Directional Differentiability of the Rotation Number for the Almost Periodic Schrödinger Equation

Rafael Obaya Miguel Paramio

1 Introduction

Let us consider the one-dimensional Schrödinger equation

$$L_E(x) = -x^{''} + g_0(t)x - Ex = 0 \tag{1}$$

with almost periodic potential g_0. The set A of energies where the Lyapunov exponent vanishes is also known to be the essential support of the absolutely continuous part of the spectral measure. This paper deals with the variation of the rotation number on A, in particular with the differentiability and Lipschitz character of this map.

Since $g_0(t)$ is a bounded function then the boundaries $\pm\infty$ are of the limit point type according to Weyl's classification. It follows that for E complex with $Im(E) > 0$, even if E belongs to the resolvent, L_E has a basis of solutions $u_+(t)$, $u_-(t)$ whithin $L^2(R_+)$ and $L^2(R_-)$. This allows to define and represent the rotation number as harmonic function on the upper half plane. This is the starting point of a theory developped by Johnson, Moser, Kotani, Simon and others. These authors obtain properties of the rotation number on the real axis taking limits following the lines $E + i\varepsilon$ as ε tends to 0^+. From this point of view the rotation number agrees with the imaginary part of the extension to the real axis of a Herglotz function. We give a new proof of some known facts and present several new results using a different technique which has some interest in itself.

Let us recall here some of the main notions and previous results needed to formulate our version. For the remaining of the paper Ω will stand for the hull of g_0. The map $T : \Omega \times R \longrightarrow R$; $(\xi, t) \longrightarrow \xi_t$, where $\xi_t(s) = \xi(t+s)$, describes a continuous flow on Ω. Thus (Ω, T) is a unique ergodic and almost periodic flow. Actually Ω is a compact group and the Haar measure its unique invariant measure. The map $J : \Omega \longrightarrow R$ assigns its value at $t = 0$ to each function of Ω.

Let E be fixed. Pastur [25] has shown that all the equations

$$-x'' + \xi(t)x = Ex \quad , \quad \xi \in \Omega \tag{2}$$

admit a commom description of the spectrum. Taking polar-symplectic coordinates $\varphi = \tan^{-1}(x/x')$, $r = 1/2\ (x^2 + (x')^2)$ we find the relations

$$\begin{aligned} \dot{\varphi} &= f_E(\xi_t, \varphi) = \cos^2\varphi - (J(\xi_t) - E)\sin^2\varphi & (3) \\ \dot{r} &= -\frac{\partial f_E}{\partial \varphi}(\xi_t, \varphi)\, r = 2(J(\xi_t) + 1 - E)\sin\varphi\cos\varphi\, r\ . & (4) \end{aligned}$$

Let $\varphi_t^E(\xi, \varphi)$ be the solution of the first equation with initial conditions (ξ, φ). The map $\Phi_t^E(\xi, \varphi) = (\xi_t, \varphi_t^E(\xi, \varphi))$ defines a continuous flow on $K = \Omega \times S^1$ and also on the projective bundle $\Sigma = \Omega \times P^1$. Once their trajectories are known the evolution of r is obtained by integration of a simple linear equation.

The solutions of (3) on R define a lift of Φ on $L = \Omega \times R$ which we denote by the same symbol. The following formula represents the rotation number:

$$\alpha(E) = \lim_{t\to\infty} \frac{\varphi\ (t, \xi, \varphi)}{t} = \int_\Sigma f_E(\xi, \varphi) d\mu \tag{5}$$

for every normalized invariant measure. The convergence of the limit above is uniform in Σ. $\alpha(E)$ is a continuous increasing function and hence it admits a derivative almost everywhere. The properties of the flow at the points of differentiability have yet to be explained.

Is also known that $\alpha(E)$ is locally log-Holder continuous; that means that there is a constant c such that

$$\mid \alpha(E_1) - \alpha(E_2) \mid \leq \frac{-c}{\log \mid E_1 - E_2 \mid} \ , \ \mid E_1 - E_2 \mid < 1\ . \tag{6}$$

Kotani and Deift-Simon prove that for almost every $(\theta, E) \in \Omega \times A$ there exist linearly independent solutions which are L^2 almost periodic with the same frequency module as g_0. Moreover Deift-Simon extend Moser's inequality $2\alpha(E)\alpha'(E) \geq 1$ to these points.

If A contains an interval I, then $\alpha(E)$ is analitic on I. This is a property of the Herglotz functions. However many examples of almost periodic Schrödinger operators with Cantor spectrum are known. See [2] and [23]. References [18], [28] contain recent results on this problem.

Our theory is based on the existence of a positive invariant measure which is absolutely continuous with respect to m. We make use of a map of Furstenberg type which transform the flow on Σ into a skew-traslation and preserves the rotation number. The idea is similar to the one used by Brunowski and Hermann for families of diffeomorphisms of the circle.

This paper is arranged as follows. In Sections 2 and 3 we present neccesary and sufficient conditions which guarantee the existence of the type of invariant measures we need. The cases of square integrable density function and continuous density function will be common in applications. Here we consider a skew-product flow defined by an abstract differential equation of the type (3). The reason is that our argument does not depend on the linearity of the equation.

