All exactly solvable quantum spin 1 chains from Hecke algebra

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Abstract

We obtain all exactly integrable spin 1 quantum chains, which satisfy the Hecke algebra and discuss their solution via Bethe Ansatz methods. We also present various generalizations for arbitrary spin S.

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1 Introduction

The study of exactly soluble models in two dimensions has been a profound inspiration and testing ground for many new ideas in several areas of physics. Eventually the straightjacket of two dimensions has to be broken through. In order to achieve this aim, the essential aspects and techniques used in solving these models, should be brought to light as clearly and economically as possible.

A royal road to obtain new exactly soluble models consists in solving the Yang-Baxter [0] factorization equations. The existence of an infinite number of conservation laws being now guaranteed, one proceeds then aiming at an exact solution. We discuss some aspects of this endeavour, such as obtaining at least part of the spectrum through Bethe Ansatz (BA) methods. The outline of this paper is as follows. In section 2, we present for the readers benefit, well known algebraic structures, allowing one to set up Yang-Baxter (YB) equations. In sections 3 and 4 we discuss the solutions of these equations wi th $U(1) \otimes U(1)$ and $U(1)$ symmetry respectively. In section 5 we exhibit the BA for the models with $U(1) \times U(1)$ sym metry.

2 Hecke algebra and its baxterization

In this section we briefly discuss, how to obtain exactly soluble models satisfying YB equations from hamiltonians obeying certain algebraic constraints, generating in this way a subset of soluble spin 1 models.These will be spin 1 generalizations of the spin 1/2 six vertex models.

First we require nearest neighbor interactions and rotational $U(1)$ invariance around the z -axis. Secondly, our hamiltonians are required to satisfy the *Hecke*-algebra. This second requirement guarantees the availability of YB equations, which in turn provide an infinite number of conservation laws.

This is not the most general algebraic setting providing exact solubility, but we restrict ourselves to this case for simplicity and leave more general situations for a future publication.

The most general nearest neighbor $U(1)$-invariant hamiltonian can be parametrized as follows :

$$\begin{gathered} g_{i+1/2} = A_1 + A_2 S_z(i) + A_3 S_z(i+1) + A_4 S_z^2(i) + A_5 S_z^2(i+1) + \\ A_6 S_z(i) S_z(i+1) + A_7 S_z^2(i) S_z(i+1) + A_8 S_z(i) S_z^2(i+1) + A_9 S_z^2(i) S_z^2(i+1) + \\ A_{10} S_-(i) S_+(i+1) + A_{11} S_+(i) S_-(i+1) + A_{12} S_-^2(i) S_+^2(i+1) + A_{13} S_+^2(i) S_-^2(i+1) + \\ A_{14} S_+(i) S_z(i+1) S_-(i+1) + A_{15} S_z(i) S_+(i) S_-(i+1) + A_{16} S_-(i) S_z(i+1) S_+(i+1) + \\ A_{17} S_-(i) S_z(i+1) S_+(i+1) + A_{18} S_z(i) S_-(i) S_+(i+1) + + A_{19} S_z(i) S_+(i) S_z(i+1) S_-(i+1), \end{gathered} \tag{1}$$

where the index on g reminds us, that $g_{i+1/2}$ operates in the three-dimensional complex vector spaces $\mathbf{C_i}$, resp. $\mathbf{C_{i+1}}$ at posi tions i and $i+1$. We use the usual spin 1 matrices

$$S_z = [1,1] - [3,3], S_- = [2,1] + [3,2], S_+ = [1,2] + [2,3],$$

where the symbol $[a,b]$ means the matrix, whose i,j element is :

$$([a,b])_{i,j} \equiv \delta_{a,i} \delta_{b,j}. \tag{2}$$

If we take the direct product of these matrices :

$$[ab|cd] \equiv [a,b] \otimes [c,d], \tag{3}$$

we may conveniently express the 9×9 matrix g as :

$$g = \sum_{a,b,c,d=1}^{3} c_{ab|cd}[ab|cd], \tag{4}$$

where the coefficients $c_{ab|cd}$ are linearly related to the A_i and due to the $U(1)$-symmetry, only 19 of them don't vanish. We now require the matrices g to satisfy the following Hecke algebra

