delta, \Delta')\big)^{d+2} F(\tau) \right|. \tag{I.1}$$

The factor $\big(1 + M^j \mathrm{dist}(\Delta, \Delta')\big)^{d+2}$ is balanced by the decay of the propagator. Now the number of terms in the sum $\sum_{\substack{\text{target legs} \\ \text{in } \Delta'}}$ is the number of target legs in Δ' rather than the total number of target legs. When applied to all contractions, this technique converts "global $n!$'s" to "local $n!$'s". For a local $n!$ to be large, there must be many fields of the same momentum slice in a single dual cube. This is prevented by the Pauli exclusion principle.

To see how the combinatorial analysis suggested in the last two paragraphs may be carried out in a manner suitable for the proof of Theorem 1, we develop a complete expansion for a toy model with a simplified propagator and a local interaction. All the complications due to the presence of the Fermi surface are removed, by hand. The proof of Theorem 1 for the true propagator and full interaction is presented in §III.

Let the dimension d be arbitrary and let $C^{(j)}$ be any propagator obeying the bounds

$$\left| \nabla^n C^{(j)}(\xi, \bar\xi) \right| \le \text{const } M^{(\frac{d+1}{2}+n)j} \big(1 + M^j(|\xi - \bar\xi|\big)^{-\gamma}. \tag{I.2}$$

for some large γ and all $n \le N$, which are typical for strictly renormalizable field theories. For example the propagator for the Gross-Neveu model [FMRS], [GK] in two space-time dimensions is of this type. The interaction of the toy model is

$$\mathcal{V}_\Lambda = \frac{1}{2} \sum_{\{\uparrow,\downarrow\}} \int_\Lambda d\tau d\mathbf{x} \;\; \bar\psi(\tau, \mathbf{x}, \sigma) \bar\psi(\tau, \mathbf{x}, \sigma') \psi(\tau, \mathbf{x}, \sigma') \psi(\tau, \mathbf{x}, \sigma).$$

We now expand

$$\langle \mathcal{A} \rangle = \frac{1}{Z_\Lambda} \int \mathcal{A} e^{-\lambda \mathcal{V}_\Lambda(\psi, \bar\psi)} d\mu_{C^{(j)}}(\psi, \bar\psi)$$

where

$$\mathcal{A} = \int \prod_{j=1}^{2p} d\xi_j \; A(\xi_1, \cdots, \xi_{2p}) \prod_{j=1}^{2p} \overset{(-)}{\psi}(\xi_j)$$

is an arbitrary monomial.

Just as in the proof of Lemma 1, Gram's inequality (or, in the unlikely event that $C^{(j)}$ does not factorize suitably, Hadamard's inequality) implies that both the numerator and denominator of $\langle\mathcal{A}\rangle$ are entire functions of λ. The denominator Z can have many j and Λ dependent zeros. But when $\lambda=0$, $Z=1$ so that $\langle\mathcal{A}\rangle$ is meromorphic on all of $\mathbb{C}$ and analytic at zero. We shall develop a formal power series expansion for $\langle\mathcal{A}\rangle$ with the property that for every N

$$\langle\mathcal{A}\rangle=\sum_{n=0}^{N}a_n(j,\Lambda)\lambda^n+O\left(\lambda^{N+1}\right).$$

A priori we do not claim that the tail $O\left(\lambda^{N+1}\right)$ is uniform in j or Λ. Nevertheless, since $\langle\mathcal{A}\rangle$ is analytic at zero we must have

$$\langle\mathcal{A}\rangle=\sum_{n=0}^{\infty}a_n(j,\Lambda)\lambda^n \tag{I.3}$$

in some, possibly j and Λ dependent, neighborhood of zero. Observe that $a_n(j,\Lambda)$ must be the sum of all connected Feynman diagrams of order n with $2p$ external legs, since (I.3) is an asymptotic expansion.

We shall also show that there exists a const , independent of j and Λ, such that

$$|a_n(j,\Lambda)|\le \text{const}^{\,n+p}M^{(d+1)pj/2}\|A\|_1.$$

As a consequence, equation (I.3) applies for all $|\lambda|<R=\text{const}^{\,-1}$. Any zeroes of Z that appear in this disk must be cancelled by zeroes of the numerator. Finally, we shall show that the limits $a_n(j)=\lim_{\Lambda\to\mathbb{R}^{d+1}}a_n(j,\Lambda)$ exist. This will prove

$$\lim_{\Lambda\to\mathbb{R}^{d+1}}\langle\mathcal{A}\rangle=\sum_{n=0}^{\infty}a_n(j)\lambda^n$$

for all $|\lambda|<R$, the analog of Theorem 1 for the toy model.

The expansion is developed inductively. At the end of step s we have a sum over a set $\mathcal{T}_s$ of terms

$$\langle\mathcal{A}\rangle=\sum_{T\in\mathcal{T}_s}\frac{\int\mathcal{A}(T,s,\Lambda)e^{-\lambda\mathcal{V}_\Lambda(\psi,\bar\psi)}d\mu_{C^{(j)}}}{\int e^{-\lambda\mathcal{V}_\Lambda(\psi,\bar\psi)}d\mu_{C^{(j)}}}. \tag{I.4}$$

Each $\mathcal{A}(T,s,\Lambda)$ is a monomial in the fields $\psi,\bar\psi$ of degree $2\pi(T,s)$. In the event that $\pi(T,s)=0$ the ratio of integrals simplifies to the number $\mathcal{A}(T,s,\Lambda)$. Thus

$$\langle\mathcal{A}\rangle=\sum_{\substack{T\in\mathcal{T}_s\\ \pi(T,s)=0}}\mathcal{A}(T,s,\Lambda)+\sum_{\substack{T\in\mathcal{T}_s\\ \pi(T,s)\neq 0}}\frac{\int\mathcal{A}(T,s,\Lambda)e^{-\lambda\mathcal{V}_\Lambda(\psi,\bar\psi)}d\mu_{C^{(j)}}}{\int e^{-\lambda\mathcal{V}_\Lambda(\psi,\bar\psi)}d\mu_{C^{(j)}}}\ . \tag{I.5}$$

The numbers in the first sum are not touched in subsequent steps of the expansion. We shall show that each ratio in the second sum is $O\left(\lambda^s\right)$.

During step $s+1$, two operations are performed. The first is integration by parts. We apply the integration by parts formulae

$$\begin{aligned}\int\psi(\xi)F(\psi,\bar\psi)d\mu_{C^{(j)}}(\psi,\bar\psi)&=\int d\bar\xi\ C^{(j)}(\xi,\bar\xi)\int\frac{\delta}{\delta\bar\psi(\bar\xi)}F(\psi,\bar\psi)d\mu_{C^{(j)}}(\psi,\bar\psi)\\ \int\bar\psi(\bar\xi)F(\psi,\bar\psi)d\mu_{C^{(j)}}(\psi,\bar\psi)&=-\int d\xi\ C^{(j)}(\xi,\bar\xi)\int\frac{\delta}{\delta\psi(\xi)}F(\psi,\bar\psi)d\mu_{C^{(j)}}(\psi,\bar\psi)\end{aligned} \tag{I.6}$$

to each of the fields appearing in the monomial $\mathcal{A}(T,s,\Lambda)$. The order is chosen arbitrarily. Of course if a derivative $\frac{\delta}{\delta\overset{(-)}{\psi}(\overset{(-)}{\xi})}$ acts on a field in $\mathcal{A}(T,s,\Lambda)$, the field disappears. Thus we need not apply the integration by parts formulae to eliminate it. If no derivative acts on the exponential, all fields downstairs disappear, producing a term of degree zero. (We use the expression "downstairs" to identify fields multiplying the exponential.) Each time a derivative $\frac{\delta}{\delta\overset{(-)}{\psi}(\overset{(-)}{\xi})}$ acts on the exponential a new vertex and new fields are brought downstairs. Since the new vertex comes with a λ we see that, by the end of step $s+1$ all terms with degree $2\pi(T,s,\Lambda)$ different from zero are $O\left(\lambda^{s+1}\right)$.

We interpret this construction in terms of graphs. Each integration by parts adds a line, that is, a propagator to the graph. The lines produced by differentiating the exponential form a spanning tree. The terms $\mathcal{A}(T,s,\Lambda)$ of degree zero are values of completely formed graphs. The other terms are gestating and will at some later stage become the values of full graphs.

Fix any $T\in\mathcal{T}_s$. When all fields of $\mathcal{A}(T,s,\Lambda)$ have been eliminated by partial integration, we apply the second operation. It implements the Pauli exclusion principle. At this point there are still fields downstairs that were generated by derivatives acting on the exponential. They will be eliminated by partial integration in step $s+2$. As in the appendix of [IM], we expand each of these fields in a Taylor polynomial of degree t around c_δ, both to be determined:

$$\overset{(-)}{\psi}(\xi) = \overset{(-)}{\psi}(c_\delta) + \ \cdots \ + \frac{1}{t!}\big((\xi - c_\delta)\cdot\nabla_\xi\big)^t \ \overset{(-)}{\psi}(c_\delta)$$
$$+ \frac{1}{t!}\int_0^1 dw(1-w)^t\big((\xi - c_\delta)\cdot\nabla_\xi\big)^{t+1} \ \overset{(-)}{\psi}(c_\delta + w(\xi - c_\delta)).$$

When, in step $s+2$ a field is eliminated by integration by parts, we obtain a Taylor expansion of the corresponding propagator. Thus lines in graphs carry these more general propagators. Note that the Taylor expansion contains

$$1 + (d+1) + (d+1)^2 + \cdots + (d+1)^t + (d+1)^{t+1} \leq 2(d+1)^{t+1}$$

terms.

We now describe how the (field dependent) expansion point c_δ is determined. Recall that K_j is a paving of $\mathbb{R}^{d+1}$ by cubes of side M^{-j}. Each interaction vertex downstairs is rewritten as the sum

$$\mathcal{V}_\Lambda = \frac{1}{2}\sum_{\{\uparrow,\downarrow\}}\sum_{\Delta\in K_j}\int_{\Lambda\cap\Delta} d\tau d\mathbf{x} \ \ \bar\psi(\tau,\mathbf{x},\sigma)\bar\psi(\tau,\mathbf{x},\sigma')\psi(\tau,\mathbf{x},\sigma')\psi(\tau,\mathbf{x},\sigma). \tag{I.7}$$

In the $s = 0$ step the only fields downstairs belong to the monomial $\mathcal{A}$. They are also expanded in terms of the paving. Multiple applications of (I.6) and (I.7) have generated from $\mathcal{A}(T, s, \Lambda)$ a sum of terms. Pick a term. Each field of this term is localized in a cube $\Delta \in K_j$. Let $\pi(s+1, \Delta, \sigma, b)$ be the number of fields that have the specified values of Δ, σ and b. Here $s+1$ reminds us that we are in the midst of step $s+1$ and b distinguishes between ψ's and $\bar\psi$'s. For each σ, b and $\Delta \in K_j$ we divide Δ into $\pi(s+1,\Delta,\sigma,b)^\epsilon$ identical cubes δ each of side $\pi(s+1,\Delta,\sigma,b)^{-\frac{\epsilon}{d+1}}M^{-j}$. The value of ϵ will be picked later. The center of δ is called c_δ and the number of fields in δ with specified values of σ and b is called $\pi(s+1,\delta,\sigma,b)$. The δ in the above Taylor expansion is, of course, that containing ξ. In particular the Taylor expansion must be done inside the ξ integrals. Note that, by the hypothesis (I.2) on the behavior of the propagator, each $(\xi - c_\delta)\cdot\nabla_\xi$ that acts on a ψ produces a factor of

$$\pi(s+1,\Delta,\sigma,b)^{-\frac{\epsilon}{d+1}}M^{-j}\cdot M^j. \tag{I.8}$$

Since the fields anticommute, any nonzero integral may contain at most one field having any given value of σ, b at any c_δ. The same is true for each derivative of the fields.

