ON IRWIN'S PROOF OF THE PSEUDOSTABLE MANIFOLD THEOREM [1]

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Abstract. We simplify and extend Irwin's proof of the pseudostable manifold theorem.

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1. Introduction

In [Ir1], Irwin introduced a very clever method to prove the stable manifold theorem near hyperbolic points. The proof was then, streamlined in [W]. Compared to previous proofs of the stable manifold theorem, the proof was technically quite simple since it only required the use of the implicit function theorem in Banach spaces. The Banach spaces considered had a very natural interpretation as spaces whose elements were orbits. This made the method very natural in the study of partially hyperbolic systems (Pesin theory) for which individual orbits are hyperbolic but there is little global hyperbolicity in the system. (See e.g. [FHY].)

Later, in [Ir2] Irwin proposed a new method to prove the pseudo-stable manifold theorem that also used spaces of sequences. (A corollary of the pseudo-stable manifold theorem is the center or the center stable manifold theorem.) Unfortunately, the resulting proof was somewhat complicated because it required the use of specialized implicit function theorems that only worked in Banach spaces of sequences. The proof in [W] does not use specialized implicit function theorems but only claims Lipschitz regularity for the manifold. Given a C^r function with r an integer, the proof in [Ir2] can only conclude that the invariant manifold is C^{r-1} and that the $r-1^{\mathrm{st}}$ derivative is bounded.

The goal of this paper is to present a proof of the pseudo-stable manifold theorem that is based on the consideration of spaces of orbits but nevertheless only uses the contraction mapping theorem in Banach spaces. The proof that we present here also produces sharp regularity results with respect to the regularity of the mapping. In particular, it can deal with the case that the map is C^r, $r \in \mathbf{Z}$

We also present some explicit examples that show that the regularity claimed in the theorem is sharp. In [Ir2], Irwin mentions the existence of such examples and attributes them to Van Strien.

We believe that Irwin's method has several advantages. Let us just mention that the idea of considering spaces of sequences with the right type of long term behavior and on which the requirement of being an orbit is imposed as an equation (which can be solved using *e.g.* fixed point theorems or variational methods) is finding an increasing number of applications. Also, since the spaces of sequences that enter into the proof are characterized by their long term behavior, they are invariant under changes of variables that respect rates of growth. For example, for maps of the torus, the spaces of sequences in the universal cover – Euclidean space – with growth slower than a prescribed rate are invariant under homeomorphisms of the torus. It follows immediately that the pseudostable manifolds constructed by Irwin's method are invariant under topological changes of variables in the torus. The argument can also be adapted to prove pseudo-stable invariant foliation theorems for Anosov systems on tori.

Some technical advantages are that it makes it somewhat easier to deal with functions with Hölder regularity than the graph transform method and that one can discuss directly dependence on parameters. The usual graph transform method uses operators which are based on composition operators, which are badly behaved, as operators between spaces of Hölder functions. Even if it is, by now well known how to cope with these problems, Irwin's method does it in a straightforward way. For the pseudostable manifold theorem, dependence on parameters can be proved easily by replacing the system with another in which the parameter is included and assigned trivial dynamics. (See e.g [RT], [La].) Nevertheless, this trick does not work with the stable manifold. Since Irwin's method only uses the soft implicit function theorem, dependence on parameters is automatic and it is possible to compute the derivatives rather explicitly.

Let us finally remark that the construction we present here is not the only one which is possible. Even if the theorems we will present include some uniqueness conclusions given rates of growth of the orbits, there are other constructions that also give rise to invariant manifolds that could be called pseudostable since they are also tangent to the pseudostable subspace **S**. In general they will not coincide with teh manifolds we construct. At the end of the paper we present a discussion of the possible pseudostable manifolds that one can define and examples that illustrate that they often differ. We will also discuss their regularity properties.

2. Notation and statement of results.

Notation.

Let $\mathcal{X}$ be a Banach space . If a is a real number bigger than 1, we define the space $\mathcal{S}^a$ as the space of sequences $\{\rho_n\}_{n=0}^{\infty}$ in $\mathcal{X}$ such that $||\rho||_a \equiv \sup_n ||a^{-n}\rho_n||_{\mathcal{X}} < \infty$. Equipped with the $|| \; ||_a$ norm, $\mathcal{S}^a$ is a Banach space. We will also observe that, when $0 < a \leq a'$, $\mathcal{S}^a \subset \mathcal{S}^{a'}$. We will denote the natural immersion of $\mathcal{S}^a$ into $\mathcal{S}^{a'}$ by $\imath_{a,a'}$

If $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces and r is not an integer, we will define $C^r(\mathcal{X},\mathcal{Y})$ – or just C^r if the context makes it clear which spaces we are referring to – as the space of functions which can be differentiated $[r]$ times at every point and for which the norm:

$$\begin{aligned}||\phi||_{C^r} &\equiv \sup_{x\in\mathcal{X}} ||\phi(x)|| + \ldots + \sup_{x\in\mathcal{X}} ||D^{[r]}\phi(x)|| \\ &+ \sup_{x,\Delta\in\mathcal{X}} ||\phi(x+\Delta) - \phi(x) - D\phi(x)\Delta - \ldots - D^{[r]}\phi(x)\Delta^{\otimes[r]}||/||\Delta||^r\end{aligned}$$

is finite.

