Solutions of Nonlinear Wave Equations and Localization Theory *

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In this note we describe a new method for constructing time periodic solutions of nonlinear wave equations of the form

$$u_{tt} = u_{xx} - g(x,u) \qquad 0 < x < \pi, \tag{1}$$

with either periodic or Dirichlet boundary conditions. For a complete description of the results and methods with detailed proofs, see [2]. This article will attempt only to explain the general ideas behind this method and some of its possible extensions.

Our main result is the following:

Theorem *Consider equation (1) with periodic boundary conditions. There is a set of nonlinearities $g(x,u)$, which is dense in $L^2_{per}(0,\pi)$ and has open intersection with the set of analytic functions, for which the following statements are true:*
There exists a Cantor set $\mathcal{C} \subset \mathbf{R}$, of positive measure, and a C^∞ curve $\rho(\Omega)$ such that for every $\Omega \in \mathcal{C}$, equation (1) has a non-trivial, analytic, time periodic solution $u(x,t;\Omega)$, with angular frequency Ω, with $\|u(\cdot,\cdot;\Omega)\|_{L^2} = \rho(\Omega)$.

Similar results hold if periodic boundary conditions are replaced with Dirichlet boundary conditions.

The conditions which determine whether or not the theorem applies to a given nonlinear term are finite in number, involve only the first three terms in the Taylor series for $g(x,u)$, and can, at least in principle, be explicitly verified for a given $g(x,u)$. As a particular example of the sort of equation to which the theorem applies, consider the Klein-Gordon equation,

$$u_{tt} = u_{xx} - m^2 u + u^3 \quad .$$

Then in the case of Dirichlet boundary conditions, we are able to show that the theorem applies to this equation for an open set of values of m of full Lesbesgue measure. (In this example, and in any cases in which the nonlinear term $g(x,u) = g(u)$, a simple analysis of the phase plane gives the existence result when periodic boundary conditions are imposed.)

Our method is perturbative in nature and thus starts from an analysis of the linearized problem. We assume that $g(x,u)$ is analytic and has a Taylor series at $u = 0$ of the form $g(x,u) = g_1(x)u + g_2(x)u^2 + \ldots$ so that to first order (1) becomes

$$v_{tt} = (\partial_x^2 - g_u(x,0))v \quad , \quad 0 < x < \pi, \tag{2}$$

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This equation is easily solved. If the eigenvectors and eigenvalues of the Sturm-Liouville operator $\left(-\frac{d^2}{dx^2} + g_u(x,0)\right)$ with the appropriate boundary conditions are $\{\psi_j(x)\}_{j=1}^{\infty}$ and $\{\omega_j^2\}_{j=1}^{\infty}$ (assume for simplicity that all the eigenvalues are positive, though this is not necessary) then $v^{(j)}(x,t) = \psi_j(x)\sin(\omega_j t)$ is a periodic solution of (2), for every $j = 1, 2\ldots$. We wish to show that there are solutions of the full non-linear equation (1), which are "close" to these solutions of the linearized equations. With only a slight loss of generality we will focus on the case $j = 1$ – i.e. we look for solutions of (1), with frequency close to ω_1. If such a solution exists, we can write it in an eigenfunction-Fourier expansion as

$$u(x,t) = \sum_{j=1}^{\infty} \sum_{k\in\mathbf{Z}} \widehat{u}(j,k)\psi_j(x)e^{i\Omega kt} \tag{3}$$

where $\Omega \approx \omega_1$. If we substitute (3) into (1), we obtain an infinite set of coupled equations for the coefficients $\{\widehat{u}(j,k)\}$, which we write schematically as

$$F(\widehat{u})(j,k) = W(\widehat{u})(j,k) + V(\Omega)\widehat{u}(j,k) = 0, \quad j = 1,2\ldots, \;\; ; \; k \in \mathbf{Z} \; . \tag{4}$$

Here, $V(\Omega)$ is the diagonal operator $V(\Omega)\widehat{u}(j,k) = (\omega_j^2 - k^2\Omega^2)\widehat{u}(j,k)$, which arises from the linearized equation (2), while $W(\widehat{u})$ is a non-diagonal operator coming from the non-linear part of the function $g(x,u)$, whose form we will comment upon below.

We will construct solutions $\{\widehat{u}(j,k)\}$ of (4) by applying Newton's method, taking as our starting point the function $\varphi(p) = p\delta(1,1) + \overline{p}\delta(1,-1)$, where $p \in \mathbf{C}$. Note that φ is a solution of the linearized problem (i.e. when $W = 0$) if $\Omega = \omega_1$. (Here, $\delta(j,k)$ is the Kronecker δ-function on the lattice $\mathbf{Z}^+ \times \mathbf{Z}$.)