Thus, for each σ, b, δ, all but $2(d+1)^t$ of the fields having this σ, b and located in δ must be Taylor remainders. That is, there are at least $\pi(s+1,\delta,\sigma,b) - 2(d+1)^t$ Taylor remainders. This completes the description of the expansion. We now prove

Lemma5

Let the propagator $C^{(j)}$ *obey* (I.2). *Then there exists a* const , *independent of* j *and* Λ, *such that*

$$|a_n(j,\Lambda)| \le \text{const}^{\,n+p} M^{(d+1)pj/2} \|A\|_1.$$

Furthermore the limits

$$a_n(j) = \lim_{\Lambda \to \mathbb{R}^{d+1}} a_n(j,\Lambda)$$

exist and obey the same bounds.

Proof : We use the "method of combinatorial factors" to keep track of the many sums in the expansion. This technique uses the elementary estimate

$$\kappa_i > 0 \ , \ \sum_i \kappa_i^{-1} \le 1 \quad \Rightarrow \quad \left|\sum_i U_i\right| \le \sup_i |\kappa_i U_i| \,. \tag{I.9}$$

to replace each sum by a supremum. To help remember the combinatorial factor κ_i multiplying the value U_i of a given diagram, the factor is assigned to a specific line or vertex of the diagram.

Here are the combinatorial factors used to control each of the operations.

(1)*Integration by parts.*

- A factor of two, assigned to the leg initiating the integration by parts suffices to decide whether or not the leg brings down a new vertex from the exponent.

When it does,

- a factor of two, assigned to the target leg, counts the number of possible target legs.

When it doesn't, we need to work harder. Vertices are continuously being added downstairs from the exponent during the $(s+1)^{\text{st}}$ step. Thus the set of possible target legs both grows and shrinks as the stage progresses. On the other hand, all source legs must come from the set of legs downstairs at the beginning of the stage. So it is easier to count the number of contractions by labeling each target leg with the name of the source leg that contracted to it rather than use the usual procedure, which is to label each source leg with the name of the target leg to which it contracts.

- A factor of two per leg suffices to decide whether or not the leg is a target leg.

For each target leg, we organize the sum over source legs according first, to the cube $\Delta \in K_j$ of the source and second, to which leg in Δ is the source. As in (I.1)

- a factor of const $\big(1 + M^j \mathrm{dist}(\Delta, \Delta')\big)^{d+2}$, assigned to the line generated, suffices to control the sum over Δ.
- A factor $\pi(s+1, \Delta, \sigma, b)$, assigned to the source leg, counts the number of possible source legs within Δ.

(2) *Implementation of the Pauli exclusion principle.*

- a factor of const $\big(1 + M^j \mathrm{dist}(\Delta, \Delta')\big)^{d+2}$ assigned to the propagator that brought a vertex down from the exponent, will control the sum over localization cubes of the vertex.

The sum over localization cubes for the fields of $\mathcal{A}$ is not controlled using combinatorial factors. It will shortly be performed explicitly.

- The Taylor expansion splits each leg into at most $2(d+1)^{t+1}$ pieces.

It remains only to bound an integral. The integration variables are the positions of the fields of $\mathcal{A}$. The integrand is the supremum, over positions and diagrams, of the product of

- the above combinatorial factors
- const $\big(1 + M^j \mathrm{dist}(\Delta, \Delta')\big)^{-\gamma}$ per line of the diagram
- $|\lambda| \int_\Delta d^{d+1}\xi \; 1 \;\le |\lambda| M^{-j(d+1)}$ per vertex of the diagram
- $|A(\xi_1, \cdots, \xi_{2p})|$
- and the factors that come from the Taylor expansions used to implement the Pauli exclusion principle.

The latter are

$$\begin{aligned}
&\prod_s \prod_{\sigma,b} \prod_{\Delta\in K_j} \prod_{\delta\in\Delta} \left[\pi(s,\Delta,\sigma,b)^{-\frac{\epsilon}{d+1}}\right]^{(t+1)\left(\pi(s,\delta,\sigma,b)-2(d+1)^t\right)} \\
&\le \prod_s \prod_{\sigma,b} \prod_{\Delta\in K_j} \left[\pi(s,\Delta,\sigma,b)^{-\frac{\epsilon}{d+1}}\right]^{(t+1)\left(\pi(s,\Delta,\sigma,b)-2(d+1)^t\pi(s,\Delta,\sigma,b)^\epsilon\right)} .
\end{aligned} \tag{I.10}$$

Pick any $\zeta > 0$. It is possible to choose ϵ and t, depending only on d and ζ so that

$$(I.10) \le \prod_s \prod_{\sigma,b} \prod_{\Delta\in K_j} \mathrm{const}\, \pi(s,\Delta,\sigma,b)^{-\zeta\pi(s,\Delta,\sigma,b)}.$$

Altogether

$$\begin{aligned}
|a_n(j,\Lambda)| &\le \mathrm{const}^{\,n+p} M^{\frac{d+1}{2}j\frac{4n+2p}{2}} M^{-(d+1)jn} \|A\|_1 \\
&= \mathrm{const}^{\,n+p} M^{\frac{d+1}{2}pj} \|A\|_1
\end{aligned}$$

The first power of M^j came from covariance bounds and the second from integration over the positions of the interaction vertices. When the Λ constraint is removed, a_n is expressed as an absolutely convergent series obeying the same bound. ∎

Our discussion of the toy model is now complete. The rest of the paper is devoted to the full model.

§II Sector Counting Lemmas

In this section we prove Lemmas 2 and 3 formulated in the introduction.

Lemma 2

Fix $m \in \mathbb{Z}^{d+1}$ *and* $\ell \ge 2$ *. Then, the number of* 2ℓ*-tuples*

$$\{S_1,\ \cdots\ S_{2\ell}\}$$

of sectors of side M^j *on the Fermi sphere for which there exist* $\mathbf{k}_i \in \mathbb{R}^d$, $i = 1,\ \cdots,\ 2\ell$ *satisfying*

$$\mathbf{k}_i' \in S_i, \quad |\mathbf{k}_i - \mathbf{k}_i'| \le \mathrm{const}\, M^j \ , \quad i = 1,\ \cdots,\ 2\ell$$

and

$$|\mathbf{k}_1 + \ \cdots\ + \mathbf{k}_{2\ell}| \le \mathrm{const}\,(1+|m|)\, M^j$$

is bounded by

$$\mathrm{const}^{\,\ell}(1+|m|)^d M^{-(2\ell-1)(d-1)j} M^j \left\{1+|j|\delta_{d,2}\delta_{\ell,2}\right\}.$$

Here, $\mathbf{k}'=\frac{\mathbf{k}}{|\mathbf{k}|}$ *denotes the projection of* $\mathbf{k}$ *onto the Fermi surface.*

Proof : For any fixed $\mathbf{k}_i,\ i=1,\ \cdots,\ 2\ell-1$ there are at most $O(1+|m|)^{d-1}$ sectors within $O\left((1+|m|)M^j\right)$ of $\mathbf{k}_1+\ \cdots\ +\mathbf{k}_{2\ell-1}$. Thus, the problem is reduced to determining the number of $(2\ell-1)$-tuples $\{S_1,\ \cdots\ ,S_{2\ell-1}\}$ of sectors for which there exist momenta $\mathbf{k}_i\in \mathbb{R}^d,\ i=1,\ \cdots,\ 2\ell-1$ with $\mathbf{k}_i'\in S_i,\ \ |\mathbf{k}_i-\mathbf{k}_i'|\le \mathrm{const}\,M^j$, such that $\mathbf{k}_1+\ \cdots\ +\mathbf{k}_{2\ell-1}$ is within $O\left((1+|m|)M^j\right)$ of the Fermi surface.

Since $|\mathbf{k}_i-\mathbf{k}_i'|\le \mathrm{const}\,M^j$ there must exist $\mathbf{k}_i'\in S_i$ such that $\mathbf{k}_1'+\ \cdots\ +\mathbf{k}_{2\ell-1}'$ is within $O\left((1+|m|+\ell)M^j\right)$ of the Fermi surface. As the projections $\mathbf{k}_i'$, $1\le i\le 2\ell-1$, vary independently over their sectors, the sum $\mathbf{k}_1'+\ \cdots\ +\mathbf{k}_{2\ell-1}'$ varies by $O(\ell M^j)$. Thus, the problem is further reduced to counting the number N_ℓ of $(2\ell-1)$-tuples $\{S_1,\ \cdots\ ,S_{2\ell-1}\}$ of sectors such that for all $\mathbf{k}_i'\in S_i$ the sum $\mathbf{k}_1'+\ \cdots\ +\mathbf{k}_{2\ell-1}'$ is within $O\left((1+|m|+\ell)M^j\right)$ of the Fermi surface. Observe that, if the volume of every sector is at least $\mathrm{const}\ M^{(d-1)j}$,

$$N_\ell\ \le\ \mathrm{const}^{\,-(2\ell-1)} M^{-(2\ell-1)(d-1)j}\ \prod_{i=1}^{2\ell-1}\left(\int_{k_F S^{d-1}} d\mathbf{k}_i'\right) f\left(\mathbf{k}_1'+\ \cdots\ +\mathbf{k}_{2\ell-1}'\right)$$

where f is a smooth function that is one when $\mathbf{k}_1'+\ \cdots\ +\mathbf{k}_{2\ell-1}'$ is within $O\left((1+|m|+\ell)M^j\right)$ of the Fermi surface and is zero when it is at least a distance $2O\left((1+|m|+\ell)M^j\right)$ from the Fermi surface.