Notice that this is a extremely strong norm. It includes a very strong control of the derivatives at infinity and also estimates on the behavior of the Taylor remainder. This norm makes $C^r(\mathcal{X},\mathcal{Y})$ a Banach space.

The definition can be changed in an obvious way to include $C^{r+\text{Lipschitz}}$ when r is an integer and we also obtain a Banach space with the obvious norms.

For the case that r is an integer, we will require that

$$\sup_x \|\phi(x+\Delta) - \phi(x) - D\phi(x)\Delta - \cdots - D^r\phi(x)\Delta^{\otimes r}\| \leq \eta_\phi(\|\Delta\|)\|\Delta\|^r$$

where $\eta_\phi : \mathbf{R}^+ \to \mathbf{R}^+$ is decreasing $\eta_\phi(0) = 0$, $\sup_{x\in\mathbf{R}^+}\eta_\phi(x) < \infty$. Unfortunately, there is no easy way to make this space into a Banach space. The obvious choice of norm does not afford any control on the uniformity of the error. In a non-compact space, it is possible to have uniform limits of uniformly continuous functions which are not uniformly continuous. The proof of the invariant manifold will also work in this case, but it will require special considerations so we will relegate it to an special section.

Notice that in this paper when we refer to C^r, r, not an integer, we include the case when $r = k+$ Lipschitz, k an integer. If we claim that a theorem is true for C^r, r not an integer, and r in a certain range that includes $k+1$, then it is valid for $C^{k+\text{Lipschitz}}$.

The main result of this paper is:

Theorem 2.1. *Let f be a C^r mapping from $\mathcal{X}$ to itself and $\mathcal{X} = \mathbf{S} \oplus \mathbf{U}$ a direct sum decomposition of $\mathcal{X}$ into two closed subspaces which are invariant under $Df(0)$. We will denote the corresponding projectors by Π^S , Π^U and assume – without any loss of generality – that the norm satisfies:*

$$||x|| = \sup\{||\Pi^{\mathbf{S}}x||, ||\Pi^{\mathbf{U}}x||\}$$

Assume that, for some $a > 1$:

(i) $f(0) = 0$

(ii) $||Df(0)|_{\mathbf{S}}|| < a$

(iii) $||Df^{-1}(0)|_{\mathbf{U}}|| < a^{-r}$

(iv) $||\tilde{f}||_{C^r}$ *is sufficiently small, where* $\tilde{f}(x) = f(x) - Df(0)x$,

Then, the set $W^c = \{x \in \mathcal{X} | \sup_{n\geq 0} |f^n(x)|a^{-n} < \infty\}$ is a C^r manifold. Moreover W^c is tangent to $\mathbf{S}$.

Remark. Notice that, by the characterization of the set W^c, it is clear that it is invariant under f. We will see later that, in general, the manifold produced by this theorem is not the only invariant smooth manifold tangent to $\mathbf{S}$.

Remark. The theorem above implies an analogue statement for differential equations defined by C^r vector fields just by taking the time-1 map. We observe, however that there are partial differential equations – e.g. semilinear parabolic partial differential equations – that define an smooth time one map even if the vector field is not even continuous.

Remark. Notice that the conclusions of the theorem are independent of the choice of norms we pick in $\mathcal{X}$ even though the hypotheses are not since they include the conditions that certain operators are contractions. It is an standard result in functional analysis that, provided that $\sigma(Df(0))$, the spectrum of $Df(0)$, satisfies:

(ii') $\sigma(Df(0)|_{\mathbf{S}}) \subset \{z \in \mathbf{C} \big| |z| < a\}$

(iii') $\sigma(Df(0)|_{\mathbf{U}}) \subset \{z \in \mathbf{C} \big| |z| > a^r\}$

where σ denotes the spectrum, then, we can choose a norm $|||\quad ||||$ in $\mathcal{X}$ equivalent to the original one and such that with respect to this new norm the conditions (ii) and (iii) are satisfied. The remaining condition (iv) will depend on the choice of norm we made. It is possible to choose the norm $|||\quad |||$ in such a way that it is also the supremum of the norms of the projections.

Remark. For Banach spaces that admit smooth functions with bounded support and identically one in a neighborhood of the origin, – usually termed bump functions – one can obtain a version of the pseudostable manifold theorem that does not involve condition iv) even if it does not recover growth conditions.

In effect, we can consider the map $\hat{f}$ defined by $\hat{f}(x) = \Phi(\eta x) f(x) + (1 - \Phi(\eta x)) Df(0)x$. where Φ is such a bump function. We observe that, by choosing η large enough, iv) will be satisfied. Moreover, since in a neighborhood U of the origin $\hat{f} = f$ the manifold W^c obtained applying Theorem 2.1 to $\hat{f}$ will be invariant under f in a small neighborhood of the origin. It is, then possible to extend it in such a way that it is invariant under f. Of course, such manifolds are also tangent to $\mathbf{S}$ at the origin. Nevertheless, the characterizations of the points by the growth of the orbits holds only with respect to $\hat{f}$, not with respect to f.

Unfortunately, there are examples that show that the family we obtain may depend on the choice of Φ. (See the examples at the end of this paper.) We also remark that there are examples ([BF], [Dev]) of infinite dimensional Banach spaces in which there are no smooth functions with compact support even if the concept of smoothness is considerably weaker than that we have considered here. On the other hand, there are many Banach spaces for which the norm, when restricted to a ball not containing the origin is smooth in the sense considered here – *e.g.* Hilbert spaces– For these spaces, cut off functions exist.

