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Local dynamics of mean-field Quantum Systems

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Abstract. In this paper we extend the theory of mean-field-dynamical semigroups given in [**DW1,Du1**] to treat the irreversible mean-field dynamics of quasi-local mean-field observables. These are observables which are site averaged except within a region of tagged sites. In the thermodynamic limit the tagged sites absorb the whole lattice, but also become negligible in proportion to the bulk. We develop the theory in detail for a class of interactions which contains the mean-field versions of quantum lattice interactions with infinite range. For this class we obtain an explicit form of the dynamics in the thermodynamic limit. We show that the evolution of the bulk is governed by a flow on the one-particle state space, whereas the evolution of local perturbations in the tagged region factorizes over sites, and is governed by a cocycle of completely positive maps. We obtain an H-theorem which suggests that local disturbances typically become completely delocalized for large times, and we show this to be true for a dense set of interactions. We characterize all limiting evolutions for certain subclasses of interactions, and also exhibit some possibilities beyond the class we study in detail: for example, the limiting evolution of the bulk may exist, while the local cocycle does not. In another case the bulk evolution is given by a diffusion rather than a flow, and the local evolution no longer factorizes over sites.

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1. Introduction.

The characteristic feature of mean-field systems can be expressed by saying that each particle or elementary subsystem interacts in an equal way with every other such subsystem, and responds to the average of these interactions. In this paper we will be concerned with the limiting dynamics of such systems as their size becomes infinite. Therefore we will consider a sequence of models comprising an increasing collection of copies of the basic subsystem. When we speak of an interaction between the subsystems, we mean that for each model in the sequence a (generally) irreversible dynamics is specified. The mean-field nature of the models entails first of all that the interaction is invariant with respect to permutation of the subsystems; the idea that each subsystem responds to an average is made precise by the property that the generator of the dynamics of a large system can be approximated by taking a generator involving only a few (often just two) subsystems, averaging it over all permutations of the subsystems, and multiplying it by the number of subsystems. This is in close analogy to lattice systems with translation invariant interaction: there one obtains the Hamiltonian for a finite region approximately by averaging terms involving only a few sites over all translations which map these sites into the given region, and by multiplying with the volume of the region. In this analogy mean-field systems are just lattice systems, whose underlying lattice has permutation symmetry rather than translation symmetry. This analogy suggests a canonical way of obtaining a "mean-field approximation" of an arbitrary lattice model with translation invariance: one merely has to take the Hamiltonians of the lattice model for some sequence of regions going to infinity in the sense of van Hove [**Rue**], and symmetrize each with respect to all permutations of the lattice sites. We do not attempt to justify this procedure as an approximation to the original lattice system. Our aim is rather to obtain as complete an analysis of the mean-field theory as possible.

The description of mean-field systems in terms of their permutation symmetry becomes more transparent if one looks at the intensive rather than the extensive observables. As described above the Hamiltonian of a mean-field system divided by the number of subsystems, i.e. the intensive variable "Hamiltonian density", has the property that for a large system it is approximately equal to the Hamiltonian density of a smaller version of the system, symmetrized over all permutations. Sequences of observables (indexed by the system size) with this property were called "approximately symmetric" in [**RW1**], and have become the central notion of a research programme on mean-field systems. The basic result in [**RW1**] concerns the thermodynamics of Hamiltonian mean-field systems, and is a formula for the free energy density in the thermodynamic limit in terms of a Gibbs variational principle in one-particle quantities. This result was later extended to "inhomogeneous mean-field systems" in which the permutation symmetry is restricted to sites with approximately equal external or random parameters [**RW2**]. If one starts from a lattice model with translation invariant interaction, the thermodynamics of its mean-field version can be written down directly by evaluating the mean energy and the mean entropy for homogeneous product states.

This prescription is often taken as the definition of the mean-field approximation. However, it would be impossible to define the dynamics "in the mean-field approximation" if this is only understood as a class of variational states. In contrast, in our programme mean-field models are treated as quantum systems in their own right. The dynamics of mean-field models was treated in [**DW1**] from the point of view that the dynamics should map the set of mean-field intensive variables, i.e. it should map the approximately symmetric sequences into itself. A corresponding study of the inhomogeneous case was undertaken in [**DRW**], and the special properties of Hamiltonian dynamics, as opposed to general irreversible dynamics, were described in [**DW2**]: in this case one obtains in the limit a flow on the state space of the one-particle algebra, which is Hamiltonian in the full sense of classical mechanics with respect to a canonical Poisson bracket structure on the state space. In earlier approaches [**Bo1**] beginning with [**HL**] this had been noted only in the case when the Hamiltonian is written in terms of the generators of a Lie group representation so that a symplectic structure can be imported from the coadjoint orbits.

The works described so far focussed entirely on the properties of the intensive observables, which in the mean-field limit become completely delocalized. This leaves open the question how the evolutes of a localized observable behave under a mean-field dynamics. Intuitively, the picture is that under a completely delocalized evolution such as a mean-field dynamics the observable would instantaneously develop a completely delocalized tail, while initially still exhibiting a strong dependence on the original localization region. For very large times one might expect that this dependence on the original localization becomes weaker, especially when the dynamics is dissipative. It is therefore natural to use a concept analogous to the approximately sequences in which the symmetrization operations leave out all the sites of the original localization region. Put differently these sites are given a "tag" and one aims to study the motion of the tagged subsystems under the averaged influence of the remaining ones. This programme has been carried out in [**Du1**] for any fixed set I of tagged sites. In this paper we further extend this approach allowing more and more tagged sites in thermodynamic limit, as long as the proportion of tagged sites goes to zero. The above intuitive picture is confirmed by our analysis.

A closely related programme for the study of mean-field systems has been based on the work of Morchio and Strocchi [**MS**]. Their aim was to show how the dynamics of a system with long range interactions can be defined in the thermodynamic limit even though the quasi-local local algebra in the usual sense cannot be invariant under such an evolution due to appearance of delocalized tails. Their proposal is to enlarge the quasi-local algebra by suitable weak limits of observables capable of describing delocalized intensive quantities. It is clear that these limits exist only with respect to a suitably chosen set of states, and consequently much of the theory centers on this choice. For the case of mean-field theories their programme was carried out by Bóna [**Bo1**] and Unnerstall [**Un1,Un2**]. In a sense their approach is dual to ours, in focussing on the states rather than on the observables. In particular, the permutation symmetry, which is as central to their approach as to ours, is built in by choosing the folium of permutation symmetric states on the quasi-local algebra, whereas in our approach it determines the connection between observables of systems of different sizes. The thermodynamic limit of observables in our approach is always taken in norm, whereas in the picture of Morchio and Strocchi it is typically taken in the s-topology associated with the chosen folium of states. Consequently, our limiting object is a C*-algebra, whereas they arrive more naturally at a W*-algebra or a von Neumann algebra.

The paper is organized as follows. In section 2 we define quasi-local mean-field observables. These are what we call the quasi-symmetric sequences of observables: those which are delocalized (i.e. site-averaged) except over local regions of tagged sites which become proportionately negligible in the thermodynamic limit. Such sequences of observables have well defined "thermodynamic limits" in a space which we construct explicitly.

In section 3 we formulate the notion of a mean-field dynamical semigroup as a sequence of dynamical semigroups which preserves the set of quasi-symmetric observables, and which furthermore gives rise to contraction semigroup on the inductive limit space. We demonstrate that a wide class of evolutions has this property, this class being considerably wider than in [**HL,Bo1,Un3**]. In particular, we include the mean-field versions of arbitrary translation invariant, possibly dissipative lattice interactions. The existence of the limiting dynamics is subject to a growth condition which is far weaker than that required for the original translation invariant interactions [**BR**] . For this class of models the limiting dynamics is shown to have the following special form: on initially localized observables it factorizes over the individual sites of the region of localization, while the global evolution of the delocalized tail is implemented by a flow on the one-site state space of the system. The non-linear differential equation for this flow is just the Hartree equation. Such a form was obtained in [**Bo1**], but only for Hamiltonian interactions between finite numbers of sites. More recently this type of dynamical evolution has been considered by Bóna [**Bo2**] as a generalization of quantum mechanics itself, and was linked to a modification of quantum mechanics recently proposed by Weinberg [**Wei**]. As a special case, our theory can be applied to classical Markov processes: the factorization of the local evolutions has been used to investigate the Poissonian approximation in queueing networks [**Du2**].

In section 4 we consider some properties of the limiting evolution in some general cases. Firstly, we show that if the finite volume dynamics is Hamiltonian, then the limiting dynamics is completely determined by the energy density function appearing in the Gibbs variational principle for the equilibrium states: as a Hamiltonian function in the sense of classical mechanics it generates the flow which describes the global evolution via a Poisson structure on the one-particle state space. Its gradient is the Hamiltonian operator (depending on the global state) generating the local unitary cocycle. This description is complete in the sense that any Hamiltonian function can be approximated by one arising from our class of models. The next level of complexity is given by the sequences of generators which can be written in Lindblad form in terms of approximately symmetric observables. Here the local dynamics is still given by a state dependent Hamiltonian. However, it can no longer be expressed as the gradient of single function. We show that up to approximation any state dependent Hamiltonian arises from a model of this type. The global flow is no longer Hamiltonian, and is essentially arbitrary in the class considered.

The flow, and indeed the whole limiting evolution in this subclass is reversible (exists for negative times), while all evolutions for finite size systems are strictly dissipative. Finally, in the full class studied in section 3 we obtain an (up to approximations arbitrary) state-dependent Lindblad generator. However, we observe that such evolutions do not exhaust the set of mean-field dynamical semigroups. This is illustrated by describing a sequence of dynamical semigroups whose mean-field limiting dynamics exists in our sense, but lacks some of the fundamental features established for the lattice class: the global limiting dynamics is given by a diffusion on the one-particle state space rather than a flow, and the evolution of local observables does not reduce to a product of one-site evolutions. In one of the classes mentioned above the local dynamics is still Hamiltonian, while the global evolution is not. The converse can also happen in the sense that any generator (e.g. a Hamiltonian one) may be perturbed in such a way that the global evolution is unchanged, but the local evolution becomes dissipative. We construct such perturbations explicitly in terms of permutation operators.

In section 5 we study the relation between the local and the global dynamics. In fact we are able to construct an example of a sequence of semigroups which is a mean-field dynamical semigroup in the global, but not local, sense. A limiting dynamics exists for the fully site averaged observables *only*. Finally, we investigate the delocalization of initially localized observables for lattice class evolutions. We prove an H-Theorem which suggests that in the dissipative case all local information should be lost as the local states are drawn towards the flow of the global state. We show that under the addition of an arbitrarily small perturbation any lattice class generator has such an evolution.

2. Quasi-symmetric Observables

In this section we describe the notion of quasi-symmetric observables, which generalizes on the one hand the usual quasi-local observables known from lattice models, and on the other hand the mean-field intensive variables introduced in [**RW1**]. In order to define the thermodynamic limit of a physical quantity it is always necessary to define the observable in question for all system sizes occurring on the way to the thermodynamic limit. For example, for the usual interactions of lattice systems it is the translation invariance of the potential which determines the connection between the energy observables at different system sizes. Quasi-symmetry as defined here is a property not of an observable of a single system of finite size but of a net of observables indexed by the size. Associated with this notion is a definition of the thermodynamic limit of a quasi-symmetric observable, and much of the work in this section will go into the identification of the space in which these limits lie.

Before taking up the formal development let us clarify the aim of this section by relating it to a standard construction in functional analysis, the inductive limit of Banach spaces. There one has a sequence $(\mathcal{A}_N)$ of spaces with a system of isometric "inclusion maps" $j_{NM} : \mathcal{A}_M \to \mathcal{A}_N$ (defined for $N \geq M$) satisfying the chain relation $j_{NR} = j_{NM} \circ j_{MR}$. The term "inclusion map" indicates that the elements $X_R \in \mathcal{A}_R$ and $j_{NR}X_R \in \mathcal{A}_N$

will eventually be identified. In other words, we are interested only in the sequence $N \mapsto X_N$, which is defined for sufficiently large N (e.g. $N \geq R$) and satisfies $X_N = j_{NM} X_M$ for all N, M for which j_{NM} and X_M are defined. The space of such sequences is then called the "union" of the $\mathcal{A}_N$ with respect to the inclusions j_{NM}. It is clear that this set of sequences forms a vector space under N-wise operations. If we work in the category of Banach spaces the limit space $\mathcal{A}_\infty$ of the system $(\mathcal{A}_N, j_{NM})$ is taken as the completion of this union. The elements of the completion can also be represented by sequences, namely by those for which $\|X_N - j_{NM} X_M\|$ becomes arbitrarily small as both N and M become sufficiently large. Note that in the trivial case where all $\mathcal{A}_N$ are equal and j_{NM} is always the identity these sequences are precisely the Cauchy sequences. So we might call sequences with this property "j-Cauchy". Sequences X, X' for which $\|X_N - X'_N\| \to 0$ represent the same element of the completion. Thus $\mathcal{A}_\infty$ is equal to the quotient of the space of j-Cauchy sequences up to equality under the seminorm $\|X\| = \lim_N \|X_N\|$.

The quasi-local algebra of a lattice system is an example of this construction. Here the $\mathcal{A}_N$ are the observable algebras of an increasing family of regions, and the embedding j_{NM} is by tensoring with the identity element on all sites of $N \setminus M$. Since the j_{NM} in this case are homomorphisms of C*-algebras, the union becomes a *-algebra, and the limit space $\mathcal{A}_\infty$ is also a C*-algebra, called the C*-inductive limit of the $\mathcal{A}_N$. $\mathcal{A}_\infty$ is usually called the quasi-local algebra of the lattice system, and we will denote it by $\mathcal{A}_{loc}$, reserving the symbol "$\mathcal{A}_\infty$" for other limit spaces to be discussed below.

A very similar construction was used in [**RW1**] to define the algebra of intensive observables of mean-field systems. Here one uses the same spaces $\mathcal{A}_N$, but the inclusions j_{NM} are modified by averaging over all permutation automorphisms of the larger region. It is easy to check that the resulting maps j_{NM} again satisfy the chain relation, but they are no longer isometric, nor even injective. Nevertheless, the notions of j-Cauchy sequences (called "approximately symmetric" in [**RW1**]) and the limit space $\mathcal{A}_\infty$ make sense even in this case. It turns out that the N-wise product of j-Cauchy sequences is again j-Cauchy so that the limit space becomes an (abelian) C*-algebra even though the j_{NM} are no longer homomorphisms. In this paper we generalize the construction still further: we will allow the chain relation to be not strictly satisfied but only asymptotically for large indices. In fact it suffices for a sensible definition of j-Cauchy sequences and the limit space to have that $\lim_{M\to\infty} \text{limsup}_{N\to\infty} \|(j_{NR} - j_{NM} \circ j_{MR}) X_R\| = 0$ for every fixed R and $X_R \in \mathcal{A}_R$. We will not, however, develop an abstract theory of "fuzzy inductive limits" along these lines, but instead will focus on the case at hand, the physical motivation for the choice of the j_{NM}, and the concrete representation of the limit space $\mathcal{A}_\infty$.

We will consider systems composed of many "particles", each of which has observables described by the same C*-algebra with unit $\mathcal{A}$. For most of the general theory we do not need any further assumptions on this algebra but in many models of interest $\mathcal{A}$ is just a finite dimensional matrix algebra describing a "spin". In section 3, in the discussion of mean-field dynamics in the full lattice class of generators we will make this assumption for simplicity. By $K(\mathcal{A})$ or simply by K we denote the state space of this algebra. We equip K with the weak* topology. The evaluation of a continuous linear functional σ on any

C*-algebra $\mathcal{B}$ on $X \in \mathcal{B}$ will be written as $\langle \sigma, X \rangle$. To each particle we associate a "site" of a lattice $\mathcal{N}$, e.g. $\mathcal{N} = \mathbb{Z}^d$ for systems on a d-dimensional cubic lattice. Denoting by $\mathcal{A}_{\{x\}}$ the isomorphic copy of $\mathcal{A}$ "at site x", we write $\mathcal{A}_I = \bigotimes_{x \in I} \mathcal{A}_{\{x\}}$ for the observable algebra of the subsystem localized in the finite subset $I \subset \mathcal{N}$. Here and below we always use the the minimal C*-tensor product, although in applications the algebras concerned are usually finite dimensional matrix algebras, for which all C*-tensor products coincide. Mappings between finite regions induce homomorphisms between the associated obervable algebras. Explicitly, if $\eta : I \to J$ is an injective map we define $\hat{\eta} : \mathcal{A}_I \to \mathcal{A}_J$ by

$$\hat{\eta}\big(A_1 \otimes A_2 \cdots \otimes A_{|I|}\big) = A_{\eta^{-1}(1)} \otimes A_{\eta^{-1}(2)} \cdots A_{\eta^{-1}(|J|)} \tag{2.1}$$

with the understanding that on the right hand side $A_{\eta^{-1}(x)} = \mathbb{1}$, whenever x is not in the range of η. Note that if η is the inclusion map of I into $J \supset I$, $\hat{\eta}$ is just the usual embedding between the subalgebras $\mathcal{A}_I$ and $\mathcal{A}_J$ used in the construction of the quasi-local algebra of the lattice system as a C*-inductive limit. Since we will be interested in yet another kind of inductive limit it will be convenient to suppress the inclusion maps $\hat{\eta} : \mathcal{A}_I \to \mathcal{A}_J$, and similarly the inclusion of each $\mathcal{A}_I$ into the quasi-local algebra $\mathcal{A}_{loc}$. Thus for $I \subset J$ we shall simply write $\mathcal{A}_I \subset \mathcal{A}_J \subset \mathcal{A}_{loc}$.