In Sections 3 and 4 we apply the previous conclusions to the Schrödinger linear equation. Let A_0 be the set of energies at which L_E has L^2 almost periodic solutions. Let us fix $E_0 \in A_0$. First we show that (Σ, Φ^{E_0}) admits an invariant measure $\mu = 1/q\, dm$ such that $1/q \in L^2(\Sigma, m)$ and $q \in L^1(\Sigma, m)$. In particular q will be continuous when the solutions of L_{E_0} are bounded.

Let Ω be the hull of an adequate almost periodic ξ and $\Gamma_0 \in C(\Omega)$ such that $\Gamma_0(\xi_t) = (g_0 - E_0)(t)$. We consider α as a continuous function from $C(\Omega)$ to R. We shall verify the Lipschitz variation of α throught Γ_0.

In Sections 5 and 6 we treat the differentiability of α at Γ_0. We introduce the set $C_{ad}(\Omega)$ of admissible directions of differentiability by requiring the invariance of a certain integral expression. Thus we avoid considering the weak variation of a family of invariant measures which appear in the representation of $\alpha(\Gamma_0 + \eta\Gamma)$. We prove that if $\Gamma \in C_{ad}(\Omega)$ the rotation number has a derivative at Γ_0 in the direction Γ. In particular we obtain the classical derivative of α with respect to E at the points of A_0 and we find an explicit formula easier to calculate . (This is one of the advantages of our method).

2 The invariant absolutely continuous measure

We set $L = \Omega \times R$ and $K = \Omega \times S^1$ where $S^1 = R/2\pi Z$. The symbols ρ and ρ_0 stand respectively for the Lebesgue measure on R and its normalized restriction on S^1. We represent the normalized invariant measure on Ω by m_0 and the completion of the σ-algebra of its Borel sets by $\mathcal{A}_0$. A new measure can be introduced on K (resp. on L) in an obvious way by taking the product measure $m = m_0 \otimes \rho_0$ (resp. $m_L = m_0 \otimes \rho$) defined on the completion of the Borel sets $\mathcal{A}$ (resp. $\mathcal{A}_L$). From now on we always refer to these measures when no other indication is made.

Let $T : R \times \Omega \to \Omega$ be a continuous unique ergodic flow on Ω. Let us assume that both maps f, $\partial f/\partial\varphi : K \to R$ are continuous. In this conditions the flow map $\Phi : R \times K \to K$ ($\Phi : R \times L \to L$), defined by the solutions of the differential equation

$$\dot{\varphi} = f(\xi_t, \varphi) \ , \ (\xi, \varphi) \in K \tag{7}$$

is continuous and measurable. In this Section we are going to discuss the existence and means of constructing a positive invariant measure under Φ which is absolutely continuous with respect to m. We start by identifying the density function of such measures with the positive measurable solutions of the functional equation

$$p(\Phi_t(\xi, \varphi)) = p(\xi, \varphi) e^{-\int_0^t \frac{\partial f}{\partial \varphi}(\Phi_s(\xi,\varphi))ds} \tag{8}$$

defined along the trajectories of (7). First we have to remember the following well kown fact of ergodic theory :

Proposition 2.1 *Assume that $p_1 : K \to R$ is a measurable function such that at every $t \in R$*

$$p_1(\Phi_t(\xi, \varphi)) = p_1(\xi, \varphi) e^{-\int_0^t \frac{\partial f}{\partial \varphi}(\Phi_s(\xi,\varphi))ds}$$

for almost every $(\xi, \varphi) \in K$. Then there exists a measurable function $p : K \to R$ such that $p_1 = p$ almost everywhere and

$$p(\Phi_t(\xi, \varphi)) = p(\xi, \varphi) e^{-\int_0^t \frac{\partial f}{\partial \varphi}(\Phi_s(\xi,\varphi))ds}$$

for every $(t, \xi, \varphi) \in R \times K$.

Proposition 2.2 *Let $p_1 \in L^1(K,m)$ be a positive function. The following statements are equivalent:*
i) The measure $\mu = p_1 dm$ is invariant under the flow Φ.
ii) There exists a measurable solution of (8) p, such $p = p_1$ almost everywhere.

Proof: Let us notice that if $q \in L^1(S^1, \rho_0)$ and $(t, \xi_0, \varphi_o) \in R \times K$ then

$$\int_{\varphi_t(\xi_0,0)}^{\varphi_t(\xi_0,\varphi_0)} q(\varphi) d\varphi = \int_0^{\varphi_0} q(\varphi_t(\xi_0, \varphi)) e^{\int_0^t \frac{\partial f}{\partial \varphi}(\Phi_s(\xi,\varphi))ds} d\varphi \ .$$