Remark. Notice that if a function has uniformly continuous derivatives on a ball around the origin, cutting off as above will produce a uniformly differentiable function. Notice that in locally compact Banach spaces – this is equivalent to finite dimensional –, it is automatic that all continuously differentiable functions are uniformly differentiable on balls.

Remark. Notice that the growth condition imposed on the orbits is not local since the behavior of the map outside a very large ball could affect the rate of growth of iterates. On the other hand we observe that the growth condition is invariant under changes of variables that are either globally Lipschitz or $\mathrm{Id} + L^\infty$. The later situation arises when considering Anosov systems of the torus – or of any manifold compact manifold whose universal cover is R^d. The conjugating homeomorphisms given by structural stability are in $\mathrm{Id} + L^\infty$ of the lift. Hence we have the following corollary which has applications to rigidity theorems.

Corollary 2.2. *Let f, g be Anosov diffeomorphisms of $\mathbf{T}^d$ sufficiently close to a linear one. Let h be a homeomorphism sufficiently C^0 close to the identity such that $f \circ h = h \circ g$. Assume that $g(p) = p$ and that the derivatives of g, f at p, $h(p)$ respectively satisfy the hypothesis of Theorem 2.1. Let $W^{c,f}$, $W^{c,g}$ be the manifolds obtained applying Theorem 2.1 to the lifts of f, g to the universal cover and projecting* *the invariant manifolds in the conclusions to the manifold. Then* $h(W^{c,g}) = W^{c,f}$.

3. Proof of Theorem 2.1 for $r \notin \mathbf{N}$

Following Irwin and Wells , we consider the mapping $\chi : \mathbf{S} \times \mathcal{S}^a \mapsto \mathcal{S}^b$, $b \geq a$ defined by:

$$\chi(x,\rho)_n = \begin{cases} \Pi^S f(\rho_{n-1}) + (\Pi^U Df(0)\Pi^U)^{-1}\left[\Pi^U \rho_{n+1} - \Pi^U \tilde{f}(\rho_n)\right] & \text{if } n > 0; \\ x + (\Pi^U Df(0)\Pi^U)^{-1}\left[\Pi^U \rho_1 - \Pi^U \tilde{f}(\rho_0)\right] & \text{if } n = 0. \end{cases} \tag{3.1}$$

The point of this definition is that for any $b \geq a$ the condition $f(\rho^*_n) = \rho^*_{n+1}$ when $n \geq 0$ and $\Pi^S \rho^*_0 = x$, can be expressed as the fixed point equation:

$$\chi(x,\rho^*)_n = \rho^* \quad . \tag{3.2}$$

The invariant manifold will be the range of the mapping $x \mapsto \rho^*_0$, where $\rho^* \in \mathcal{S}^a$ solves the equation above, hence it will be an orbit that does not grow too fast under iteration.

Notice that since $\mathcal{S}^b \subset \mathcal{S}^{b'}$ in a natural way whenever $b' \geq b$, we can consider this equation as an equation in any $\mathcal{S}^b$, $b \geq a$, space provided that we prove uniqueness of solutions in $\mathcal{S}^b$.

What we want to do is to apply the implicit function theorem to (3.2) for a conveniently chosen b (which will turn out to be, roughly, a^r) The reason why we do not want to take $b = a$ is that χ will not be differentiable in that case. The conditions (ii) and (iii) in Theorem 2.1 will be used to show that some auxiliary mappings in those spaces are contractions. The last claim of the theorem will be proved by computing explicitly the derivative with respect to x of $x \mapsto \rho^*(x)$ when $x = 0$.

The following propositions will make all this more precise.

Proposition 3.1. *Let* $b > a$. *If* k *is any noninteger number such* $1 < k < r$, *then* $\chi : \mathbf{S} \times \mathcal{S}^b \mapsto \mathcal{S}^{b^k}$ *is* C^k *(with respect to* ρ*). Moreover, the derivatives can be computed by termwise differentiation.*

Proof. Fix $n > 0$. By substituting the definition of χ and applying the Taylor formula with remainder for f we get:

$$\begin{aligned} &\chi(x,\rho+\gamma)_n = \chi(x,\rho)_n + \\ &\Pi^S Df(\rho_{n-1})\gamma_{n-1} + \left(\Pi^U Df(0)\Pi^U\right)^{-1}\left[\Pi^U \gamma_{n+1} - \Pi^U D\tilde{f}(\rho_n)\gamma_n\right] + \\ &\Pi^S D^2 f(\rho_{n-1})\gamma_{n-1}^{\otimes 2} - \left(\Pi^U Df(0)\Pi^U\right)^{-1}\Pi^U D^2\tilde{f}(\rho_n)\gamma_n^{\otimes 2} + \\ &\dots \\ &\Pi^S D^{[k]} f(\rho_{n-1})\gamma_{n-1}^{\otimes [k]} - \left(\Pi^U Df(0)\Pi^U\right)^{-1}\Pi^U D^{[k]}\tilde{f}(\rho_n)\gamma_n^{\otimes [k]} + R_n \quad . \end{aligned} \tag{3.3}$$

where $[k]$ denotes the integer part of $[k]$ By the uniformity of the Taylor expansion of f that was built into the definition of C^r, when k is not an integer, the norm of the remainder R can be bounded by: $(||\gamma_n||_{\mathcal{X}})^k + (||\gamma_{n-1}||_{\mathcal{X}})^k + (||\gamma_{n+1}||_{\mathcal{X}})^k$.

