$gl(\infty)$, $diff_A\, S^2$, $diff_A\, T^2$ AND $su(N)$- APPROXIMATION

Martin Schlichenmaier

University of Mannheim

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1. Introduction.

In the following, I like to report on joint work with M. Bordemann, J. Hoppe, and P. Schaller [1]. The starting point of our work was the observation that in the context of membrane theory the algebra $diff_A\, S^2$ (the algebra of infinitesimal area preserving diffeomorphisms, resp. the algebra of divergence free vector fields of the sphere S^2) and the algebra $diff_A\, T^2$ (the analogous algebra for the torus) can be described as a $su(N)$ -limit for $N \to \infty$, in the sense that in certain specific basis of $su(N)$ the structure constants of $su(N)$ converge to the structure constants of the above mentioned algebras. Let me just cite Hoppe [2] for the sphere and Fairlie, Fletcher and Zachos [3] for the torus. For more references, see our paper [1].

Now the story got a little bit confusing, because it was tried to make a naive identification of $diff_A\, S^2$ and $diff_A\, T^2$ with the algebra $su_+(\infty)$ (or $su(\infty)$). In our paper we showed that the algebras $diff_A\, S^2$, $diff_A\, T^2$ (or better certain dense subalgebras of them) and $gl(\infty)$ are pairwise non isomorphic. It was even proved by Hoppe and Schaller [4] that there exits a infinite family of algebras, including $diff_A\, T^2$, each of them non isomorphic but nevertheless each of them can be approximated by $su(N)$, $N \to \infty$.

Note, the relation between an infinite dimensional algebra and a series of finite dimensional algebras is also important in other contexts. For example, it appears also in quantization schemes, semiclassical limit and so on.

Because the time is limited, I will not be able to cover every aspect in detail. Hence, I like to explain the main points in rough terms and concentrate on some special points later on.

The term "limit" above was a rather flabby notion. We suggested a rigourous definition of limit and approximation. This we call an L_α−approximation. We showed that the above limits can be covered by this concept.

As already remarked, non isomorphic algebras can have the same sequence of approximating finite dimensional algebras. The situation is even more general. We start from a compact Kähler manifold. If we consider this Kähler manifold as a classical phase space we can apply a geometric quantization procedure to approximate the algebra of infinitesimal symplectic transformations by a sequence of antihermitean quantum operators. The rough idea is to realize the quantum operators as operators on the space of holomorphic sections of increasing tensor powers of the prequantum line bundle.

The algebra of infinitesimal symplectic transformations is in relation to the algebra of infinitesimal area preserving diffeomorphism (with respect to the symplectic volume). In the case of dimension 2 they are the same. We expect this approximation to be a $su(N)-$ approximation in our sense. We showed this to be true for the real $2n-$dimensional torus. The Kähler form ω defines on the torus the structure of an $n-$dimensional complex manifold and by the prequantization condition it is automatically an abelian variety (i.e. a complex tori which admits an embedding into projective space). In this case everything can be calculated in terms of theta functions.

Let me remark, that A.S. Schwarz [5] also mentions such connections and that there is an relation to Berezin's coherent states.

2. $diff_A\, S^2$, $diff_A\, T^2$ and $gl(\infty)$

I believe one reason of the general confusion was that people were talking of different algebras and nevertheless were using the same name. To avoid this, let me introduce the following algebras

$$gl(\infty), \quad gl_+(\infty), \quad L_\Lambda, \quad diff'_A T^2, \quad diff'_A S^2$$

and certain related algebras.

$gl(\infty)$ is the Lie algebra of complex $\infty-$dimensional matrices with finite support, i.e.

$$gl(\infty) := \{\, (a_{ij})_{i,j\in\mathbb{Z}} \mid a_{ij} \in \mathbb{C}, \text{all but a finite number of the } a_{ij} = 0\,\}\ .$$

The Lie bracket is the usual matrix commutator. A basis is given by the elementary matrices E_{ij}. The matrix E_{ij} has 1 as the (i,j)-th entry and 0 as all other entries. Here (i,j) ranges over $\mathbb{Z}\times\mathbb{Z}$. The commutator of the basis elements is

$$[E_{ij}, E_{kl}] = \delta_{j,k}E_{il} - \delta_{i,l}E_{kj}\ .$$

If we replace $\mathbb{Z}$ by $\mathbb{N}$ we obtain the algebra $gl_+(\infty)$ by the analogous definitions. Any bijective map $\mathbb{N}\cong\mathbb{Z}$ induces an isomorphism of $gl_+(\infty)$ with $gl(\infty)$. (Nevertheless, we will distinguish them, because there exists no canonical isomorphism between them.) Due to the finite support of the matrices the trace is well-defined and the subalgebras $sl(\infty)$, resp. $sl_+(\infty)$ can be obtained by restricting oneself to matrices with trace equal zero.