$$\begin{aligned} g_{i+1/2}\, g_{i+3/2}\, g_{i+1/2} &= g_{i+3/2}\, g_{i+1/2}\, g_{i+3/2}, \\ g^2_{i+1/2} + I_9 &= x\, g_{1+1/2}, \\ [g_{i+1/2}, g_{j+1/2}] &= 0, \qquad for\ \ |i-j| \geq 2 \end{aligned} \tag{5}$$

where I_9 is the 9-dimensional unit matrix and x is a *deformation* parameter of the algebra, allowing for the presence of a coupling constant occurring in the hamiltonian. We will also use the parametrization $x = q/\imath + \imath/q$, $q = \exp(\imath\gamma)$, where γ is in general a complex number.

It is also convenient to express the operators $g_{i+1/2}$ in terms of *Temperley-Lieb* operators :

$$e_{i+1/2} = q - \imath g_{i+1/2}, \tag{6}$$

satisfying

$$\begin{aligned} e^2_{i+1/2} &= \beta e_{i+1/2}, \\ e_{i+1/2}\ e_{i+3/2}\ e_{i+1/2} - e_{i+1/2} &= e_{i+3/2}\ e_{i+1/2}\ e_{i+3/2} - e_{i+3/2}, \\ [e_{i+1/2}, e_{j+1/2}] &= 0, \qquad for\ |i-j| \geq 2, \end{aligned} \tag{7}$$

where $\beta = q + 1/q$. When both sides of equ.(7) vanish, the Hecke algebra degenerates into the Temperley-Lieb algebra.

We will take our hamiltonian for an N-site chain to be :

$$H(\beta) = -\sum_{j=1}^{N} e_{j+1/2}, \tag{8}$$

which obviously commutes with $\sum_{i=1}^{N} S_z(i)$. This may differ from other authors conventions by at most a linear transformation. It is interesting to point out here, that the transformation $\beta \to -\beta$ implements the change from *ferromagnetic* to *anti-ferromagnetic* behavior.

Once with a solution at hand for g, i.e. the parameters $A_a, a = 1, \ldots, 19$ have been adjusted so that g satisfies the Hecke algebra, we have at our disposal a standard way to *baxterize* g. This means the introduction of a spectral parameter u, which is necessary to write down the YB equations.

When γ is real, the solutions to the YB equations will be written in t he *trigonometric* regime ($\beta < 2$) and when γ is pure imaginary ($\beta > 2$) , they are in the *hyperbolic* regime. Based on all examples we know of, we do expect all models in the trigonometric regime ($\beta \leq 2$) to be critical (massless), while for $\beta > 2$ they will be massive.

The baxterization recipe is : define a 9×9 matrix $X(u)$ by

$$X_{i+1/2}(u) = \sin(\gamma + u) - \sin(u) e_{i+1/2}. \tag{9}$$

Due to the Hecke algebra $X(u)$ satisfies

$$X_{i+1/2}(u)\, X_{i+3/2}(u+v)\, X_{i+1/2}(v) = X_{i+3/2}(v)\, X_{i+1/2}(u+v)\, X_{i+3/2}(u), \tag{10}$$

$$[X_{i+1/2}(u), X_{j+1/2}(v)] = 0, \qquad for \;\; |i-j| \geq 2.$$

These equations are equivalent to the YB equations for S-matrices (or Boltzmann weights $\mathcal{L}$), if we define them as

$$X_{i+1/2}(u) = \sum_{a,b,c,d=1}^{3} S_{ab}^{cd}(u) I^{(1)} \otimes I^{(2)} \otimes \ldots \otimes [b,c]^{(i)} \otimes [a,d]^{(i+1)} \otimes \ldots . \tag{11}$$

Here $S_{ab}^{cd}(u)$ describes the scattering of particles $a + b \to c + d$ with *colors* a, b, c, d and relative rapidity $\imath u$. (Row-to-row) Monodromy and transfer matrices are built using the S-matrices as Boltzmann weights. Define the local transition matrix $\mathcal{L}$, acting on *horizontal* indices a and c and *vertical* indices b and d as :

$$\mathcal{L}_{ac|bd}(u) = S_{cd}^{ab}(u). \tag{12}$$

As usual, we will supress the vertical indices and regard the matrix elements of $\mathcal{L}$ (in horizontal space) to be operators in the vertical space. The monodromy matrix T is :

$$T_{a,\{b\}}^{c,\{d\}}(u) = \sum_{a_i} \mathcal{L}_{aa_1}(u) \otimes \mathcal{L}_{a_1 a_2}(u) \otimes \ldots \otimes \mathcal{L}_{a_{N-1} c}(u). \tag{13}$$

The transfer matrix $\tau(u)$ is the (horizontal) trace of the monodromy matrix : $\tau_{\{b\}}^{\{d\}}(u) = T_{a,\{b\}}^{a,\{d\}}(u)$.