Since $||\gamma_n||_{\mathcal{X}} \leq ||\gamma||_b b^n$ we have $||R_n||_{\mathcal{X}} \leq K||\gamma||_b^k b^{nk}$. where K depends only on b,k and $||f||_{C^k}$ In other words, if we set the derivatives of χ of order up to $[k]$ to the expressions obtained by termwise differentiation, $||R||_{b^k} \leq K||\gamma||_b^k$, which establishes the claim in the proposition.

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Proposition 3.2. *χ is C^∞ with respect to x.*

Proof. x only enters in one of the coefficients and it appears linearly.

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Proposition 3.3. *Given a strong enough smallness assumption in the hypothesis of Theorem 2.1, $\chi(x,.)$ maps $\mathcal{S}^b$ into $\mathcal{S}^b$ and is a contraction for all b, $a \leq b \leq a^r$, all $x \in \mathbf{S}$. The contraction factor is independent of x,b.*

Proof. Using the definition of the n^{th} component of χ we have:

$$\begin{aligned}&|\Pi^S(\chi(x,\rho)_n - \chi(x,\gamma)_n)| \leq \\ &\qquad ||Df(0)|||\rho_{n-1} - \gamma_{n-1}| + \sup_x ||D\tilde{f}(x)||\ |\rho_{n-1} - \gamma_{n-1}|\end{aligned}$$

if $n > 0$ and, obviously, equal to 0 if $n = 0$. Using the definition of $||\quad||_{\mathcal{S}^b}$, we can bound $|\rho_n - \gamma_n| \leq ||\gamma - \rho||_{\mathcal{S}^b} b^n$. Therefore:

$$|\Pi^S(\chi(x,\rho)_n - \chi(x,\gamma)_n)| \leq (||Df(0)|_{\mathbf{S}}|| + \epsilon')\, b^{n-1}||\rho - \gamma||_{\mathcal{S}^b}$$

where $\epsilon' = \sup_x ||D\tilde{f}||$.

Analogously, we can bound:

$$\begin{aligned}&|\Pi^U(\chi(x,\rho)_n - \chi(x,\gamma)_n)| \leq \\ &\qquad ||Df(0)|_{\mathbf{U}}^{-1}||\left(\Pi^U(\rho_{n+1} - \gamma_{n+1})\right)| + \sup_x ||D\tilde{f}(x)|||\rho_n - \gamma_n| \quad \text{if } n > 0 \\ &|\Pi^U(\chi(x,\rho)_n - \chi(x,\gamma)_n)| \leq \\ &\qquad ||Df(0)|_{\mathbf{U}}^{-1}||\left[|\Pi^U(\rho_1 - \gamma_1)| - ||\Pi^U|| \sup_x ||D\tilde{f}(x)||\quad |\rho_0 - \gamma_0|\right] \text{ if } n = 0\end{aligned}$$

Using again the definition of $||\quad||_{\mathcal{S}^b}$, we can bound

$$|\Pi^U(\chi(x,\rho)_n - \chi(x,\gamma)_n)| \leq (||Df(0)|_{\mathbf{U}}^{-1}|| + \epsilon'')b^{n+1}||\rho - \gamma||_{\mathcal{S}^b}$$

where $\epsilon'' = b^{-1}||Df(0)|_{\mathbf{U}}^{-1}|| \sup_x ||D\tilde{f}(x)||$.

Notice that both ϵ',ϵ'' can be made arbitrarily small by assuming that the smallness assumptions in Theorem 2.1 are strong enough.

Using the fact that the norm of a vector in $\mathcal{X}$ is the supremum of the norm of the projections, we obtain that:

$$|\chi(x,\rho)_n - \chi(x,\gamma)_n| \le b^n \max\left(b(||Df(0)|_{\mathbf{S}}|| + \epsilon'), b^{-1}(||Df(0)|_{\mathbf{U}}^{-1}|| + \epsilon'')\right) ||\rho - \gamma||_{\mathcal{S}^b}$$

Using the definition of $||\quad||_{\mathcal{S}^b}$, this means that the Lipschitz constant of $\chi(x,.)$ is less than $\max\left(b(||Df(0)|_{\mathbf{S}}|| + \epsilon'), b^{-1}(||Df(0)|_{\mathbf{U}}^{-1}|| + \epsilon'')\right)$ which can be made strictly less than one, uniformly in x and b under the hypothesis of the lemma.

A simpler version of these estimates shows that $\chi(x,.)$ maps $\mathcal{S}^b$ onto itself. (It suffices to observe that $\chi(x,0)$ is in $\mathcal{S}^b$ and estimate as here $\chi(x,\rho) - \chi(x,0)$.)