There are $|N|!/(|N| - |M|)!$ injective maps from a set of $|M|$ elements into a set of $|N| \geq |M|$ elements. In **[RW1,DW1]** the identification between the intensive mean-field observables at different system sizes was made by the average of all $\hat{\eta}$, where η runs over all injective maps. In contrast, only a single map (namely the natural injection $\eta : M \hookrightarrow N$) is used in the construction of the quasi-local algebra. Here we will use an average over a subset of injective maps, which generalizes both of these possibilities: for $I \subset M \subset N$ we define $\mathcal{J}^I_{NM}$ as the set of all injective maps $\eta : M \to N$ such that $\eta(i) = i$ for all $i \in I$, which is a set of $|N \setminus M|!/|M \setminus I|!$ elements. The corresponding average is

$$j^I_{NM} = \frac{|M \setminus I|!}{|N \setminus M|!} \sum_{\eta \in \mathcal{J}^I_{NM}} \hat{\eta} \quad : \mathcal{A}_M \to \mathcal{A}_N \quad . \tag{2.2}$$

Thus for $I = \emptyset$ we recover the map used in the "global" theory of mean-field systems **[RW1,DW1]**, and for $I = M$ we get the injection used for the quasi-local algebra. The family j^I_{NM} for fixed I was used in **[Du1]** to set up a theory of mean-field systems with a fixed set I of "tagged particles". In this paper we go one step further, by allowing the set of tagged particles to become infinite in the thermodynamic limit.

Thus we will take the limit not only over an increasing family of regions, we will also consider in each region a subset of tagged sites, such that in the limit every site of the lattice eventually becomes tagged. We formalize this by using the notion of tagged sets: a **tagged set** is a finite subset $N \subset \mathcal{N}$ of the lattice under consideration, together with a subset $N^\top \subset N$ of "tagged sites". Rather than denoting a tagged set by the pair $(N, N^\top)$ we will just use the symbol N, in much the same way as a vector space is usually denoted by the same letter as its underlying set, without explicit reference to the operations defined on it. For tagged sets we define an inclusion relation $M \Subset N$ as "$M \subset N$ and $M^\top \subset N^\top$". For tagged sets $M \Subset N$ we now define

$$j_{NM} = j_{NM}^{M^\top} : \mathcal{A}_M \to \mathcal{A}_N \quad . \tag{2.3}$$

This is the basic family of inclusions on which our inductive limit construction is built. In applications one usually does not take the observables to be defined for all regions N, but only along some subsequence of regions (e.g. cubes). Therefore we will assume some net $(N_\alpha)_{\alpha\in\aleph}$ of tagged sets to be given, and we will only consider limits along this net. Allowing only sequences at this point would not introduce a simplification in anything we do in this paper. On the other hand it is convenient to be able to state the theory for a general net of regions in $\mathcal{N}$ going to the lattice in the sense of van Hove, without being forced to specify a particular enumeration. Therefore we allow the index set $\aleph$ to be an arbitrary directed set. Readers who feel more at home with sequences are invited to take $\aleph = \mathbb{N}$, and to substitute "sequence" for "net" throughout. This will be sufficient (though perhaps not convenient) for all applications. Our only assumptions on the net $(N_\alpha)_{\alpha\in\aleph}$ are that it is increasing with respect to the relation $\Subset$, that the tagged subsets absorb the lattice, i.e. $\bigcup_\alpha N_\alpha^\top = \mathcal{N}$, and that in the limit the tagged sites are relatively few, i.e.

$$\lim_\alpha \frac{|N_\alpha^\top|}{|N_\alpha|} = 0 \quad . \tag{2.4}$$

Since the net of regions will be fixed once and for all there is no ambiguity in writing $N \to \infty$ for $\alpha \to \infty$, and $\lim_N f(N)$ for $\lim_\alpha f(N_\alpha)$ for the limit of any N-dependent quantity. We will adopt this convention from now on, so in the sequel we will never refer to the labels α or the set $\aleph$.

We now single out the j-Cauchy nets in the sense mentioned in the introduction to this section. These nets $N \mapsto X_N$ with $X_N \in A_N$ are the basic observables we consider. X_N will be symmetrized over most sites in N, i.e. over all sites with the exception of the relatively small subset $N^\top$. Intuitively, X_N is a local observable with a symmetrized (or completely delocalized) tail. One should think of X_N as a net of observables "converging to a quasi-local mean-field limit". Our formal definition is given below, together with the corresponding notion [**Du1**] for a fixed set of tagged sites.

2.1 Definition. *Let $X_N \in \mathcal{A}_N$ for every N in the given fixed net of tagged sets. Then*

(1) the net $N \mapsto X_N$ is called a **quasi-symmetric**, *or a quasi-symmetric observable, if*
$$\lim_{M\to\infty} \limsup_{N\to\infty} \|X_N - j_{NM} X_M\| = 0 \quad .$$
The set of such nets will be denoted by $\mathcal{Y}$.

(2) the net $N \mapsto X_N$ is called **I-symmetric**, *if*
$$\lim_{M\to\infty} \limsup_{N\to\infty} \|X_N - j_{NM}^I X_M\| = 0 \quad .$$
The set of such nets will be denoted by $\mathcal{Y}^I$.

As noted before the crucial property of the maps j for making quasi-symmetry a notion of "convergent net" is the approximate chain relation $j_{NR} \approx j_{NM} \circ j_{MR}$. This relation will now be proven together with some other basic combinatorial facts.

2.2 Proposition. *Let $I \subset J \subset R \subset M \subset N \subset \mathcal{N}$. Then*

(1) $j^I_{NR} = j^I_{NM} \circ j^J_{MR}$.

(2) $\left\| j^I_{NR} - j^J_{NM} \circ j^I_{MR} \right\| \leq 2|R|\,|J| \dfrac{|N| + |M|}{|N|\,|M|} \leq 4|R| \dfrac{|J|}{|M|}$.

(3) $\lim_{M\to\infty} \operatorname{limsup}_{N\to\infty} \| j_{NR} - j_{NM} \circ j_{MR} \| = 0$.

Proof: All maps appearing in (1) and (2) act like the identity on $\mathcal{A}_I$, and like their counterparts with $I = \emptyset$ on the remining sites. Therefore it suffices to show (1) for $I = \emptyset$. Suppose (2) had been proven for this special case. Then we would obtain for the general case a bound of the same form, but with $|I|$ subtracted from the numbers appearing in it. The bound as stated then follows from the monotonicity of the function $x \mapsto (a+x)(b+x)(c+x)^{-1}$ when $(a+x)$ and $(b+x)$ are positive, and $c \geq \max\{a, b\}$. It therefore suffices to show both (1) and (2) only in the case $I = \emptyset$.

(1) $j^\emptyset_{NM}(A)$ can be computed by taking $\hat\eta(A)$ for any injective map $\eta : M \to N$ and then symmetrizing over all permutations of N. It follows that $j^\emptyset_{NM} \circ \hat\eta = j^\emptyset_{NR}$ for *any* injective $\eta : R \to M$. Equation (1) thus follows by taking the appropriate average over η.

(2) Consider the map $\tilde\jmath_{NR}$ (resp. $\tilde\jmath_{MR}$) defined as the equal-weight averages over all $\hat\eta$ with $\eta : R \to N$ (resp. $\eta : R \to M$) such that in addition $\eta(R) \cap J = \emptyset$. Let $p^J_N = j^J_{NN}$ denote the average over all permutation automorphisms of $\mathcal{A}_N$ of permutations leaving J pointwise fixed. Then $\tilde\jmath_{NR} = p^J_N \circ \hat\eta$ and $j^J_{NM} = p^J_N \circ \hat\eta_1$, where η and η_1 are any of the maps over which $\tilde\jmath_{NR}$ and j^J_{NM} are averages. Hence $\tilde\jmath_{NR} = p^J_N \circ \hat\eta_1 \circ \hat\eta_2 = j^J_{NM} \circ \hat\eta_2$, where $\eta_2 : R \to M$ is injective with $\eta_2(R) \cap J = \emptyset$. By averaging over all η_2 we find

$$\tilde\jmath_{NR} = j^J_{NM} \circ \tilde\jmath_{MR} \quad .$$

The rest of the proof consists in establishing the estimate

$$\left\| \tilde\jmath_{NR} - j^\emptyset_{NR} \right\| \leq 2 \frac{|R|\,|J|}{|N|} \quad .$$

Applying the same estimate to $\tilde\jmath_{MR}$, and inserting into the above equation then yields the result. The second form of the estimate follows because $|M| \leq |N|$.

Let $\mathcal{J} \equiv \mathcal{J}^\emptyset_{NR}$ denote the set of all injective $\eta : R \to M$, and $\tilde{\mathcal{J}}$ the subset with $\eta(R) \cap J = \emptyset$. Note that for large N the "probability" $\eta(R)$ meeting J goes to zero. More precisely, by Lemma IV.1 of **[RW1]**we have that

$$\varepsilon \equiv \frac{|\mathcal{J} \setminus \tilde{\mathcal{J}}|}{|\mathcal{J}|} \leq \frac{|R|\,|J|}{|N|} \quad .$$

Now both $j^\emptyset_{NR}$ and $\tilde\jmath_{NR}$ are averages of $\hat\eta$ with different weights. Since $\|\hat\eta\| = 1$ for all η we can estimate their norm difference by the sum of the absolute differences of these weights. For $\eta \in \tilde{\mathcal{J}}$ the weight in $j^\emptyset_{NR}$ is $|\mathcal{J}|^{-1}$, and in $\tilde\jmath_{NR}$ it is $|\tilde{\mathcal{J}}|^{-1}$. The difference is $\varepsilon|\tilde{\mathcal{J}}|^{-1}$.

Thus multiplied with the number $|\tilde{\mathcal{J}}|$ of terms we get the contribution ε to the error. For the remaining $|\mathcal{J} \setminus \tilde{\mathcal{J}}| = \varepsilon|\mathcal{J}|$ terms the weight in $j^{\emptyset}_{NR}$ is still $|\mathcal{J}|^{-1}$, but is zero in $\tilde{\jmath}_{NR}$. Hence these terms also contribute ε to the error estimate, and putting the contributions of these two types of terms together, we obtain the required estimate for $\|\tilde{\jmath}_{NR} - j^{\emptyset}_{NR}\|$.

(3) Taking $J = M^\top$ and $I = N^\top$ in (2) we get $\limsup_N \|j_{NR} - j_{NM} \circ j_{MR}\| \leq 2|R|\frac{|M^\top|}{|M|}$, which goes to zero as $M \to \infty$ by our standing assumption (2.2) on the net of tagged sets.

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In the following Lemma we establish a standard way of showing that a given net $X_\bullet$ is quasi-symmetric, namely by showing that X_N can be uniformly approximated for large N by a net of the special form $N \mapsto j_{NR}Y$ for $Y \in \mathcal{A}_R$. We will call such nets **basic nets**, and denote the set of such nets by $\mathcal{Y}_{bas}$. In an ordinary inductive limit $\mathcal{Y}_{bas}$ corresponds to the union $\bigcup_N \mathcal{A}_N$, which is dense in the limit Banach space $\mathcal{A}_\infty$ by definition. This density statement carries over to general "fuzzy inductive limits", that is, whenever the chain relation holds approximately. Here we establish it first on the level of nets. Since by Proposition 2.2(1) the chain relation holds for j^I_{NM} with fixed I we can hence apply the same reasoning to the inductive system $(\mathcal{A}_N, j^I_{NM})$.

2.3 Lemma. *Let $X_N \in \mathcal{A}_N$ for all N in the given net of tagged sets. Then $X_\bullet$ is quasi-symmetric iff for all $\varepsilon > 0$ there are a tagged set R and $Y \in \mathcal{A}_R$ such that*

$$\limsup_N \|X_N - j_{NR}Y\| \leq \varepsilon \quad .$$

$X_\bullet$ is I-symmetric iff in addition one can choose $R^\top = I$.

Proof: (1) Let $X_\bullet$ be quasi-symmetric. Then by definition there is for any $\varepsilon > 0$ some tagged set M such that $\limsup_N \|X_N - j_{NM}X_M\| \leq \varepsilon$. Hence we can set $R = M$ and $\hat{Y} = X_M$. Conversely, suppose that $\|X_N - j_{NR}Y\| \leq \varepsilon$ for $N \supset N_\varepsilon$. Then $\|X_N - j_{NM}X_M\| \leq 2\varepsilon + \|j_{NR}Y - j_{NM} \circ j_{MR}Y\|$ for $N \supset M \supset N_\varepsilon$. Taking in this estimate the limit $\limsup_M \limsup_N$ and using the approximate chain relation Lemma 2.4(2) we find that this limit is less than 2ε for any ε. Exactly the same arguments work for I-symmetric nets, with all j_{NM} replaced by j^I_{NM}.

■

With the help of this Lemma we can clarify the relations between quasi-symmetry and I-symmetry for different values of I. Intuitively, $\mathcal{Y}$ is the limit of $\mathcal{Y}^I$ of $I \nearrow \mathcal{N}$, i.e. the limit of allowing more and more tags. It will be useful also to have a systematic way of removing tags, i.e. to include sites previously exempted from all symmetrizations back into the bulk. The operator of "removing all tags except those in I" is given by

$$p^I_N := j^I_{NN} : \mathcal{A}_N \to \mathcal{A}_N \quad . \tag{2.5}$$

By Proposition 2.2(1) p^I_N clearly is a projection. $p^{\emptyset}_N$ is the operation of removing all tags.

2.4 Proposition.

(1) For $I \subset J$, $\mathcal{Y}^I \subset \mathcal{Y}^J \subset \mathcal{Y}$.

(2) The map $p^I : X_\bullet \mapsto (p^I_\bullet X_\bullet)$ projects $\mathcal{Y}$ onto $\mathcal{Y}^I$.

Proof: (1) The inclusion $\mathcal{Y}^I \subset \mathcal{Y}$ for any I is obvious from Lemma 2.3. What remains to be shown is that any basic net of the form $N \mapsto j^I_{NR}Y$ can be approximated by one of the form $j^J_{NM}\tilde{Y}$. By Proposition 2.2(2) we can set $\tilde{Y} = j^I_{NM}Y$ for some $M \subset \mathcal{N}$, and get $\sup_N \left\| j^I_{NR}Y - j^J_{NM}\tilde{Y} \right\| \leq 2(|R|\,|J|)/|M|$, which can be made arbitrarily small by taking M large enough.

(2) It is evident that the operation $p^I_\bullet$ on nets is a projection. By Proposition 2.2(1) with $N = M$ we have $p^I_N \circ j^J_{NM} = j^I_{NM}$ for $I \subset J$. Hence on basic nets $j^J_{NR}Y$ with $J \supset I$ the projection operation produces again basic nets. Since we can approximate any quasi-symmetric net by basic nets j_{NR} with $R^\top = J$ sufficiently large, Lemma 2.3 says that $p^I_\bullet$ maps $\mathcal{Y}$ into $\mathcal{Y}^I$. Taking $I = J$ it is clear that basic I-symmetric nets are invariant under the projection, hence $p^I_\bullet(\mathcal{Y}) = \mathcal{Y}^I$.

■

We can now proceed to identify the inductive limit space of the system $(\mathcal{A}_N, j_{NM})$. We will use the following notation: for any tagged set N, and any $\rho \in K$ we introduce the conditional expectation $\mathbb{E}^\rho_{N\setminus N^\top} : \mathcal{A}_N \to \mathcal{A}_{N^\top}$ with respect to the product state $\rho^{N\setminus N^\top}$ on the untagged sites. Thus

$$\langle \sigma, \mathbb{E}^\rho_{N\setminus N^\top}(A) \rangle = \langle \sigma \otimes \rho^{N\setminus N^\top}, A \rangle \quad , \tag{2.5}$$

where σ is an arbitrary state of $\mathcal{A}_{N^\top}$, and $A \in \mathcal{A}_N$. Since we identify $\mathcal{A}_{N^\top}$ with a subalgebra of $\mathcal{A}_N$ we can consider $\mathbb{E}^\rho_{N\setminus N^\top}$ as a projection of norm one on $\mathcal{A}_N$, i.e. a conditional expectation in the sense of Umegaki [**Ume**]. If we identify $\mathcal{A}_N$ in turn with a subalgebra of $\mathcal{A}_{loc}$ we can also consider $\mathbb{E}^\rho_{N\setminus N^\top}$ as a map $\mathbb{E}^\rho_{N\setminus N^\top} : \mathcal{A}_N \to \mathcal{A}_{loc}$. This is the point of view taken in the following Theorem. We recall at this point that K, being the state space of a unital C*-algebra, is weak*-compact. For any C*-algebra $\mathcal{B}$, $\mathcal{C}(K, \mathcal{B})$ will denote the space of weak*-continuous functions on K, taking values in $\mathcal{B}$, and topologized with the supremum norm $\|f\| = \sup_{\rho \in K} \|f(\rho)\|_{\mathcal{B}}$.