$i \Rightarrow ii)$ Let $A \subset K$ be a measurable set and $A_t = \Phi_t(A)$. According to the above remark the following identities hold:

$$\int_{A_t} p_1 dm = \int_\Omega (\int_{S^1} p_1(\xi,\varphi)\chi_{A_t}(\xi,\varphi) d\rho_0) dm_0 =$$

$$\int_\Omega (\int_{S^1} p_1(\xi, \varphi_t(\xi_{-t},\varphi)) e^{\int_0^t \frac{\partial f}{\partial \varphi}(\Phi_s(\xi_{-t},\varphi))ds} \chi_{A_t}(\xi_{-t},\varphi) d\rho_0) dm_0 =$$

$$\int_\Omega (\int_{S^1} p_1(\xi_t, \varphi_t(\xi,\varphi)) e^{\int_0^t \frac{\partial f}{\partial \varphi}(\Phi_s(\xi,\varphi))ds} \chi_A(\xi,\varphi) d\rho_0) dm_0 =$$

$$\int_A p_1(\xi_t, \varphi_t(\xi,\varphi)) e^{\int_0^t \frac{\partial f}{\partial \varphi}(\Phi_s(\xi,\varphi))ds} dm = \int_A p_1(\xi,\varphi) dm \ .$$

Therefore

$$p_1(\Phi_t(\xi,\varphi)) e^{\int_0^t \frac{\partial f}{\partial \varphi}(\Phi_s(\xi,\varphi))ds} = p_1(\xi,\varphi)$$

almost everywhere. Proposition (2.1) allows us to obtain a solution of (8).
$ii \Rightarrow i)$ If p satisfies (8) the above identities show that $\int_{A_t} p_1 dm = \int_A p_1 dm$. This means that $\mu = p_1 dm$ is invariant under Φ.

The solutions of (8) satisfy a linear differential equation along the trajectories defined by Φ. Unfortunately the local theory cannot be applied and all the solutions must be constructed globally.

Let us represent the set of continuous functions on K by $C(K)$ and the set of Borel measures by $M(K)$. By the Riesz representation theorem we can identify $M(K)$ with the dual space of $C(K)$. Its unit ball B is weakly compact and metrizable . Let us consider the average measures $(\mu_T)_{T>0}$ defined by

$$\mu_T(A) = \frac{1}{2T}\int_{-T}^{T} m(\Phi_t(A)) dt = \frac{1}{2T}\int_A \int_{-T}^{T} e^{\int_0^t \frac{\partial f}{\partial \varphi}(\Phi_s(\xi,\varphi))ds} dt dm$$

for each measurable Borel subset $A \subset K$. If A is an invariant set we have $\mu_T(A) = m(A)$. In particular $\mu_T(K) = m(K)$. Hence the family of measures $(\mu_T)_{T>0}$ possesses some weakly convergent subsequence. Moreover μ_T is absolutely continuous with respect to m and

$$p_T(\xi, \varphi) = \frac{1}{2T} \int_{-T}^{T} e^{\int_0^t \frac{\partial f}{\partial \varphi}(\Phi_s(\xi,\varphi))ds} dt \tag{9}$$

is its density function. We are going to check that if $\{p_T\}_{T>0}$ converges almost everywhere then its limit p satisfies the equation (8).

For every $0 < l < T$ we have

$$p_T(\Phi_l(\xi, \varphi)) = \frac{1}{2T} \int_{-T}^{T} e^{\int_0^t \frac{\partial f}{\partial \varphi}(\Phi_{s+l}(\xi,\varphi))ds} dt =$$

$$\frac{1}{2T} \int_{-T}^{T} e^{\int_l^{t+l} \frac{\partial f}{\partial \varphi}(\Phi_s(\xi,\varphi))ds} dt = \frac{1}{2T} \int_{-T}^{T} e^{\int_0^{t+l} \frac{\partial f}{\partial \varphi}(\Phi_s(\xi,\varphi))ds} e^{-\int_0^l \frac{\partial f}{\partial \varphi}(\Phi_s(\xi,\varphi))ds} dt \ .$$

Therefore

$$\frac{T-l}{T} p_{T-l}(\xi, \varphi) \leq p_T(\Phi_l(\xi, \varphi)) e^{\int_0^l \frac{\partial f}{\partial \varphi}(\Phi_s(\xi,\varphi))ds} \leq \frac{T+l}{T} p_{T+l}(\xi, \varphi) \ .$$

Taking limits as $T \to \infty$ we obtain

$$p(\Phi_l(\xi, \varphi)) = p(\xi, \varphi) e^{-\int_0^l \frac{\partial f}{\partial \varphi}(\Phi_s(\xi,\varphi))ds}$$

almost everywhere. This means that there is an invariant set $K_0 \subset K$ with $m(K_0) = 1$ such that there is convergence for all $(\xi, \varphi) \in K_0$. Defining p as zero in $K - K_0$ we get a solution of (8).