Let

$$V := \langle T_{\vec{m}} \mid \vec{m} \in \mathbb{Z}^2 \rangle_{\mathbb{C}}$$

be the $\mathbb{C}$-vector space generated by the basis $T_{\vec{m}}$. This vector space carries different Lie algebra structures, e.g. the family of sine-algebras [3]. They are defined as follows: for $\Lambda \in \mathbb{R}$ with $\Lambda \neq 0$ we set

$$[T_{\vec{m}}, T_{\vec{n}}]^\Lambda = \left(\frac{1}{2\pi\Lambda} \sin 2\pi\Lambda(\vec{m} \times \vec{n}) \right) T_{\vec{m}+\vec{n}} \ ,$$

for $\Lambda = 0$ we set

$$[T_{\vec{m}}, T_{\vec{n}}] := [T_{\vec{m}}, T_{\vec{n}}]^0 := (\vec{m} \times \vec{n}) T_{\vec{m}+\vec{n}} \ .$$

Here we use the notation $\vec{m} \times \vec{n} = \vec{m} \wedge \vec{n} = m_1 n_2 - m_2 n_1$, where $\vec{m} = (m_1, m_2)$, $\vec{n} = (n_1, n_2)$. We denote these Lie algebras by $\widetilde{L}_\Lambda = (V, [..,..]^\Lambda)$. Obviously, the algebras $\widetilde{L}_\Lambda$ are direct sums (of Lie algebras)

$$\widetilde{L}_\Lambda = \langle T_{(0,0)} \rangle \oplus \langle \ T_{\vec{m}} \mid \vec{m} \in \mathbb{Z}^2 \setminus \{(0,0)\} \ \rangle \ .$$

The first summand consists of multiples of the central element $T_{(0,0)}$. The second summand we call L_Λ. The Lie algebra L_0 is (by some abuse of notation) also called $diff'_A T^2$, due to its relation with the complexified Lie algebra of the area preserving diffeomorphisms of T^2. Of course, $diff'_A T^2$ is only the subalgebra of nonconstant vector fields generated (as vector space) by finite linear combination of the generators $T_{\vec{m}}$, $(\vec{m} \neq 0)$. The element $T_{(0,0)}$ in $\widetilde{L}_0$ does not correspond to a vector field.

Our last infinite dimensional Lie algebra shall be the algebra generated by the elements

$$Y_{lm}, \quad \text{with} \quad l \in \mathbb{N}, \quad m = -l, \ldots, 0, \ldots, +l$$

with the Lie bracket (Here and in the following summation convention is assumed)

$$[Y_{lm}, Y_{l'm'}] = g_{lm,l'm'}^{l''m''} Y_{l''m''}$$

where the structure constants are given by explicit formulas which I do not want to give here. The only thing which is of importance here, is that they will be nonvanishing only for

$$m'' = m + m' \quad \text{and} \quad |l - l'| \leq l'' \leq l + l' - 1 \ .$$

This Lie algebra we will call $diff'_A S^2$. Again, $diff'_A S^2$ is only the (complexified) subalgebra of $diff_A S^2$ generated by finite linear combinations of the vector fields corresponding to Y_{lm} (see [2]). In the following, it will sometimes be convenient to consider also the trivial central extension $diff'_A S^2 \oplus \mathbb{C} \cdot Y_{00}$ by an additional element Y_{00}.

Coming now to our problem of the limit. Let us start with $gl_+(\infty)$. Induced by a numbering of the basis of the vector space on which $gl_+(\infty)$ is operating we get an embedding of the algebra $gl(N)$ into $gl(\infty)$ by considering the operations involving only the first N basis elements. This embedding we call the standard embedding. By increasing N one obtains a chain of subalgebras

$$gl(N) \ \subset \ gl(N+1) \ \subset \ gl(N+2) \ \subset \ \ldots \ .$$

As every element of $gl_+(\infty)$ lies in some $gl(N)$ we can call $gl_+(\infty)$ a '$gl(N),\ N \to \infty$ limit'. In fact, $gl_+(\infty)$ is the 'direct limit' of the standard embedding in the sense of the language of categories.