3 $U(1) \otimes U(1)$-symmetric solutions

The solutions of equs.(5) may be classified according to their symmetry. In this section, we discuss solutions with an extra $U(1)$ symmetry, requiring the hamiltonian to commute also with $\sum_{i=1}^{N} S_z^2(i)$. In the case of spin 1, we may associate three colors with the spin states and the extra $U(1)$-symmetry refers to color conservation. We thus look for S-matrices, which do not allow processes like $1 + 2 \to 1 + 3$ etc. Our S-matrices are therefore of the form

$$S_{ab}^{cd}(u) = \left\{ S_{aa}^{aa}(u),\; S_{ab}^{ab}(u),\; S_{ab}^{ba}(u) \right\} \tag{14}$$

One immediately sees, that this requires

$$c_{12|32} = c_{32|12} = c_{21|23} = c_{23|21} = 0. \tag{15}$$

To simplify the classification of all possible solutions, we will first scale our matrices g or e to a standard form. For this, we note the 9×9 matrix g is a direct product of 3×3 matrices and can thus be written in terms of the matrices of equ.(2) : $[a, b] \otimes [c, d]$. Since these satisfy

$$[a, b] * [c, d] = \delta_{b,c}[a, d], \tag{16}$$

we may rescale them as

$$\overline{[a, b]} = \rho_{a,b}\, [a, b] \tag{17}$$

and provided

$$\rho_{a,a} = 1,\ \ \rho_{b,a} = 1/\rho_{a,b},\ \ \ \rho_{a,b}\, \rho_{b,c} = \rho_{a,c}. \tag{18}$$

The matrices $\overline{[a, b]}$ satisfy the same algebra as $[a, b]$. In the present case, when equ.(15) is satisfied, we are free to rescale such that $|c_{12|21}| = |c_{21|12}| = |c_{13|31}| = |c_{31|13}| = |c_{23|32}| = |c_{32|23}| = 1$. In order to satisfy Hecke, we put

$$c_{12|21} = c_{21|12} = c_{13|31} = c_{31|13} = c_{23|32} = c_{32|23} = \imath. \tag{19}$$

Moreover, if $c_{12|32} = c_{32|12} = 0$ (resp. $c_{21|23} = c_{23|21} = 0$), the spectrum of g from equ.(4) will not depend upon the values of $c_{21|23}$ and $c_{23|21}$ (resp. $c_{12|32} and c_{32|12}$). In this way we generate models that, although aparently violating the $U(1) \otimes U(1)$-symmetry, do nevertheless share the same spectrum with the corresponding $U(1) \otimes U(1)$-symmetric models with $c_{12|23} = c_{23|21} = 0 (resp. c_{12|32} = c_{23|21} = 0)$ [1].

After this rescaling, we have basically two types of solutions, from which all the others may be obtained by simple manipulations (like changing the color indices etc.). We now state e-matrices for these two cases. The first family depends on three free parameters $\nu_a; a = 1, 2, 3$, which may assume the values 0 or 1 :

$$e^{(1)}(\beta; \nu_1, \nu_2, \nu_3) = \sum_a \beta\nu_a [aa|aa] + \sum_{a<b} ([aa|bb]\, q + [bb|aa]/q) \tag{20}$$

$$+ \sum_{a \neq b} [ab|ba].$$

The second family depends on only two free parameters $\nu_a, a = 1, 2; (\nu_a = 0, 1)$ and is given by :

$$e^{(2)}(\beta; \nu_1, \nu_2) = \beta\nu_1 ([11|11] + [11|22] + [22|11] + [22|22]) \tag{21}$$

$$+\beta\nu_2[33|33] + [11|33]\, q + [33|11]/q + [22|33]\, q + [33|22]/q +$$

$$\sum_{a \neq b} ([ab|ba] - [12|21] - [21|12])\,.$$

We see, that this second solution can be generated from the first one of equ.(20) by simply decoupling colors 1 and 3. This interesting decoupling phenomenon will, as we shall see in section 5, reduce the nested BA, which solves model equ.(20), to an unnested one.