On the other hand, if we define

$$q_T(\xi, \varphi) = \frac{1}{2T} \int_{-T}^{T} e^{-\int_0^t \frac{\partial f}{\partial \varphi}(\Phi_s(\xi,\varphi))ds} dt \tag{10}$$

then

$$\frac{1}{q_T(\xi, \varphi)} \leq \frac{2T}{\int_{-T}^{T} e^{-\int_0^t \frac{\partial f}{\partial \varphi}(\Phi_s(\xi,\varphi))ds} dt} \leq \frac{1}{2T} \int_{-T}^{T} e^{\int_0^t \frac{\partial f}{\partial \varphi}(\Phi_s(\xi,\varphi))ds} dt \leq p_T(\xi, \varphi) \ . \tag{11}$$

In a similar way, if the family $\{q_T\}_{T>0}$ converges almost everywhere to a function q, not identically zero, then

$$q(\Phi_t(\xi,\varphi)) = q(\xi,\varphi)e^{\int_0^t \frac{\partial f}{\partial \varphi}(\Phi_s(\xi,\varphi))ds} . \tag{12}$$

The set $C = \{(\xi,\varphi) \in K / q(\xi,\varphi) > 0\}$ is measurable, invariant and moreover $m(C) > 0$. Defining

$$q_1(\xi,\varphi) = \begin{cases} 0 & \text{if } (\xi,\varphi) \in K - C \\ \dfrac{1}{q(\xi,\varphi)} & \text{if } (\xi,\varphi) \in K \end{cases}$$

q_1 satisfies

$$q_1(\Phi_t(\xi,\varphi)) = q_1(\xi,\varphi)e^{-\int_0^t \frac{\partial f}{\partial \varphi}(\Phi_s(\xi,\varphi))ds}$$

and we obtain a measurable solution of (8).

Retaining the previous notation we will prove the following

Proposition 2.3 *i) If $p = \lim_{T\to\infty} p_T$ exists in an invariant set C with $m(C) > 0$, where p does not vanish, then $p\chi_C \in L^1(K,m)$ and $p\chi_C dm$ is an invariant measure under the flow.*
ii) If $q = \lim_{T\to\infty} q_T$ exists in an invariant set C with $m(C) > 0$, where q does not vanish, then $q_1\chi_C \in L^2(K,m)$ and $q_1\chi_C dm$ is an invariant measure under the flow.

Proof: $i)$ Actually we only have to check that $p\chi_C \in L^1(K,m)$. This is a consequence of Fatou's lemma: one has

$$0 < \frac{1}{\lambda} = \int p\chi_C dm = \int \lim_{T\to\infty} \inf p_T\chi_C dm \leq \lim_{T\to\infty} \inf \int p_T\chi_C dm = m(C) .$$

In these conditions $\mu = \lambda p\chi_C dm$ is an invariant measure absolutely continuous with respect to m and moreover it is equivalent to m in C. It follows from this fact that the family $\{p_T\}_{T>0}$ is uniformly integrable.

As a matter of fact, for every $\epsilon > 0$ there exists $\delta_1 > 0$ such that if $A \subset C$ is a measurable set with $\mu(A) < \delta_1$ then $m(A) < \epsilon$.

In an analogous way there exists $\delta_2 > 0$ such that if $A \subset C$ is a measurable set with $m(A) < \delta_2$ then $\mu(A) < \delta_1$. Hence if $A \subset C$ and $m(A) < \delta_2$ then $\mu(A) = \mu(\Phi_t(A)) < \delta_1$ and finally $m(\Phi_t(A)) < \epsilon$ for every $t \in R$.

This proves that $\int_A p_T dm < \epsilon$ for every $T \in R$. We may apply now the theorem of Vitali to get

$$0 < \frac{1}{\lambda} = \int p\chi_C dm = \lim_{T\to\infty} \int p_T \chi_C dm = m(C) \ .$$

This also shows that μ preserves the original measure of each invariant subset of C.
ii) Using the inequality (11) and repeating the above argument we get that $p_1\chi_C \in L^1(K,m)$. Hence, if $1/\lambda = \int_C q_1 dm$ then $\mu = \lambda q_1 \chi_C dm$ is a normalized invariant measure. Besides, according to Birkhoff's ergodic theorem we have

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} q_1(\Phi_T(\xi,\varphi))\chi_C(\xi,\varphi)dt =$$

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} q_1(\xi,\varphi)\chi_C(\xi,\varphi)e^{-\int_0^t \frac{\partial f}{\partial \varphi}(\Phi_s(\xi,\varphi))ds}dt = \chi_C(\xi,\varphi)$$

for almost every $(\xi,\varphi) \in K$. Since $\chi_C \in L^1(K,\mu)$ we finally conclude that $q_1\chi_C \in L^2(K,\mu)$ and $\int_C q_1^2 dm = \int_C q_1 dm \leq 1$
Notice that the last identity holds only if $C = K$, $q_1 = 1$ and therefore m is an invariant measure.

The converse statements are also true:

Proposition 2.4 *i) Let us assume that $\mu_1 = p_1 dm$ is a normalized invariant measure with $p_1 \in L^1(K,m)$. Then there exists $p = \lim_{T\to\infty} p_T$ in an invariant set C with $m(C) > 0$ where p does not vanish.*
ii) Let us assume that $\mu_2 = p_2 dm$ is a normalized invariant measure with $p_2 \in L^2(K,m)$. Then there exists $q = \lim_{T\to\infty} q_T$ in an invariant set C with $m(C) > 0$ where q does not vanish.