If we choose for every N another basis $\{T_a^N \mid a = 1, \ldots, n^2\}$ in $gl(N)$, then we can describe the Liealgebra $gl(N)$ by its structure constants

$$f_{a,b}^{c,N}, \qquad [T_a^N, T_b^N] = f_{a,b}^{c,N} T_c^N \ .$$

If we asume that $f_{ab}^{c,N}$ has a well defined limit $f_{a,b}^c := \lim_{N\to\infty} f_{a,b}^{c,N}$ for all a, b, c and that for fixed a and b the set

$$\{\, c \in \mathbb{N} \mid \text{there exists a } N \text{ such that } f_{a,b}^{c,N} \neq 0 \,\}$$

is finite then we can define a Lie algebra generated by elements $\{\, T_a \mid a \in \mathbb{N}\,\}$ with the bracket $[T_a, T_b] = f_{a,b}^c T_c$.

Nevertheless the above condition does not imply that the family of base transformations $C^{(N)}$ has to define a base transformation C also in the limit. For this we would have to additionally require that (for fixed i, j) the element E_{ij} is only a finite(!) linear combination of the T_a^N such that the number of elements is bounded independent of N, and vice versa! Of course, if this condition is fulfilled, the limit will be isomorphic to $gl_+(\infty)$. However, in most of the interesting examples this will not be the case. Hence, we take the convergence of the structure constants as the starting point. The resulting algebra will then in general not be isomorphic to $gl_+(\infty)$ as we will see.

Let me introduce a specific example

Let $N, M \in \mathbb{N}$, N odd, $1 \le M < N$ with M and N relatively prime and $\omega := \mathbf{e}^{4\pi\,\mathbf{i}\,M/N}$ be a primitive $N-$th root of unity. Define the $N^2 \times N^2$- matrix C^N (To avoid cumbersome notation we drop the superscript N in the following.) with indices $a = \overrightarrow{m} = (m_1, m_2)$, $m_1, m_2 = -\frac{N-1}{2}, \ldots, +\frac{N-1}{2}$; $\ i, j = 1, \ldots, N$

$$C_{\overrightarrow{m},ij} := \frac{\mathbf{i}\,N}{4\pi M}\omega^{\frac{1}{2}m_1 m_2 + (i-1)m_1}\delta_{i+m_2, j \bmod N}$$

where $\delta_{k,l \bmod N}$ is equal to 1 if $k \equiv l \bmod N$ and 0 otherwise. The matrix C is the base change matrix

$$T_a^N = C_{a,ij}^N E_{ij} \qquad a = 1, 2, \ldots, N^2 \ .$$

With respect to the new basis the structure constants can be given as

$$f^{\vec{a}}_{\vec{m},\vec{n}} = \frac{N}{2\pi M} \sin \frac{2\pi M}{N} (\vec{m} \times \vec{n})\, \delta_{\vec{m}+\vec{n},\vec{a} \bmod N} \ .$$

($\delta_{\vec{m},\vec{n} \bmod N}$ is the obvious generalization of $\delta_{m,n \bmod N}$ to two dimension.)
Let $M, N \to \infty$ in such a way that $M/N \to \Lambda \in \mathbb{R}$. Then we obtain as 'limit' the infinite dimensional algebra $\widetilde{L}_\Lambda$. For $M = 1$ and $N \to \infty$ we get as a special case the 'limit' $\widetilde{L}_0 = diff'_A T^2 \oplus \mathbb{C}$. This example shows, that the 'limit' is not unique. To see this, let Λ and Λ' be two different irrational numbers with $0 < \Lambda, \Lambda' < \frac{1}{4}$. We can approximate them by two sequences

$$\frac{M_k}{N_k} \quad \text{resp.} \quad \frac{M'_k}{N'_k} \quad \text{with} \quad N_k = N'_k \ .$$

Hence we have as the $k-$th element of the 'sequence' the algebra $gl(N_k)$. But it was proven by Hoppe and Schaller [4] that $\widetilde{L}_\Lambda$ is not isomorphic to $\widetilde{L}_{\Lambda'}$ for $\Lambda \neq \Lambda'$.

In a similar way we obtain the algebra $diff'_A S^2 \oplus \mathbb{C}$ as another $gl(N)$, $N \to \infty$ limit.

3. $L_\alpha-$Approximations

We introduced the following concept of an approxiamtion, resp. limit. Given a family of real or complex Lie algebras

$$(L_\alpha, [..,..]_\alpha, d_\alpha, \alpha \in I)$$

where we denote by $[..,..]_\alpha$ the Lie bracket in L_α, d_α a metric in L_α and by I either $\mathbb{N}$, $\mathbb{R}$ or some other subset of $\mathbb{R}$ with ∞ as accumulation point. Now let $(L, [..,..])$ be another Lie algebra satisfying the

condition 1. *(i) There exists a surjective map $p_\alpha : L \to L_\alpha$ for every $\alpha \in I$.*

(ii) For each $x, y \in L$ the following holds: If $d_\alpha(p_\alpha(x), p_\alpha(y)) \to 0$ for $\alpha \to \infty$ then $x = y$.