Rewriting, and recalling that the Fourier transform of the $(d-1)$-sphere $\delta(|\mathbf{k}|-k_F)$ is

$$\mathrm{const}\ \left(k_F|\mathbf{p}|\right)^{1-\frac{d}{2}} J_{\frac{d}{2}-1}(k_F|\mathbf{p}|)$$

we have

$$\prod_{i=1}^{2\ell-1}\left(\int_{k_F S^{d-1}} d\mathbf{k}_i'\right) f\left(\mathbf{k}_1'+\ \cdots\ +\mathbf{k}_{2\ell-1}'\right)$$

$$=\mathrm{const}^{2\ell-1}\int \prod_{i=1}^{2\ell-1} d^d\mathbf{k}_i\ \prod_{i=1}^{2\ell-1}\ \delta(|\mathbf{k}_i|-k_F)\ f\left(\mathbf{k}_1+\ \cdots\ +\mathbf{k}_{2\ell-1}\right)$$

$$= \mathrm{const}^{2\ell-1} \int d^d\mathbf{t} \prod_{i=1}^{2\ell-1} d^d\mathbf{k}_i \prod_{i=1}^{2\ell-1} d^d\mathbf{p}_i \prod_{i=1}^{2\ell-1} (k_F|\mathbf{p}_i|)^{1-\frac{d}{2}} J_{\frac{d}{2}-1}(k_F|\mathbf{p}_i|)$$

$$\prod_{i=1}^{2\ell-1} e^{i<\mathbf{k}_i,\mathbf{p}_i>} \hat{f}(\mathbf{t}) e^{i<\mathbf{t},\mathbf{k}_1+\cdots+\mathbf{k}_{2\ell-1}>}$$

$$= \mathrm{const}^{2\ell-1} \int d^d\mathbf{t} \prod_{i=1}^{2\ell-1} d^d\mathbf{p}_i \prod_{i=1}^{2\ell-1} (k_F|\mathbf{p}_i|)^{1-\frac{d}{2}} J_{\frac{d}{2}-1}(k_F|\mathbf{p}_i|) \prod_{i=1}^{2\ell-1} \delta(\mathbf{p}_i+\mathbf{t}) \hat{f}(\mathbf{t})$$

$$= \mathrm{const}^{2\ell-1} \int d^d\mathbf{t} \left\{(k_F|\mathbf{t}|)^{1-\frac{d}{2}} J_{\frac{d}{2}-1}(k_F|\mathbf{t}|)\right\}^{2\ell-1} \hat{f}(\mathbf{t}) \ .$$

The classical estimates

$$J_\alpha(r) \sim \mathrm{const}\ r^\alpha \ , \quad r \to 0$$

$$J_\alpha(r) = O\left(r^{-\frac{1}{2}}\right) \ , \quad r \to \infty$$

imply that

$$(k_F|\mathbf{t}|)^{1-\frac{d}{2}} J_{\frac{d}{2}-1}(k_F|\mathbf{t}|) = \begin{cases} O(1) & \text{for small } |\mathbf{t}| \\ O\left(|\mathbf{t}|^{\frac{1-d}{2}}\right) & \text{for large } |\mathbf{t}| \end{cases}$$

Consequently,

$$\hat{f}(\mathbf{t}) = \mathrm{const} \int d^d\mathbf{p}\ e^{-i<\mathbf{p},\mathbf{t}>} f(|\mathbf{p}|) \ = \ \mathrm{const} \int_0^\infty dr\ r^{d-1} \int_{S^{d-1}} d\mathbf{p}'\ e^{-i<r\mathbf{p}',\mathbf{t}>} f(r)$$

$$= \mathrm{const} \int_0^\infty dr\ r^{d-1} \ (r|\mathbf{t}|)^{1-\frac{d}{2}} J_{\frac{d}{2}-1}(r|\mathbf{t}|)\ f(r)$$

$$= \mathrm{const}\ (1+|m|+\ell)\ M^j \int_0^\infty dr\ r^{d-1} \ (r|\mathbf{t}|)^{1-\frac{d}{2}} J_{\frac{d}{2}-1}(r|\mathbf{t}|)\ \frac{f(r)}{\|f\|_1}$$

$$= \begin{cases} O\left((1+|m|+\ell)M^j|\mathbf{t}|^{\frac{1-d}{2}}\right) & \text{for } |\mathbf{t}| \ \geq \ 1 \\ O\left((1+|m|+\ell)M^j\right) & \text{for } |\mathbf{t}| \ \leq \ 1 \end{cases}$$

We now obtain

$$N_\ell \ \leq \ \mathrm{const}^\ell\ M^{-(2\ell-1)(d-1)j} \int d^d\mathbf{t} \left\{(k_F|\mathbf{t}|)^{1-\frac{d}{2}} J_{\frac{d}{2}-1}(k_F|\mathbf{t}|)\right\}^{2\ell-1} \hat{f}(\mathbf{t})$$

$$\leq \ \mathrm{const}^\ell\ (1+|m|)\ M^{-(2\ell-1)(d-1)j} M^j \left\{\int_{|\mathbf{t}|\ \geq\ 1} d^d\mathbf{t}\ |\mathbf{t}|^{-\ell(d-1)} \ + \ \int_{|\mathbf{t}|\ \leq\ 1} d^d\mathbf{t}\right\}$$

$$\leq \ \begin{cases} \mathrm{const}^\ell\ (1+|m|)\ M^{-(2\ell-1)(d-1)j} M^j & \text{for } \ell(d-1) \ > \ d \\ \text{logarithmically divergent} & \text{for } \ell(d-1) = d \end{cases}$$

If $\ell(d-1)=d$, then $\ell=d=2$ and we consider the integral

$$\int d^d\mathbf{t}\ \left\{(k_F|\mathbf{t}|)^{1-\frac{d}{2}}\,J_{\frac{d}{2}-1}(k_F|\mathbf{t}|)\right\}^3\ \widehat{f}(\mathbf{t})$$

Of course $f(t)$ is Schwartz class and hence so is $\widehat{f}$. The convergence of the integral is not in question. However each derivative of f produces a factor of M^{-j} . Thus, we may gain a power of $|\mathbf{t}|^{-2}$ in our estimate of $\widehat{f}(\mathbf{t})$ only at the cost of one M^{-2j} . Taking the geometric mean between

$$|\widehat{f}(\mathbf{t})| \le \text{const}\,(1+|m|)\ M^j|\mathbf{t}|^{-\frac{1}{2}}$$

and

$$|\widehat{f}(\mathbf{t})| \le \text{const}\,(1+|m|)\ M^{-j}|\mathbf{t}|^{-\frac{5}{2}}$$

one obtains for every $\epsilon>0$

$$|\widehat{f}(\mathbf{t})| \le \text{const}\,(1+|m|)\ M^{(1-2\epsilon)j}|\mathbf{t}|^{-(\frac{1}{2}+2\epsilon)}\ .$$

The constant in the last line is independent of ϵ . Finally, for $d=2$

$$\begin{aligned} N_2 &\le \text{const}\ (1+|m|)\ M^{-3j}\ M^j\ \left\{M^{(-2\epsilon)j}\int_{|\mathbf{t}|\,\ge\,1} d^2\mathbf{t}\ |\mathbf{t}|^{-(2+2\epsilon)}\ +\ 1\right\} \\ &= \text{const}\,(1+|m|)\ M^{-3j}\ M^j\ \left\{M^{(-2\epsilon)j}\ \text{const}\ \epsilon^{-1}+\ 1\right\}\ . \end{aligned}$$

Taking the limit $\epsilon\to 0$ gives the desired result. ■

We now turn to the proof of Lemma 3. Actually we shall prove the stronger

Lemma $3'$

Let $d=2$ and $\ell\ge 4$. Let $\Omega_2,\ldots,\Omega_\ell$ be intervals on the Fermi circle each of length $\Omega\ge M^{j/2}$. Let Σ_1 be a fixed sector. The number of $(\ell-1)$-tuples of sectors $\{\Sigma_2,\cdots,\Sigma_\ell\}$ for which there exist $\mathbf{k}_i\in\mathbb{R}^2,\ i=1,\cdots,\ell$ satisfying

$$\mathbf{k}_i'\in\Sigma_i\cap\Omega_i,\quad |\mathbf{k}_i-\mathbf{k}_i'|\le\text{const}\,M^j\ ,\quad i=1,\cdots,\ell$$

and

$$|\mathbf{k}_1+\ \cdots\ +\mathbf{k}_{2\ell}|\le\text{const}\ (1+|m|)\,M^j$$

is bounded by

$$\mathrm{const}^{\,\ell}\,(1+|m|)^2\left(\Omega M^{-j/2}\right)^{\ell-3}\quad .$$

Proof : Denote by $\mathbf{r}_n$ the center of the sector containing $\mathbf{k}_n$. Renumber $\mathbf{k}_2,\cdots,\mathbf{k}_\ell$ so that $|\mathbf{r}_\ell\cdot\mathbf{r}_{\ell-1}|$ is minimal amongst all $\left\{|\mathbf{r}_n\cdot\mathbf{r}_p|\ \middle|\ n,p\neq 1\right\}$. In other words $\phi=\angle(\mathbf{k}'_{\ell-1},\mathbf{k}'_\ell)$ is as close to $\pi/2$ as possible. All other $\angle(\mathbf{k}'_n,\mathbf{k}'_p)$'s with $n,p\geq 2$ must be within $\phi+O(M^{j/2})$ of either 0 or π.

When $M^i\leq\phi\leq M^{i+1}$ or $M^i\leq\pi-\phi\leq M^{i+1}$ the number of accessible $(\ell-1)$-tuples of sectors is bounded by

$$N_\ell\prod_{i=2}^{\ell-2}\min\left\{\Omega,M^i\right\}M^{-j/2}$$

where N_ℓ is the number of sectors accessible to the last two $\mathbf{k}$'s once the sectors for $\mathbf{k}_2,\cdots,\mathbf{k}_{\ell-2}$ have been fixed. We shall shortly show that $N_\ell\leq\mathrm{const}^{\,\ell}\,(1+|m|)^2$. The sum over those i's with $M^i\leq\Omega$ is bounded by

$$\mathrm{const}^{\,\ell}\sum_{\substack{i\\ M^i\leq\Omega}}(1+|m|)^2M^{i(\ell-3)}M^{-j(\ell-3)/2}\leq\mathrm{const}^{\,\ell}\,(1+|m|)^2\left(\Omega M^{-j/2}\right)^{\ell-3}$$

provided $\ell\geq 4$. Now consider $M^i>\Omega$. Once the sectors of all the $\mathbf{k}_i$'s except the $\ell^{\rm th}$ have been selected there can be at most one i consistent with $\mathbf{k}_\ell$ falling in Ω_ℓ. For this one value of i

$$N_\ell\prod_{i=2}^{\ell-2}\min\left\{\Omega,M^i\right\}M^{-j/2}\leq\mathrm{const}^{\,\ell}\,(1+|m|)^2\left(\Omega M^{-j/2}\right)^{\ell-3}$$

It suffices to consider i obeying $M^i>\mathrm{const}^{\,\ell}(1+|m|)M^{j/2}$ so fix any such i and $\Sigma_1,\cdots,\Sigma_{\ell-2}$. We now compute $\mathbf{a}$ and ϵ, defined by

$$\begin{aligned}\mathbf{a}&=-\mathbf{r}_1-\ \cdots\ -\mathbf{r}_{\ell-2}\\ \mathbf{a}+\epsilon&=\mathbf{k}'_{\ell-1}+\mathbf{k}'_\ell\\ &=-\mathbf{k}_1-\ \cdots\ -\mathbf{k}_{\ell-2}+O\big((1+|m|)M^j\big)\ .\end{aligned}$$