The ferro- and anti-ferromagnetic regimes of the solutions equs.(20) and (21) are also related :

$$e^{(1)}(\beta; \nu_1, \nu_2, \nu_3) = e^{(1)}(-\beta; 1 - \nu_1, 1 - \nu_2, 1 - \nu_3)\ +\ \beta, \tag{22}$$

[1] This feature of the spectrum was checked numerically.

$$e^{(2)}(\beta;\nu_1,\nu_2) = e^{(2)}(-\beta; 1-\nu_1, 1-\nu_2) \,+\, \beta.$$

Expressed in terms of the spin 1 variables, these solutions have the form:

$$\begin{gathered}
e^{(1)} = \nu_2\beta + \frac{1}{2}(q-1/q)\left(S_z(i) - S_z(i+1)\right) - \\
\left(\nu_2\beta - \frac{1}{2}(q+1/q)\right)\left(S_z^2(i) + S_z^2(i+1)\right) + \\
\frac{1}{4}\left((\nu_1+\nu_3)\beta - (q+1/q)\right) S_z(i)S_z(i+1) + \\
\frac{1}{4}\left((\nu_1+4\nu_2+\nu_3)\beta - 3(q+1/q)\right) S_z^2(i)S_z^2(i+1) + \\
\frac{1}{4}\left((\nu_1-\nu_3)\beta + (q-1/q)\right) S_z^2(i)S_z(i+1) + \frac{1}{4}\left((\nu_1-\nu_3)\beta - (q-1/q)\right) S_z(i)S_z^2(i+1) + \\
\left(S_z(i) - S_z(i+1)\right)\left(S_+(i)S_-(i+1) - S_-(i)S_+(i+1)\right) + S_-^2(i)S_+(i+1) + \\
S_+^2(i)S_-^2(i+1) + 2\ S_z(i)S_z(i+1)\left(S_-(i)S_+(i+1) + S_+(i)S_-(i+1)\right)
\end{gathered} \tag{23}$$

and

$$\begin{gathered}
e^{(2)} = \nu_1\beta + \frac{1}{2}(\nu_1\beta - 1/q)S_z(i) - -\frac{1}{2}(\nu_1\beta - q)S_z(i+1) - \\
\frac{1}{2}(\nu_1\beta - 1/q)S_z^2(i) - \frac{1}{2}(\nu_1\beta - q)S_z^2(i+1)) + \\
\frac{1}{4}\left((\nu_1+\nu_2)\beta - (q+1/q)\right)\left(S_z(i)S_z(i+1) + S_z^2(i)S_z^2(i+1)\right) - \\
\frac{1}{4}\left((\nu_1+\nu_2)\beta - (q+1/q)\right)\left(S_z^2(i)S_z(i+1) + S_z(i)S_z^2(i+1)\right) + \\
S_-^2(i)S_+^2(i+1) + S_+^2(i)S_-^2(i+1) - S_+(i)S_z(i+1)S_-(i+1) - \\
S_z(i)S_-(i)S_+(i+1) + S_z(i)S_-(i)S_z(i+1)S_+(i+1) + S_z(i)S_+(i)S_z(i+1)S_-(i+1).
\end{gathered} \tag{24}$$

The solution $e^{(1)}(\beta;0,0,0)$ corresponds to the exactly integrable anisotropic $SU(3)$ chain [0][0][0], whose isotrop ic point $q=1$ is known to satisfy Hecke [0]. For general values of ν_1,ν_2,ν_3 the solution $e^{(1)}$ is related to the hamiltonian version of the spin 1 Perk-Schultz model [0]:

$$H^{P-S}(\lambda;\epsilon_1,\epsilon_2,\epsilon_3) = \sum_i h_i^{P-S}(\lambda;\epsilon_1,\epsilon_2,\epsilon_3) \tag{25}$$

with

$$h_i^{P-S}(\lambda;\epsilon_1,\epsilon_2,\epsilon_3) = \sum_a \epsilon_a \cos\lambda[aa|aa] + \sum_{a\neq b}\left([ab|ba] + \mu_{ab}[aa|bb]\right),$$