The most difficult part of this process is to control the inverse of the operator

$$D_{\tilde{u}}F = D_{\tilde{u}}W + V(\Omega) \tag{5}$$

which occurs when we linearize about an approximate solution $\widetilde{u}$. We control this operator in two stages. The first problem we encounter is that when $\Omega = \omega_1$, the linear operator $V(\omega_1)$ has a two-dimensional null space $\ell^2(N)$, where $N = \{(1,1),(1,-1)\}$. This is a common problem in bifurcation theory and we treat it using a standard method, the Lyapunov-Schmidt method. Define Q = orthogonal projection onto $\ell^2(N)$ and $P = \mathbf{1} - Q$. Then (4) is equivalent to a pair of equations,

$$QF(\widehat{u}) = QW(\widehat{u})(j,k) + (\omega_1^2 - \Omega^2)\widehat{u}(j,k) = 0, \quad (j,k) \in N \;\; , \tag{6}$$

and

$$PF(\widehat{u}) = PW(\widehat{u})(j,k) + (\omega_j^2 - k^2\Omega^2)\widehat{u}(j,k) = 0, \quad (j,k) \notin N. \tag{7}$$

Note that the "Q-equation" (*i.e.* (6)) is a finite dimensional problem (in fact two dimensional) and can be handled by ordinary implicit function theorem arguments. Thus, we will concentrate in the remainder of this note on how one can solve the "P-equation" (7).

Equation (7) is solved inductively by finding better and better approximate solutions, on larger and larger subsets of the lattice, $\mathbf{Z}^+ \times \mathbf{Z}$. In the limit, the subset converges to the whole lattice and the solution converges to a true solution of (7). In order to start the induction we make certain assumptions about finitely many of the frequencies $\{\omega_j\}$, which is equivalent to making assumptions about the first order term in the Taylor expansion of the non-linear term $g(x,u)$ in (1). We comment below on just how restrictive these assumptions are.

The initial "box" on which we solve (7) is the set $B_0 = \{(j,k) \in \mathbf{Z}^+ \times \mathbf{Z} \mid j + |k| \leq L_0\}$, with L_0 a large constant. We then assume that for (j,k) in B_0 the frequencies of the linearized problem satisfy (i) $|\omega_j^2 - k^2\omega_1^2| \geq d_0$ and (ii) $|k\omega_1 - j| \geq c_0(j+|k|)^{-\tau}$, where d_0, c_0 and τ are positive constants. Furthermore we assume that the nonlinearity satisfies a "twist" condition. This condition ensures that the bifurcation curve $\rho(\Omega)$ has non-zero curvature at $\Omega = \omega_1$, and it can be explicitly computed from a formula involving the second and third order terms $g_2(x)$ and $g_3(x)$ of the Taylor series of the nonlinearity and the eigenfunctions of the linearized operator. When restricted to B_0, (7) becomes a finite dimensional problem and conditions (i) and (ii) guarantee that it can be solved by the ordinary implicit function theorem. One finds further that the coefficients $\{u^{(0)}(j,k)\}$ of this solution decay exponentially in $(|j|+|k|)$, which means that $u^{(0)}$ fails to be a true solution of (7) (i.e. a solution on the whole lattice $\mathbf{Z}^+ \times \mathbf{Z}$ rather than just on B_0) only by an amount of $\mathcal{O}(e^{-\sigma L_0})$, for some $\sigma > 0$. Thus, if L_0 is large we have a very good approximate solution.

To improve this solution we use Newton's method. If we restrict (7) to a new, larger lattice region $B_1 \supset B_0$, where B_1 is defined analagously to B_0, but with L_0 replaced by $L_1 > L_0$, and try to solve $PF(u^{(0)} + v^{(0)}) \mid_{B_1} = 0$ by Newton's method, we find

$$v^{(0)} = -\left\{P(D_{u^{(0)}}W + V(\Omega)) \mid_{B_1} P\right\}^{-1} PF|_{B_1}(u^{(0)}). \tag{8}$$

From the comments above, we know that $PF|_{B_1}$ is small. Thus, the only difficulty we encounter is controlling the inverse of the operator $P(D_{u^{(0)}}W + V(\Omega))|_{B_1}P$. We have an explicit form for the diagonal term $V(\Omega)$, so this causes no problems. To understand the off-diagonal term $D_{u^{(0)}}W$, we note that for the example of the Klein-Gordon equation, $u_{tt} = u_{xx} - m^2u + u^3$, one can compute D_uW exactly if one takes u to be the solution of the linearized problem. One finds that $D_uW + V(\Omega) = \gamma\Delta + V(\Omega)$, where γ is a constant, and Δ is the lattice laplacian. Thus, in this example, the task of inverting $D_{u^{(0)}}W + V(\Omega)$ is the same as controlling the inverse of a lattice Schrödinger operator. This is a problem that has been studied in depth in the physics literature. In particular, Fröhlich and Spencer [3] have shown that if one can insure that the "resonances" in $V(\Omega)$ are not too close together and not too severe, then the matrix elements of $(\gamma\Delta + V(\Omega))^{-1}$ will decay exponentially fast as one moves away from the diagonal. For more general non-linearities than that in the Klein-Gordon equation, or if we evaluate the derivative at a point other than the solution of the linearized equation, one will not obtain $D_{u^{(0)}}W = \gamma\Delta$. However, $D_{u^{(0)}}W$ will still be a bounded operator with matrix elements which decay exponentially fast as one moves away from the diagonal, and the methods of Fröhlich and Spencer are still applicable.