$\blacksquare$

Remark. Notice that $|D^k \tilde{f}(x)\gamma^{\otimes k}| \le \|\tilde{f}\|_{C^k} |\gamma|^k$. A calculation similar to the one we performed to show that $\chi(x,.)$ was a contraction in S^b $a \le b \le a^r$ shows that $D_2^k \chi(x,\rho)$ is a bounded operator from S^b to S^{b^k} $\alpha \le b \le \beta$ where α and β can be made arbitrarily small and arbitrarily large respectively by assuming that $\|D^{[k]}\tilde{f}\|_{C^{k-[k]}}$ is sufficiently small.

If b is such that $a \le b \le b^r$, Applying the contraction mapping theorem, whose hypothesis are verified because of Proposition 3.3, we obtain for every x there exist a $\rho^*(x) \in S^b$ which solves (3.2). Moreover such $\rho^*(x)$ is the only solution in S^b.

Since $S^b \subset S^{b'}$ if $b' > b$, the existence part of the conclusions is stronger the smaller the b is, while the uniqueness part of the conclusion is stronger the larger b is.

Notice also that the elementary contraction mapping principle shows that the map $x \to \rho^*(x)$ is C^r $r \le$ Lipschitz if χ is. Hence we have established Theorem 2.1 except for the regularities greater than Lipschitz.

The following result completes the proof.

Lemma 3.4. *The mapping $\rho^* : \mathcal{S} \to S^{a^s}$ defined by requiring that ρ^* solves (3.2) is C^s when $s \le r$, s not an integer.*

Proof. We have already established the result for $s \le$ Lipschitz.

To prove the existence of higher derivatives we will derive heuristically a formula for the derivatives and then, show that they indeed satisfy the estimates that establish that they are derivatives.

If we take derivatives formally in (3.2), we obtain

$$D_1\chi(x,\rho^*) + D_2\chi(x,\rho^*)D_x\rho^* = D_x\rho^* \tag{3.4}$$

Hence, we guess that the derivative of ρ^* should be

$$\Delta \equiv -\big(D_2\chi(x,\rho^*) - Id\big)^{-1} D_1\chi(x,\rho^*) \tag{3.5}$$

Notice that since $D_2\chi : S^a \to S^a$ is a contraction, this is an element of S^a.

To prove that (3.5) is indeed a derivative and that ρ^* is $C^{1+\beta}$ it suffices to show that:

$$\|\rho^*(x+y) - \rho^*(x) - \Delta y\|_{S^{a^{1+\beta}}} \le C|y|^{1+\beta} \tag{3.6}$$

Since $\rho^*(x+y)$ is by definition the solution of $\chi(x+y,\rho^*) = \rho^*$, and χ is a uniform contraction, (3.6) follows from

$$\|\chi(x+y, \rho^*(x+y) - \rho^*(x) - \Delta y\|_{S^{a^{1+\beta}}} \le C|y|^{1+\beta} \quad ,$$

which can established by remembering that, by Proposition 3.1 we have:

$$\|\chi(x+y,\rho^*(x)+\Delta y) - \chi(x,\rho^*(x)) - D_1\chi(x,\rho^*(x))y - D_2\chi(x,\rho^*(x))\Delta y\|_{S^{a^{1+\beta}}} \le C|y|^{1+\beta}$$

and using the definition of Δ.

This establishes Lemma 3.4 for $s \le 1+$ Lipschitz.

Higher derivatives can be obtained by induction. If we have proved the theorem for $s \le 1 = 1+$ Lipschitz we can obtain a guess for the i^{th} derivative by taking i derivatives of (3.2).

A simple calculation shows that:

$$D_2\chi(x,\rho^*)D_x^i\rho^* + R_{i-1} = D_x^i\rho^* \tag{3.7}$$

Where R_{i-1} is a symmetric multilinear operator from $\mathbf{S}^{\otimes i}$ to S^a whose expression involves tensor products of derivatives of ρ up to order $i-1$.

Hence, it is natural to guess that the i^{th} derivative will be

$$\Delta_i = -(D_2\chi(x,\rho^*) - Id)^{-1} R_{i-1} \tag{3.8}$$

We now we interpret $D_2\chi(x,\rho^*)$ as a bounded operator from S^{a^i} to itself. By Proposition 3.3 this operator is a contraction and, hence $(D_2\chi(x,\rho^*) - Id)^{-1}$ exists as a bounded operator from S^{a^i} to itself.

As before, to show that this is indeed a bona fide derivative it suffices to obtain estimates

$$\begin{aligned}\|\chi(x+y,\rho^*(x) + D\rho^*(x)y + \cdots + D^{i-1}\rho^*(x)y^{\otimes i-1} + \Delta_i y^{\otimes i})& \\ - (\rho^*(x) + D\rho^*(x)y + \cdots + D^{i-1}\rho^*(x)y^{\otimes i-1} + \Delta_i y^{\otimes i})\|_{S^{a^{i+\beta}}}& \\ \le C|y|^{i+\beta}&\end{aligned} \tag{3.9}$$

We recall that by the construction of R_{i-1} if χ is $C^{i+\beta}$ we have

$$\begin{aligned}
&\|\chi(x+y, \rho^*(x) + D\rho^*(x)y + \cdots + D^{i-1}\rho^*(x)y^{\otimes i-1} + \Delta_i y^{\otimes i}) \\
&\qquad - [D_2\chi(x,\rho^*)\Delta_i y^{\otimes i} + R_{i-1}y^{\otimes i}]\|_{\mathcal{S}^{a^{1+\beta}}} \\
&\le C|y|^{i+\beta}
\end{aligned} \tag{3.10}$$

If we substitute the expression (3.8), for Δ_i, into (3.10) we obtain (3.9) and the theorem is established.