where λ is the anisotropic coupling,$\mu_{ab} = \imath\sin\lambda\left(sign(a-b) - (a-b)/3\right)$ and $\epsilon_a = \pm 1, a = 1,2,3$.It is straightforward to check, that these models are related by the rather involved correspondence:

$$h_i^{P-S}(\lambda;\epsilon_1,\epsilon_2,\epsilon_3) = e^{(1)}(-2\cos\lambda; \frac{1-\epsilon_1}{2}, \frac{1-\epsilon_2}{2}, \frac{1-\epsilon_3}{2}) - \tag{26}$$

$$\frac{\beta}{2} I_3 \otimes [1,1] \frac{1}{6}(q - 1/q) \left([2,2] \otimes I_3 + 2[3,3] \otimes I_3\right) -$$
$$\frac{1}{3}(2q + 1/q)\, I_3 \otimes [2,2] - \frac{1}{6}(5q + 1/q)\, I_3 \otimes [3,3],$$

where I_3 is the the three-dimensional unit matrix. However, if we impose periodicity on both models, the above relation simplifies to:

$$\sum_{i=1}^{N} h_i^{P-S}(\lambda; \epsilon_1, \epsilon_2, \epsilon_3) = \sum_{i=1}^{N} \left(e^{(1)}(-2\cos\lambda; \frac{1-\epsilon_1}{2}, \frac{1-\epsilon_2}{2}, \frac{1-\epsilon_3}{2}) + \cos\lambda \right). \tag{27}$$

The above relations show that, analogously to the XXZ-chain, in order to satisfy the Hecke algebra, the model defined with open ends must contain boundary terms with coupling constants, which depend on the deformation parameter β.

The second family of solutions $e^{(2)}$, which has only two free parameters, is to our knowledge not related to any known integrable quantum hamiltonian.

Before closing this section, let us present generalizations of the $U(1) \otimes U(1)$-solutions, satisfying the Hecke algebra for arbitrary spin $S > 1$. In this case we have $n = 2s + 1$ colors and $[a, b]; a, b = 1, \ldots, n$ are n-dimensional matrices. The generalization of $e^{(1)}$ is simply given by equ.(20) and in the periodic case is related to the n-color Perk-Schultz model by an expression similar to equ.(27).

For $S > 1 (n \geq 4)$ there are several generalizations of equ.(21), since now we can decouple more than two colors. Decoupling always in pairs, whose number is denoted by k,we have for the $e^{(2)}$ operator:

$$e^{(2)}(\beta; \nu_1, \ldots, \nu_{n-k}) = \sum_{a,b=1; a\neq b}^{n} [ab|ba] + \sum_{a<b}^{n} \left([aa|bb]\, q + [bb|aa]/q\right) - \tag{28}$$

$$\sum_{a=1}^{k} \left([2a-1, 2a|2a, 2a-1] + [2a-1, 2a-1|2a, 2a]\, q + [2a, 2a|2a-1, 2a-1]/q\right) +$$

$$\nu_1 \sum_{a=1}^{k} \left([2a-1, 2a-1] + [2a, 2a]\right) \otimes \left([2a-1, 2a-1] + [2a, 2a]\right) + \sum_{a=2k+1}^{n} \nu_{a-2k+1}[aa|aa]$$

4 Solutions with $U(1)$-symmetry

In this case there is only one solution [2], given by:

$$e^{(3)}(\beta) = [11|33]\, \hat{q}^2 + [33|11]/\hat{q}^2 + [22|22] + \tag{29}$$

$$\left([12|32] + [21|23]\right) \hat{q} + \left([23|21] + [32|12]\right)/\hat{q},$$

where now $\beta = \hat{q}^2 + 1/\hat{q}^2 + 1$. This model does in fact obey the Temperley-Lieb algebra and its hamiltonian commutes with the quantum group $U_q sl(2)$. This solution has been presented in [0] and [0]. The g-matrix of this model coincides with the $R(u)$-matrix of the

[2]Solving the Hecke algebra for this case, one finds a solution depending on two free parameters and four signs. After rescaling, all have the same spectrum, up to at most a linear transformation.