Proof: i) Let $D = \{(\xi,\varphi) \in K/p_1(\xi,\varphi) > 0\}$. In our conditions is obvious that $1/p_1\ \chi_D \in L^1(K,\mu_1)$. Birkhoff's ergodic theorem guarantees the existence of a measurable function $p_1^\star \in L^1(K,\mu_1)$ such that

$$p_1^\star(\xi,\varphi) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} (\frac{1}{p_1}\chi_D)(\Phi_t(\xi,\varphi))dt = \lim_{T\to\infty} \frac{p_T(\xi,\varphi)}{p_1(\xi,\varphi)}\chi_D(\xi,\varphi) \ .$$

This implies that for almost every $(\xi,\varphi) \in K$

$$\lim_{T\to\infty} p_T(\xi,\varphi)\chi_D(\xi,\varphi) = p_1^\star(\xi,\varphi)p_1(\xi,\varphi)$$

and moreover

$$\int_D p_1^\star p_1 dm = \int_D p dm = m(D) \ .$$

In fact the last equality holds in every invariant subset $D' \subset D$. If the set C contains all the points of convergence where p does not vanish then

$$m(D) = \int_D p_1^\star p_1 dm = \int_C p_1^\star p_1 dm = m(C)$$

which proves the claim.
ii) Let $D = \{(\xi,\varphi) \in K/p_2(\xi,\varphi) > 0\}$. Repeating the above argument we guarantee the existence of a measurable function $p_2^\star \in L^1(K,\mu_2)$ such that

$$p_2^\star(\xi,\varphi) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} (p_2\chi_D)(\Phi_t(\xi,\varphi))dt = p_2(\xi,\varphi) \lim_{T\to\infty} q_T(\xi,\varphi)\chi_D(\xi,\varphi)$$

This implies that for almost every $(\xi,\varphi) \in K$

$$\lim_{T\to\infty}(q_T\chi_D)(\xi,\varphi) = (\frac{p_2^\star}{p_2}\chi_D)(\xi,\varphi)$$

and moreover

$$\int_D p_2^\star p_2 dm = \int_D p_2^2 dm$$

As before, if C contains the points of convergence where $p_2^\star$ does not vanish then $m(C) = m(D)$. Thus (ii) is established.

These facts point out some ways to reach invariant absolutely continuous measures. Actually the existence of such measures depends on the behaviour of the families of functions $\{p_T\}_{T>0}$, $\{q_T\}_{T>0}$. Of course if $\{q_T\}_{T>0}$ converges to a non identically zero limit, so does $\{p_T\}_{T>0}$. In this case the invariant measure $\mu = \lambda q_1 dm$ has a density function within $L^2(K,m)$. We will show that if μ is not ergodic there are many other invariant measures whose density functions do not belong to $L^2(K,m)$.

Indeed, using a result by Fustenberg [12] , we know that there are invariant sets of arbitrary $\mu-$measure. In fact we can choose a sequence $(D_n)_{n\in N}$ of dijoints sets such that $\mu(D_n) = 1/2^n$ for every $n \in N$. Thus we have

$$\int_{D_n} q_1 dm = \int_{D_n} q_1^2 dm = \frac{1}{\lambda 2^n} \ .$$

Taking $\lambda_n = 2^{\frac{n}{2}}$ for each $n \in N$ and $q_2 = \sum_{n=1}^{\infty} \lambda_n q_1 \chi_{D_n}$ then

$$\int q_2 dm = \sum_{n=1}^{\infty} \int_{D_n} \lambda_n q_1 dm = \sum_{n=1}^{\infty} \frac{1}{\lambda 2^{\frac{n}{2}}} < \infty$$

but

$$\int q_2^2 dm = \sum_{n=1}^{\infty} \int_{D_n} \lambda_n^2 q_1 dm = \infty \ .$$

This proves that $q_2 dm$ is an invariant measure but q_2 is not square integrable.

3 The continuous density function

Let (X, d) be a compact metric space and Φ a continuous flow on X. Let us remember the following concepts

Definition 3.1 *We say that the flow (X, Φ) is distal when for each pair x, y of different elements of X there is a $\delta > 0$ such that $d(\Phi_t(x), \Phi_t(y)) > \delta$ for every $t \in R$.*

Definition 3.2 *We say that the flow (X, Φ) is almost periodic when for every $\epsilon > 0$ there is a $\delta > 0$ such that if x, y are elements of X with $d(x, y) < \delta$ then $d(\Phi_t(x), \Phi_t(y)) < \epsilon$ for every $t \in R$.*

If (X, Φ) is almost-periodic it is distal. The converse is not true, even if (X, Φ) is minimal and distal it need not to be almost-periodic.