■

Remark. Notice that, in the formula for χ, x enters only linearly, so it is very easy to compute derivatives with respect to x.

The condition (ii) of Theorem 2.1 (together with smallness assumptions in $\tilde{f}$) implies that $\mathcal{S}^a$ gets mapped into itself.

This finishes the proof of Theorem 2.1 except for the claim of the manifold being tangent to the space $\mathbf{S}$. Substituting the explicit formula for the derivative of the map with respect to x we find that $D_x\rho^*(0) = 0$. This finishes the proof of Theorem 2.1.

■

Remark. Out of this method of proof it is very easy to conclude smooth dependence on parameters for Irwin's manifolds. If we consider that f_λ is an smooth family of C^r maps, the same arguments that we have used before to check differentiability of χ with respect to x, can be used to establish differentiability of χ_λ with repect to λ.

4. Proof of Theorem 2.1 for $r \in \mathbf{N}$

When $r \in \mathbf{N}$, it is not true that $\chi \in C^r$, hence, the previous argument cannot establish that $\rho^* : \mathbf{S} \mapsto \mathcal{S}^{a^r}$ is C^r. Nevertheless, we observe that the regularity of the pseudostable manifold only requires regularity of the mapping $\rho_0^* : \mathbf{S} \mapsto \mathcal{X}$ obtained by taking the zeroth component of the mapping ρ^*. We will be able to establish this regularity by considering topologies on the spaces of sequences weaker than those induced by the $\| \quad \|_{\mathcal{S}^a}$ norms we considerd before. In particular, componentwise convergence will play a role.

We will assume in the rest of this section that the hypotheses of Theorem 2.1 hold and that r is an integer.

The following proposition is trivial.

Proposition 4.1. *For every $n \in \mathbf{N}$, the map $\chi_n : \mathbf{S} \times \mathcal{S}^a \mapsto \mathcal{X}$ that produces the n^{th} component of $\chi(x, \rho)$ is C^r.*

Proof. Just observe that each component depends only on a finite number of components and that for a fixed number of components, the $\mathcal{S}^a$ norm is stronger than the norm in each of the components.

■

Notice that the derivative will depend only on a finite number of components.

An inmediate corollary of the previous result is:

Lemma 4.2. *Given $N \in \mathbf{N}$ and $\epsilon > 0$, it is possible to find $\delta > 0$ such that if $0 < ||\gamma||_{\mathcal{S}^a} \le \delta$, then,*

$$\sup_{n \le N} \left|\chi_n(x, \rho + \gamma) - D\chi_n(x, \rho)\gamma - \cdots - D^r\chi_n(x, \rho)\gamma^{\otimes r}\right| \le \epsilon ||\gamma||_{\mathcal{S}^a} \tag{4.1}$$

We also have:

$$\sup_{n \le N} \left|\chi_n(x, \rho + \gamma) - D\chi_n(x, \rho)\gamma - \cdots - D^r\chi_n(x, \rho)\gamma^{\otimes r}\right| \frac{1}{||\gamma||_{\mathcal{S}^a}} \le K < \infty \tag{4.2}$$

Proof. The last inequality is a consequence of the hypothesis $f \in C^r$ with our definition of C^r, which included the hypothesis that the modulus of continuity η_f was bounded.

Notice that applying the results of the previous section, we can conclude that ρ^* is C^{r-1}. We also can obtain a guess for what the r^{th} derivative should be in the same way that we did before, that is, taking r formal derivatives of (3.2) and solving the resulting equation that the r^{th} derivative should solve as in (3.8). To check that this procedure is well defined, notice that the computation of R_{i-1} in (3.7) only requires the existence of $r-1$ derivatives. The existence of the inverse in (3.8) can be guaranteed provided that $D\chi$ is a contraction from $\mathcal{S}^{a^r}$ to $\mathcal{S}^{a^r}$. The later is the case if we assume that $||Df(0)|_{\mathbf{S}}|| + ||D\tilde{f}|| < a$ $||Df(0)|_{\mathbf{U}}^{-1}|| + ||D\tilde{f}|| < a^r$, which we can assume by if we assume that $||D\tilde{f}||$ is small enough.

Unfortunately, this guess fails to be a derivative in the sense we considered before because if we substitute it as before in (3.2), we do not obtain a residual which is small in the $\mathcal{S}^{a^r}$ sense. This is due to the fact that χ is not C^r as a mapping into $\mathcal{S}^{a^r}$. Nevertheless, using the fact that χ is differentiable componentwise, we can obtain that the residual has an arbitrarily large number of arbitrrily small components.

Lemma 4.3. *Denote by $\rho^{\text{guess}}(x, y) = \rho^*(x) + D\rho^*(x)y + \cdots \Delta_r y^{\otimes r}$, where Δ_r, is obtained as in (3.8). Then, it is possible to find $K < \infty$ such that*

$$\sup_n |\chi_n(x + y, \rho^{\text{guess}}(x, y)) - \rho_n^{\text{guess}}(x, y)| \le K|y|^r \tag{4.3}$$

Moreover, given $\epsilon > 0$, $N \in \mathbf{N}$, it is possible to find $\delta > 0$ such that $|y| \le \delta$ implies:

$$\sup_{n \le N} |\chi_n(x+y, \rho^{\text{guess}}(x,y)) - \rho_n^{\text{guess}}(x,y)| \le \epsilon |y|^r \tag{4.4}$$

Proof.