Chose a coordinate system in which $\mathbf{r}_2=(k_F,0)$. Then, since $\theta_n=\angle(\mathbf{r}_2,\mathbf{r}_n)$, is within $2M^{i+1}$ of 0 or π the x and y coordinates of every $\mathbf{r}_n$ and $\mathbf{k}_n$, $2\leq n\leq\ell$, obey

$$\begin{aligned}x_n&=\pm[k_F+O\big(M^j\big)]\cos O\big(M^i\big)=\pm k_F+O\big(M^{2i}\big)\\ y_n&=[k_F+O\big(M^j\big)]\sin O\big(M^i\big)=O\big(M^i\big)\end{aligned}$$

respectively. Consequently $\mathbf{k}_1 = -\mathbf{k}_2 - \cdots - \mathbf{k}_\ell$ and hence $\mathbf{r}_1$ obey

$$\begin{aligned} x_1 &= \pm k_F + O\big((|m|+\ell)M^{2i}\big) \\ y_1 &= O\big((|m|+\ell)M^{i}\big) \end{aligned}$$

We need to know with greater precision how much $\mathbf{k}_n$ can wiggle.

$$\begin{aligned} \mathbf{k}_n - \mathbf{r}_n &= \mathbf{k}'_n - \mathbf{r}_n + O\big(M^j\big) \\ &= k_F\Big(\cos\big(\theta_n + O(M^{j/2})\big), \sin\big(\theta_n + O(M^{j/2})\big)\Big) - k_F\Big(\cos\theta_n, \sin\theta_n\Big) + O\big(M^j\big) \\ &= O\big((M^i M^{j/2}, M^{j/2})\big)\ . \end{aligned}$$

Summing the individual wiggles yields

$$\begin{aligned} \epsilon &= \sum_{n=1}^{\ell-2}(\mathbf{r}_n - \mathbf{k}_n) + O\big((1+|m|)M^j\big) \\ &= (|m|+\ell)\, O\big((M^i M^{j/2}, M^{j/2})\big) \end{aligned}$$

and

$$\begin{aligned} \mathbf{a} &= -\mathbf{r}_1 - \cdots - \mathbf{r}_{\ell-2} \\ &= \mathbf{k}_{\ell-1} + \mathbf{k}_\ell - \sum_{n=1}^{\ell}\mathbf{k}_n + \sum_{n=1}^{\ell-2}(\mathbf{k}_n - \mathbf{r}_n) \\ &= N(2k_F, 0) + O\big((M^{2i}, M^i)\big) + O\big((1+|m|)M^j\big) + \ell\, O\big((M^i M^{j/2}, M^{j/2})\big) \\ &= N(2k_F, 0) + O\big((M^{2i}, M^i)\big) \end{aligned}$$

where $N \in \{1, 0, -1\}$.

We are now in a position to bound N_ℓ. There are two cases to be considered. First, suppose $|N| = 1$. Rotate the coordinate system by $\pi\delta_{N,-1} + O\big(M^i\big)$ to make $\mathbf{a}$ run along the positive x axis. In the new coordinates,

$$\epsilon = (|m|+\ell)\, O\big((M^i M^{j/2}, M^{j/2})\big)\ .$$

is still obeyed.

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Then the two components of

$$k_F\big(\cos\alpha, \sin\alpha\big) + k_F\big(\cos(\phi-\alpha), -\sin(\phi-\alpha)\big) = \mathbf{a} + \epsilon$$

are

$$\cos\alpha + \cos(\phi-\alpha) = \frac{|\mathbf{a}|}{k_F} + (|m|+\ell)\, O\big(M^i M^{j/2}\big)$$
$$\sin\alpha - \sin(\phi-\alpha) = (|m|+\ell)\, O\big(M^{j/2}\big)\ .$$

The y component implies that

$$|2\alpha-\phi| = (|m|+\ell)\, O\big(M^{j/2}\big)$$

so that there are const $(|m|+\ell)$ sectors accessible to $\mathbf{k}_\ell$ (once $\mathbf{r}_{\ell-1}$ has been fixed) and

$$\alpha = \frac{1}{2}\phi + (|m|+\ell)\, O\big(M^{j/2}\big)\ .$$

is bounded above and below by const M^i. Since

$$\begin{aligned}\cos\alpha + \cos(\phi-\alpha) &= \cos\alpha + \cos\alpha\cos(\phi-2\alpha) - \sin\alpha\sin(\phi-2\alpha)\\ &= \big[2 + (|m|+\ell)^2\, O\big(M^j\big)\big]\cos\alpha + (|m|+\ell)\, O\big(M^i M^{j/2}\big)\end{aligned}$$

the x component gives

$$\alpha = \cos^{-1}\left(\frac{|\mathbf{a}|}{2k_F}\right) + (|m|+\ell)\, O\left(\frac{M^i M^{j/2}}{M^i}\right)$$

Thus, there are const $(|m|+\ell)$ sectors accessible to $\mathbf{k}_{\ell-1}$ as well.

Finally, suppose that $N=0$. This time rotate the coordinate system by $O\big(M^i\big)$ or $\pi + O\big(M^i\big)$ so that $\mathbf{k}_{\ell-1}$ runs along the negative x axis.

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The angle ϕ is determined by

$$\begin{aligned}\sin\left(\frac{\pi-\phi}{2}\right) &= \frac{|\mathbf{a}+\epsilon|}{2k_F}\\ &= \frac{|\mathbf{a}|}{2k_F} + (|m|+\ell)\, O\big(M^{j/2}\big)\ .\end{aligned}$$

Thus

$$\phi = \pi - 2\sin^{-1}\left(\frac{|\mathbf{a}|}{2k_F}\right) + (|m|+\ell)\, O\big(M^{j/2}\big)$$

and $\mathbf{k}_\ell$ has access to $O\big(|m|+\ell\big)$ sectors when $\mathbf{r}_{\ell-1}$ is held fixed. The angle θ is determined by

$$\begin{aligned}\left|\sin\left(\theta-\frac{\phi}{2}\right)\right| &= \frac{\left|\epsilon\cdot\left(\cos\left(\frac{\pi-\phi}{2}\right), \sin\left(\frac{\pi-\phi}{2}\right)\right)\right|}{|\mathbf{a}|}\\ &\le \frac{(|m|+\ell)\, O\big(M^i M^{j/2}\big)}{M^i - (|m|+\ell)\, O\big(M^{j/2}\big)}\\ &\le (|m|+\ell)\, O\big(M^{j/2}\big)\ .\end{aligned}$$

In the second last line we used the hypothesis that $M^i \le \phi \le M^{i+1}$. This forces

$$\begin{aligned} |\mathbf{a}+\epsilon| &= 2k_F \sin\left(\frac{\pi-\phi}{2}\right) \\ &\ge \text{const } M^i \ . \end{aligned}$$

Once again there are at most $O\big(|m|+\ell\big)$ sectors accessible to $\mathbf{k}_{\ell-1}$. ■

§III The Full Expansion

In this Section we elaborate on the expansion presented in the introduction to prove a lemma and the two theorems. All are of the form of Lemma 5, but treat the true propagator and interaction. They are restricted to three space-time dimensions. In Lemma 6, we bound Schwinger functions. Theorem 1 constructs the effective potential. Finally in Theorem 2, we generalize the expansion to accommodate ν-body interactions, $\nu \ge 2$, of the type generated by the renormalization group flow of [FT2].

Throughout this section we expand the true propagator

$$C^{(j)} = \sum_{\ell=1}^{M^{-j/2}} C^{(j,\ell)}$$

where

$$C^{(j,\ell)}(\xi,\bar\xi) = \delta_{\sigma,\bar\sigma} \int \frac{d^3p}{(2\pi)^{d+1}} \frac{e^{i<p,\xi-\bar\xi>_-}}{ip_0 - e(\mathbf{p})} f_j(p)\,\zeta_\ell(\mathbf{p}) \ .$$

The partition of unity

$$1 = \sum_{\ell=1}^{M^{-j/2}} \zeta_\ell(\mathbf{p}), \qquad \zeta_\ell(\mathbf{p}) = \zeta_\ell\left(\frac{\mathbf{p}}{|\mathbf{p}|}k_F\right)$$

divides the Fermi circle into long sectors Σ_ℓ. We denote by r_ℓ the center of Σ_ℓ. Recall that

$$< p,\xi >_- = \mathbf{p}\cdot\mathbf{x} - p_0\tau \ ,$$

$$e(\mathbf{p}) = \frac{\mathbf{p}^2}{2m} - \mu.$$

and

$$f_j(p) = f\left(M^{-2j}\left(p_0^2 + e(\mathbf{p})^2\right)\right) .$$

The corresponding decomposition of the fields is

$$\psi(\xi) = \sum_{\ell=1}^{M^{-j/2}} \psi^{(\ell)}(\xi) \qquad\qquad \bar\psi(\xi) = \sum_{\ell=1}^{M^{-j/2}} \bar\psi^{(\ell)}(\xi) \ .$$

For each sector Σ_ℓ we introduce a lattice $\mathbf{D}_\ell$ of rectangular parallelepipeds (called boxes for short) that pave $\mathrm{I\!R}^3$. These boxes have three axes. One axis is the fixed time direction and the length of any box in that direction is M^{-j}. In the orthogonal $\mathrm{I\!R}^2$ plane, one of the axes is r_ℓ, the center of Σ_ℓ, and the length in this direction is M^{-j}. The third axis is orthogonal to r_ℓ and has length $M^{-j/2}$.

Finally, the two-body interaction in volume $\Lambda \subset \mathrm{I\!R}^3$ is

$$\mathcal{V}_\Lambda = \frac{1}{2}\int d^3\xi \prod_{i=1}^{4} d^3\eta_i \ \chi_\Lambda(\xi) V(\eta_1,\eta_2,\eta_3,\eta_4)\delta(\eta_1{+}\eta_2{+}\eta_3{+}\eta_4)\bar\psi(\xi{+}\eta_1)\psi(\xi{+}\eta_3)\bar\psi(\xi{+}\eta_2)\psi(\xi{+}\eta_4).$$

It is convenient for us to restrict only the center of mass ξ to Λ.

Lemma 6

Let

$$\mathcal{A} = \int \prod_{j=1}^{2p} d\xi_j \ A(\xi_1,\cdots,\xi_{2p}) \prod_{j=1}^{2p} \overset{(-)}{\psi}(\xi_j)$$

be a monomial of degree $2p$. Define

$$\langle \mathcal{A} \rangle = \frac{1}{Z_\Lambda}\int \mathcal{A} e^{-\lambda\mathcal{V}_\Lambda(\psi,\bar\psi)} d\mu_{C^{(j)}}(\psi,\bar\psi)$$

where

$$Z_\Lambda = \int e^{-\lambda\mathcal{V}_\Lambda(\psi,\bar\psi)} d\mu_{C^{(j)}}(\psi,\bar\psi) \ .$$

Then, there exists a const *, independent of j and Λ, such that the perturbation series*

$$\langle \mathcal{A} \rangle = \sum_{n=0}^{\infty} a_n(j,\Lambda)\lambda^n$$

converges for all $|\lambda| < R = (\mathrm{const}\, |\!|\!| V |\!|\!|)^{-1}$ and the coefficients satisfy

$$|a_n(j,\Lambda)| \le \mathrm{const}^{\,n+p} M^{pj/2} \|A\|_1 |\!|\!| V |\!|\!|^n \ .$$

Furthermore the limits

$$a_n(j) = \lim_{\Lambda \to \mathbb{R}^3} a_n(j, \Lambda)$$

exist.

In particular the infinite volume single slice Schwinger functions exist and obey good j-dependent bounds.