In this Section we assume that (Ω, T) represent a distal and unique ergodic flow. We are going to deal with the case where continuous solutions of the equation (8) exist and hence invariant measures with continuous density functions can be constructed. In fact we are interested in positive solutions, so if we take logarithms in (8) and write $k = \log p$ we obtain the new relation

$$k(\Phi_t(\xi, \varphi)) - k(\xi, \varphi) = \int_0^t \frac{\partial f}{\partial \varphi}(\Phi_s(\xi, \varphi)) ds \ . \tag{13}$$

Several conditions assuring the existance of continuous solutions of (13) appear in the literature. Let us note the following result which comes from Gottschalk-Hendlund [14] and is also proved in [13] and [16].

Proposition 3.1 *Let assume that (X, Φ) is a minimal flow. Let $h \in C(X)$, $x_0 \in X$ and suppose that $\int_0^t h(\Phi_s(x_0))ds$ is bounded. Then there exists $k \in C(X)$ such that*

$$k(\Phi_t(x)) - k(x) = \int_0^t h(\Phi_s(x))ds$$

for every $x \in X, t \in R$.

We derive the next version of this proposition which works for non minimal fibred systems defined by (7). The basic results on topological dynamics we use can be found in Ellis [10]. However all of them are direct consequences of our argument, so we think the following one is a self-contained proof. We make use of the classical properties of the diffeomorfims of the circle.

Theorem 3.1 *The following statements are equivalent:*
$i)$ There exists a continuous function $k : K \to R$, satisfying

$$k(\Phi_t(\xi, \varphi)) - k(\xi, \varphi) = \int_0^t \frac{\partial f}{\partial \varphi}(\Phi_s(\xi, \varphi))ds \ .$$

$ii)$ There exists $\xi_0 \in \Omega$, such that the family of functions

$$\begin{array}{rcl} F_t : S^1 & \longrightarrow & R \\ \varphi & \to & \int_0^t \frac{\partial f}{\partial \varphi}(\Phi_s(\xi_0, \varphi))ds \end{array}$$

is a conditionally compact subset of $C(S^1)$.

Proof: If the flow (K, Φ) is minimal, proposition (3.1) applies directly, so we do not consider this option. The implication $ii) \Rightarrow i)$ is trivial. As for the other one, we first show that the functions $(F_t)_{t \in R}$ are uniformly bounded and equicontinuous. Notice that

$$\frac{\partial \varphi}{\partial \varphi_0}(t, \xi_0, \varphi_0) = e^{\int_0^t \frac{\partial f}{\partial \varphi}(\Phi_s(\xi_0, \varphi_0))ds} \ .$$

This assures the existence of constants m, M such that

$$m \mid \varphi_1 - \varphi_2 \mid \leq \mid \Phi_t(\xi_0, \varphi_1) - \Phi_t(\xi_0, \varphi_2) \mid \leq M \mid \varphi_1 - \varphi_2 \mid \ . \qquad (14)$$

Let $(t_n)_{n=1}^{\infty}$ be a sequence of real numbers. We define

$$\begin{array}{rcl} \Psi_n : S^1 & \to & S^1 \\ \varphi & \to & \varphi_{t_n}(\xi_0, \varphi) \end{array}$$

By the Arzela-Ascoli theorem the family of diffeomorphisms $(\Psi_n)_{n=1}^{\infty}$ possesses a convergent subsequence in the $C^1(S^1)$ topology. Its limit Ψ is a diffeomorphism which also preserves the orientation.

Let us fix $(\xi_1, \varphi_1) \in K$. Since (Ω, T) is minimal there is a sequence $(t_n)_{n=1}^{\infty}$ such that $\xi_1 = \lim_{n\to\infty} T_{t_n}(\xi_0)$. Moreover we can assume that $\varphi_1 = \lim_{n\to\infty} \varphi_{t_n}(\xi_0, \varphi_0)$ for an adequate $\varphi_0 \in S^1$.

Let η be a positive constant with $\mid \int_0^t \partial f/\partial\varphi\,(\Phi_s(\xi_0, \varphi))ds \mid \leq \eta$ for every $\varphi \in S^1, t \in R$. Then

$$\int_0^t \frac{\partial f}{\partial \varphi}(\Phi_{t_n+s}(\xi_0, \varphi))ds = \int_{t_n}^{t+t_n} \frac{\partial f}{\partial \varphi}(\Phi_s(\xi_0, \varphi))ds =$$

$$\int_0^{t+t_n} \frac{\partial f}{\partial \varphi}(\Phi_s(\xi_0, \varphi))ds - \int_0^{t_n} \frac{\partial f}{\partial \varphi}(\Phi_s(\xi_0, \varphi))ds$$

hence

$$\mid \int_0^t \frac{\partial f}{\partial \varphi}(\Phi_s(\xi_1, \varphi_1))ds \mid = \lim_{n\to\infty} \mid \int_0^t \frac{\partial f}{\partial \varphi}(\Phi_{s+t_n}(\xi_0, \varphi))ds \mid \leq 2\eta\ .$$

This extends the inequality (14) and proves that there are global constants m, M such that

$$m \mid \varphi_1 - \varphi_2 \mid \leq \mid \Phi_t(\xi, \varphi_1) - \Phi_t(\xi, \varphi_2) \mid \leq M \mid \varphi_1 - \varphi_2 \mid \tag{15}$$

for every pair of points $(\xi, \varphi_1), (\xi, \varphi_2)$ from K.