2.5 Theorem.

(1) Let X be a quasi-symmetric net. Then for all $\rho \in K$ the norm limit

$$X_\infty(\rho) \equiv \lim_N \mathbb{E}^\rho_{N\setminus N^\top}(X_N) \quad \in \mathcal{A}_{loc}$$

exists uniformly for $\rho \in K$ and $\rho \mapsto X_\infty(\rho)$ is weak-to norm continuous.*

(2) The map $X \in \mathcal{Y} \mapsto X_\infty \in \mathcal{C}(K, \mathcal{A}_{loc})$ is onto, and isometric in the sense that

$$\|X_\infty\| = \lim_N \|X_N\| \quad .$$

It is also a homomorphism taking the N-wise product of nets into the product of $\mathcal{C}(K, \mathcal{A}_{loc})$.

(3) A quasi-symmetric net X is I-symmetric if and only if $X_\infty(\rho) \in \mathcal{A}_I \subset \mathcal{A}_{loc}$ for all ρ.

Proof: The core of this result has been proven in section IV of [**RW1**]. There "approximately symmetric nets" (in our terminology "$\emptyset$-symmetric" nets) were allowed to take values in a net of algebras of the form $\mathcal{B} \otimes \mathcal{A}_N$ for a fixed "initial algebra" $\mathcal{B}$, and $\mathcal{A}_N$ as above. Symmetrizations were only to be applied to the tensor factors of $\mathcal{A}_N$, and not to $\mathcal{B}$. But taking $\mathcal{B} = \mathcal{A}_I$, this is precisely a description of I-symmetric nets. Therefore we can immediately apply the results of [**RW1**](Compare also Theorem 2.1 in [**Du1**]). Thus for I-symmetric nets the limit in (1) exists, and is a weak*-continuous function $X_\infty : K \to \mathcal{B} \equiv \mathcal{A}_I \hookrightarrow \mathcal{A}_{loc}$. Moreover, every $f \in \mathcal{C}(K, \mathcal{A}_I)$ is of the form $f = X_\infty$ for some I-symmetric X. The isometry and homomorphism properties are also shown in [**RW1**].

Since every quasi-symmetric net is uniformly approximated by I-symmetric ones with finite I, existence and continuity of the limit, isometry property and homomorphism property immediately carry over from the I-symmetric case. It remains to prove (3) and that $X \mapsto X_\infty$ is onto. We have already seen that on I-symmetric nets this map is onto $\mathcal{C}(K, \mathcal{A}_I)$. Hence suppose that X is quasi-symmetric and $X_\infty \in \mathcal{C}(K, \mathcal{A}_I)$. Hence there is an I-symmetric net Y such that $X_\infty = Y_\infty$. By (2) this means that $\|X_\infty - Y_\infty\| = \lim_N \|X_N - Y_N\| = 0$. Hence X is approximated uniformly for large N by an I-symmetric net, and must be I-symmetric by Lemma 2.3.

To see that $X \mapsto X_\infty$ is onto, let $f \in \mathcal{C}(K, \mathcal{A}_{loc})$. Since $\bigcup_I \mathcal{C}(K, \mathcal{A}_I)$ is dense in $\mathcal{C}(K, \mathcal{A}_{loc})$ we can find for any summable sequence ε_ν a sequence of tagged sets R_ν and $X^\nu \in \mathcal{A}_{R_\nu}$ such that

$$f = \sum_\nu j_{\infty R_\nu} X^\nu \qquad \text{with} \qquad \|j_{\infty R_\nu} X^\nu\| \leq \varepsilon_\nu \quad ,$$

where $j_{\infty R} X_R$ denotes the limit Y_∞ for the basic net $Y_\bullet = j_{\bullet R} X_R$. The idea of the proof is to pick a sequence S_ν of tagged sets which increases sufficiently fast, and to set

$$X_N = \sum_{\substack{\nu \\ S_\nu \Subset N}} j_{N R_\nu} X^\nu \quad .$$

Note that every ν is eventually included in this sum because the tagged subsets $N^\top$ absorb $\mathcal{N}$ as $N \to \infty$. Since $\|j_{\infty R_\nu} X^\nu\| = \lim_N \|j_{N R_\nu} X^\nu\|$ we can pick S_ν such that for $N \Supset S_\nu$ we have $\|j_{N R_\nu} X^\nu\| \leq 2\varepsilon_\nu$. The sum defining X_N is then convergent for every N. For later use we note that the numbers

$$\delta_N = \sum_{\substack{\nu \\ S_\nu \Subset N}} \|j_{N R_\nu} X^\nu\|$$

converge to a finite limit.

We now have to show that for sufficiently rapidly growing S_ν the net X becomes quasi-symmetric. With the estimate Proposition 2.2(2) we get

$$\begin{aligned} \|X_N - j_{NM}X_M\| &\leq \sum_{\substack{\nu \\ S_\nu \Subset M}} \left\| \left(j_{NR_\nu} - j_{NM} \circ j_{MR_\nu}\right) X^\nu \right\| + \sum_{\substack{\nu \\ S_\nu \Subset N;\, S_\nu \not\Subset M}} \|j_{NR_\nu} X^\nu\| \\ &\leq \sum_{\substack{\nu \\ S_\nu \Subset M}} \|X^\nu\| \cdot 4|R_\nu| \frac{\left|M^\top\right|}{|M|} + (\delta_N - \delta_M) \quad . \end{aligned}$$

If S_ν is chosen large enough the ν^{th} term in the sum is only present if M is large in the sense of the basic net along which we take all limits. Since $\left|M^\top\right|/|M| \to 0$ as $M \to \infty$ in that net, we can pick S_ν such that the ν^{th} term is bounded by ε_ν for all N, M. Hence the sum converges absolutely, and vanishes in the limit $\lim_{M\to\infty} \operatorname{limsup}_{N\to\infty}$. The second term vanishes because the δ_N converge. It is evident from the construction that $X_\infty = \lim_N j_{\infty N} X_N = \sum_\nu j_{\infty R_\nu} X^\nu = f$. Hence $X \mapsto X_\infty$ is surjective.

■

3. The dynamics of quasi-symmetric observables.

In the previous section we have identified the quasi-symmetric nets as the appropriate mean-field nets of observables. Suppose a dynamics for the mean-field system is given. By this we mean that for each N in our fixed net of subregions of $\mathcal{N}$ there is specified a semigroup $T_{t,N} : t \geq 0$ of completely positive unit preserving linear maps on $\mathcal{A}_N$. We can say that the dynamics has good mean-field properties if at least it maps the set of quasi-symmetric nets into itself. In the first part of this section we shall formalize the notion of a mean-field dynamical semigroup as a dynamics which in addition gives rise to a well-defined limiting semigroup in the inductive limit space $\mathcal{A}_\infty$. The dynamical semigroups considered in **[Du1]** had the prima facie weaker property that they preserved only I-symmetry for each finite $I \subset \mathcal{N}$. We will show that this is in fact an equivalent property to the preservation of quasi-symmetry under the additional hypothesis that each $T_{t,N}$ is permutation symmetric.

In physical models it is a set of generators G_N of the dynamics $T_{t,N} = e^{tG_N}$ which will usually be provided; this by way of a net of Hamiltonians or a net of dissipative maps. Thus one will want to determine whether a given net of generators exponentiates to form a mean-field dynamical semigroup, and in that case to compute the limiting semigroup on $\mathcal{A}_\infty$.

Our aim in this section is to demonstrate that a wide class of dissipative interactions in quantum lattice systems do indeed generate mean-field dynamical semigroups. These can be thought of as the mean-field version of interactions with infinite range, but subject to a relatively weak decay condition. Indeed, we are able to show that the decay conditions required for the existence of a limiting dynamics are strictly weaker those required for the corresponding translation invariant interaction. Of course, this class includes interactions involving no more than a fixed finite number of sites as a special case. Apart from proving the existence of the limiting dynamics for the class of lattice models, we obtain a form for the limiting dynamics which shows that observables living on different tagged sites evolve independently according to the (time-dependent) average state of the system. This conforms with the intuitive physical picture of mean-field dynamics. We stress, however, that mean-field dynamical limits need not in general have this property. Indeed, in section 4.5 of this paper we construct examples of mean-field dynamical limits which do not.

We will start the section by generalizing the mean-field dynamics of I-symmetric sequences as described in [**Du1**] to that of quasi-symmetric nets. We then describe the dynamics of quasi-symmetric nets under the influence of generators of a fixed polynomial degree, and demonstrate the factorization property of the dynamics in the thermodynamic limit. Finally, we show that the dynamics of the lattice class of models can be approximated by those with polynomial generators (i.e. those in which only a finite number of sites interact) and show that the factorization of the dynamics is preserved by this approximation.

We will call a net of operators $T_\bullet$ **quasi-symmetry preserving** if it maps the set of quasi-symmetric nets onto itself, that is if $X_\bullet \in \mathcal{Y} \;\Rightarrow\; T_\bullet X_\bullet \in \mathcal{Y}$. The proof of the following Lemma is a straightforward modification of Lemma 2.2 of [**DW1**].

Lemma 3.1. *Let $T_\bullet$ be a uniformly bounded net operators which is quasi-symmetry preserving. Then there exists a unique operator T_∞ on $\mathcal{A}_\infty$ such that for all quasi-symmetric nets $X_\bullet$ $(T_\bullet X_\bullet)_\infty = T_\infty X_\infty$.*

Definition 3.2. *A net $T_{t,\bullet}$: $t \geq 0$ of completely positive unital (i.e. identity preserving) contractions is called a* **mean-field dynamical semigroup** *if*

(1) for each $t \geq 0$, $T_{t,\bullet}$ is quasi-symmetry preserving,

(2) $[0,\infty) \ni t \mapsto T_{t,\infty}$ is a stongly continuous contraction semigroup on $\mathcal{A}_\infty$.

The requirement of strong continuity for the limit semigroup $T_{t,\infty}$ can be seen as a statement about uniformity of the continuity of the $T_{t,N}$ with N. Indeed, it can be shown (c.f. Theorem 2.3 of [**DW1**]) that 3.2(2) is implied by 3.2(1) under the additional requirement that

$$\lim_{\substack{t\to 0\\ N\to\infty}} \|T_{t,N}X_N - X_N\| = 0$$

for all $X_\bullet \in \mathcal{Y}$.

For any I-symmetric net $X_\bullet$ (for example, a net which is J-symmetric for some $J \subset I$), we will find it useful to refer explicitly to its mean-field limit as an element of $\mathcal{C}(K,\mathcal{A}_I)$, rather than the injection into $\mathcal{C}(K,\mathcal{A}_{loc})$. We will use the symbol X_∞^I for this purpose.

Corresponding to Lemma 3.1 we have for each finite $I \subset \mathcal{N}$ a notion of I-symmetry preservation for nets of maps. Moreover, as is detailed in [**Du1**], a uniformly bounded I-symmetry preserving net of maps $T_\bullet$ has a unique limit T^I_∞ on $\mathcal{C}(K, \mathcal{A}_I)$ such that for all I-symmetric nets $X_\bullet$, $(T_\bullet X_\bullet)^I_\infty = T^I_\infty X^I_\infty$. For $I \subset R$, $j^I_{\infty R} X_R$ will denote the limit function X^I_∞ corresponding to the basic I-symmetric net $j^I_{\bullet R} X_R$.

Suppose that a net of maps $T_\bullet$ is I-symmetry preserving for all finite $I \subset \mathcal{N}$. Since we view $\mathcal{A}_I$ as a subalgebra of $\mathcal{A}_{loc}$, we canonically regard T^I_∞ as a map on the subalgebra $\mathcal{C}(K, \mathcal{A}_I) \subset \mathcal{C}(K, \mathcal{A}_{loc}) \equiv \mathcal{A}_\infty$. Now the union over I of the subalgebras $\mathcal{C}(K, \mathcal{A}_I)$ is dense in $\mathcal{A}_\infty$. Thus we might expect to construct from the maps T^I_∞ a map T_∞ as a limit of quasi-symmetry preserving maps on $\mathcal{Y}$.

It will be the case in all examples which we treat that $T_\bullet$ is permutation symmetric in the sense that for all tagged sets N, T_N commutes with any automorphism $\hat{\pi}$ of $\mathcal{A}_N$ induced by a permutation π of N. Note that this means that $T_{t,N}$ is independent of the tagging $N^\top$. With permutation invariance the notions of "quasi-symmetry preservation" and "I-symmetry preservation for all finite $I \subset \mathcal{N}$" become equivalent.

Theorem 3.3. *Let $T_\bullet$ be a net of unital permutation-symmetric contractions. Then the following are equivalent:*

(1) $T_\bullet$ is I-symmetry preserving for each finite $I \subset \mathcal{N}$

(2) $T_\bullet$ is quasi-symmetry preserving.

Proof: (1)$\Longrightarrow$(2) Since $T_\bullet$ is I-symmetry preserving for all finite $I \subset \mathcal{N}$, it is quasi-symmetry preserving on the dense subset of basic nets in $\mathcal{Y}$. Approximating any quasi-symmetric net as closely as desired by a basic net we see that $T_\bullet$ is quasi-symmetry preserving on the whole of $\mathcal{Y}$.

(2)$\Longrightarrow$(1) Let $X_\bullet$ be I-symmetric. Then $X_\bullet$ and hence $T_\bullet X_\bullet$ are quasi-symmetric. But by permutation symmetry of $T_\bullet$ we have $T_\bullet X_\bullet = T_\bullet p^I_\bullet X_\bullet = p^I_\bullet T_\bullet X_\bullet$, which by Proposition 2.4(2) is I-symmetric.

■

It is worth remarking at this point by analogous reasoning to that used in the proof of the above Theorem, one can compare the I-symmetric limits and J-symmetric limits of $T_\bullet X_\bullet$ for an I-symmetric net when $I \subset J$. Since $T_\bullet X_\bullet$ is I-symmetric, it is also J-symmetric with limit $(T^I_\infty X^I_\infty) \otimes \mathbb{1}_{J \setminus I}$. But from Proposition 2.4(1) X is J-symmetric and $X^J_\infty = X^I_\infty \otimes \mathbb{1}_{J \setminus I}$. Thus the family of operators T^I_∞ obeys the **consistency relation**

$$T^J_\infty (X^I_\infty \otimes \mathbb{1}_{J \setminus I}) = T^I_\infty X^I_\infty \otimes \mathbb{1}_{J \setminus I} \quad .$$

Corollary 3.4. *Replace definition 3.2 by the weaker statement that for all finite $I \subset \mathcal{N}$, $T_{t,\bullet}$ is I-symmetry preserving and has a strongly continuous limit $T^I_{t,\infty}$ on $\mathcal{C}(K, \mathcal{A}_I)$. If each $T_{t,N}$ is permutation symmetric, then $T_{t,\bullet}$ is a mean-field dynamical semigroup.*

Proof: By Theorem 3.3, for each $t \geq 0$, $T_{t,\bullet}$ is quasi-symmetric preserving. Since for each finite I, $t \mapsto T^I_{t,\infty}$ is strongly continuous, $T_{t,\infty}$ is strongly continuous on the dense set $\cup_I \mathcal{C}(K, \mathcal{A}_I)$; and since $\|T^I_{t,\infty}\| \leq 1$, $T_{t,\infty}$ extends to a strongly continuous contraction semigroup on the whole of $\mathcal{A}_\infty$.

■

We now turn to the question of finding nets of operators which generate mean-field dynamical semigroups. We deal first with perhaps the simplest class of generators: those which are constructed for each N by resymmetrization of an interaction of a fixed finite number of sites, and rescaled by the system size $|N|$. For any C*-algebra $\mathcal{V}$ let $\mathcal{B}(\mathcal{V})$ denote the set of bounded linear operators on $\mathcal{V}$. Define the symmetrization operator $\mathrm{Sym}_N : \bigcup_{M \subset N} \mathcal{B}(\mathcal{A}_M) \to \mathcal{B}(\mathcal{A}_N)$ by setting setting $\mathrm{Sym}_N G_M$ to be the average over all bijective maps $\eta : N \to N$ of $\hat{\eta}^{-1}(G_M \otimes \mathrm{id}_{N \setminus M})\hat{\eta}$. Thus $\mathrm{Sym}_N G_M$ is the average over the copies $G_{\eta(M)}$ of G_M acting on all possible subsets $\mathcal{A}_{\eta(M)}$ of $\mathcal{A}_N$.

Definiton 3.5. *A net of operators $G_\bullet$ will be called a* **bounded polynomial generator** *of* **degree** *R if for some $R \subset \mathcal{N}$ and all $N \supset R$,*

$$G_N = \frac{|N|}{|R|} \mathrm{Sym}_N G_R \quad ,$$

where G_R is the generator of a semigroup of completely positive unital maps on $\mathcal{A}_R$, and $\|G_R\| \equiv \gamma < \infty$.

One sees by use of the Trotter product formula that each $T_{t,N} = e^{tG_N}$ is completely positive.