Proof : As in the introduction our expansion is developed inductively. Once again, at the end of step s we have a sum over a set $\mathcal{T}_s$ of terms

$$\begin{aligned}\langle \mathcal{A} \rangle &= \sum_{T \in \mathcal{T}_s} \frac{\int \mathcal{A}(T, s, \Lambda) e^{-\lambda \mathcal{V}_\Lambda(\psi, \bar{\psi})} d\mu_{C^{(j)}}}{\int e^{-\lambda \mathcal{V}_\Lambda(\psi, \bar{\psi})} d\mu_{C^{(j)}}} \\ &= \sum_{\substack{T \in \mathcal{T}_s \\ \pi(T,s)=0}} \mathcal{A}(T, s, \Lambda) + \sum_{\substack{T \in \mathcal{T}_s \\ \pi(T,s)\neq 0}} \frac{\int \mathcal{A}(T, s, \Lambda) e^{-\lambda \mathcal{V}_\Lambda(\psi, \bar{\psi})} d\mu_{C^{(j)}}}{\int e^{-\lambda \mathcal{V}_\Lambda(\psi, \bar{\psi})} d\mu_{C^{(j)}}} \quad .\end{aligned} \tag{III.1}$$

Recall that each $\mathcal{A}(T, s, \Lambda)$ is a monomial in the fields $\psi, \bar{\psi}$ of degree $2\pi(T, s)$. We used the fact that the ratio of integrals simplifies to the number $\mathcal{A}(T, s, \Lambda)$ when $\pi(T, s) = 0$. Each ratio in the second sum will be $O(\lambda^s)$.

We now outline the $(s+1)^{\text{th}}$ step in the induction. The details will be presented shortly. First pick any field of $\mathcal{A}(T, s, \Lambda)$. Apply the integration by parts formula (I.6). In the event that the derivative brings a new vertex down from the exponent, expand the new vertex as a sum over momentum sectors ℓ and position space boxes $\Delta \in \mathbf{D}_\ell$. Here, this step is considerably more complicated than (I.7). Next, pick any other field of the original monomial $\mathcal{A}(T, s, \Lambda)$ and repeat the construction. When all the fields of $\mathcal{A}(T, s, \Lambda)$ have been exhausted, substitute Taylor expansions much as in §I.

The sum over sectors and conjugate boxes is complicated because conservation of momentum must be exploited to restrict the sums over sectors at every vertex. We now consider this in more detail. The volume cutoff interaction is

$$
\begin{aligned}
\mathcal{V}_\Lambda &= \frac{1}{2}\int d^3\xi \prod_{i=1}^4 d^3\eta_i\ \chi_\Lambda(\xi)V(\eta_1,\eta_2,\eta_3,\eta_4)\delta(\Sigma\eta_i)\bar\psi(\xi+\eta_1)\psi(\xi+\eta_3)\bar\psi(\xi+\eta_2)\psi(\xi+\eta_4)\\
&= \frac{1}{2}\int d^3\xi \prod_{i=1}^4 \left(d^3\eta_i\frac{d^3k_i}{(2\pi)^3}\right)\ \chi_\Lambda(\xi)V(\eta_1,\eta_2,\eta_3,\eta_4)\delta(\Sigma\eta_i)\bar\psi(k_1)\psi(k_3)\bar\psi(k_2)\psi(k_4)\\
&\qquad\qquad e^{-i<k_1,\xi+\eta_1>_-}e^{i<k_3,\xi+\eta_3>_-}e^{-i<k_2,\xi+\eta_2>_-}e^{i<k_4,\xi+\eta_4>_-}\\
&= \frac{1}{2}\int \prod_{i=1}^4 \frac{d^3k_i}{(2\pi)^3}\ \tilde\chi_\Lambda(-k_1-k_2+k_3+k_4) < k_1,k_2|V|k_3,k_4 > \bar\psi(k_1)\psi(k_3)\bar\psi(k_2)\psi(k_4)\\
&= \frac{1}{2}\sum_{m\in\mathbf{Z}^3}\int \prod_{i=1}^4 \frac{d^3k_i}{(2\pi)^3}\ \chi(k_1+k_2-k_3-k_4+M^jm)\tilde\chi_\Lambda(-k_1-k_2+k_3+k_4)\\
&\qquad\qquad < k_1,k_2|V|k_3,k_4 > \bar\psi(k_1)\psi(k_3)\bar\psi(k_2)\psi(k_4)\ .
\end{aligned}
$$

Here

$$
1 = \sum_{m\in\mathbf{Z}^3}\chi(M^{-j}k+m)
$$

is a partition of momentum space by C_0^∞ functions supported on cubes of side M^j and

$$
\begin{aligned}
< k_1,k_2|V|k_3,k_4 > = \int \prod_{i=1}^4 d^3\eta_i V(\eta_1,\eta_2,\eta_3,\eta_4)\delta(\eta_1+\eta_2+\eta_3+\eta_4)\\
e^{-i<k_1,\eta_1>_-}e^{i<k_3,\eta_3>_-}e^{-i<k_2,\eta_2>_-}e^{i<k_4,\eta_4>_-}\ .
\end{aligned}
$$

Reversing the calculation above expresses

$$
\begin{aligned}
\mathcal{V} &= \sum_{m\in\mathbf{Z}^3}\frac{1}{2}\int d^3\xi \prod_{i=1}^4 d^3\eta_i\ \chi_{m,\Lambda}(\xi)V(\eta_1,\eta_2,\eta_3,\eta_4)\delta(\Sigma\eta_i)\bar\psi(\xi+\eta_1)\psi(\xi+\eta_3)\bar\psi(\xi+\eta_2)\psi(\xi+\eta_4)\\
&= \sum_{m\in\mathbf{Z}^3}\frac{1}{2}\int \prod_{i=1}^4 d^3\xi_i\ \chi_{m,\Lambda}\left(\frac{\xi_1+\xi_2+\xi_3+\xi_4}{4}\right)V(\xi_1,\xi_2,\xi_3,\xi_4)\bar\psi(\xi_1)\psi(\xi_3)\bar\psi(\xi_2)\psi(\xi_4)
\end{aligned}
\tag{III.2}
$$

where $\chi_{m,\Lambda}$ is the inverse Fourier transform of $\chi(M^{-j}k+m)\tilde\chi_\Lambda(k)$. Even if the interaction V is of compact support, with respect to the center of mass, $\chi_{m,\Lambda}V$ is no longer supported in a compact neighborhood of Λ because χ is of compact support in momentum space. This is no problem. The vertex is already connected by propagators to the external generalized vertex $\mathcal{A}$, so integration over the center of mass variable is taken care of.

Only two properties of $\chi_{m,\Lambda}$ are required. The first is

$$
\begin{aligned}
\sup_{\xi\in\mathbb{R}^3}|\chi_{m,\Lambda}(\xi)| &\le \sup_\xi\left|M^{3j}\int d^3\eta\ e^{i<\eta,M^jm>_-}\tilde\chi(M^j\eta)\chi_\Lambda(\xi-\eta)\right|\\
&\le \mathrm{const}_N\,(1+|m|)^{-N}
\end{aligned}
\tag{III.3}
$$

independent of j, provided derivatives acting on χ_Λ produce, at worst, M^j's. In other words χ_Λ must decay from 1 down to zero in a distance $O(M^{-j})$. This is a consequence of the standard integration by parts trick, as in the proof of Lemma 4, and

$$\left|\nabla_\eta{}^b \widetilde{\chi}(\eta)\right| \leq \text{const}^{\,b} \, \frac{1}{(1+|\eta|)^4} \ .$$

Once conservation of momentum at a vertex has been implemented by (III.2), each field at the vertex is expanded in sectors. Recall that the vertex was produced by an application of the integration by parts formulae (I.6) to a source field. So, one of its legs inherits its sector number from this source field. A priori, decomposing each of the three remaining fields into a sum over sectors could produce $M^{-3j/2}$ terms. The second property of $\chi_{m,\Lambda}$ is that there are only $O(|m|^2 M^{-j/2})$ nonzero terms because of the constraints imposed by conservation of momentum and the fact that we are in two space dimensions. See Lemma 3.

Finally, each field $\overset{(-)}{\psi}{}^{(\ell)}$ is expanded in boxes $\Delta \in \mathbf{D}_\ell$. Altogether, the "sum over sectors and conjugate boxes" operation replaces (I.7) by

$$\mathcal{V} = \sum_{m\in\mathbf{Z}^3} \sum_{\ell_1,\ell_2,\ell_3,\ell_4} \frac{1}{2} \prod_{i=1}^{4} \left(\sum_{\Delta_i \in \mathbf{D}_{\ell_i}} \int_{\Delta_i} d^3\xi_i \right) \ \chi_{m,\Lambda}\left(\frac{\xi_1+\xi_2+\xi_3+\xi_4}{4}\right)$$
$$\times\, V(\xi_1,\xi_2,\xi_3,\xi_4)\bar{\psi}^{(\ell_1)}(\xi_1)\psi^{(\ell_3)}(\xi_3)\bar{\psi}^{(\ell_2)}(\xi_2)\psi^{(\ell_4)}(\xi_4) \tag{III.4}$$

In step $s = 0$ the only fields downstairs are those belonging to the monomial $\mathcal{A}$. One decomposes each field in sectors Σ_ℓ and then in conjugate boxes $\Delta \in \mathbf{D}_\ell$. There is no need for the sum over m associated with conservation of momentum.

We describe the Taylor expansions in more detail. Now there are sectors to be taken into account. Let $\pi(s+1,\ell,\Delta,\sigma,b)$ be the number of fields with the specified values of σ, b, ℓ and $\Delta \in \mathbf{D}_\ell$. For each ℓ,σ,b and $\Delta \in \mathbf{D}_\ell$ we divide Δ into $\pi(s+1,\ell,\Delta,\sigma,b)^\epsilon$ identical sub-boxes (really rectangular parallelepipeds) δ each similar to Δ. Thus each little box has dimensions

$$\pi(s+1,\ell,\Delta,\sigma,b)^{-\frac{\epsilon}{3}} M^{-j} \times \pi(s+1,\ell,\Delta,\sigma,b)^{-\frac{\epsilon}{3}} M^{-j} \times \pi(s+1,\ell,\Delta,\sigma,b)^{-\frac{\epsilon}{3}} M^{-j/2}.$$

The value of ϵ will be picked later. As in §I the center of δ is called c_δ and the number of fields in δ with specified values of ℓ, σ and b is called $\pi(s+1,\ell,\delta,\sigma,b)$.