Applying the methods of Fröhlich and Spencer places restrictions on the allowed frequencies of our periodic orbits. In the present problem the "resonances" correspond to points (j,k) at which the potential $V(\Omega)(j,k)$ approaches zero. A simple calculation using the fact that asymptotically the linear frequencies in this problem satisfy $\omega_j \sim j$, shows that if we eliminate frequencies Ω which fail to satisfy the diophantine inequality

$|k\Omega - j| \geq c(j + |k|)^{-\tau}$, then for any remaining frequency, Ω, two lattice sites at which $V(\Omega)$ is small, must be widely separated. One must also insure that at any remaining sites where $V(\Omega)$ is small, the linearized operator $(D_u W + V(\Omega))$ does not have points in its spectrum that are too close to zero. This entails a further excision of frequencies Ω, so that at the conclusion of this induction argument we are left with a family of periodic orbits, but the frequencies in this family form a Cantor set of positive measure, not an interval as one typically finds in ordinary finite dimensional bifurcation theory. Once one makes these excisions of frequencies, the inverse of the linearized operator in Newton's method is controllable, and the induction argument converges to give a family of periodic orbits. We remark that methods related to ours have been applied to construct periodic solutions for some nonlinear versions of the Anderson localization model by Albanese, Fröhlich and Spencer [1].

There are several related problems which we think can be treated by this method. First of all, there is the existence of quasi-periodic (in time) orbits for (1). One can again make an eigenfunction-Fourier expansion similar to (3), but with n-frequencies in time – i.e. the exponential becomes $\exp\{i(\sum_{\ell=1}^{n} \Omega_\ell k_\ell)t\}$. In this case the original partial differential equation is reduced to a problem on the lattice $\mathbf{Z}^+ \times \mathbf{Z}^n$. One can then apply Newton's method just as in the periodic case. While formally the same, the problems one encounters when studying the resonances of the linearized problem are more complicated than in the periodic case, and we have not yet finished this analysis. However, we believe that this construction will also quasi-periodic solutions of (1). We note that if successful this method would also give a new construction of quasi-periodic orbits of finite dimensional hamiltonian systems of the form $H(p, q) = \sum_{j=1}^{N} \frac{1}{2} p_j^2 + V(q)$, in the neighborhood of an elliptic equilibrium point which is different from the usual construction using the KAM Theorem.

Another case which seems difficult to treat with KAM methods, but which we believe the present methods can treat, is the case of resonances in the linear problem. Suppose, for instance, that the linear frequencies ω_1 and ω_2 satisfy $\omega_2 = 2\omega_1$. In this case when we convert the problem to one on the lattice we find the null space of the diagonal operator $V(\Omega)$ is four dimensional rather than two dimensional. We can still proceed with the Lyapunov Schmidt procedure, however, and the "P-equation" (7) is solved just as in the non-resonant case. This leaves us with the "Q-equation", (6). This is now a finite (in this example, four) dimensional problem. Although the resonance makes the ordinary implicit function theorem inapplicable, we believe other methods will allow us to solve this problem — either perturbative techniques like those of Schmidt [6], or variational methods like those used by Weinstein [8] and Moser [5] to extend Lyapunov's center theorem to the case in which the linear problem contains resonances. The advantage that the present method offers, in our opinion, is that it allows one to treat quite independently the problems arising from the linear resonance ("Q-equation") and the infinite dimensionality of the problem ("P-equation").

We conclude with a short discussion of how these methods compare with KAM approaches to the problem which have been used by Kuksin [4] and Wayne [7]. The principle difference is in the sort of non-resonance conditions that are imposed on the frequencies of the linear problem. The conditions imposed by the present method require essentially that no multiple of the frequency of the periodic orbit we are constructing resonate with any of the other linearized frequencies. This condition is an infinite dimensional generalization of the hypothesis of the Lyapunov center theorem. In contrast, the KAM type methods require in addition that certain non-resonance conditions hold among those linear frequencies that we are not perturbing. These additional conditions arise because in the KAM method one transforms the hamiltonian of the problem to some normal form. The present method is not a transformation theory in this sense, but rather a more direct implicit function theorem. In contrast to previous approaches like that of Zehnder [9], which used the "group structure" given by the transformation theory to invert the linearized operator in Newton's method explicitly, in the present method we obtain bounds on this inverse without using a transformation theory. Indeed, even the hamiltonian nature of the problem plays little explicit role in this approach, so that we hope it may be applicable to a wide variety of problems.

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