The scaling $(|N|/|R|)$ in Definition 3.5 means that for each N, each site responds to a mean of its interaction with all other sites. For example if $|R| = 2$ then for all $A \in \mathcal{A}$,

$$G_N(A \otimes \mathbb{1}_{N \setminus \{1\}}) = \frac{1}{2(|N|-1)} \sum_{x \in N} \mathbb{1}_{N \setminus \{1,x\}} \otimes (G_{\{1,x\}} + G_{\{x,1\}})(A \otimes \mathbb{1}) \quad .$$

The I-symmetric properties of semigroups with bounded polynomial generators have been investigated in [**Du1**]. We can extend these as follows.

Theorem 3.6. *Let $G_\bullet$ be a bounded polynomial generator of degree R, and set $T_{t,\bullet} = e^{tG_\bullet} : t \geq 0$. Then*

(1) T is a mean-field dynamical semigroup.

(2) T has the **disjoint homomorphism property**, *namely, for all finite $I \subset \mathcal{N}$*

$$T^I_{t,\infty} = \bigotimes_{i \in I} T^{\{i\}}_{t,\infty} \quad ,$$

where the tensor product is to be understoood in the range space $\mathcal{A}_I$ of $\mathcal{C}(K, \mathcal{A}_I)$, and each $T^{\{i\}}_{t,\infty}$ is an isomorphic copy of the same map.

(3) The restriction of $T_{t,\infty}$ to the intensive (i.e. $\emptyset$-symmetric) observables is implemented by a weak-continuous flow $\mathcal{F}_t : t \geq 0$ on K, i.e. for X_∞ intensive and $t \geq 0$,*

$$T_{t,\infty} X_\infty = X_\infty \circ \mathcal{F}_t \quad .$$

where $K \times [0, \infty) \ni (\rho, t) \mapsto \mathcal{F}_t \rho \in K$ is jointly continuous and $\mathcal{F}_t \mathcal{F}_s = \mathcal{F}_{t+s}$.

Proof: (1) By section 5 of [**Du1**], for each finite $I \subset \mathcal{N}$, $t \mapsto T_{t,\bullet}$ is I-symmetry preserving with a strongly continuous limit $t \mapsto T^I_{t,\infty}$on $\mathcal{C}(K, \mathcal{A}_I)$. Thus by Corollary 3.4 $T_{t,\bullet}$ is a mean-field dynamical semigroup.

(2) is proved in section 5 of [**Du1**] and (3) in Proposition 3.4(4) of [**DW1**].

■

We now come on to discuss the exact form of $T_{t,\infty}$ when $T_{t,\bullet}$ has a bounded polynomial generator $G_\bullet$ of degree R. For each $\rho \in K$ and $R \ni y$ define the bounded linear operator $L^\rho_{\{y\}}$ on $\mathcal{A}_{\{y\}}$ by

$$L^\rho_{\{y\}} A = \mathbb{E}^\rho_{R \setminus \{y\}} G_R (A \otimes \mathbb{1}_{R \setminus \{y\}}) \qquad \text{and set} \qquad L^\rho = L^\rho_{\{1\}} \quad . \tag{3.1}$$

Thus for a fixed ρ the $L^\rho_{\{y\}}$ are isomorphic copies of the operator L^ρ on $\mathcal{A}$. In Proposition 3.4 of [**DW1**] it was seen that L^ρ is the generator of the implementing flow $\mathcal{F}$, i.e.

$$\frac{d}{dt} \mathcal{F}_t \rho = \mathcal{F}_t \rho \circ L^{\mathcal{F}_t \rho} \quad .$$

This is the sense in which it is said in [**AM**] that L^ρ is the generator of a non-linear dynamical semigroup for mean-field models. But we now observe that L^ρ plays a more general role: it generates *local* dynamics in mean-field models. For let $X_N = j_{NR} X_R$, making $X_\bullet$ I-symmetric for any $I \subset R^\top$. Then according to Proposition 5.2 of [**Du1**],

$$(G^I_\infty X^I_\infty)(\rho) = (j^I_{\infty R} \sum_{x \in R} L^\rho_{\{x\}} X_R)(\rho) = \mathbb{E}^\rho_{R \setminus I} \sum_{x \in R} L^\rho_{\{x\}} X_R \quad .$$

We shall prove below that $t \mapsto L^{\rho_t}$ is the generator of what we term the **local cocycle** $t \mapsto \Lambda^\rho_t$ in $\mathcal{B}(\mathcal{A})$ which (i) implements the flow $\mathcal{F}_t \rho = \rho \circ \Lambda^\rho_t$; and (ii) has products which implement the local evolutions: $(T^I_{t,\infty} X^I_\infty)(\rho) = (\Lambda^\rho_t)^I (X^I_\infty)(\mathcal{F}_t \rho)$. We start be considering the cocycle. In Lemma 3.7 we establish the existence of solutions to the differential equation $\dot{\Lambda}^\rho_t = \Lambda^\rho_t \circ L^{\mathcal{F}_t \rho}$. The topological Lemma 3.8 is required to determine continuity of the solution in Proposition 3.9.

Lemma 3.7.

(1) The equation

$$\frac{d}{dt}\Lambda_t^\rho = \Lambda_t^\rho \circ L^{\mathcal{F}_t\rho} \quad ,$$

with initial condition $\Lambda_0^\rho = \mathrm{id}$ *has a unique solution* $[0,\infty) \times K \ni (t,\rho) \mapsto \Lambda_t^\rho \in \mathcal{B}(\mathcal{A})$.

(2) The local cocycle Λ *of (1) has the composition law*

$$\Lambda_s^\rho \circ \Lambda_t^{\mathcal{F}_s\rho} = \Lambda_{s+t}^\rho \quad .$$

Proof: (1) $\|L^\rho\| \le \gamma$. Thus, existence and uniqueness of a norm-continuous solution of the integral equation

$$\Lambda_t^\rho = \mathrm{id} + \int_0^t ds \Lambda_s^\rho L^{\mathcal{F}_s\rho} \tag{3.2}$$

follows by standard methods (see e.g. **[HS]**). We clearly have the norm estimates

$$\|\Lambda_t^\rho\| \le e^{\gamma t} \qquad \text{and} \qquad \lim_{t\to 0} \sup_{\rho\in K} \|\Lambda_t^\rho - \mathrm{id}\| = 0 \quad . \tag{3.3}$$

(2) For all $\rho \in K$ and $t \ge s \ge 0$ define $\Gamma^\rho(s,t) = \Lambda_s^\rho \Lambda_{t-s}^{\mathcal{F}_s\rho}$ Then

$$\frac{d}{dt}\Gamma^\rho(s,t) = \Gamma^\rho(s,t) L^{\mathcal{F}_t\rho} \qquad \text{and} \quad \Gamma^\rho(s,s) = \Lambda_s^\rho \quad .$$

So for fixed s and ρ we have that for $t \ge s$ the map $t \mapsto \Gamma^\rho(s,t)$ obeys the same differential equation as Λ_t^ρ, and has the same boundary value at the point $t = s$. Thus by uniqueness in part (1) above, $\Gamma^\rho(s,t) = \Lambda_t^\rho$ for all $t \ge s$.

■

Lemma 3.8. *Let* Ω_0 *be a compact set in* $\mathcal{A}$. *Then there exists a compact set* $\Omega \supset \Omega_0$ *such that for any* $\gamma' > \gamma$,

$$\rho \in K,\ A \in \Omega \Longrightarrow L^\rho \in \gamma'\Omega \quad .$$

Proof: Since for any $X \in \mathcal{A}_R$, $\rho \mapsto \mathbb{E}^\rho_{R\setminus\{1\}} X$ is weak*-to-norm continuous and bounded, $K \times \mathcal{A} \ni (\rho, A) \mapsto L^\rho A$ is jointly continuous. Thus the set $\Omega_1 = \{L^\rho A |\, \rho \in K,\ A \in \Omega_0\}$, being the continuous image of the compact set $K \times \Omega_0$, is compact. Furthermore, $\sup_{A\in\Omega_1} \|A\| \le \gamma \sup_{A\in\Omega_0} \|A\|$.

Proceed by iteration and construct in a like manner the sequence of compact sets Ω_2, Ω_3 and so on. For any $\gamma' > \gamma$, construct the set

$$\tilde{\Omega} = \left\{ A \in \mathcal{A} \,\middle|\, A = \sum_{1=i}^{\infty} (\gamma')^{-i} t_i A_i :\ A_i \in \Omega_i,\ t_i \in [0,1] \right\} \quad .$$

Then $\tilde{\Omega}$ is bounded and $\left\{ L^\rho \tilde{\Omega} \mid \rho \in K \right\} \subset \tilde{\Omega}$. Furthermore, by construction, $\tilde{\Omega}$ can be approximated to within ε by finite sums from the compact sets $(\Omega_n)_{n\in\mathbb{N}}$ and is hence pre-compact. Taking the closure Ω of $\tilde{\Omega}$ we obtain the required set.

■

Proposition 3.9. *For each $A \in \mathcal{A}$ the map $(\rho, t) \mapsto \Lambda_t^\rho A$ is jointly continuous.*

Proof: Since by eq (3.3) $t \mapsto \Lambda_t^\rho$ is norm-continuous, uniformly in ρ, it is enough to prove that for each t, $\rho \to \Lambda_t^\rho A$ is weak*-to-norm continuous. Now $(t, \rho) \mapsto \mathcal{F}_t \rho$ and $(\rho, A) \mapsto L^\rho A$ are both jointly continuous. Thus by composition $(t, \rho, A) \mapsto L^{\mathcal{F}_t \rho} A$ is jointly continuous.

For a fixed $A \in \mathcal{A}$, let Ω be the compact set Ω corresponding to $\Omega_0 = \{A\}$ in Lemma 3.8. Then since

$$(\Lambda_t^\rho - \Lambda_t^\sigma)A = \int_0^t ds (\Lambda_s^\rho - \Lambda_s^\sigma) L^{\mathcal{F}_s \rho} A + \Lambda_s^\sigma (L^{\mathcal{F}_s \rho} - L^{\mathcal{F}_s \sigma}) A$$

we have that

$$\sup_{B \in \Omega} \|(\Lambda_t^\rho - \Lambda_t^\sigma) B\| \leq \gamma \int_0^t ds \sup_{B \in \Omega} \|(\Lambda_s^\rho - \Lambda_s^\sigma) B\| + \gamma^{-1}(e^{\gamma t} - 1)\varepsilon_t(\rho, \sigma)$$

where $\varepsilon_t(\rho, \sigma) = \sup_{0 \leq s \leq t} \sup_{B \in \Omega} \left\|(L^{\mathcal{F}_s \rho} - L^{\mathcal{F}_s \sigma}) B\right\|$. Thus, by Gronwall's Lemma (see e.g. **[HS]**)

$$\sup_{B \in \Omega} \|(\Lambda_t^\rho - \Lambda_t^\sigma) B\| \leq \gamma^{-1}(e^{\gamma t} - 1)\varepsilon_t(\rho, \sigma) \quad .$$

Since Ω, K and $[0, t]$ are compact, then by the joint continuity of $(\rho, t, A) \mapsto L^{\mathcal{F}_t \rho} A$, $\varepsilon_t(\rho, \sigma) \to 0$ as $\sigma \to \rho$ weak*. Thus $(\Lambda_t^\rho - \Lambda_t^\sigma) A \to 0$ as $\sigma \to \rho$ weak*.

■

Now according to Theorem 3.6(2) $T_{t,\infty}^I$ is constructed as a tensor product (in $\mathcal{A}_{loc}$) of $T_{t,\infty}^{\{i\}}$: $i \in I$. Thus to know $T_{t,\infty}$ it suffices to calculate one of the $T_{t,\infty}^{\{i\}}$. The purpose of Proposition 3.9 is that it enables us to verify that a possible candidate for $T_{t,\infty}^{\{i\}}$ is indeed a strongly continuous contraction semigroup on $\mathcal{C}(K, \mathcal{A})$. With no loss of generality we take $i = 1$. We define for each finite $I \subset \mathcal{N}$ the algebra

$$\mathcal{P}^I = \bigcup_{R \subset \mathcal{N}} \{ j_{\infty R}^I X \,\big|\, X \in \mathcal{A}_R \} \quad .$$

Thus $\mathcal{P}^I$ can be thought of as a dense polynomial subalgebra of $\mathcal{C}(K, \mathcal{A}_I)$ comprising the mean-field limits of basic I-symmetric nets.

Theorem 3.10. *Let $X_\bullet$ be $\{1\}$-symmetric. Then*

$$(T_{t,\infty}^{\{1\}} X_\infty^{\{1\}})(\rho) = \Lambda_t^\rho X_\infty^{\{1\}}(\mathcal{F}_t \rho)$$

Proof: Define

$$(\hat{T}^{\{1\}}_{t,\infty} X^{\{1\}}_\infty)(\rho) = \Lambda^\rho_t X^{\{1\}}_\infty(\mathcal{F}_t\rho)$$

We show that $\hat{T}^{\{1\}}_{t,\infty}$ is a strongly continuous contraction semigroup on $\mathcal{C}(K,\mathcal{A})$. By the joint continuity of $(\rho,t)\mapsto\Lambda^\rho_t$ into the strong-operator topology on $\mathcal{B}(\mathcal{A})$, and the joint continuity of $(\rho,t)\mapsto\mathcal{F}_t\rho$, then for each $X^{\{1\}}_\infty\in\mathcal{C}(K,\mathcal{A})$ we have that $(\rho,t)\mapsto\Lambda^\rho_t X^{\{1\}}_\infty(\mathcal{F}_t\rho)$ is jointly continuous, uniformly for $\rho\in K$ compact and t in compacta. Hence we conclude that $\hat{T}^{\{1\}}_{t,\infty}X^{\{1\}}_\infty\in\mathcal{C}(K,\mathcal{A})$ and that $t\mapsto\hat{T}^{\{1\}}_{t,\infty}$ is strongly continuous. Furthermore we have the composition law

$$(\hat{T}^{\{1\}}_{t,\infty}\hat{T}^{\{1\}}_{s,\infty}X^{\{1\}}_\infty)(\rho) = \Lambda^\rho_t\Lambda^{\mathcal{F}_t\rho}_s X^{\{1\}}_\infty(\mathcal{F}_t\rho) = \Lambda^\rho_{t+s}X^{\{1\}}_\infty(\mathcal{F}_{t+s}\rho) = (\hat{T}^{\{1\}}_{t+s,\infty}X^{\{1\}}_\infty)(\rho) \quad ,$$

where the second equality uses the composition law of Λ. Since $\left\|\hat{T}^{\{1\}}_{t,\infty}\right\| \le \|\Lambda^\rho_t\| \le e^{\gamma t}$ we conclude from Proposition 1.17 of [**Dav**] that $\hat{T}^{\{1\}}_{t,\infty}:\ t\ge 0$ is a strongly continuous semigroup on $\mathcal{C}(K,\mathcal{A})$.

We calculate the action of the generator of $t\mapsto\hat{T}^{\{1\}}_{t,\infty}$ on a $\{1\}$-symmetric basic function of degree $R\ni 1$. By Lemma 3.7, $t\mapsto\Lambda^\rho_t$ is differentiable uniformly in ρ, and by Proposition 3.4 of [**DW1**], so is $t\mapsto\mathcal{F}_t\rho$ (in the weak* sense). So we can differentiate:

$$\begin{aligned}\frac{d}{dt}(\hat{T}^{\{1\}}_{t,\infty}X^{\{1\}}_\infty)(\rho)\Big|_{t=0} &= \frac{d}{dt}\Lambda^\rho_t\mathbb{E}^{\mathcal{F}_t\rho}_{R\setminus\{1\}}X_R\Big|_{t=0}\\ &= \mathbb{E}^\rho_{R\setminus\{1\}}\sum_{x\in R}L^\rho_{\{x\}}X_R\\ &= (G^{\{1\}}_\infty X^{\{1\}}_\infty)(\rho) \quad .\end{aligned}$$

Thus the generator, $\hat{G}^{\{1\}}_\infty$, of $t\mapsto\hat{T}^{\{1\}}_{t,\infty}$ agrees with $G^{\{1\}}_\infty$ on $\mathcal{P}^{\{1\}}$. Since $\left\|\hat{T}^{\{1\}}_{t,\infty}\right\|\le e^{\gamma t}$, any $\kappa>\gamma$ lies in the resolvent set of $\hat{G}^{\{1\}}_\infty$. For such κ, $(\kappa-\hat{G}^{\{1\}}_\infty)\mathcal{P}^{\{1\}} = (\kappa-G^{\{1\}}_\infty)\mathcal{P}^{\{1\}}$. But it is proved in Proposition 5.3(3) of [**Du1**] that $\mathcal{P}^{\{1\}}$ is a core for $G^{\{1\}}_\infty$, and consequently $(\kappa-\hat{G}^{\{1\}}_\infty)\mathcal{P}^{\{1\}}$ must be dense in $\mathcal{C}(K,\mathcal{A})$. By Proposition 2.1 of [**Dav**], $\mathcal{P}^{\{1\}}$ is also as core for $\hat{G}^I_\infty$. Since the generators $\hat{G}^{\{1\}}_\infty$ and $G^{\{1\}}_\infty$ agree on a core, they are equal, and so $\hat{T}^{\{1\}}_{t,\infty}=T^{\{1\}}_{t,\infty}$ for all $t\ge 0$.