Izergin-Korepin model [0], at a particular value of their spectral parameter u. The models are different though, since the Izergin-Korepin models is a 19-vertex model, whereas equ.(29) involves only 15 vertices.

In terms of spin variables our model describes the anisotropic version of the bi-quadratic spin-1 model [0] and is given by:

$$
\begin{aligned}
e(\beta) = &\left(\vec{S}(i)\cdot\vec{S}(i+1)\right)^2 - 1 + \frac{\beta-3}{4}\left(S_z^2(i)S_z^2(i+1) - S_z(i)S_z(i+1)\right) + \\
&\frac{\hat{q}^2-\hat{q}^{-2}}{4}\left(S_z(i)S_z^2(i+1) - S_z^2(i)S_z(i+1)\right) + \\
&(\hat{q}-1)\left(S_z(i)S_+(i)S_z(i+1)S_-(i+1) + S_-(i)S_z(i)S_+(i+1)S_z(i+1)\right) + \\
&(\hat{q}^{-1}-1)\left(S_+(i)S_z(i)S_-(i+1)S_z(i+1) + S_z(i)S_-(i)S_z(i+1)S_+(i+1)\right),
\end{aligned}
\tag{30}
$$

where $\vec{S}(i)\cdot\vec{S}(i+1)$ denotes the standard rotationally invariant scalar product.

Similarly to the $U(1)\otimes U(1)$ case, we also find here higher spin generalizations. In fact, for $S>1$ we present two families of solutions $e^{(m)}(S,\beta), m=1,2$. They all satisfy the Temperley-Lieb algebra and reduce to $e^{(3)}$, when $S\to 1$ and are given by:

$$
e^{(m)}(S,\beta) = \sum_{k=0}^{2S}\sum_{l=0}^{2S} D^{(m)}_{k,l}\,[1+k, 1+l|2S+1-k, 2S+1-l],
\tag{31}
$$

where the coefficients $D^{(m)}$ for the two families are given by:

$$
D^{(1)}_{k,l} = -(-\hat{q})^{2S-l-k}, \qquad l,k = 0,1,\dots,2S
\tag{32}
$$

with β being

$$
\beta = -\sum_{l=0}^{2S}(\hat{q})^{2S-l}.
$$

and

$$
D^{(2)}_{k,l} = \hat{q}^{(\delta_{k,0}+\delta_{l,0}-\delta_{k,2S}-\delta_{l,2S})/2}
\tag{33}
$$

and

$$
\beta = \hat{q} + 1/\hat{q} + 2S - 1.
$$

The first family coincides with the one reported earlier in ref.[0], but the second one, which correponds to a different deformation, was not reported previously, at least to our knowledge.

5 Bethe-Ansatz

All the models presented above can be *solved* by the BA - or some variant thereof - , although for lack of space we will present here only the cases with $U(1)\otimes U(1)$-symmetry, leaving the rest for a future publication.

We will apply the BA in it's algebraic version pioneered by Takhtadjian and Fadeev [0]. In order to find the eigenvalues and eigenvect ors of the transfer matrix $\tau(u)$, one proceeds in two steps. First find a *reference* state, which is a convenient eigenstate $|\Omega\rangle$ of $\tau(u)$, such that the monodromy matrix $L_{a,b}(u)$ applied to $|\Omega>$ takes a simple form. That is, applying $L_{a,b}(u)$ to $|\Omega>$, we obtain either zero or a state proportional to $|\Omega>$ or a new vector. The matrix elements of $L_{a,b}(u)$ producing a new vector will be called creation operators and those giving zero are destruction operators. One also has to make sure, that the YB equations give usable commutation relations for these craetion and destruction operators. Second, since our BA will normally be of the nested type, one has to decide where to start the BA hierarchy. Since all this depends on the details of the model, it is an art to be exercised on each model (or set of models) in turn.