The action of derivatives, given by Lemma 4.b, differs from (I.2) in several respects. In particular the appearance of the phase $e^{-i<r_\ell,\xi-\bar{\xi}>_-}$ forces us to modify the Taylor expansion. Based on this remark we define

$$\begin{aligned}\Psi^{(\ell)}(\xi) &= e^{-i<\mathbf{r}_\ell,\xi>}\psi^{(\ell)}(\xi)\\ \bar{\Psi}^{(\ell)}(\xi) &= e^{i<\mathbf{r}_\ell,\xi>}\bar{\psi}^{(\ell)}(\xi).\end{aligned}$$

We expand each $\Psi^{(\ell)}(\xi)$, $\bar{\Psi}^{(\ell)}(\xi)$ downstairs in a Taylor polynomial

$$\begin{aligned}\overset{(-)}{\Psi}{}^{(\ell)}(\xi) = \overset{(-)}{\Psi}{}^{(\ell)}(c_\delta) + \ \cdots\ &+ \frac{1}{t!}\big((\xi-c_\delta)\cdot\nabla_\xi\big)^t\ \overset{(-)}{\Psi}{}^{(\ell)}(c_\delta)\\ &+ \frac{1}{t!}\int_0^1 dw(1-w)^t\big((\xi-c_\delta)\cdot\nabla_\xi\big)^{t+1}\ \overset{(-)}{\Psi}{}^{(\ell)}(c_\delta+w(\xi-c_\delta))\end{aligned} \tag{III.5}$$

with δ being the box that contains ξ. As before, the Taylor expansion is done inside the ξ integrals. One might worry that the restriction to a very small box δ weakens conservation of momentum and consequently increases the number of nonzero terms in the sector sums. However, the sector sums have already been cut down, so there is no problem.

Each Taylor expansion contains

$$1+3+3^2+\cdots+3^t+3^{t+1}\le 2\times 3^{t+1}$$

terms. Note that, by Lemma 4.b, each $(\xi-c_\delta)\cdot\nabla_\xi$ that acts on a $\overset{(-)}{\Psi}{}^{(\ell)}$ produces a factor of

$$\pi(s+1,\ell,\Delta,\sigma,b)^{-\frac{\epsilon}{3}}\left[M^{-j}M^j+M^{-j}M^j+M^{-j/2}M^{j/2}\right]=3\pi(s+1,\ell,\Delta,\sigma,b)^{-\frac{\epsilon}{3}}.$$

Since the fields anticommute, any nonzero integral may contain at most one field having any given value of ℓ,σ,b at any c_δ. The same is true for each derivative of the fields. Thus, for each ℓ,σ,b,δ, all but 2×3^t of the fields having this ℓ,σ,b and located in δ must be Taylor remainders. That is, there are at least $\pi(s+1,\ell,\delta,\sigma,b)-2\times 3^t$ Taylor remainders. This completes the description of the expansion.

We use the same strategy for performing the estimates as in Lemma 5. Here are the combinatorial factors used to control each of the three operations.

(1) *Integration by parts.*

- A factor of two, assigned to the leg initiating the integration by parts suffices to decide whether or not the leg brings down a new vertex from the exponent.

When it does,

- a factor of two, assigned to the target leg, counts the number of possible target legs.

When it doesn't, we count the number of possible source legs for each target leg by applying the rules of Lemma 5 in each sector ℓ. Precisely, for each target leg, we organize the sum over source legs according first, to the cube $\Delta \in \mathbf{D}_\ell$ of the source and second, to which leg in Δ is the source.

- A factor of two per leg suffices to decide whether or not the leg is a target leg.
- A factor of $\text{const}\, \rho^{(j,\ell)}(\Delta, \Delta')^4$, assigned to the line generated, suffices to control the sum over Δ.
- A factor $\pi(s+1, \ell, \Delta, \sigma, b)$, assigned to the source leg, counts the number of possible source legs within Δ.

(2)*Sums over sectors and conjugate boxes.* The sum $\sum_{m\in\mathbb{Z}^3} \cdots \chi_{m,\Lambda}$ is controlled by

- a factor $\text{const}\, |m|^4$ assigned to the vertex.

When a vertex first moves downstairs from the exponent the sector of one of its legs is fixed by the source leg that initiates the process. By Lemma 3 the number of sectors accessible to the remaining legs is bounded by

- a factor $\text{const}\,(1+|m|^{4/3})M^{-j/2}$ assigned to the vertex.
- If the external vertex does not impose any constraint on the sector sums of its $2p$ legs then the number of sectors accessible is M^{-pj}.

The sums $\sum_{\Delta_i \in D_\ell}$ are not controlled by combinatorial factors. They will be performed explicitly.

(3)*Pauli exclusion principle.*

- The Taylor expansion splits each leg into at most $2\times 3^{t+1}$ pieces.

It remains only to bound an integral. The integration variables are the positions of the fields of $\mathcal{A}$ and the interaction vertices $\mathcal{V}$ that have been brought down from the exponent. The integrand is the supremum, over positions and diagrams, of the product of

- the above combinatorial factors

- const $\rho^{(j,\ell)}(\Delta,\Delta')^{-\gamma}$ per line of the diagram
- $|\lambda|\left(\sup_{\xi\in\mathbb{R}^3}|\chi_{m,\Lambda}(\xi)|\right)|V(\xi_1,\xi_2,\xi_3,\xi_4)| \le \mathrm{const}|\lambda|\,(1+|m|)^{-N}\,|V(\xi_1,\xi_2,\xi_3,\xi_4)|$ per vertex of the diagram (see (III.3))
- $|A(\xi_1,\cdots,\xi_{2p})|$
- and the factors that come from the Taylor expansions used to implement the Pauli exclusion principle.

The latter are

$$\begin{aligned}&\prod_s\prod_{\ell,\sigma,b}\prod_{\Delta\in\mathbf{D}_\ell}\prod_{\delta\in\Delta}\left[3\pi(s,\ell,\Delta,\sigma,b)^{-\frac{\epsilon}{3}}\right]^{(t+1)\left(\pi(s,\ell,\delta,\sigma,b)-2\times 3^t\right)}\\ &\le\prod_s\prod_{\ell,\sigma,b}\prod_{\Delta\in\mathbf{D}_\ell}\left[3\pi(s,\ell,\Delta,\sigma,b)^{-\frac{\epsilon}{3}}\right]^{(t+1)\left(\pi(s,\ell,\Delta,\sigma,b)-2\times 3^t\pi(s,\ell,\Delta,\sigma,b)^{\epsilon}\right)}\ .\end{aligned}\tag{III.6}$$

Pick any $\zeta>0$. It is possible to choose ϵ and t, depending only on ζ so that

$$\text{(III.6)}\le\prod_{\mathrm{s}}\prod_{\ell,\sigma,\mathrm{b}}\prod_{\Delta\in\mathbf{D}_\ell}\mathrm{const}\,\pi(\mathrm{s},\ell,\Delta,\sigma,\mathrm{b})^{-\zeta\pi(\mathrm{s},\ell,\Delta,\sigma,\mathrm{b})}.$$

Altogether

$$|a_n(j,\Lambda)|\le\mathrm{const}^{\,n+p}\int\prod_i d\xi_i|A|\left(\prod|V|\right)\left(\prod\rho^{(j,\ell)}(\Delta,\Delta')^{-\gamma+4}\right)M^{\frac{3}{2}j\frac{4n+2p}{2}}M^{-jn/2}M^{-pj}\tag{III.7}$$

where the second product is over vertices and the third is over lines. The first power of M^j came from covariance bounds and the second and third from sector sums for fields at interaction vertices and $\mathcal{A}$ respectively. After discarding some of the decay factors $\rho^{(j,\ell)}$, we can view the integrand as a connected tree with generalized vertices V and A and lines $\rho^{(j,\ell)}$. Integrate starting at the extremities of the tree and working towards the root A. Suppose that the extremal vertex V is connected to the tree by the argument ξ_1. The integral over ξ_2,ξ_3,ξ_4 with ξ_1 held fixed produces a $\|V\|$. A decay factor $\rho^{(j,\ell)}$ is enough to fix the box in which ξ_1 lives. The integral over ξ_1 within that box costs $M^{-5j/2}$, the volume of the box. Repeat for all the other V's. Finally, the integral over the arguments of $\mathcal{A}$ gives $\|A\|_1$. In conclusion

$$\begin{aligned}|a_n(j,\Lambda)|&\le\mathrm{const}^{\,n+p}\|A\|_1\|V\|^nM^{-5jn/2}M^{3j(n+p/2)}M^{-jn/2}M^{-pj}\\&=\mathrm{const}^{\,n+p}\|A\|_1\|V\|^nM^{-pj/2}\ .\end{aligned}$$

When the finite volume, i.e. Λ, constraint is removed, a_n is expressed as an absolutely convergent series obeying the same bound. ■

The next order of business is the effective potential

$$\mathcal{G}^{(j)}_{\Lambda}(\psi^e,\bar\psi^e)=\log\frac{1}{Z^{(j)}_{\Lambda}}\int e^{-\lambda\mathcal{V}_{\Lambda}(\psi+\psi^e,\bar\psi+\bar\psi^e)}d\mu_{C^{(j)}}(\psi,\bar\psi).$$

Let $G^{(j,\Lambda)}_p$ be the p-point Green's function generated by $\mathcal{G}^{(j)}_{\Lambda}$. We now prove the

Theorem 1

Let $d=2$. There exists a const *, independent of j and Λ, such that*

$$|||g_n(p,j,\Lambda)|||\le \mathrm{const}^{\,n+p}M^{(2-5p/2)j}|||V|||^n$$

where

$$|||g_n(p,j,\Lambda)|||=\max_k\ \sup_{\xi_k}\int\prod_{i\neq k}d\xi_i\ |g_n(p,j,\Lambda)(\xi_1,\cdots,\xi_{2p})|.$$

Furthermore the limits

$$g_n(p,j)=\lim_{\Lambda\to\mathbb{R}^3}g_n(p,j,\Lambda)$$

exist and the infinite volume Green's functions at scale j

$$G^{(j)}_p=\sum_{n=0}^{\infty}g_n(p,j)\lambda^n$$

are analytic in $|\lambda|<R=(\mathrm{const}\,|||V|||)^{-1}$.

Proof : To get started apply a single functional derivative

$$\frac{\delta}{\delta\psi^e}\mathcal{G}^{(j)}_{\Lambda}=-\lambda\frac{\int\left(\frac{\delta}{\delta\psi^e}\mathcal{V}_{\Lambda}(\psi+\psi^e,\bar\psi+\bar\psi^e)\right)e^{-\lambda\mathcal{V}_{\Lambda}(\psi+\psi^e,\bar\psi+\bar\psi^e)}d\mu_{C^{(j)}}(\psi,\bar\psi)}{\int e^{-\lambda\mathcal{V}_{\Lambda}(\psi+\psi^e,\bar\psi+\bar\psi^e)}d\mu_{C^{(j)}}(\psi,\bar\psi)}\ .\qquad\text{(III.8)}$$

Now expand as in Lemma 6. However, before each application of the integration by parts formulae, decompose the downstairs of the numerator into a polynomial in $\psi,\psi^e,\bar\psi,\bar\psi^e$. That is, multiply out products of $\psi+\psi^e$ and $\bar\psi+\bar\psi^e$. The integration by parts formulae are then applied to the $\psi,\bar\psi$'s. The expansion terminates for those monomials that are independent of $\psi,\bar\psi$, i.e. that are functions of the external fields $\psi^e,\bar\psi^e$ alone. For these terms the monomial factors out of the numerator, leaving the quotient of two identical integrals that cancel.

The rest of the expansion and most of the estimates are the same as before. However, the external legs enter the effective potential in a way different from the Schwinger functions of Lemma 6.