It follows that the flow is distal and semisimple. This means that the compact set K is a disjoint union of an uncountable collection $(K_j)_{j\in J}$ of minimal compact subsets. According to the proposition (3.1) a continuous solution of (13) can be constructed in each of these sets K_j. The main problem consists of making a good choice which allows us to define a continuous function on K. In order to solve this problem we analize now how these subsets cover the entire space K.

Let us consider $j_0 \in J$ such that $(\xi_0, 0) \in K_{j_0}$ and let us identify S^1 with $\{\xi_0\} \times S^1 \subset K$. Let $A = K_{j_0} \cap S^1 = \{\varphi_i / i \in I_{j_0}\}$. For every $i \in I_{j_0}$ there exists a sequence of real numbers $(t_n^i)_{n=1}^{\infty}$ such that the diffeomorphisms $\Psi_n^i : S^1 \to S^1; \varphi \to \varphi_{t_n^i}(\xi_0, \varphi)$ converge to a diffeomorphism Ψ^i with $\varphi_i = \Psi^i(0) = \lim_{n\to\infty} \varphi_{t_n^i}(\xi_0, 0)$. The families $\mathcal{F} = \{\Psi^i\}, \mathcal{G} = \{(\Psi^i)^{-1}\}$ are equicontinuous and $\Psi^i(A) = A$ for each $i \in I_{j_0}$.

This shows that if A' does not vanish then $A' = A$. In these conditions for all $\epsilon > 0$ there is $\delta(\epsilon) > 0$ such that for every $\varphi \in A$ can be found $\varphi' \in A$ with $\delta < \varphi - \varphi' < \epsilon$. We set $\delta_1 = m\delta$ and $\epsilon_1 = M\epsilon$, there exists a finite chain $\varphi_0, \varphi_1, \cdots \varphi_N$ of elements of A such that $\varphi_0 = 0, \varphi_N > 2\pi$ and $\delta_1 < \varphi_{i+1} - \varphi_i < \epsilon_1$, $i = 0 \cdots N-1$. Indeed, if $\varphi_0, \cdots \varphi_k$ are the first $k+1$ elements of the chain we choose $\varphi^\star_{k-1} \in A$ with $\delta < \varphi^\star_{k-1} - \varphi_{k-1} < \epsilon$ and we take a diffeomorphism Ψ with $\Psi(\varphi_{k-1}) = \varphi_k$ and $\delta_1 < \Psi(\varphi^\star_{k-1}) - \varphi_k < \epsilon_1$. Defining $\varphi_{k+1} = \Psi(\varphi^\star_{k-1})$ we obtain the next element. This proves that $A' = S^1$ and this situation only arises when the flow is minimal.

We conclude that $A' = \emptyset$ and $A = \{\varphi_0 = 0, \varphi_1, \cdots \varphi_N = 2\pi\}$ is a finite set. If $C_i = [\varphi_i, \varphi_{i+1}) \subset S^1$, and Ψ^i denote the diffeomorphism of $\mathcal{F}$ with $\Psi^i(0) = \varphi_i$, then $\Psi^i(C_0) = C_i$ for $i = 0 \cdots N-1$. This shows that C_i only contains one element of each subset K_j. Hence K_j is a finite cover of Ω.
Let $\varphi_j \in C_0 \cap K_j$, for each $j \in J$. We denote by k_{j_0} the solution of (13) on K_{j_0} satisfying $k_{j_0}(0) = 0$ and assume that $k_{j_0}(\varphi_1) = \eta$. We denote by k_j the solution of (13) on K_j satisfying $k_j(\varphi_j) = \varphi_j/\varphi_1\,\eta$. All the $(k_j)_{j\in J}$ together define a function k which is solution of (13) on K. Our last step consist in proving that k is continuous.

If $\{(\xi_n, \varphi_n)\}_{n=1}^\infty$ is a sequence of elements of K with limit (ξ', φ') there exist indices $(j_n)_{n=1}^\infty$ and real numbers $(t_n)_{n=1}^\infty$ such that $(\xi_n, \varphi_n) \in M_{j_n}$, $d(\Phi_{t_n}(\xi_0, \varphi_{j_n}), (\xi_n, \varphi_n)) < 1/n$ and $\mid k(\Phi_{t_n}(\xi_0, \varphi_{j_n})) - k(\xi_n, \varphi_n) \mid < 1/n$.
We may suppose that $(\varphi_{j_n})_{n=1}^\infty$ is a convergent sequence (otherwise we would choose an adequate subsequence) with limit φ'_0. The points (ξ', φ') and (ξ_0, φ'_0) belong to the same minimal subset where k is continuous. Moreover $(\xi', \varphi') = \lim_{n\to\infty} \Phi_{t_n}(\xi_0, \varphi'_0)$. So we immediately deduce that $k((\xi', \varphi')) = \lim_{n\to\infty} k(\Phi_{t_n}(\xi_0, \varphi'_0))$. Now we can write

$$\mid k(\xi', \varphi') - k(\xi_n, \varphi_n) \mid \leq \frac{1}{n} + \mid k(\xi', \varphi') - k(\Phi_{t_n}(\xi_0, \varphi_{j_n})) \mid \leq$$

$$\frac{1}{n} + \mid k(\xi', \varphi') - k(\Phi_{t_n}(\xi_0, \varphi'_0)) \mid + \mid k(\Phi_{t_n}(\xi_0, \varphi'_0)) - k(\Phi_{t_n}(\xi_0, \varphi_{j_n})) \mid \ .$$