Let us take the $U(1)\otimes U(1)$ model of family equ.(20) with $\nu_1 = \nu_2 = \nu_3 = 1$. The vacuum to be chosen is

$$|\Omega_2>= \prod_i^N \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} . \tag{34}$$

The local transition matrix $\mathcal{L}$, when applied to this vacuum, takes the form :

$$\mathcal{L}(u)|\Omega_2>= \begin{pmatrix} c(u) & 0 & 0 \\ * & a_+(u) & * \\ 0 & 0 & c(u) \end{pmatrix} |\Omega_2> . \tag{35}$$

Here the stars $*$ stand for nonzero entries not proportional to $|\Omega_2>$, so that we are actually somewhat abusing the notation. The functions are :

$$a_\pm(u) = \sin(\gamma \pm u), \qquad c(u) = \frac{\sin(u)}{\sin(\gamma + u)}. \tag{36}$$

It follows, that $|\Omega_2>$ is an eigenstate of the transfer-matrix,since

$$T_1^1(u)|\Omega_2>= T_3^3(u)|\Omega_2>= c^N(u)|\Omega_2>$$

$$T_2^2(u)|\Omega_2>= a_+^N(u)|\Omega_2> . \tag{37}$$

Starting from element $T_2^2(u)$, we can now unleash the nested BA machinery an d obtain the following set of eigenvalues $\tau(u)$:

$$\tau(u) = \prod_{l=1}^{m_1} \frac{1}{c(u-\lambda_l)} + c^N(u) \prod_{l=1}^{m_1} \frac{1}{c(\lambda_l - u)} \tau_1(u),$$

$$\tau_1(u) = \prod_{k=1}^{m_2} \left(\frac{1}{c(u-\Lambda_k)} + \frac{1}{c(\Lambda_k - u)} \prod_{j=1}^{m_1} c(u-\lambda_j) \right) , \tag{38}$$

where the λ_l and Λ_k, satisfy the equations :

$$\prod_{l=1}^{m_1} c(\Lambda_k - \Lambda_l) = \prod_{j=1, j\neq l}^{m_1} \frac{c(\Lambda_j - \Lambda_k)}{c(\Lambda_k - \Lambda_j)}$$

$$c^N(\Lambda_l) \prod_{k=1}^{m_2} \frac{1}{c(\lambda_l - \Lambda_k)} = \prod_{j=1, j\neq l}^{m_1} \frac{c(\lambda_j - \Lambda_l)}{c(\Lambda_l - \Lambda_j)}. \tag{39}$$

The corresponding eigenvectors are given by linear combinations of vectors, obtained by applying $B_1(u)$ and $B_2(u)$ to $|\Omega_2>$, where $B_1(u) = T_2^1(u)$ and $B_2(u) = T_3^1(u)$. In detail :

$$|_1\{\lambda_l, a_l\}_{m_1} >= \sum_{a_l=1}^{2} F_1(a_1, \ldots, a_{m_1}) \prod_{l=1}^{m_1} B_{a_l}(\lambda_l)|\Omega_2>, \tag{40}$$

where $F_1(a_1, \ldots, a_{m_1})$ is an now eigenvector of the six vertex transfer -matrix $T_1(u)$, constructed form local transition matrices

$$l_{ac|bc}(u) = c(u)\delta_{ab}\delta_{cd} + b_c(u)\delta_{ad}\delta_{bc} \ , \qquad a \neq b \neq c \neq d, \tag{41}$$

$$l_{11}^{11}(u) = l_{22}^{22}(u) = 1.$$

Here

$$\begin{aligned} b_1(u) &= \sin(\gamma)e^{\imath u} \\ b_2(u) &= \sin(\gamma)e^{-\imath u}. \end{aligned} \tag{42}$$

The same scheme may now be applied to the corresponding decoupled model of equ. (21). In this case the two-step nested BA simplifies to a single step affair. This happens, because after executing the first step, the re sulting auxiliary six vertex model has a trivial $l(u)$ matrix, proportional to the unit matrix. The eigenvalues are then given by:

$$\tau(u, \{\lambda_i\}) = \prod_{l=1}^{m} \left(\frac{a^N(u)}{c(\lambda_l - u)} + \frac{c^N(u)}{c(u - \lambda_l)} \right), \tag{43}$$

the λ_l satisfying:

$$c^N(\lambda_k) = \prod_{j=1, j\neq k}^{m} \frac{c(\lambda_k - \lambda_j)}{c(\lambda_j - \lambda_k)}. \tag{44}$$

Thus the eigenvalues are identical to the ones of the six vertex model and out o f all eigenvalues the BA yields only this subset. the question of how to retrieve the remaining eigenvalues and the BA for the $U(1)$-symmetric case is left for a forthcoming publication.

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