We call $(L_\alpha, [..,..]_\alpha, d_\alpha, \alpha \in I)$ an *approximating sequence* for $(L, [..,..])$ induced by $(p_\alpha, \alpha \in I)$ and L an $L_\alpha-$*quasilimit* if the following condition is also valid

Condition 2. *For each $x, y \in L$,*

$$d_\alpha(p_\alpha[x,y], [p_\alpha x, p_\alpha y]_\alpha) \to 0 \quad (\alpha \to \infty) \ .$$

Of course the definition depends on the metrics d_α chosen. If we replace the family of metrics (d_α) by a uniformly equivalent family of metrics (d'_α) we obtain that the

property of beeing an approximating sequence is the same, whether we take it with respect to the metrics d or d'.
(By an uniform equivalent family of metric we understand that there exist positive $a, b \in \mathbb{R}$ such that

$$a \cdot d_\alpha(x_\alpha, y_\alpha) \ \le \ d'_\alpha(x_\alpha, y_\alpha) \ \le \ b \cdot d_\alpha(x_\alpha, y_\alpha) \qquad \forall \alpha \in I, \ \forall x_\alpha, y_\alpha \in L_\alpha \ .)$$

We will immediately see, that the same sequence of algebras can approximate non isomorphic algebras. However we have a weak uniqueness theorem implying, that the Lie structure on the underlying vector space L of an $L_\alpha-$quasilimit is unique once the linear maps $(p_\alpha, \alpha \in I)$ is specified.

In many examples also the following condition is fulfilled

Condition 3. *There exists a family of linear maps* $(i_\alpha : L_\alpha \to L, \alpha \in I)$ *and an index* $\alpha_0 \in I$ *such that for* $\alpha \ge \alpha_0$

$$p_\alpha \circ i_\alpha = id_\alpha \quad \textit{and} \quad i_\alpha(L_\alpha) \subseteq i_\beta(L_\beta) \quad \beta \ge \alpha \ .$$

In this case we call (L_α) a splitting $L_\alpha-$ approximation.

Let me make two examples illustrating the concept:

Example 1. Let L and $(L_\alpha, \alpha \in I)$ be different Lie algebras with the same underlying vector space V. If we choose as p_α and as i_α the identity map and as d_α a fixed metric d on V then the condition 1 and 3 are clearly fulfilled. condition 2 reads as

$$d([x,y],[x,y]_\alpha) \to 0 \quad (\alpha \to \infty) \ .$$

This reflects the approximation of the structure constants. To make the example more concrete let L (and hence all L_α) be generated by T_n with $n \in \mathbb{N}$. By $\langle T_n, T_m \rangle = \delta_{n,m}$ we get a scalar product on V. If we choose $d(x,y) := \sqrt{\langle x-y, x-y \rangle}$ this implies for the structure constants f_{nm}^k, $f_{nm}^{k,\alpha}$ defined by

$$[T_n, T_m] = f_{nm}^k T_k \quad \text{resp.} \quad [T_n, T_m]_\alpha = f_{nm}^{k,\alpha} T_k \ .$$

convergency

$$\lim_{\alpha \to \infty} f_{nm}^{k,\alpha} = f_{nm}^k \ .$$

Conversely, if for fixed n and m the set

$$\{\, k \in \mathbb{N} \ \mid \text{there exists a } \alpha \text{ such that } f_{n,m}^{k,\alpha} \ne 0 \,\}$$

is finite then the convergence of the structure constants implies condition 2.

Example 2: This is again the torus algebra $diff'_A T^2$. We start with the algebras $\widetilde{L}_\Lambda$ introduced in section 2. Here we are especially interested in $\Lambda = 0$ (the torus algebra) and $\Lambda = \frac{1}{N}$. We use L for $\widetilde{L}_0$ and L^N for $\widetilde{L}_{1/N}$. The subspace

$$J^N := \langle T_{\overrightarrow{m}} - T_{\overrightarrow{m}+N\cdot\overrightarrow{a}} \mid \overrightarrow{m}, \overrightarrow{a} \in \mathbb{Z}^2 \rangle \ \mathbb{C}$$

is an ideal in L^N. Hence we can define the factor algebra $L^{(N)} := L^N / J^N$ with $\varphi_N : L^N \to L^{(N)}$ the canonical projection map. This Lie algebra has dimension N^2. A basis is given by the N^2 elements

$$\varphi_N(T_{\vec{m}}), \qquad \vec{m} = (p,q) \qquad 0 \le p,q < N \ .$$

By definition of the factor algebra, the Lie product is given as

$$[\varphi_N(T_{\vec{m}}), \varphi_N(T_{\vec{n}})]^{(N)} = \frac{N}{2\pi} \sin \frac{2\pi}{N} (\vec{m} \times \vec{n})\, \varphi_N(T_{\vec{m}+\vec{n} \bmod N}) \ .$$

Hence we see that for N odd $L^{(N)}$ is exactly the Lie algebra $gl(N)$ written in the basis as introduced in section 2, example 2.