The only combinatorial factors that change are those that count sectors. Lemma 3, the main tool for sector counting, does not apply to vertices containing external legs, since the momentum of the external leg need not be near the Fermi circle. It is still true that when a vertex first moves downstairs from the exponent the sector of one of its legs is fixed by the source leg that initiates the process. By Lemma 3 the number of sectors accessible to the remaining legs is bounded by

- a factor $\mathrm{const}\,(1+|m|^{4/3})M^{-j/2}$, assigned to the vertex, when the vertex contains no external legs.
- If the vertex does contain an external leg, there are at most two other internal legs. There are at most M^{-j} sectors accessible to these legs.
- The first external vertex was created by a functional derivative, rather than integration by parts. There are at most $M^{-3j/2}$ sectors accessible to its internal legs.

The main bound

$$\begin{aligned}&\|g_n(p,j,\Lambda)\|\\&\le \mathrm{const}^{\,n+p}\int\prod_i d\xi_i\ \Big(\prod|V|\Big)\Big(\prod\rho^{(j,\ell)}(\Delta,\Delta')^{-\gamma+4}\Big)M^{\frac{3}{2}j\frac{4n-2p}{2}}M^{-jn/2}M^{-pj}M^{-j/2}\end{aligned}\tag{III.7$'$}$$

is obtained from (III.7) by deleting $|A|$ and adjusting the powers of M^j. External lines are amputated, so that the number of lines $\frac{4n+2p}{2}$ becomes $\frac{4n-2p}{2}$, accounting for the first factor of M^j. The remaining powers of M^j come from the sector counts. We have included one sector sum per vertex, plus one "extra" sector sum per external leg, plus an additional supplement for the first vertex. Finally, by definition of the norm $\|\ \cdot\ \|$ the position of one argument ξ_j

of one vertex is held fixed, removing one volume integral $M^{-5j/2}$. Thus

$$\begin{aligned} \|g_n(p,j,\Lambda)\| &\le \text{const}^{\,n+p} \|V\|^n M^{-5j(n-1)/2} M^{3j(n-p/2)} M^{-jn/2} M^{-pj} M^{-j/2} \\ &= \text{const}^{\,n+p} \|V\|^n M^{(2-5p/2)j} \ . \end{aligned} \tag{III.9}$$

■

Under a renormalization group flow, the effective potential of scale j becomes the interaction at scale $j-1$. If we are to use an expansion like that of Theorem 1 in such a setting, we must allow the interaction to contain monomials of degree $2q$ for all $q \ge 1$, not just $q = 2$. On the other hand we must retain the memory that the $2q$-legged monomial began life as a bunch of four-legged monomials.

So, let's collect together some additional consequences, not stated in Theorem 1. First each graph contributing to $g(q,j)$ must have at least $\max\{1, q-1\}$ (four-legged) vertices so it comes with a power of at least $\lambda^{\max\{1,q-1\}}$, though the estimates will eat up a portion of this.

Second, the magnitude of $g(q,j)$ reflects the the power counting of a graph built from four-legged vertices. Consider any graph having v four-legged vertices and $2q$ amputated external legs. This graph has $2v-q$ propagators. The supremum of a single sector propagator in position space is $M^{3j/2}$. Each vertex, save one which is held fixed to break translation invariance, is integrated over all $\mathbb{R}^3$. Each such integral gives $M^{-5j/2}$. When $g(q,j)$ acts as an interaction at scale j or lower, the momenta of its external legs are restricted to lie within M^j of the Fermi surface. Then the sector counting Lemma 3 applies to both internal and external vertices. Roughly speaking, the sector counting Lemma says that the legs of a four-legged vertex are paired and that the sector of either leg of the pair determines that of the other leg of the pair. As the sector of one leg at each vertex is fixed by conservation of momentum, there is one sum over $M^{-j/2}$ sectors per vertex. This gives a total power counting factor, including the sector sums for the external legs, of

$$M^{(2v-q)3j/2} M^{-(v-1)5j/2} M^{-vj/2} = M^{(5-3q)j/2} \ .$$

The sector sums for the (amputated) external legs are only performed when propagators are later hooked onto them. It is convenient to save up the sector sums for external legs in a way that avoids having to distinguish between external/internal pairings and external/external pairings. So we leave the sum over sectors for each external *vertex* explicit, instead of bounding it by the number of terms times the size of the maximum term. Each term in the resulting expansion for the $2q$-point function then has the sectors of all external *legs* fixed. But it also has the sectors of all internal propagators hooked to the vertex fixed.

Third, approximate conservation of momentum increases the number of available sectors by a factor that depends on the degree of approximateness in the conservation of momentum. The degree to which $g(q,j)$ conserves momentum depends on the degree to which its internal vertices conserve momentum. Suppose that the original model is supported in a volume of size $|\Lambda| = M^{-3J}$ and that the cutoff function decays smoothly to zero in a distance M^{-J}. When we apply (III.4) each vertex v in $g(q,j)$ is assigned a number E_v with the property that the vertex is zero unless the sum of the momenta feeding into it is bounded by $E_v M^J$. Consequently, $g(q,j)$ is zero unless the sum of the momenta feeding into it is bounded by $\sum_v E_v M^J$.

Based on the motivation of the last paragraph we now consider interactions of the form

$$\mathcal{V}_\Lambda = \sum_{j'>j} \sum_{q=1}^{\infty} \int \prod_{i=1}^{q} d\xi_i d\bar{\xi}_i \; V_{\Lambda,q}^{(j')}(\xi_1, \bar{\xi}_1, \cdots, \xi_q, \bar{\xi}_q) \prod_{i=1}^{q} \bar{\psi}(\bar{\xi}_i)\psi(\xi_i) \tag{III.10a}$$

when evaluating the effective potential with covariance of scale j. The term $\mathcal{V}_{\Lambda,2}^{(0)}$ in this sum is the ordinary vertex (III.2) with four fields. Each kernel is expressed as a sum

$$V_{\Lambda,q}^{(j')} = \sum_{m \in \mathcal{M}_q} V_{\Lambda,q,m}^{(j')} \; . \tag{III.10b}$$

The index j' gives the scale at which the graphs contributing to $V_{\Lambda,q,m}^{(j')}$ were formed. You should think of m as measuring the extent to which exact conservation of momentum fails as well as giving the sectors of some internal lines. In particular,

- there is an $E_{q,m}^{(j')} \in \mathbb{Z}$ such that $V_{\Lambda,q,m}^{(j')}$ is zero unless the sum of the momenta flowing into it is bounded by $E_{q,m}^{(j')} M^J$ (III.10c)

We assume that there is an $0 < \alpha < 1$ such that, when all the arguments of $V_{\Lambda,q}^{(j')}$ have momentum within const M^j of the Fermi surface and when one argument lies in a fixed sector of scale j,

- the remaining $2q-1$ legs of each nonzero $V^{(j')}_{\Lambda,q,m}$ are supported in $C_1^q \exp\left\{\left(E^{(j')}_{q,m}M^{J-j}\right)^\alpha\right\} M^{(2q-3)(j'-j)/2}K^{(j')}_{q,m}$ sector $(2q-1)$-tuples of scale j.

(III.10d)

That is, $K^{(j')}_{q,m}$ is the number of sector $(2q-1)$-tuples at scale j' when the kernel was first generated. However, a sector that is introduced with width $M^{j'/2}$ is subdivided into $M^{(j'-j)/2}$ sectors at scale j. The number of accessible sectors of scale j for the $2q-1$ remaining external legs given their sectors at scale j' is bounded by $const^q(1+E^{(j')}_{q,m}M^{J-j})^2M^{(2q-3)(j'-j)/2}$ in Lemma 3. It is convenient for us to use the multiplicative property of the exponential so we bound $(1+EM^{J-j})^2 \le \text{const}\ \exp\left\{\left(EM^{J-j}\right)^\alpha\right\}$.

We further assume that the kernel is analytic in the region $|\lambda| < R$ and obeys the bound

$$\sum_{m\in\mathcal{M}_q} \exp\left\{\sum_{\imath=J}^{j'-1}\left(E^{(j')}_{q,m}M^{J-\imath}\right)^\alpha\right\} K^{(j')}_{q,m}|||V^{(j')}_{\Lambda,q,m}||| \le K_1|\lambda|^{q/2}M^{\frac{1}{2}(5-3q)j'} \quad . \tag{III.10e}$$

throughout that region. Note that λ times the vertex (III.2) is of this form, with

$$\begin{aligned}
V^{(j')}_{\Lambda,q} &= 0 \text{ for all } q\ne 2,\ j'\ne 0\\
\mathcal{M}_2 &= \mathbb{Z}^3\\
V^{(0)}_{\Lambda,2,m} &= \frac{1}{2}\chi_{m,\Lambda}V\\
K^{(0)}_{2,m} &= \text{const}\\
E^{(0)}_{2,m} &= |m_1|+|m_2|+|m_3|\\
|||V^{(0)}_{\Lambda,2,m}||| &\le |\lambda|\text{const}\ e^{-\text{const}\ |m|^{1/2}}|||V|||\\
\alpha &= \frac{1}{3}\\
R &= \infty
\end{aligned}$$

We are making a somewhat stronger hypothesis on the cutoff function $\chi_{m,\Lambda}$ here than in (III.3). The standard C^∞ compactly supported functions obey this stronger hypothesis. To be precise let

$$\begin{aligned}
\chi_1(x) &= \begin{cases} 0 & x\le 0\\ e^{-x^{-2}}e^{-(x-1)^{-2}} & 0<x<1\\ 0 & x\ge 1\end{cases}\\
\chi_2(x) &= \left[\int_0^1 \chi_1(t)dt\right]^{-1}\int_0^x \chi_1(t)dt\\
\chi_3(x) &= \chi_2(x+2)\chi_2(-x-2)
\end{aligned}$$

and finally

$$\chi_\Lambda(\xi) = \prod_{i=0}^{2} \chi_3(M^J \xi_i) \ . \tag{III.11}$$

Then

$$\sup_{x\in\mathbb{R}} \left| \frac{d^n}{dx^n} \chi_3(x) \right| \le \text{const}^{\,n}\, n^{3n/2}$$

so that the Fourier transform

$$|\widetilde{\chi}_3(k)| \le 4 \frac{\text{const}^n\, n^{3n/2}}{k^n}$$

for all even $n \ge 0$. Let $\beta > 3/2$. Choosing n to be the even integer nearest $|k|^{1/\beta}$ yields

$$\begin{aligned} |\widetilde{\chi}_3(k)| &\le \text{const}^n \, \frac{n^{3n/2}}{n^{\beta n}} \\ &\le e^{-(\beta-\frac{3}{2})n \ln n + n \ln \text{const}} \\ &\le \text{const}\, e^{-(\beta-\frac{3}{2})n} \\ &\le \text{const}\, e^{-\text{const}\,|k|^{1/\beta}} \ . \end{aligned}$$

We know from the perturbative analysis of [FT2] that two and four point interaction vertices which have internal scales j' higher than the external scale j have to be renormalized. This problem will be treated in a later paper. But we can already state a rigorous result if we limit ourselves to the part of the theory containing only convergent graphs. This part of the model is the sum of all graphs that have no two or four point subgraphs with the scales assigned to their internal lines higher than the scales assigned to their external lines. This was called completely positive power counting case in [Ri]. To isolate this part of the model it suffices to require that, with the exception of $q = 2,\ j' = 0$, all $V^{(j')}_{\Lambda,q}$ with $q \le 2,\ j' > j$ be zero.