We recall that Δ_r was chosen precisely in such a way that if we expand in powers of y $\chi(x+y, \rho^{\text{guess}}(x,y)) - \rho^{\text{guess}}(x,y)$, the Taylor expansion up to order r vanishes. Even if it is impossible to estimate the remainder in the sense of $\mathcal{S}^{a^r}$, it is possible to estimate it component by component. In each component, we obtain that the remainder can be bounded by the remainder of the Taylor expansion of f. Hence, using the uniformity of the modulus of continuity of the derivative, we obtain (4.3). Moreover, (4.4) follows because it is a bound in the Taylor remainder of a finite number of components.

■

We observe that in our situation, Proposition 3.3 still applies and, hence we have $\rho^*(x,y) = \lim_{i\to\infty} \chi^i(x+y, \rho^{\text{guess}}(x,y))$ where we denote by χ^i the application of $\chi(x+y, .)$ i times and the limit is understood in the $\mathcal{S}^{a^r}$ sense $-i.$ $e.$ componentwise.

Hence, we can estimate

$$|\rho_n^*(x+y) - \rho_n^{\text{guess}}(x,y)| \le \sum_{i=0}^{\infty} |\chi_n^{i+1}(x+y, \rho^{\text{guess}}) - \chi_n^i(x+y, \rho^{\text{guess}})| \tag{4.5}$$

Lemma 4.4. *Let $\alpha = \max(||Df(0)|_{\mathbf{S}}, ||Df(0)|_U U^{-1}||) + ||D\tilde{f}||_{C^0}$, which, we will assume according to the hypothesis of Theorem 2.1 is strictly less than 1.*

Let $\rho, \gamma \in \mathcal{S}^{a^r}$ be such that for some $0 < \epsilon < K$, $N \in \mathbf{N}$ we have:

$$\sup_{n \le N} |\rho_n - \gamma_n| \le \epsilon$$
$$\sup_{n} |\rho_n - \gamma_n| \le K$$

Then

$$\sup_{n \le N-1} |\chi_n(x,\rho) - \chi_n(x,\gamma)| \le \epsilon\alpha$$
$$\sup_{n} |\chi_n(x,\rho) - \chi_n(x,\gamma)| \le K\alpha$$

Proof. If $n > 0$, we estimate

$$\begin{aligned} &\left|\Pi^{\mathbf{S}}(\chi_n(x,\rho) - \chi_n(x,\gamma))\right| \le \\ &\left|\Pi^{\mathbf{S}} Df(0)\Pi^{\mathbf{S}}(\rho_{n-1} - \gamma_{n-1}) + \Pi^{\mathbf{S}}(\tilde{f}(\rho_{n-1}) - \tilde{f}(\gamma_{n-1}))\right| \\ &\le \alpha|\rho_{n-1} - \gamma_{n-1}| \end{aligned} \tag{4.6}$$

And

$$\begin{aligned}&\left|\Pi^{\mathbf{U}}(\chi_n(x,\rho)-\chi_n(x,\gamma))\right| \leq \\ &||Df(0)|_{\mathbf{U}}^{-1}|||\Pi^{\mathbf{U}}(\rho_{n+1}-\gamma_{n+1})|+|\Pi^{\mathbf{U}}(\tilde{f}(\rho_n)-\tilde{f}(\gamma_n))| \leq \\ &|Pi^{\mathbf{U}}(\rho_{n+1}-\gamma_{n+1})|+||D\tilde{f}||_{C^0}|\rho_n-\gamma_n|\end{aligned} \tag{4.7}$$

If $0 < n \leq N-1$ both (4.6), (4.7) can be bounded by $\alpha\epsilon$. If $n \geq N$ both terms can be estimated by αK.

The case $n = 0$ is easier and is left to the reader.

■

By applying repeatedly the lemma we derive the following corollary.

Lemma 4.5. *Let α, ϵ, N, ρ, γ be as in Lemma 4.4. Let $n \in \mathbf{N}$ be such that $n < N$. Then*

$$\sum_{i=0}^{\infty} |\chi_n^i(x,\rho)-\chi_n^i(x,\gamma)| \leq \frac{\epsilon}{1-\alpha}+\frac{K\alpha^{N-n}}{1-\alpha}$$

Now we can prove that $\rho_0^*(x)$ is C^r and that $[\Delta_r]_0$ is a bona-fide derivative. (Where we denote by $[\Delta_r]_0 y^{\otimes r} = (\Delta_r y^{\otimes r})_0$.)

This amounts to showing that given $\epsilon > 0$ we can find $\delta > 0$ such that if $|y| \leq \delta$, then

$$|\rho_0^*(x+y)-\rho_0^*(x)-D\rho_0^*(x)y-\cdots-[\Delta_r]_0 y^{\otimes r}| \leq \epsilon|y|^r \tag{4.8}$$

(The fact that we can find a K such that $|\rho_0^*(x+y)-\rho_0^*(x)-D\rho_0^*(x)y-\cdots-[\Delta_r]_0 y^{\otimes r}| \leq K|y|^r$ that we included in the definition of C^r follows very easily by observing that $[\Delta_r]_0$ is uniformly bounded.)