Now we define an L_α approximation for our Lie algebra L. As index set I we take the natural numbers. We use as L_αs the algebras $L^{(N)}$, as p_α the canonical map φ_N and as metric on $L^{(N)}$ the norm induced by the standard scalar product

$$\langle \varphi_N(T_{\vec{m}}), \varphi_N(T_{\vec{n}}) \rangle = \delta_{m_1,n_1} \cdot \delta_{m_2,n_2} \ . \tag{3-6}$$

By setting $i_N(\varphi_N(T_{\vec{n}}))) := T_{\vec{n} \bmod N}$ we obain a linear map $L^{(N)} \to L$ which obeys $\varphi_N \circ i_N = id$. By direct calculation it can be seen that all the conditions are fulfilled.

In a completely similar way we obtain that not only the algebra $diff'_A\, T^2 \oplus \mathbb{C}T_{00}$ but also the algebra $diff'_A\, S^2 \oplus \mathbb{C}Y_{00}$ bis a $L_\alpha-$quasilimit having the same sequence of $gl(n)$ as approximating sequence.

4. The Non-Isomorphy

Theorem.

$$gl_+(\infty), \quad diff'_A\, S^2 \quad and \quad diff'_A\, T^2$$

are pairwise non-isomorphic.

In the proof one can always replace $gl_+(\infty)$ by its subalgebra $sl_+(\infty)$. without changing the argument. The same is true if we replace $diff'_A T^2$ (resp. $diff'_A S^2$) by its trivial central extensions. Let me also remind you that $gl_+(\infty)$ is isomorphic to $gl(\infty)$.

Let me indicate just a few steps in the proof. Assume the existence of a Lie algebra isomorphism

$$\Phi : \{Y_{lm}\} \ \to \ \{E_{ij}\} \ .$$

Let Φ_{lm} denote $\Phi(Y_{lm})$ then we obtain the relation

$$[\Phi_{10}, \Phi_{lm}] = m\sqrt{\frac{3}{4\pi}}\,\Phi_{lm},$$

$$[\Phi_{11}, \Phi_{l,-1}] = -\sqrt{\frac{3}{8\pi}}\sqrt{l(l+1)}\,\Phi_{l0} \ .$$

Because all Φ_{lm} are finite linear combinations of the E_{ij} Φ_{10} and Φ_{11} will be zero outside some upper left block of size $J \times J$. Now $(a_{ij}) := [\Phi_{10}, E_{kl}]$ will have vanishing entries for if both indices i and j are bigger than J. Hence, this will also be true for Φ_{lm} Hence Φ can not be surjective.

This shows the non isomorphy for one pair. To compare $diff'_A\, S^2$ and $diff'_A\, T^2$ we showed that the adjoint action of Y_{11} on $diff'_A\, S^2$ is locally nilpotent. But $diff'_A\, T^2$ does not allow locally nilpotent actions.

Note however, we do not claim, that there is no embedding of $diff'_A\, T^2$ or $diff'_A\, S^2$ into one of the following algebra

$$\overline{gl}(\infty) = \{\ (a_{ij})_{i,j\in\mathbb{Z}} \mid \text{ there is an } r \text{ such that } \ a_{ij} = 0 \ \text{ if } |i-j| > r\ \}$$
$$\overline{gl_+}(\infty) = \{\ (a_{ij})_{i,j\in\mathbb{N}} \mid \text{ there is an } r \text{ such that } \ a_{ij} = 0 \ \text{ if } |i-j| > r\ \}\ .$$

(This algebras are not isomorphic anymore)

In this context it is interesting to note that Floratos [24] was able to show that L_Λ for $\Lambda \neq 0$ can be embedded into $\overline{gl}(\infty)$. The question whether this is true also for $L_0 = diff'_A T^2$ remains still open.

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Fakultät für Mathematik, Universität Mannheim, P.O. Box 103462, D-6800 Mannheim 1, Fed. Rep. of Germany