■

Using our formalism the positivity and flow-implementing properties of Λ^ρ_t follow straightforwardly.

Proposition 3.11.

(1) Each Λ^ρ_t is completely positive and unital.

(2) $\mathcal{F}_t\rho = \rho\circ\Lambda^\rho_t$.

Proof: (1) For any R with $1\in R$

$$\Lambda^\rho_t X = (T^{\{1\}}_{t,\infty}j_{\infty\{1\}}RX\otimes\mathbb{I}_{R\setminus\{1\}})(\rho) = \lim_{N\to\infty}\mathbb{E}^\rho_{N\setminus\{1\}}T_{t,N}(X\otimes\mathbb{I}_{N\setminus\{1\}}) \quad .$$

Since $X \mapsto X \otimes \mathbb{I}_{\mathcal{N}\setminus\{1\}}$, $T_{t,N}$ and $\mathbb{E}^{\rho}_{\mathcal{N}\setminus\{1\}}$ are all completely positive unital maps, Λ^{ρ}_t, as a limit of such maps, is also completely positive and unital.

(2) For $A \in \mathcal{A}$,

$$\begin{aligned}\langle \rho \circ \Lambda^{\rho}_t, A\rangle &= \langle \rho, \Lambda^{\rho}_t(j^{\{1\}}_{\infty\{1\}}A)(\mathcal{F}_t\rho)\rangle = \langle \rho, (T^{\{1\}}_{t,\infty} j_{\infty\{1\}}A)(\rho)\rangle \\ &= \lim_{N\to\infty} \langle \rho^N, T_{t,N}\; j_{N\{1\}}A\rangle = (j_{\infty\{1\}}A)(\mathcal{F}_t\rho) = \langle \mathcal{F}_t\rho, A\rangle\end{aligned}$$

■

Before extending Theorem 3.10 to treat the evolution of quasi-symmetric observables, note that since each Λ^{ρ}_t is completely positive and unital, then by Theorem 4.23 of [**Tak**] the product map $\Lambda^{\rho}_t \otimes \ldots \otimes \Lambda^{\rho}_t$ (with $|I|$ factors) on the $|I|$-fold algebraic tensor product $\mathcal{A}^{\odot I}$ extends to a completely positive unital map on $\mathcal{A}_I$. We denote this latter map by $(\Lambda^{\rho}_t)^I$. Being positive and unital $\left\|(\Lambda^{\rho}_t)^I\right\| = 1$. We can extend each $(\Lambda^{\rho}_t)^I$ to $\mathcal{B}(\mathcal{A}_{loc})$ by tensoring with the identity map, and construct the infinite tensor product $(\Lambda^{\rho}_t)^{\infty} = \lim_{I\nearrow\mathcal{N}}(\Lambda^{\rho}_t)^I$, the limit being in the strong operator topology of $\mathcal{B}(\mathcal{A}_{loc})$. Our final theorem for bounded polynomial generators is as follows.

Theorem 3.12. *Let $T_{t,\bullet} = e^{tG_\bullet}$ with $G_\bullet$ a bounded polynomial generator. Then Λ_t* **locally implements** *$T_{t,\infty}$ in the sense that for all $X \in \mathcal{Y}$,*

$$\big(T_{t,\infty}X_{\infty}\big)(\rho) = (\Lambda^{\rho}_t)^{\infty} X_{\infty}(\mathcal{F}_t\rho) \quad . \tag{3.4}$$

Proof: Combining Theorem 3.10 with Theorem 3.6(3) we see that equation (3.4) holds for I-symmetric nets $X_\bullet$ with limits of the form $X^I_{\infty} = A^{i_1}_{\infty} \otimes \ldots Z^{i_{|I|}}_{\infty}$. Since $(\Lambda^{\rho}_t)^I$ is bounded, one obtains the stated result for any function in $\mathcal{C}(K, \mathcal{A}_I)$ by approximation with limits of sums of such terms. The final form is obtained by approximating nets in $\mathcal{Y}$ by basic nets.

■

Recalling that $(T_{t,\infty}X_{\infty})(\rho) = \lim_{N\to\infty} \mathbb{E}^{\rho}_{\mathcal{N}\setminus N^{\top}} T_{t,N}X_N \in \mathcal{A}_{loc}$, the form of $T_{t,\infty}$ given above shows that Λ^{ρ}_t implements the one-site evolution of tagged sites when the bulk (of untagged sites) is in the product state formed from ρ. In the remainder of this section we extend Theorem 3.12 beyond the bounded polynomial generators. Consider the following nets of generators.

Definition 3.13. *A net of operators $G_\bullet$ will be called* **lattice class** *if for each finite $M \subset \mathcal{N}$ there exists net $N \mapsto \Gamma^N_M \in \mathcal{B}(\mathcal{A}_M)$ such that following condtions hold.*

(1) $\Gamma^N_M = 0$ for all $N \subset M$.

(2) $\Gamma_M = \lim_{N\to\infty} \Gamma^N_M$ exists in the strong operator topology.

(3) The bounds $\gamma_M \equiv \sup_{N\supset M} \left\|\Gamma^N_M\right\|$ are summable so that $\sum_M |M|\gamma_M \equiv \gamma < \infty$.

(4) For each N

$$G_N \equiv \sum_{M \subset N} \frac{|N|}{|M|} \,\mathrm{Sym}_N(\Gamma_M^N)$$

is the generator of a norm-continuous semigroup of completely positive unital contractions on $\mathcal{A}_N$.

This definition makes sense not only for nets of generators, but also of general bounded operators on $\mathcal{A}_N$. For G_N to generate it is sufficient, but by no means necessary, that each Γ_M^N generates on $\mathcal{A}_M$. The polynomial generators (resp. operators) are the special case, where Γ_M^N is non-zero only for some M, and independent of N. The next level of complexity is to allow the N-dependence, but to retain only one fixed M. A generator constructed in this way is asymptotically equal to the polynomial generator constructed from $\Gamma_M = \lim_M \Gamma_M^N$. In this case the "lattice class bound" γ is $\gamma = |M| \sup_N \left\|\Gamma_M^N\right\|$. If for each i in some index set G^i is a lattice class net of operators on $\mathcal{B}(\mathcal{A})$ with lattice class bounds γ^i such that $\sum_i \gamma^i < \infty$ the sum $G_N \equiv \sum_i G_N^i$ exists for all N, and defines again a lattice class net with bound $\gamma \leq \sum_i \gamma^i$. It is useful to note that the sets M in this definition enter only via their cardinality: due to the symmetrization over M implicit in Sym_N the labelling of the set M becomes completely irrelevant. By adding up all terms coming from M's of the same cardinality we can reduce the sum over M to a sum over only one standard set M, say $\{1, \ldots, |M|\}$.

The lattice class generators can be seen to arise in the following way. Let $\mathcal{N} = \mathbb{Z}^d$, and let the fixed net of regions be such that $N \to \infty$ in the sense of Van Hove [**Rue**]. $\mathcal{S}$ will denote the set of finite subsets of $\mathbb{Z}^d$. Suppose that a translation invariant family of generators $M \mapsto \Gamma_M \in \mathcal{B}(\mathcal{A}_M)$ is specified. Construct the generator net

$$\hat{G}_N = \sum_{M \ni 0} \frac{1}{|M|} \sum_{\substack{x \in \mathbb{Z}^d \\ M+x \subset N}} \Gamma_{M+x} \quad .$$

$\hat{G}_\bullet$ is, of course, translation invariant rather than permutation invariant. When $G_M(\cdot) = i[\Phi_M, \cdot]$ for some family (Φ_M) of self adjoint potentials, it can be shown [**BR**] that a limiting dynamics exists provided that $\sum_{M \in \mathcal{S}} e^{|M|} \|\Gamma_M\|$ is finite. But it is shown in [**DW1**] that the symmetrized version of this interaction $N \mapsto G_N = \mathrm{Sym}_N \hat{G}_N$ is lattice class. For lattice class interactions it is then proved in [**DW1**] that a limiting dynamics for intensive (i.e. $\emptyset$-symmetric) observables exists. We see from Definition 3.13(3) of the lattice class that this means that this dynamics exists under the condition that $\gamma = \sum_{M \in \mathcal{S}} |M| \|\Gamma_M\|$ is finite, a considerably weaker condition than that of [**BR**].

In the remainder of this section we will show that for lattice class generators, the limiting dynamics exists for *all* quasi-symmetric nets, and furthermore that this dynamics is locally implemented as in Theorem 3.12.

With the Γ_M as in Definition 3.13, define the bounded polynomial generator net $G_\bullet^M$ by

$$G_N^M = \sum_{\hat{M}\subset M} \frac{|N|}{|\hat{M}|} \operatorname{Sym}_N \Gamma_{\hat{M}} \quad .$$

We aim to show that $G_\bullet$ generates a mean-field dynamical semigroup by showing that it can be approximated by those generated by the $G_\bullet^M$. When we assume that $\mathcal{A}$ is finite dimensional this turns out to be quite easy to prove. In view of the calculation of the $\emptyset$-symmetric mean-field dynamics for lattice class generators in section 4 of [**DW1**], we expect that the proof for $\mathcal{A}$ infinite dimensional is possible, albeit lengthy.

Denote by $\Lambda_t^{M,\rho}$ the cocycle which locally implements the mean-field dynamical semigroup generated by $G_\bullet^M$, and denote by $L^{M,\rho}$ its generator. $\mathcal{F}_t^M$ will be the corresponding flow on K.

Lemma 3.14. *Let $G_\bullet$ be lattice class, and let $\mathcal{A}$ be finite dimensional. Then the norm limits*

$$L^\rho A = \lim_{M\to\infty} L^{M,\rho}A = \sum_M \mathbb{E}^\rho_{M\setminus\{1\}}\Gamma_M(A\otimes \mathbb{I}_{M\setminus\{1\}})$$

and $\Lambda_t^\rho \equiv \lim_{M\to\infty}\Lambda_t^{M,\rho}$ exist, are continuous functions of ρ, and satisfy equation (3.2). Λ_t^ρ is completely positive and unital.

Proof: Summing the terms in $G_\bullet^M$ we see by comparison with equation (3.1) that

$$L^{M,\rho}A = \sum_{\hat{M}\subset M} \mathbb{E}^\rho_{\hat{M}\setminus\{1\}} G_{\hat{M}}(A\otimes \mathbb{I}_{\hat{M}\setminus\{1\}}) \quad .$$

$\left\|L^{M,\rho}A\right\| \le \sum_{M'\subset M}\|\Gamma_{M'}\|\,\|A\|$. By 3.13(3) this is bounded uniformly in M and ρ by γ, and the tail $\sum_{M'\supset M}\|\Gamma_{M'}\| \to 0$ as $M\to\infty$. Hence $L^{M,\rho}A$ is convergent as $M\to\infty$ to the form of $L^\rho A$ given. Since convergence is uniform in ρ and for each M $\rho\mapsto L^{M,\rho}$ is continuous, then $\rho\to L^\rho A$ is continuous. According to Theorem 4.11 and Proposition 4.6(2) of [**DW1**], the flows $\mathcal{F}_t^M\rho$ converge weak* as $M\to\infty$, uniformly for t in compacta, to some $\mathcal{F}_t\rho\in K$, where $t\mapsto\mathcal{F}_t$ is a weak*-continuous flow on K. Since $\mathcal{A}$ is finite dimensional this holds in the norm topology of K as well. It is now a straightforward matter to show that $\Lambda_t^{M,\rho}$ converges uniformly to the unique norm-continuous solution Λ_t^ρ of the equation $\Lambda_t^\rho = \mathrm{id}+\int_0^t ds\Lambda_s^\rho L^{\mathcal{F}_s\rho}$. Since convergence is uniform, $\rho\mapsto\Lambda_t^\rho$ is continuous. As a limit of completely positive unital maps, Λ_t^ρ is completely positive and unital.

■

Theorem 3.15. *Let $G_\bullet$ be of lattice class, with $\mathcal{A}$ finite dimensional. Then $G_\bullet$ is the generator of mean-field dynamical semigroup which is locally implemented by the Λ_t^ρ of Lemma 3.14, and which hence has the disjoint homomorphism property.*

Proof: Since we work in the norm topology of K it is a simple matter to show that for all finite $I \subset \mathcal{N}$, $(\hat{T}^I_{t,\infty} f) = (\Lambda^\rho_t)^I f(\mathcal{F}_t\rho)$ defines a strongly continuous contraction semigroup on $\mathcal{C}(K, \mathcal{A}_I)$. One differentiates to find the action of its generator $\hat{G}^I_\infty$ on basic I-symmetric nets $X_\bullet = j^I_{\bullet R} X_R$ with $I \subset R$ as

$$(\hat{G}^I_\infty X^I_\infty)(\rho) = \mathbb{E}^\rho_{R\setminus I} \sum_{x\in R} L^\rho_{\{x\}} X_R \quad .$$

But this is equal to $(G^I_\infty X^I_\infty)(\rho)$. For $G_\bullet j_{\bullet R} X_R = \sum_M Y^{(M)}_\bullet$ where for for each M, $Y^{(M)}_\bullet$ is the quasi-symmetric net $N \mapsto Y^{(M)}_N = (|N|/|M|)(\mathrm{Sym}_N \Gamma^N_M) j_{NR} X_R \ : \ N \supset M$. By 3.13(3), $M \mapsto \|Y^{(M)}\|$ is summable, so that for each $\varepsilon > 0$ there exists M_ε such that $\|G_\bullet j_{\bullet R} X_R - \sum_{M\subset M_\varepsilon} Y^{(M)}_\bullet\| < \varepsilon$. Hence $G_\bullet j_{\bullet R} X_R$ is quasi-symmetric and

$$\begin{aligned}
(G_\bullet j_{\bullet R} X_R)^I_\infty(\rho) &= \lim_{M\to\infty} \sum_{M'\subset M} Y^{(M'),I}_\infty(\rho) \\
&= \lim_{M\to\infty} \mathbb{E}^\rho_{R\setminus I} \sum_{x\in R} L^{\rho,M} X_R \\
&= \lim_{M\to\infty} \mathbb{E}^\rho_{R\setminus I} \sum_{x\in R} L^\rho X_R \\
&= (\hat{G}_\bullet j_{\bullet R} X_R)^I_\infty(\rho) \quad .
\end{aligned}$$

In Proposition 3.16 below we show that $\mathcal{P}^I$ is a core for $\hat{G}^I_\infty$. Then the above argument shows that for $s \in \mathbb{C} : \ \Re e(s) > 0$, $((s - G_\bullet)\mathcal{Y}_{bas})^I_\infty = (s - G^I_\infty)\mathcal{P}^I = (s - \hat{G}^I_\infty)\mathcal{P}^I$ is dense in $\mathcal{Y}^I$. So by the implication (4)⇒(5) of Theorem 2.3 of [**DW1**], and Theorem 3.2 of [**Du1**], G^I_∞ is well-defined and $G_\bullet$ has an I-symmetry preserving mean-field limit which is generated by G^I_∞. This is true for all I, thus $G_\bullet$ generates a mean-field dynamical semigroup $T_{t,\infty}$, and $(T_{t,\infty} X_\infty)(\rho) = (\Lambda^\rho_t)^\infty X(\mathcal{F}_t\rho)$.

■

It remains to show that $\mathcal{P}^I$ is a core for $\hat{G}^I_\infty$. Our strategy is to express $\hat{G}^I_\infty$ in terms of a derivative on $\mathcal{C}(K, \mathcal{A}_I)$, and then use standard methods to show firstly that the set of differentiable functions is preserved by $\hat{T}^{\{1\}}_{t,\infty}$ and is hence a core for $\hat{G}^I_\infty$, and secondly that each differentiable function can be approximated, along with its derivatives, by an element of $\mathcal{P}^I$.

For a unital C*-algebra $\mathcal{V}$ and $f \in \mathcal{C}(K, \mathcal{V})$ we define the gradient $\mathrm{d}f(\rho)$ of f at $\rho \in K$ by

$$\langle \sigma - \rho, \mathrm{d}f(\rho)\rangle \equiv \frac{d}{dt} f(t\sigma - (1-t)\rho)\Big|_{t=0} \quad , \tag{3.5}$$

and say that f is differentiable whenever this exists as a continuous function on K. Equation (3.5) must be understood as being $\mathcal{V}$-valued in the sense that the duality $\langle \cdot, \cdot \rangle$ is between $\mathcal{A}$ and K, leaving $\langle \sigma - \rho, \mathrm{d}f(\rho)\rangle \in \mathcal{V}$. Equation (3.5) fixes the gradient only up to a multiple of the identity. We remove this ambiguity and fix $\mathrm{d}f$ as an element of $\mathcal{C}(K, \mathcal{V} \otimes \mathcal{A})$ by imposing the convention that $\langle \rho, \mathrm{d}f(\rho)\rangle = 0$. $\mathcal{C}^1(K, \mathcal{V})$ will denote the set of differentiable functions in $\mathcal{C}(K, \mathcal{V})$. Clearly $\mathcal{P}^I \subset \mathcal{C}^1(K, \mathcal{A}_I)$.