Theorem 2

Let $\mathcal{V}$ obey (III.10) *and let* $V^{(j')}_{\Lambda,q} = 0$ *for* $q = 1,\ j' > j$ *and* $q = 2,\ 0 > j' > j$. *Then, if M is large enough, there is an $R_0(M, K_1) > 0$ such that the effective potential $\mathcal{G}$ obeys* (III.10) *for all* $|\lambda| < \min(R, R_0)$.

Proof : The expansion and bounds are the same as for Theorem 1, with the exception that the vertices are more complicated and that we leave the sums associated with external vertices explicit. We use a tilde to designate index sets and constants that refer to the effective potential. There is, of course the trivial graph in which a $V^{(j')}_{\Lambda,q}$ gets fed directly into the effective potential as a single vertex graph. All nontrivial contributions get put into $\widetilde{V}^{(j)}_{\Lambda,p,m}$'s. The new index set

$$\widetilde{\mathcal{M}}_p = \mathbb{Z}^{>0} \bigtimes_{v=1}^{2p} \bigcup_{j_v > j} \bigcup_{q_v=0}^{\infty} \bigcup_{\substack{m_v \in \mathcal{M}_{q_v} \\ I \subset \{1,\cdots,2q_v\}}} \{\text{sector assignments to the lines } I \text{ of } V^{(j_v)}_{\Lambda,q_v,m_v}\}$$

Here v labels the external vertices of terms contributing to $\widetilde{V}^{(j)}_{\Lambda,p,m}$. In the event that there are fewer than $2p$ external vertices, the extra q_v's are set to zero. The set I selects the legs of the vertex v that will be internal to the $2p$-point function. We shall denote by $\imath_v$ and e_v the number of legs of the vertex v that end up being internal and external legs, respectively, of $\widetilde{V}^{(j)}_{\Lambda,p,m}$. Suppose, for example, that the first seven vertices brought down from the exponent end up being internal, but that the eighth ends up being a $V^{(j_1)}_{\Lambda,q_1,m_1}$ with the legs in I_1, a proper subset of $\{1, \cdots, 2q_1\}$, being internal and of sectors $\ell_1, \cdots, \ell_{|I_1|}$. Then this term will contribute to a $\widetilde{V}^{(j)}_{\Lambda,p,m}$ with the second component (out of $2p+1$ components) of m being $(j_1, q_1, m_1, I_1, \ell_1, \cdots, \ell_{|I_1|})$. The new conservation of momentum index is

$$E^{(j)}_{p,m} = E + \sum_v E^{(j_v)}_{q_v,m_v}$$

where E, the first of the $2p+1$ components of m, is the sum of the conservation of momentum indices of all internal vertices.

The main bound (III.9) of Theorem 1 is replaced by

$$\|\widetilde{V}^{(j)}_{\Lambda,p,m}\| \le \prod_{\substack{\text{internal}\\ \text{vertices}}} \left(\sum_{j_v>j} \sum_{q_v} \sum_{m_v \in \mathcal{M}_{q_v}} C_1^q \exp\left\{ E^{(j_v)^\alpha}_{q_v,m_v} M^{(J-j)\alpha} \right\} M^{(q_v-\frac{3}{2})(j_v-j)} K^{(j_v)}_{q_v,m_v} \|V^{(j_v)}_{\Lambda,q_v,m_v}\| \right)$$
$$\left(\prod_{\substack{\text{external}\\ \text{vertices}}} \|V^{(j_v)}_{\Lambda,q_v,m_v}\| \right) M^{-5j(n-1)/2} M^{\frac{3}{2}j(\Sigma 2q_v-2p)/2} K_2^{\Sigma 2q_v-2p} K_3^{\Sigma 2q_v} \tag{III.9$'$}$$

The constant K_2 includes the constants arising from bounding the propagator and summing or taking the supremum of $\rho^{(j,\ell)}(\Delta,\Delta')^{-\gamma+4}$. The constant K_3 includes the combinatorial factors associated with propagators and external legs. The latter include a factor of two for deciding whether or not a leg contracted to the exponent, a factor of two to decide which leg was the target leg, in the event that there was contraction to the exponent (note that $q \le 2^q$) and a factor of $(2\times 3^{t+1})^2$ from the Taylor expansion. The differences between K_2 and K_3 are that the former applies only to propagators, i.e. internal legs, while the latter is independent of M.

Moving around the powers of M

$$\begin{aligned}
&M^{-5j(n-1)/2} M^{\frac{3}{2}j(\Sigma 2q_v-2p)/2} \prod_{\substack{\text{internal}\\ \text{vertices}}} M^{(q_v-\frac{3}{2})(j_v-j)} \\
&= M^{\frac{1}{2}(5-3p)j} \prod_{\substack{\text{internal}\\ \text{vertices}}} M^{(q_v-\frac{3}{2})(j_v-j)} M^{-\frac{1}{2}(5-3q_v)j} \prod_{\substack{\text{external}\\ \text{vertices}}} M^{-\frac{1}{2}(5-3q_v)j} \\
&= M^{\frac{1}{2}(5-3p)j} \prod_{\substack{\text{internal}\\ \text{vertices}}} M^{\frac{1}{2}(2-q_v)(j_v-j)} M^{-\frac{1}{2}(5-3q_v)j_v} \prod_{\substack{\text{external}\\ \text{vertices}}} M^{-\frac{1}{2}(5-3q_v)j}
\end{aligned}$$

we end up with

$$\|\widetilde{V}^{(j)}_{\Lambda,p,m}\| \le \prod_{\substack{\text{internal}\\ \text{vertices}}} \left(\sum_{j_v>j} \sum_{q_v} \sum_{m_v \in \mathcal{M}_{q_v}} M^{\frac{1}{2}(2-q_v)(j_v-j)} M^{-\frac{1}{2}(5-3q_v)j_v} \exp\left\{ E^{(j_v)^\alpha}_{q_v,m_v} M^{(J-j)\alpha} \right\} \right.$$
$$\left. (C_1 K_2^2 K_3^2)^{q_v} K^{(j_v)}_{q_v,m_v} \|V^{(j_v)}_{\Lambda,q_v,m_v}\| \right)$$
$$\left(\prod_{\substack{\text{external}\\ \text{vertices}}} M^{-\frac{1}{2}(5-3q_v)j} K_2^{\imath_v} K_3^{2q_v} \|V^{(j_v)}_{\Lambda,q_v,m_v}\| \right) M^{\frac{1}{2}(5-3p)j}$$

Now summing over $m \in \widetilde{\mathcal{M}}_p$ entails, for each external vertex, a factor of two per line (to decide whether it is internal or not), a sum over j_v and m_v and a sum over sector assignments to the lines of $V^{(j_v)}_{\Lambda,q_v,m_v}$ that ended up being internal to $\widetilde{V}^{(j)}_{\Lambda,p,m}$. Including a factor $\widetilde{K}^{(j)}_{p,m}$ accounts for a sum over sector assignments to the lines that ended up external. So there is a sum over assignments to all legs and, since

$$E^{(j)^\alpha}_{q,m} \le \sum_{\substack{\text{all}\\ \text{vertices}}} E^{(j_v)^\alpha}_{q_v,m_v}\ ,$$

the left hand side of (III.10a) for $\widetilde{V}^{(j)}_{\Lambda,p,m}$ is bounded by

$$\begin{aligned}
&M^{-\frac12(5-3p)j}\sum_{m\in\widetilde{\mathcal{M}}_p}\exp\left\{\sum_{\imath=J}^{j-1}\left(E^{(j)^\alpha}_{q,m}M^{J-\imath}\right)\right\}\widetilde{K}^{(j)}_{p,m}|||\widetilde{V}^{(j)}_{\Lambda,p,m}|||\\
&\quad\le\prod_{\substack{\text{all}\\ \text{vertices}}}\Bigg(\sum_{j_v>j}\sum_{q_v}\sum_{m_v\in\mathcal{M}_{q_v}}M^{\frac12(2-q_v)(j_v-j)}M^{-\frac12(5-3q_v)j_v}\exp\left\{\sum_{\imath=J}^{j}\left(E^{(j_v)^\alpha}_{q_v,m_v}M^{J-\imath}\right)\right\}\\
&\qquad\qquad K_2^{\imath_v}(2C_1K_3^2)^{q_v}K^{(j_v)}_{q_v,m_v}|||V^{(j_v)}_{\Lambda,q_v,m_v}|||\Bigg)\\
&\quad\le\prod_{\substack{\text{all}\\ \text{vertices}}}\left(\sum_{j_v>j}\sum_{q_v}M^{\frac12(2-q_v)(j_v-j)}K_1K_2^{\imath_v}(2C_1K_3^2)^{q_v}|\lambda|^{q_v/2}\right)\\
&\quad\le|\lambda|^{p/2}\prod_{\substack{\text{all}\\ \text{vertices}}}\left(\sum_{j_v>j}\sum_{q_v}M^{-\frac14(j_v-j)}M^{\frac14(2-q_v)}K_1K_2^{\imath_v}(2C_1K_3^2)^{q_v}|\lambda|^{\imath_v/4}\right)\\
&\quad\le|\lambda|^{p/2}\prod_{\substack{\text{all}\\ \text{vertices}}}K_1M^{1/2}|\lambda|^{1/8}\left(\sum_{j_v>j}\sum_{q_v}M^{-\frac14(j_v-j)}M^{-q_v/4}K_2^{\imath_v}(2C_1K_3^2)^{q_v}|\lambda|^{\imath_v/8}\right)
\end{aligned}$$

For the second inequality we used $j_v > j$. For the third we also used $q_v \ge 3$. Since C_1 and K_3 are independent of M we can choose M so that $2C_1K_3^2 \le M^{1/8}$. Then, if $|\lambda|^{1/8} \le K_2^{-1}$,

$$\begin{aligned}
&M^{-\frac12(5-3p)j}\sum_{m\in\widetilde{\mathcal{M}}_p}\exp\left\{\sum_{\imath=J}^{j-1}\left(E^{(j)^\alpha}_{q,m}M^{J-\imath}\right)\right\}\widetilde{K}^{(j)}_{p,m}|||\widetilde{V}^{(j)}_{\Lambda,p,m}|||\\
&\qquad\le|\lambda|^{p/2}\prod_{\substack{\text{all}\\ \text{vertices}}}K_1M^{1/2}|\lambda|^{1/8}\left(\sum_{j_v>j}\sum_{q_v}M^{-\frac14(j_v-j)}M^{-q_v/8}\right)\\
&\qquad\le|\lambda|^{p/2}\prod_{\substack{\text{all}\\ \text{vertices}}}K_1M^{1/2}\,\frac{M^{-\frac14}}{1-M^{-\frac14}}\,\frac{M^{-\frac58}}{1-M^{-\frac18}}\,|\lambda|^{1/8}\\
&\qquad\le|\lambda|^{p/2}
\end{aligned}$$

provided $|\lambda|$ is small enough. ■

Using Theorem 2 inductively, we conclude that the sum over all scales of the completely convergent graphs contributing to any given Green's function is analytic in λ at $\lambda = 0$.

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