To prove that we can satisfy (4.8), we observe that by Lemma 4.3, we can find K in such a way that $|\chi_n(x+y,\rho^{\text{guess}}(x,y))-\rho_n^{\text{guess}}(x,y)| \leq K|y|^r$.

Choose N big enough that $\alpha^N(1-\alpha)^{-1}K \leq \epsilon/2$ and take the δ provided that Lemma 4.3 that guarantees that for $|y| \leq \delta$ we have:

$$\sup_{n\leq N} |\chi_n(x+y,\rho^{\text{guess}}(x,y))-\rho_n^{\text{guess}}(x,y)| \leq |y|^r(1-\alpha)\epsilon/2$$

Applying Lemma 4.5 with $\rho = \rho^{\text{guess}}(x,y)$, $\gamma = \chi(x+y,\rho^{\text{guess}}(x,y))$, we obtain (4.8) and, hence, the theorem is proved.

■

5. Examples

The following examples shows that in general the manifolds constructed are not more differentiable than the claim of Theorem 2.1. The two examples are quite instructive since they show that there are different obstructions to differentiablility.

Example 5.1. *Consider the mapping f of $\mathbf{R}^2 \mapsto \mathbf{R}^2$ defined by:*

$$(x, y) \mapsto (2x, 3y + \varphi(x))$$

where φ is a C^∞ function with support in the interval $[0.9, 1]$. If φ is not identically 0, for any $\epsilon > 0$, the pseudo stable manifold obtained by taking $a = 2 + \epsilon$ and r such that $a^r = 3 - \epsilon$ in Theorem 2.1 is not $C^{\frac{\log 3}{\log 2} + \epsilon}$.

Proof. In this case, we can construct the manifold almost explicitly.

Since for sufficiently large x , the mapping f agrees with the linear transformation, if $x > 1$, the only point p of the form $p = (x, y)$ such that $f^n(p)(3 - \epsilon)^{-n}$ remains bounded is precisely $p = (x, 0)$.

We can construct the whole invariant manifold by iterating backwards this manifold that we have so, we can see it will be the graph of the function:

$$\sum_{n=0}^{\infty} 3^{-n} \varphi(2^n x)$$

$\blacksquare$

Notice, however that the map f in Example 5.1 has invariant manifolds tangent to the x axis which are C^∞. Such manifolds can be readily constructed by declaring that $[-0.8, 0.8]$ is in the manifold and determining the rest of the manifold in such a way that the invariance property holds. It will be the graph of the function

$$\sum_{n=0}^{\infty} 3^{n} \varphi(2^{-n} x)$$

However most of the the points of this smooth invariant manifold have orbits that grow asymptotically with the largest eigenvalue.

Notice further that if we cut-off the function as indicated in the remarks after Theorem 2.1 we could obtain just the linear map $(x, y) \to (2x, 3y)$ for which the invariant manifold is just the coordinate axis. The invariant manifold produced for the original map would then be the one produced in the previous remark, and not the one produced by direct application of the theorem.

Unfortunately, it is not true that all maps have smooth pseudo-stable manifolds as the following example shows:

Example 5.2. *The map $f : \mathbf{R}^2 \mapsto \mathbf{R}^2$ $f(x, y) = (2x, 4y + x^2)$ does not leave invariant any C^2 manifold tangent to the x axis in any neighborhood of the origin.*

Proof. If such a manifold existed it would be possible to write it as the graph of a mapping w from the x axis to the y axis.

The invariance of the manifold is equivalent to the map w satisfying: $w(2x) = 4w(x)+x^2$. If the map w were twice differentiable, we would have: $4w''(2x) = 4w''(x) + 2$, which evaluated at $x = 0$ produces a contradiction.

■

Notice that the local obstruction studied in Example 5.2 would not have worked for the f in Example 5.1. We used crucially that $4 = 2^2$. It turns out that it is possible to show that given non–resonance conditions, one gets locally invariant manifolds which are smooth.

6. References

[AM] R. Abraham, J. Marsden: "*Foundations of mechanics*", Benjamin (1978).

[BF] R. Bonic, J. Frampton: Smooth functions on Banach manifolds. Jour. Math. Mech. **16**, 877–898 (1966).

[Dev] R. Deville: Geometric implications of the existence of very smooth bump functions in Banach spaces. Isr. Jour. Math. **67**, 1–22 (1989).

[FHY] A.Fathi , M. Herman , J–C. Yoccoz: A proof of Pesin's stable manifold theorem. In "*Lec. Notes in Math.* ". Springer **1007**, (1983).

[Ir1] M.C. Irwin: On the stable manifold theorem. Bull. London Math. Soc. **2**, 196–198 (1970).

[Ir2] M.C. Irwin: A new proof of the pseudostable manifold theorem. Jour. London Math. Soc. **21**, 557–566 (1980).

[La] O. E. Lanford III: Bifurcation of periodic solutions into invariant tori: the work of Ruelle and Takens. In " *Lect. Notes in Math.* ". Springer **322**, (1973).

[RT] D. Ruelle, F. Takens: On the nature of Turbulence. Comm. Math. Phys. **20**, 167-192 (1971).

[W] J. C. Wells: Invariant manifolds of non-linear operators. Pac. Jour. Math **62**, 285-293 (1976).