This notion of a derivative also lifts to $\mathcal{B}(\mathcal{V})$. Let $H \in \mathcal{C}(K, \mathcal{B}(\mathcal{V}))$. Then we define $\mathrm{d}H$ to be the element of $\mathcal{C}(K, \mathcal{B}(\mathcal{V}))$ such that $(\mathrm{d}H)X = \mathrm{d}(HX)$ for each $X \in \mathcal{V}$. For example, take $\mathcal{V} = \mathcal{A}$, and let $\hat{L}$ be the local generator corresponding to a bounded polynomial generator $G_\bullet$ of degree M. Let $A \in \mathcal{A}$, $\rho, \sigma \in K$, and for $h \in [0,1]$ set $\rho_h = \rho + h(\sigma - \rho)$. Then

$$\begin{aligned}\langle \sigma, \mathrm{d}\hat{L}^\rho A\rangle &= \lim_{h\to 0}(\mathbb{E}^{\rho_h}_{M\setminus\{1\}} - \mathbb{E}^{\rho}_{M\setminus\{1\}})G_M(A \otimes \mathbb{1}_{M\setminus\{1\}}) \\ &= (|M|-1)\mathbb{E}^{\sigma-\rho}_{\{2\}}\mathbb{E}^{\rho}_{M\setminus\{1,2\}}G_M(A \otimes \mathbb{1}_{M\setminus\{1\}}) \quad .\end{aligned}$$

According to Theorem 4.11 and Proposition 4.3 of [**DW1**], the limit flow $\mathcal{F}_t\rho = \rho\circ\Lambda^\rho_t = \rho\circ\lim_M \Lambda^{M,\rho}_t$ is differentiable and hence preserves the set of differentiable complex-valued functions. In particular

$$\mathrm{d}(f \circ \mathcal{F}_t)(\rho) = J^\rho_t(\mathrm{d}f)(\mathcal{F}_t\rho)$$

for a suitable Jacobian $J^\rho_t \in \mathcal{B}(\mathcal{A})$, and furthermore there exists a bound $\|J^\rho_t\| \le e^{\gamma t}$. We require now to prove a similar result for Λ^ρ_t. Since we work with $\mathcal{A}$ finite dimensional, the proof is quite simple. Item (5) of the following proposition also provides the last remaining step in the proof of Theorem 3.15.

Proposition 3.16. *Let $\mathcal{A}$ be finite dimensional. Then*

(1) L is differentiable

(2) Λ_t is differentiable for all $t \ge 0$.

(3) For all finite $I \subset \mathcal{N}$, $\mathcal{C}^1(K, \mathcal{A}_I)$ is invariant under $\hat{T}^I_{t,\infty}$ for all $t \ge 0$.

(4) For all finite $I \subset \mathcal{N}$, $\mathcal{C}^1(K, \mathcal{A}_I)$ is a core for $\hat{G}^I_\infty$.

(5) For all finite $I \subset \mathcal{N}$, $\mathcal{P}^I$ is a core for $\hat{G}^I_\infty$.

Proof: (1)

$$\begin{aligned}h^{-1}\left\|(L^{\rho_h} - L^\rho) - (L^{M,\rho_h} - L^{M,\rho})\right\| &\le h^{-1}\sum_{M'\supset M}(|M'|-1)\left\|(\mathbb{E}^{\rho_h}_{M'\setminus\{1\}} - \mathbb{E}^{\rho}_{M'\setminus\{1\}})\Gamma_{M'}\right\| \\ &\le \|\sigma - \rho\|\sum_{M'\supset M}|M'|\|\Gamma_{M'}\| \quad . \end{aligned} \tag{3.6}$$

By Definition 3.13(3) this bound is the tail of a covergent sum. Thus the limit of the LHS of inequality (3.6) as $M \to \infty$ is zero, uniformly in h. We showed above that each L^M is differentiable, and so $\mathrm{d}L^\rho$ exists and is equal to $\lim_{M\to\infty}\mathrm{d}L^{M,\rho}$.

(2) Since $\mathcal{A}$ is finite dimensional, we consider $t \mapsto (\mathcal{F}_t\rho, \Lambda^\rho_t)$ as an integral curve of the vector field $(\dot{\rho}, \dot{\Lambda}) = (\rho\circ L^\rho, \Lambda\circ L^\rho)$ on the Banach space $K\times\mathcal{B}(\mathcal{A})$ with norm $\|(\rho,\Lambda)\| = \|\rho\|+\|\Lambda\|$. Since L is bounded and $\rho \mapsto L^\rho$ is differentiable, one sees (from section 4.1 of [**AMR**])

that $\rho \mapsto \Lambda_t^\rho$ is differentiable, at least locally in time. In fact, since $\left\|(\dot{\rho}, \dot{\Lambda})\right\| \leq \gamma \|(\rho, \Lambda)\|$, then in fact these integral curves exists for all time and are differentiable.

(3) Let $f \in \mathcal{C}^1(K, \mathcal{A}_I)$. Then clearly

$$(\mathrm{d}\hat{T}^I_{t,\infty} f)(\rho) = (\mathrm{d}(\Lambda_t^\rho)^I) f(\mathcal{F}_t \rho) + (\Lambda_t^\rho)^I J_t^\rho \mathrm{d}f(\mathcal{F}_t \rho) \quad .$$

(4) Let $f \in \mathcal{C}^1(K, \mathcal{A}_I)$. Then

$$\begin{aligned} \hat{G}^I_\infty f(\rho) &= \frac{d}{dt} (\Lambda_t^\rho)^I f(\mathcal{F}_t \rho)\Big|_{t=0} \\ &= \mathbb{E}^\rho_{R\setminus I} \sum_{x \in I} L^\rho_{\{x\}} f \langle \rho \circ L^\rho, \mathrm{d}f(\rho) \rangle \quad . \end{aligned} \tag{3.7}$$

Thus $\mathcal{C}^1(K, \mathcal{A}_I)$ is a subset of $\mathrm{dom}(\hat{G}^I_\infty)$, which by (2) and (3) is $\hat{T}^I_{t,\infty}$ invariant. Furthermore, $\mathcal{C}^1(K, \mathcal{A}_I)$ is dense in $\mathcal{C}(K, \mathcal{A}_I)$ (it contains the dense subset of polynomials $\mathcal{P}^I$) and so it is a core for $\hat{G}^I_\infty$.

(5) We complete the proof by showing any $f \in \mathcal{C}^1(K, \mathcal{A}_I)$ there is a sequence of polynomials $(f_n)_{n \in \mathbb{N}} \subset \mathcal{P}^I$ such that $\lim_{n \to \infty} f_n = f$ and $\lim_{n \to \infty} \mathrm{d}f_n = \mathrm{d}f$. For then from equation (3.7) one sees that $\lim_{n \to \infty} \hat{G}^I_\infty f_n = \hat{G}^I_\infty f$ and so $\mathcal{P}^I$ is a core for $\hat{G}^I_\infty$.

Consider the set $\mathcal{L}$ of linear functions $\left\{\rho \mapsto \mathbb{E}^\rho_{\{1\}} A \,\middle|\, A \in \mathcal{A}_{|I|+1}\right\}$ in $\mathcal{P}^I$. Clearly the algebra generated by $\mathcal{L}$ is dense in $\mathcal{P}^I$ and hence dense in $\mathcal{C}(K, \mathcal{A}_I)$. Furthermore for $\rho \neq \sigma \in K$ we can choose and g and h in $\mathcal{L}$ such that $g(\rho) \neq 0$ and $\langle \sigma - \rho, \mathrm{d}h(\rho) \rangle \neq 0$. So, by Nachbin's Theorem stated in Theorem 1.2.1 of **[Lla]**, the algebra generated by $\mathcal{L}$ is dense in $\mathcal{C}^1(K, \mathcal{A}_I)$ in the norm $\|f\|_1 = \|f\| + \|\mathrm{d}f\|$, as required.

■

4. Properties of the the limiting evolution

4.1. Hamiltonian systems

In many examples the semigroups $T_{t,N}$ are reversible in the sense that the generator is of the form

$$G_N(X) = |N| i [H_N, X] \tag{4.1}$$

with a Hamiltonian density $H_N = H_N^* \in \mathcal{A}_N$. For the thermodynamics of mean-field systems it is sufficient for H to be $\emptyset$-symmetric **[RW1]**. For the dynamics one needs to assume more, e.g. that the generator be of lattice class as in Definition 3.13. This is readily written in terms of H: we want that

$$H_N = \sum_{M\subset N} j^{\emptyset}_{NM} H^N_M \qquad \text{with} \quad H^N_M \in \mathcal{A}_M \tag{4.2}$$

$$\text{such that} \qquad \sum_M |M|^2 \sup_N \left\| H^N_M \right\| < \infty$$

$$\text{and} \qquad H^\infty_M = \|\cdot\| - \lim_N H^N_M \qquad \text{exists.}$$

Then Definition 3.13 is satisfied with $\Gamma^N_M(\cdot) = |M| i[H^N_M, \cdot\,]$.

For Hamiltonian dynamics each $T_{t,N}$ is an automorphism. Since the N-wise products of quasi-symmetric nets are again quasi-symmetric we conclude immediately that $T_{t,\infty}(X_\infty Y_\infty) = \big(T_{t,\bullet}(X_\bullet Y_\bullet)\big)_\infty = \big(T_{t,\bullet}(X_\bullet) T_{t,\bullet}(Y_\bullet)\big)_\infty = T_{t,\infty}(X_\infty) T_{t,\infty}(Y_\infty)$. Thus $T_{t,\infty}$ is a homomorphism. Within the lattice class of generators we can say more: the local evolutions are themselves Hamiltonian, with a ρ-dependent Hamiltonian:

$$L^\rho(A) = i[H^\rho, A] \qquad \text{with} \quad H^\rho = \sum_M |M| \, \mathbb{E}^\rho_{M\setminus\{1\}}(H^\infty_M) \tag{4.3}$$

The growth condition on $\sup_N \left\| H^N_M \right\|$ ensures that $\|H^\rho\|$ is bounded on K, and H^ρ has continuous first derivatives with respect to ρ. This form of L^ρ has the consequence that each Λ^ρ_t is unitarily implemented: we have

$$\Lambda^\rho_t(A) = U^\rho_t A U^{\rho *}_t \qquad \text{with} \quad \frac{d}{dt} U^\rho_t = i U^\rho_t H^{\mathcal{F}_t \rho} \quad \text{and} \quad U^\rho_0 = 0 \quad . \tag{4.4}$$

The Hamiltonian H^ρ is closely related to the energy density function $H_\infty : K \to \mathbb{R}$, which enters the Gibbs variational principle for the limiting free energy of the mean-field system [**RW1**]. In the Euler-Lagrange equations for this variational principle one needs the gradient of this function, i.e. the derivatives along directions in the state space. The gradient $\mathrm{d}H_\infty(\rho)$ in the sense of equation (3.5) is an element of $\mathcal{A}$, also called the "effective Hamiltonian". The thermal equilibrium states are then infinite product states with a one-particle state ρ which is an equilibrium state for H^ρ. This amounts to an implicit non-linear equation for ρ known as the "gap equation" [**RW1,Wer**]. Assuming H_N to be of the form (4.2) we obtain

$$\begin{aligned}
\langle \sigma - \rho, \mathrm{d}H_\infty(\rho) \rangle &= \frac{d}{dt} H_\infty(t\sigma - (1-t)\rho)\Big|_{t=0} && (4.5)\\
&= \frac{d}{dt} \sum_M \langle (t\sigma - (1-t)\rho)^M, H^\infty_M \rangle\Big|_{t=0} \\
&= \sum_M |M| \langle (\sigma - \rho) \otimes \rho^{|M|-1}, H^\infty_M \rangle \\
&= \sum_M |M| \langle \sigma - \rho, \mathbb{E}^\rho_{N\setminus\{1\}} H^\infty_M \rangle \\
&= \langle \sigma - \rho, H^\rho \rangle \quad . && (4.6)
\end{aligned}$$

Here the first equality in (4.5) is the definition of the gradient as an element $\mathrm{d}H_\infty(\rho) \in \mathcal{A}$, and the last line shows that H^ρ satisfies this definition. It is clear, however, that equation (4.5) fixes the gradient only up to a multiple of the identity. As in section 3, we can get rid of this ambiguity by imposing the convention $\langle \rho, \mathrm{d}H_\infty(\rho)\rangle = 0$. Then the above equation becomes $\mathrm{d}H_\infty(\rho) = H^\rho - \langle \rho, H^\rho\rangle \mathbb{1}$.

The identification of H^ρ with the gradient of H_∞ is also important for establishing an important property of the flow $\mathcal{F}_t$ in the Hamiltonian case: it is itself Hamiltonian in the sense of classical mechanics [**DW2**]. In order to make sense of this statement we have to introduce a symplectic structure on the state space K. The state space itself has no natural symplectic structure (it may be odd dimensional). However, each of the leaves of the foliation of the state space into unitary equivalence classes of states allows a non-degenerate symplectic stucure [**DW2**]. Since Λ_t^ρ is unitarily implemented we already know that the flow $\mathcal{F}_t\rho = \rho \circ \Lambda_t^\rho$ respects this foliation. The easiest way to define the symplectic strucure on all leaves simultaneously is to define the Poisson bracket of two differentiable functions $f, g : K \to \mathbb{R}$. Using the definition (4.5) of the gradient we set

$$\{f, g\}(\rho) = \langle \rho, i[\mathrm{d}f(\rho), \mathrm{d}g(\rho)]\rangle \quad . \tag{4.7}$$

Note that the convention for the gradient is irrelevant here, since multiples of the identity drop out of the commutator anyway. One now checks easily [**DW2**] that the flow satisfies Liouville's equation in the form

$$\frac{d}{dt} f(\mathcal{F}_t\rho)\Big|_{t=0} = \{H_\infty, f\}(\rho) \quad . \tag{4.8}$$

The possibility of writing the limiting evolution as a classical Hamiltonian flow was noticed a long time ago in [**HL**]. However, in order to state this, Hepp and Lieb used the natural symplectic structure on the coadjoint orbits of a Lie group. Therefore the Hamiltonian had to be written as a function of the generators of a group representation. This approach was also adopted by [**Bo1**]. It has the disadvantage of introducing an additional auxiliary object (the group representation) which becomes unnecessary as soon as the symplectic structure is established on the state space itself. For the dissipative evolutions discussed below the disadvantage becomes even more pronounced.

To summarize: if each $T_{t,N}$ is generated by a Hamiltonian, then the global dynamics is given by a *Hamiltonian* flow, and the local dynamics is also generated by a *Hamiltonian*.

4.2. Lindblad generators from symmetric nets

It is well known [**Lin**] that the generator of a dynamical semigroup can be written as a sum of a commutator and terms of the form $G(X) = V^*[X, V] + [V^*, X]V$. If we want to turn this into a net of generators a natural possibility is to insert for V a $\emptyset$-symmetric net like the Hamiltonians in the previous subsection, and to multiply the result by the system size. It is this class that we would like to study here. We mention that the only type of dissipative inter-particle interaction included in some previous work **[Un3]** was a single term of this type.

More precisely, we demand that the generators are of the form

$$
\begin{aligned}
G_N(X) &= |N| \sum_\alpha V_{\alpha,N}^*[X, V_{\alpha,N}] + [V_{\alpha,N}{}^*, X]V_{\alpha,N} \\
\text{where} \qquad V_{\alpha,N} &= \sum_{M\subset N} j^{\emptyset}_{NM} V^N_{\alpha,M} \\
\text{where} \qquad \gamma_{\alpha,M} &= \sup_N \left\| V^N_{\alpha,M} \right\| < \infty \quad , \\
V^\infty_{\alpha,M} &= \lim_N V^N_{\alpha,M} \text{exists in norm} \quad , \\
\text{and} \qquad & \sum_\alpha \left(\sum_M |M|^2 \gamma_{\alpha,M} \right)^2 \left(\sum_M \gamma_{\alpha,M} \right) < \infty \quad .
\end{aligned}
\tag{4.9}
$$

It is clear that under these circumstances the nets $V_{\alpha,\bullet}$ are $\emptyset$-symmetric, and

$$
V_{\alpha,\infty}(\rho) = \sum_M \langle \rho^M, V^\infty_{\alpha,M} \rangle \quad . \tag{4.10}
$$

Moreover, the functions $V_{\alpha,\infty} : K \to \mathbb{C}$ are differentiable, and $\mathrm{d}V_{\alpha,\infty}(\rho) = \sum_M \mathbb{E}^\rho_{M\setminus 1}(V^\infty_{\alpha,M}) \in \mathcal{A}$. We can then compute the local dynamics as follows:

4.1 Proposition. *Generators of the form (4.9) are lattice class in the sense of definition 3.13, and hence define a mean-field dynamical semigroup. The generator of the local dynamics is*

$$
\begin{aligned}
L^\rho(A) &= i[H^\rho, X] \quad , \\
\textit{where} \qquad H^\rho &= \sum_\alpha \frac{1}{i} \left(V_{\alpha,\infty} \mathrm{d}V^*_{\alpha,\infty} - V^*_{\alpha,\infty} \mathrm{d}V_{\alpha,\infty} \right) \quad .
\end{aligned}
$$

Proof: By the remarks after Definition 3.13 it suffices to consider a single term α. Hence we will simply omit α from the above formulas. Moreover, we may assume that V^N_M is non-zero only for some standard set $\{1, \ldots, |M|\}$ for each cardinality of M. Now each of the two terms in $G_N = |N| V^*_N[X, V_N] + [V_N{}^*, X]V_N$ involves a double sum over M, M' of terms of the type

$$
G_N^{(M,M')} X = |N| \, (j^{\emptyset}_{NM} V_M)^* [X, (j^{\emptyset}_{NM'} V_{M'})] \quad .
$$

We claim that $G^{M,M'}$ is a lattice class net of operators with a lattice class bound $(|M| + |M|')^2 \gamma_M \gamma_{M'}$. By the remarks after 3.13 this will be enough to complete the proof, since $\sum_{m,m'} (m + m')^2 \gamma_m \gamma_{m'} \leq 4 (\sum_m m^2 \gamma_m)(\sum_m \gamma_m)$.

The expression for $G_N^{(M,M')}$ is the average over all pairs (π,π') of permutations of $\{1,\ldots,|N|\}$ of $|N|\ \hat\pi(V_M)^*[X,\hat\pi'(V_{M'})]$, where we have identified $V_M, V_{M'}$ with elements of $\mathcal{A}_N$ living at the sites indicated. Substituting $\pi'=\pi^{-1}\pi''$ we can thus write

$$G_N^{(M,M')} = |N|\,\mathrm{Sym}_N\left(\frac{1}{N!}\sum_\pi V_M^*[\,\bullet\,,\hat\pi(V_{M'})]\right) \quad .$$

It is easy to see that under the outer symmetrization all terms coincide, for which the "overlap" $M\cap\pi(M')$ has the same number of elements. Let $N!c_k(N)$ denote the number of permutations of $\{1,\ldots,|N|\}$ with $|M\cap\pi(M')|=k$. By definition we have $\sum_k c_k(N)=1$. Then we can write $G_N^{(M,M')}=|N|/(|M|+|M'|)\,\mathrm{Sym}_N\,\Gamma_M^N$ with Γ_M^N an operator on $\mathcal{A}_{M\&M'}$, where $M\&M'$ is a set of cardinality $|M|+|M'|$, say $\{1,\ldots,|M|+|M'|\}$, and

$$\Gamma_{M\&M'}^N = |M\&M'|\sum_k c_k(N)(V_M\otimes\mathbb{1}^{|M'|})^*[\,\bullet\,,\mathbb{1}^{|M|-k}\otimes V_{M'}\otimes\mathbb{1}^k] \quad .$$

This expression makes sense only for $|N|\geq|M\&M'|=(|M|+|M|')$, but we can choose any definition of Γ_M^N for the finitely many exceptional N without changing the validity of our claim. Now by Lemma IV.1 of **[RW1]** we have $c_0=1-\mathbf{O}(N^{-1})$, and hence $\Gamma_{(M,M')}=\lim_N\Gamma_{(M,M')}^N=(V_M\otimes\mathbb{1}^{M'})^*[\,\bullet\,,\mathbb{1}^M\otimes V_{M'}]$. It remains to compute the the limiting generator L^ρ. We could do this by adding up the contributions $L_{(M,M')}^\rho$ from all pairs (M,M').

A quicker way to see the result is to use the results of the previous section: since $V_{\alpha,N}$ satisfies the conditions (4.2) (apart from hermiticity) we know (by splitting $V_{\alpha,N}$ into hermitian and skew-hermitian part) that $N\mapsto X_N\equiv|N|[V_{\alpha,N},A\otimes\mathbb{1}^{|N|-1}]$ is a $\{1\}$-symmetric net with $X_\infty(\rho)=[\mathrm{d}V_{\alpha,\infty}(\rho),A]$. Multiplying this $\{1\}$-symmetric net with the $\emptyset$-symmetric net $V_{\alpha,N}^*$ we get a $\{1\}$-symmetric net with limit $\overline{V_{\alpha,\infty}(\rho)}[\mathrm{d}V_{\alpha,\infty}(\rho),A]$. Adding the contribution from the conjugate term in the Lindblad form, and summing over α we find that $G_N(A\otimes\mathbb{1}^{|N|-1})$ is $\{1\}$-symmetric with limit $L^\rho(A)$ as stated in the Proposition. ∎

Since the local dynamics is generated by a Hamiltonian it might be suspected that this forces the global evolution to be Hamiltonian as well, but this is not so. We demonstrate this with the following elementary example:

Example: Let $\mathcal{A}$ be the algebra of 2×2-matrices, and set $V_N=j_{N1}\sigma^+$, where $\sigma^+=\begin{pmatrix}0&1\\0&0\end{pmatrix}$. Then from the Proposition one readily verifies that

$$H^\rho=\frac{1}{i}\begin{pmatrix}0&-\rho_{12}\\ \rho_{21}&0\end{pmatrix} \quad . \tag{4.11}$$

The flow is determined from the differential equation $\dot\rho=i[H^\rho,\rho]$. This equation can be written in terms of the variables $x=\rho_{11}-\rho_{22}$, $y=|\rho_{12}|^2$, and the argument of ρ_{12}. The latter is constant, and we can furthermore eliminate y from the fact that $\mathcal{F}_t\rho$ is unitarily equivalent to ρ, and consequently $2\,\mathrm{tr}(\rho^2)-1=x^2+4y\equiv\lambda^2$ is a constant of the motion.

The resulting equation $\dot{x} = x^2 - \lambda^2$ is readily solved, and gives $x(t) = -\lambda \tanh(\lambda(t - t_0))$, where t_0 is determined from the initial condition. For $t \to \infty$ we get $x(t) \to -\lambda$, and consequently $|\rho_{12}|^2 = y \to 0$. Thus in the state space, which is identified with a ball in 3 dimensions, the flow moves along the meridians on concentric spheres to the southern half of the axis. It is thus certainly not Hamiltonian.

In this example, although the flow $\mathcal{F}_t$ is no longer Hamiltonian, it is reversible in the sense that it also exists for negative times. This is no coincidence. In fact, if we replace $V_{\alpha,N}$ by $\tilde{V}_{\alpha,N} = V^*_{\alpha,N}$ we obtain another generator $\tilde{G}_\bullet$ of the form (4.9), and from Proposition 4.1 we immediately get the local Hamiltonian as $\tilde{H}^\rho = -H^\rho$. Thus in spite of the fact that for finite N no $T_{t,N}$ needs to have a positive inverse, $T_{t,\infty}$ does.

We have seen that for the generators studied in this subsection the local dynamics is generated by a state-dependent Hamiltonian H^ρ. It is natural to ask whether any more can be said about the generators of the form (4.9), or whether *any* function $\rho \mapsto H^\rho$ can occur. Since we have not attempted to find exhaustive conditions under which the mean-field limit of a net of generators exists, we cannot be expected to show the latter result. However, we will show the only slightly weaker statement that any function $\rho \mapsto H^\rho$ may be approximated by local Hamiltonians arising from generators satisfying (4.9). In particular, any ordinary differential equation respecting unitary equivalence classes is approximately the equation determining the flow $\mathcal{F}_t$ of some mean-field dynamical semigroup. This makes it unnecessary for us to provide examples of various types of possible behaviour of the flow: any structurally stable phase portrait of dynamical systems, stable and unstable points and limit cycles, as well as chaotic behaviour can occur.

The proof that approximately all H^ρ occur is simple. It is useful for this purpose to think of $\rho \mapsto H^\rho$ as a 1-form on K. This is permissible since gradients, 1-forms and local Hamiltonians are all defined only up to multiples of the identity. By Proposition 4.1 H^ρ is a sum of terms of the form $\frac{1}{i}(V_{\alpha,\infty} \mathrm{d}V^*_{\alpha,\infty} - V^*_{\alpha,\infty} \mathrm{d}V_{\alpha,\infty})$. It is useful to write $V_{\alpha,\infty} = f + ig$. Then the contribution to the Hamiltonian is $2(g\mathrm{d}f - f\mathrm{d}g)$.

In this expression f and g can now be chosen as arbitrary real valued polynomials on K, or even sums of polynomials converging in $\mathcal{C}^2$-norm. (We do not need the latter fact, it suffices to use the polynomials for the approximation argument). In particular, setting $f = gh$, any 1-form $g^2\mathrm{d}h$ with polynomial g, h can be realized. Since on a compact set any differentiable function (of finitely many variables) can be approximated uniformly together with its derivatives by polynomials [**Lla**], we can drop the constraint that g and h should be polynomials. Since we can write any bounded function as a difference of two squares (take the first square as a constant larger than the upper bound), we conclude that by taking sums we can uniformly approximate any 1-form.

To summarize, in the class of mean-field dynamical semigroups studied in this subsection the local dynamics is still *Hamiltonian*. The flow $\mathcal{F}_t$ thus respects unitary equivalence classes and is reversible, but *not Hamiltonian*. On any one equivalence class essentially any flow is possible.

4.3. General lattice class

In the previous section we demonstrated that essentially any function $\rho \mapsto H^\rho$ can occur as the local Hamiltonian of the local dynamics in a suitable mean-field model in the class described. Here we address the same question for the lattice class: we will show that the functions $\rho \mapsto L^\rho$, which can arise from mean-field dynamical semigroups with lattice class generators is dense in the set of continuous functions associating with each state ρ a generator L^ρ of some dynamical senmigroup on $\mathcal{A}$. The purpose of this question is to verify that we have not missed some structure theorem for the local dynamics which would put a constraint on this function. For simplicity we will always assume that $\mathcal{A}$ is finite dimensional.

4.2 Proposition. *Let $\mathcal{A}$ be finite dimensional, and let $\mathcal{C}(K, \mathcal{B}(\mathcal{A}))$ denote the space of continuous functions on K with values in the operators on $\mathcal{A}$. Consider the cone $\mathcal{G}$ of functions $L \in \mathcal{C}(K, \mathcal{B}(\mathcal{A}))$ such that for all ρ, L^ρ generates a dynamical semigroup, and the subcone $\mathcal{G}_{pg} \subset \mathcal{G}$ of local generators $\rho \mapsto L^\rho$ arising from polynomial generators. Then $\mathcal{G}_{pg}$ is norm dense in $\mathcal{G}$.*

Proof: We consider first polynomial generators $G_N = (|N|/|R|) \operatorname{Sym}_N G_R$ with G_R extremal in the cone of permutation symmetric Lindblad generators on $\mathcal{A}_R$, i.e. we consider the form $G_R(\cdot) = |R| \operatorname{Sym}_R V^*[\,\cdot\,, V] + [V^*, \,\cdot\,]V$ with $V \in \mathcal{A}_R$. Note that we do not require V itself to be permutation symmetric. As a convenient expression for L^ρ in terms of V we use

$$L_V^\rho(A) = \sum_{x\in R} \mathbb{E}^\rho_{R\setminus\{x\}} \Big\{ V^*[\hat\eta_x(A), V] + [V^*, \hat\eta_x(A)]V \Big\} \quad , \tag{4.12}$$

where $\hat\eta_x$ embeds an $\mathcal{A}$ as the copy of $\mathcal{A}$ at site x. ¿From this expression it is clear that $L^\rho_{V\otimes 1\!\!1} = L^\rho_V$, and more generally

$$L^\rho_{V\otimes W} = \langle \rho^R, V^*V\rangle L^\rho_W + \langle \rho^S, W^*W\rangle L^\rho_V \quad , \tag{4.13}$$

where $V \in \mathcal{A}_R$ and $W \in \mathcal{A}_S$. Note that the coefficient of L^ρ_W depends on V and conversely. We want to get rid of this dependence by finding suitable W for which the first term becomes negligible, while $\langle \rho^S, W^*W\rangle$ approximates any desired function. A subclass of the generators discussed in the previous subsection precisely meets this description: we set $W_S = j^{\emptyset}_{SS'} F$ with $F = F^* \in \mathcal{A}_{S'}$. Then by Proposition 4.1 we have $\lim_S L^\rho_{W_S}(A) = i[H^\rho, A]$ with $iH^\rho = W^*_\infty \mathrm{d}W_\infty - W_\infty \mathrm{d}W^*_\infty$. But since F is hermitian, W_∞ is a real function, and hence $H^\rho = 0$. On the other hand, $\lim_S \langle \rho^S, W^*W\rangle = |j_{\infty S'} F|^2$, which is the square of an arbitrary real polynomial on K. By this we can approximate an arbitrary positive continuous function, and consequently the closure of $\mathcal{G}_{pg}$ contains all functions of the form $\rho \mapsto f(\rho) L^\rho_V$ with $f \in \mathcal{C}(K)$, $f \geq 0$, and $L^\rho_V \in \mathcal{G}_{pg}$. Any constant function $L^\rho \equiv L$ is in $\mathcal{G}_{pg}$, since we can take the corresponding one-site generator $G_N = |N| \operatorname{Sym}_N L$.

Given now an arbitrary function $L \in \mathcal{G}$ we can choose a sufficiently fine continuous partition of the identity, i.e. $f_\alpha \in \mathcal{C}(K)$, $f_\alpha \geq 0$, $\sum_\alpha f_\alpha \equiv 1$, such that f_α has its support only near some ρ_α, such that L^ρ is uniformly close to $\sum_\alpha f_\alpha L^{\rho_\alpha}$. We have just shown that the latter expression is in the closure of $\mathcal{G}_{pg}$. Hence $\mathcal{G}_{pg}$ is dense in $\mathcal{G}$.

■

4.4 Lindblad generators from permutation operators

For finite dimensional $\mathcal{A}$ any net of generators is of the form $G_N(X) = |N| i[H_N, X] + |N| \sum_\alpha \bigl(V^*_{\alpha,N}[X, V_{\alpha,N}] + [V_{\alpha,N}{}^*, X] V_{\alpha,N}\bigr)$. In this subsection we suppose that $\mathcal{A}$ is the algebra of $d\times d$-matrices, and that H_N and each $V_{\alpha,N}$ is a linear combination of permutation operators. Then G_N vanishes on any operator X commuting with permutations, and dually $\rho^N \circ G_N = 0$ for any state $\rho \in K$. Thus every homogeneous product state ρ^N is invariant under the semigroups $T_{t,N}$. Since the generator of the flow is expressed by evaluating G_N in such states, it is clear that if the G_N define a quantum dynamical semigroup, every state ρ will be invariant under the associated flow. Hence the flow $\mathcal{F}_t$ is trivial. This does *not* mean, however, that the local dynamics is also trivial. Indeed, we know from the previous section that approximately we can realize any local generator $\rho \mapsto L^\rho$, and in particular any L^ρ such that $\rho \circ L^\rho = 0$. However, for the mean-field dynamical semigroups discussed in this section we do not have to invoke this approximate result: the flow is exactly constant.

As a first example, consider the Hamiltonian case. For simplicity we choose a polynomial generator of degree R, i.e. we set $H_N = j^\emptyset_{NR} \hat{H} = j^\emptyset_{NR} \sum_{\pi \in \mathcal{S}_R} h(\pi) U_\pi$, where $\mathcal{S}_R$ denotes the group of permutations of the sites R, U_π the unitary operator implementing the permutation π, and h is any function on $\mathcal{S}_R$. The operator $j^\emptyset_{NR}$ implies an averaging over all permutations hence we may suppose without loss of generality that $\hat{H}$ is itself permutation invariant. Equivalently, h can be taken as an invariant function ($h(\pi\pi') = h(\pi'\pi)$), i.e. it is in the center of the group algebra. The complete information about the dynamics is contained in the energy density function

$$H_\infty = \langle \rho^R, \hat{H} \rangle = \sum_\pi h(\pi) \langle \rho^R, U_\pi \rangle \quad . \tag{4.14}$$

Since every unitary $U \otimes U \cdots U = U^{\otimes R}$ commutes with U_π, $\pi \in \mathcal{S}_R$ it is clear that $H_\infty(\rho) = H_\infty(\rho \circ \mathrm{ad}_U)$. Thus H_∞ is constant on each unitary equivalence class. The flow on each of the symplectic submanifolds of K is thus generated by a constant Hamiltonian, i.e. the flow is constant in accordance with the general remarks made above. In order to evaluate (4.14) more explicitly we use the formula $\mathrm{tr}(A_1 \cdots A_N) = \mathrm{tr}\bigl((A_1 \otimes \cdots A_N) U_\pi\bigr)$, for π the cyclic permutation of $\{1, \ldots n\}$, which is readily shown by expanding both sides with respect to the same basis. We get

$$\langle \rho^R, U_\pi \rangle = \prod_k \bigl(\mathrm{tr}(\rho^k)\bigr)^{n_k(\pi)} \quad , \tag{4.15}$$

where $n_k(\pi)$ is the number of cycles of length k appearing in the cycle decomposition of $\pi \in \mathcal{S}_R$, and where we have used the symbol ρ for both the state and its density matrix. Thus H_∞ is a polynomial in the $|R|$ variables $\mathrm{tr}(\rho^k)$. Put differently, H_∞ is a symmetric polynomial in the eigenvalues of ρ. It is easy to check that all such polynomials can occur.

The Hamiltonian for the local dynamics is $H^\rho = \mathrm{d}H_\infty(\rho)$. This is non-zero, so the local dynamics is not trivial. From the form of H_∞ it is clear that H^ρ is a polynomial in ρ and the numbers $\mathrm{tr}(\rho^k)$. In particular, $[H^\rho, \rho] = 0$, confirming once again that the flow is constant.

The simplest, though physically quite interesting example of this kind of Hamiltonian is the mean-field version of the Heisenberg model. There we have $d = 2$, $R = \{1, 2\}$, and the Hamiltonian is $H_R = \sum_{\nu=1}^{3} \sigma^\nu \otimes \sigma^\nu = 2F - \mathbb{1}$, where σ^ν denotes the Pauli matrices, and $F \equiv U_{(12)}$ denotes the flip operator. Then $H_\infty(\rho) = 2\,\mathrm{tr}(\rho^2)$, and $H^\rho = 4\rho$.

In the context of the class studied in subsection 4.2 the assumptions made at the beginning of the present subsection amount to postulating that each $V^M_{\alpha,N}$ is a linear combination of permutation operators. Thus $V_{\alpha,\infty}$ can be chosen as an arbitrary polynomial in the variables $\mathrm{tr}(\rho^k)$. Repeating the arguments in 4.2 we find that H^ρ is now an arbitrary polynomial in ρ whose coefficients are symmetric polynomials in the eigenvalues of ρ. Taking the flip F and $V_N = j^{\emptyset}_{N2}F$ gives a trivial dynamics because $F = F^*$, as noted in the previous subsection. So one has to go to higher order permutations.

The next possibility is to use directly formula (4.12) for general polynomial generators. With $V = F$ it is easily evaluated using the formula $\mathrm{tr}(A \otimes BF) = \mathrm{tr}(AB)$. This gives $\mathrm{tr}\big(\sigma \otimes \rho L^\rho_F(A)\big) = \mathrm{tr}\,\sigma \otimes \rho\big(2F^2(\mathbb{1} \otimes A) - F^2(A \otimes \mathbb{1}) - (A \otimes \mathbb{1})F^2\big) = 2\big(\mathrm{tr}(\sigma)\,\mathrm{tr}(\rho A) - \mathrm{tr}(\sigma A)\,\mathrm{tr}(\rho)\big)$. Hence

$$L^\rho(A) = 2(\rho(A) - A) \tag{4.16a}$$
$$\Lambda^\rho_t(A) = e^{-2t}A + (1 - e^{-2t})\rho(A)\mathbb{1} \quad , \tag{4.16b}$$

i.e. the local evolution contracts exponentially fast to multiples of the identity.

4.5 Failure of the disjoint homomorphism property

We have shown in section 3 that for a net of generators to generate a mean-field dynamical semigroup in the sense of Definition 3.2 it is sufficient that they be of lattice class. Here we give some simple examples to show that this condition is by no means necessary. These examples also show that some of the characteristic features of the limiting semigroups derived above are not valid for arbitrary mean-field dynamical semigroups, but are consequences of the special lattice class form.

There is a standard way of obtaining a dynamical semigroup from a Hamiltonian evolution: for any Hamiltonian $H = H^*$ we may consider the generator

$$G^H(A) = H^*[A,H] + [H^*,A]H = i[H,i[H,A]] \tag{4.17}$$

Thus G^H is nothing but the square of the generator $i[H,\cdot\,]$ of the Hamiltonian evolution. It is well known (see Theorem 2.31 of [**Dav**]) that squaring the generator of a group of isometries on a Banach space produces the generator of a contraction semigroup, which is just the integral of the group of isometries with respect to the convolution semigroup of the heat equation. Explicitly, we have

$$\begin{aligned} e^{tG^H}(A) &= \int_{-\infty}^{+\infty} ds\ \mu_t(s) e^{isH} A e^{-isH} \\ \text{with} \qquad \mu_t(s) &= (4\pi t)^{-1/2} e^{-\frac{s^2}{4t}} \quad . \end{aligned} \tag{4.18}$$

It is important to note that in this integral both positive and negative s enter. Thus squaring the generator of a non-reversible quantum dynamical semigroup will not in general produce the generator of another.

We now apply this construction to a mean-field dynamical semigroup, generated by a net H_N of Hamiltonian densities satisfying (4.2). Let us denote the resulting mean-field dynamical group by $S_{t,N}(A) = \exp(it|N|H_N)A\exp(-it|N|H_N)$. We now square the generator for each N, getting

$$\begin{aligned} G_N(A) &= |N|^2 \Big(H_N^*[A,H_N] + [H_N{}^*,A]H_N \Big) \\ T_{t,N}(A) &= \int ds\ \mu_t(ds)\, S_{s,N}(A) \end{aligned} \tag{4.19}$$

Now let $X \in \mathcal{Y}$ be quasi-symmetric. Then so is $S_{s,\bullet}(X_\bullet)$. Using the strong continuity of $S_{\bullet,\bullet}$ we then find that $T_{t,\bullet}(X_\bullet)$ is again quasi-symmetric. Hence $T_{t,\bullet}$ preserves quasi-symmetry. We can take the limit $N \to \infty$ under the integral and obtain

$$T_{t,\infty} = \int ds\ \mu_t(ds)\, S_{s,\infty} \quad . \tag{4.20}$$

Hence $T_{t,\bullet}$ is a mean-field dynamical semigroup. The generator $G_\bullet$ is clearly not of lattice class, since $\|G_N\|$ grows like $|N|^2$ rather than like $|N|$. We know that the evolution described by S_∞ on the intensive variables $\mathcal{C}(K)$ is given by a Hamiltonian flow. The generator of this flow is a first order differential operator. Its square, which generates the restriction of $T_{t,\infty}$ to $\mathcal{C}(K)$ is hence a second order differential operator. We may put this in probabilistic terms saying that the evolution of intensive variables under $T_{t,\infty}$ is given by a diffusion on K rather than a flow. More precisely, we get a diffusion along the orbits of the flow generated by H_∞. We could also add several generators like $G_\bullet$ and obtain diffusions along higher dimensional submanifolds in K [**DW1**]. We note that the generator (4.17) is very similar to the form considered in section 4.2: There we would have taken $|N|\big(H_N^*[A,H_N]+[H_N{}^*,A]H_N\big)=|N|^{-1}G_N$. Since G_N has a well defined limit it is clear that $|N|^{-1}G_N$ goes to zero. We have noted this consequence of the hermitian nature of H_N before and used it in the proof of Proposition 4.2.

The integral formula (4.20) not only gives the evolution of the intensive observables but also the local evolution. It can no longer be given by a local cocycle Λ_t^ρ, because the equation determining Λ_t^ρ (Lemma 3.7) presupposes the existence of the flow. The root of this difficulty is the failure of the disjoint homomorphism property (Theorem 3.6(2))for $T_{t,N}$, which is easily verified from the form of the squared generator of S_∞.

5. Local and Global Evolutions.

5.1 Global mean-field dynamical semigroups need not be local.

The notion of mean-field dynamical semigroup which we have used in this paper, namely a limiting evolution of quasi-symmetric nets, is *a priori* stronger than the original formulation of **[DW1]** as a limiting evolution for the subset of intensive (i.e. $\emptyset$-symmetric) observables only. We constrast these by saying that the latter comprises an evolution of global or fully site-avearged quantities only, which the former gives the evolution in local regions as well.

So far we have given examples of operator nets which generate in the stronger local sense. In fact we can adapt section 4.4 to demonstrate an operator net which for which there is a limiting global evolution, but *not* a limiting local evolution. Thus the present notion of a mean-field dynamical semigroup is indeed stronger than the former notion.

Assume for the fixed net $(N_\alpha)_{\alpha\in\aleph}$ that $|N|$ takes odd and even values infinitely often. We shall call N itself odd or even accordingly. ¿From the operator $H_{\{1,2\}}=2F-\mathbb{1}$ of section 4.4, form the bounded polynomial generator $\hat{G}_N(\cdot)=|N|\,\mathrm{Sym}_N[H_{\{1,2\}}\ ,\ \cdot\ ]$, and set $G_N=(-1)^{|N|}\hat{G}_N$. Thus, $G_\bullet$ is like a bounded polynomial generator, except that the N^{th} element is multiplied by the alternating quantity $(-1)^{|N|}$. Clearly the two nets

$$T_{t,\bullet}^{\text{odd}}=\{T_{t,N}\mid N\text{ odd }\}\qquad\text{and}\qquad T_{t,\bullet}^{\text{even}}=\{T_{t,N}\mid N\text{ even }\}$$

are mean-field dynamical semigroups in the local sense, although on different nets of regions. But the local generators for the odd and even net are $L^{\rho,\text{odd}}=-4i\mathrm{ad}\rho$ and $L^{\rho,\text{even}}=4i\mathrm{ad}\rho$ respectively. Hence the full net $T_{t,\bullet}$ can have no local mean-field limit. On the other hand, examining the global evolution one sees that the limitng flow is in both cases trivial since $\rho\circ L^{\rho,\text{odd}}=\rho\circ L^{\rho,\text{even}}=0$; so $T_{t,\infty}^{\text{odd}}X_\infty=T_{t,\infty}^{\text{even}}X_\infty=X_\infty$ for any $\emptyset$-symmetric net $X_\bullet$ and $t\in\mathbb{R}$. Since for $\emptyset$-symmetric $X_\bullet$ the subnets $T_{t,N}X_N$ for N odd and N even are $\emptyset$-symmetric, we need only compare odd and even terms in the full net in order to demonstrate $\emptyset$-symmetry for the full net. But

$$\lim_{N\text{ odd }\to\infty}\ \lim_{M\text{ even }\to\infty}\left\|T_{t,N}X_N-j_{NM}^{\emptyset}T_{t,M}X_M\right\|=\left\|T_{t,\infty}^{\text{even}}X_\infty-T_{t,\infty}^{\text{odd}}X_\infty\right\|=0$$

as required.

5.2 Dynamical stability of local evolutions.

As we have stressed earlier, for mean-field dynamical semigroups with the disjoint homomorphism property the implementing map Λ plays a dual role. It implements the evolution of local states $\sigma \mapsto \sigma \circ \Lambda_t^\rho$ on the state spaces of tagged algebras, and also the flow $\mathcal{F}$ via the equation $\mathcal{F}_t\rho = \rho \circ \Lambda_t^\rho$. Now we have seen that initially localized observables (i.e. nets of the form $j^I_{\bullet R}X_R$) develop in time a symmetrized tail in the algebra over the untagged sites. Suppose that in the limit as $t \to \infty$, this tail in fact becomes dominant, so that the time developed observable loses all information about its initial localization. Working in the dual picture with an intial state ρ on each of the untagged algebras, this would mean that any initial local state σ on a tagged algebra $\mathcal{A}$ would evolve through $\sigma \mapsto \sigma \circ \Lambda_t^\rho$ towards the mean-field state $\mathcal{F}_t\rho$. This motivates the following definition.

Definition 5.1. *We shall say that a local cocycle is* **asymptotically global** *in a topology τ of K if for each $\rho, \sigma \in K$,*

$$\tau - \lim_{t\to\infty} \sigma \circ \Lambda_t^\rho - \mathcal{F}_t\rho = 0 \quad .$$

Of course, when the local generator is Hamiltonian one would not expect this type of asymptotic result. However, it is relatively easy to find an H-Theorem for the joint evolution of local and global states. (In [**DW1**] we were able to prove an H-Theorem for the flow alone, but only under the assumption that for some $\rho \in K$ and all N, ρ^N is an invariant state for $T_{t,N}$). We shall show that the relative entropy (recalled below) of an arbitrary local state $\sigma \circ \Lambda_t^\rho$ with respect to the global state $\mathcal{F}_t\rho$ is non-increasing in time.

In the following we let $S(\omega_1, \omega_2)$ denote the entropy of $\omega_2 \in K$ relative to $\omega_1 \in K$ as defined for normal states on a von Neumann algebra in [**Ara**], and extended to states on C*-algebras in [**PW,Kos**] and also in [**Pet**]. The crucial property we shall need here is that if $\gamma : \mathcal{A}^n \to A^n$ is a completely positive unital map, then $S(\omega_1, \omega_2) \geq S(\omega_1 \circ \gamma, \omega_2 \circ \gamma)$. In the particular case where both states are given by non-singular densities D_{ω_1} and D_{ω_2} with respect to a trace Tr,

$$S(\omega_1, \omega_2) = \mathrm{Tr}\big(D_{\omega_2}(\log D_{\omega_2} - \log D_{\omega_1})\big) \quad .$$

Proposition 5.2. *Let $T_{t,\bullet}$ be a mean-field dynamical semigroup whose limit has the disjoint homomorphism property with local cocyle Λ and implementing flow $\mathcal{F}$. Then for each $(\rho, \sigma) \in K \times K$, the function $[0, \infty) \ni t \mapsto S(\mathcal{F}_t\rho, \Lambda_t^\rho\sigma)$ is non-increasing.*

Proof: Since $\mathcal{F}_t\rho = \rho\circ\Lambda_t^\rho$, and since by Lemma 3.14 Λ_t^ρ is completely positive and unital, we have that

$$S(\mathcal{F}_t\rho, \sigma\circ\Lambda_t^\rho) = S(\rho\circ\Lambda_t^\rho, \sigma\circ\Lambda_t^\rho) \leq S(\rho,\sigma) \quad .$$

■

Since $t \mapsto S(\mathcal{F}_t\rho, \sigma\circ\Lambda_t^\rho)$ is only shown to be non-increasing, rather than strictly decreasing, we are unable to infer that Λ is asymptotically global. In fact, in the purely Hamiltonian case discussed in section 4.1 $S(\mathcal{F}_t\rho, \sigma\circ\Lambda_t^\rho)$ is even a constant of the motion. Hence we have to make do with the intuitive picture that the trajectories of the local state at least remain in a neighbourhood of the global state. Furthermore, nothing is said about the stability, asymptotic or otherwise, of the global state itself. Thus even *with* an asymptotically global cocycle, it can happen that trajectories of the flow take wild paths. In order to obtain an example, we can take a generator with chaotic flow, which is possible by the completeness result at the end of section 4.2. The proof of the following then Theorem shows that we may find an arbitrarily small perturbation which leaves the flow unchanged, but modifies the cocycle to an asymptotically global one.

Theorem 5.3. *The set of generators whose local cocycles are norm-asymptotically global is dense in $\mathcal{G}$.*

Proof: Let $L\in\mathcal{G}$ generate a local cocycle Λ. For any $\varepsilon > 0$ let Δ be the local cocycle generated by $L+\varepsilon W$, where W is (proportional to) the generator of equation (4.16a): $W^\rho A = \langle\rho, A\rangle \mathbb{1} - A$ for any $A\in\mathcal{A}$. Since $\rho\circ W^\rho = 0$, the flows generated by L and $L+\varepsilon W$ are identical. We denote this flow by $\mathcal{F}$. Our claim is that $L+\varepsilon W$ is norm-asymptotically global for all $\varepsilon > 0$.

It useful to introduce for any $\rho\in K$ the projection $P^\rho : \mathcal{A}\to\mathcal{A}$ with $P^\rho(A) = \langle\rho, A\rangle\mathbb{1}$. Thus $W^\rho = -(\mathrm{id} - P^\rho)$. Since $\Lambda_t^\rho\mathbb{1} = \Delta_t^\rho\mathbb{1} = \mathbb{1}$, and dually $\rho\circ\Lambda_t^\rho = \rho\circ\Delta_t^\rho = \mathcal{F}_t\rho \equiv \rho_t$ we have the relations

$$P^{\rho_t} = P^\rho\Lambda_t^\rho = P^\rho\Delta_t^\rho = \Lambda_t^\rho P^{\rho_t} = \Delta_t^\rho P^{\rho_t} \quad . \tag{5.1}$$

We can therefore restrict Δ_t^ρ to the range of the projection $\mathrm{id} - P^{\rho_t}$. More formally, we introduce the operators

$$X_t^\rho = \Delta_t^\rho - P^{\rho_t} = \Delta_t^\rho(\mathrm{id} - P^{\rho_t}) = (\mathrm{id} - P^\rho)\Delta_t^\rho \quad .$$

¿From equation (5.1) we find that $\frac{d}{dt}P^{\rho_t} = P^{\rho_t}L^{\rho_t}$. Hence X_t^ρ satisfies the differential equation

$$\frac{d}{dt}X_t^\rho = (X_t^\rho + P^{\rho_t})(L^{\rho_t} - \varepsilon(\mathrm{id} - P^{\rho_t})) - P^{\rho_t}L^{\rho_t} = X_t^\rho(L^{\rho_t} - \varepsilon\,\mathrm{id})$$

with the initial condition $X_0^\rho = (\mathrm{id} - P^\rho)$. Clearly, this is the same equation satisfied by $e^{-\varepsilon t}\Lambda_t^\rho(\mathrm{id} - P^{\rho_t})$, and by uniqueness we conclude that

$$\Delta_t^\rho = (1 - e^{-\varepsilon t})P^{\rho_t} + e^{-\varepsilon t}\Lambda_t^\rho \quad .$$

As $t\to\infty$ the second term goes to zero, so that Δ_t^ρ is norm-asymptotically global.

■

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