On the other hand

$$\mid k(\Phi_{t_n}(\xi_0, \varphi'_0)) - k(\Phi_{t_n}(\xi_0, \varphi_{j_n})) \mid \leq$$

$$\mid k(\xi_0, \varphi'_0) - k(\xi_0, \varphi_{j_n}) \mid + \mid \int_0^{t_n} \frac{\partial f}{\partial \varphi}(\Phi_s(\xi, \varphi_{j_n}))ds - \int_0^{t_n} \frac{\partial f}{\partial \varphi}(\Phi_s(\xi_0, \varphi_0))ds \mid =$$

$$| k(\xi_0,\varphi_0') - k(\xi_0,\varphi_{j_n}) | + | F_{t_n}(\varphi_{j_n}) - F_{t_n}(\varphi_{j_0}) | \ .$$

Taking limits as $n \to \infty$, we obtain that $k(\xi',\varphi') = \lim_{n\to\infty} k(\xi_n,\varphi_n)$. This guarantees the continuity of k and completes the proof.

Obviously the solution of (13) is not unique. Two different continuous solutions of this equation differ in a constant on each subset K_j. Conversely every continuous function $\kappa : [0,\varphi_1] \to R$ with $\kappa(\varphi_1) - \kappa(0) = \eta$ allows us to find a solution of (13).

When the methods of this and the previous Sections are applied they give invariant measures under Φ which can be different. We now discuss when the families $\{p_T\}$, $\{q_T\}$ converge to a continuous limit. This fact depends on the set of points where there is convergence.

Proposition 3.2 *i) For every $\epsilon > 0$ there is a measurable subset $\Omega_\epsilon \subset \Omega$ with $m(\Omega_\epsilon) > 1-\epsilon$ such that $\{p_T\}$ and $\{q_T\}$ converge uniformly in $\Omega_\epsilon \times S^1$.*
ii) The functions $\{p_T\}$, $\{q_T\}$ converge almost everywhere for every invariant measure under the flow.

Proof: Let h be a continuous function defined on K and

$$h_T(\xi,\varphi) = \frac{1}{2T}\int_{-T}^{T} h(\Phi_t(\xi,\varphi))dt \ . \tag{16}$$

The family $\{h_T\}$ converges almost everywhere to a measurable limit $h^\star$. Let C be the set of points of convergence and C_ξ its section at ξ. We are going to show that C_ξ is closed for every $\xi \in \Omega$.
Let $(\varphi_n)_{n=1}^\infty$ be a sequence of points of C_ξ with limit φ. We derive from (15) and from the continuity of h that for every $\epsilon > 0$ there is n_0 such that if $n \geq n_0$ then

$$d(\Phi_{t_n}(\xi,\varphi_n), \Phi_{t_n}(\xi,\varphi)) \leq \epsilon$$

and

$$| h(\Phi_t(\xi,\varphi_n)) - h(\Phi_{t_n}(\xi,\varphi)) | \leq \epsilon \ .$$

Moreover there exists T such that if $T_1 > T_2 > T$ then

$$| \frac{1}{2T_1}\int_{-T_1}^{T_1} h(\Phi_t(\xi,\varphi_{n_0}))dt - \frac{1}{2T_2}\int_{-T_2}^{T_2} h(\Phi_t(\xi,\varphi_{n_0}))dt | \leq \frac{\epsilon}{3} \ .$$

Combining the last two inequalities we get

$$\mid \frac{1}{2T_1}\int_{-T_1}^{T_1} h(\Phi_t(\xi,\varphi))dt - \frac{1}{2T_2}\int_{-T_2}^{T_2} h(\Phi_t(\xi,\varphi))dt \mid \leq \epsilon \; .$$

This means that h_T converges at (ξ,φ) .

If $\xi \in \Omega$ and $\rho_0(C_\xi) = 1$ then $C_\xi = S^1$. Since $m(\Omega) = 1$ there exists Ω_0 with $m(\Omega_0) = 1$ such that $C_\xi = S^1$ for every $\xi \in \Omega_0$.

Let $\xi \in \Omega_0$. The functions $h_T(\xi,\varphi)$ are uniformly bounded and equicontinuous on S^1 where they converge uniformly. Hence $h^\star$ is continuous on $\{\xi\} \times S^1$. The functions $H_T : \Omega \to R$ defined by

$$H_T(\xi) = \begin{cases} \sup\limits_{\varphi\in S^1} \mid h_T(\xi,\varphi) - h^\star(\xi,\varphi) \mid & \text{if } \xi \in \Omega_0 \\ 0 & \text{otherwise} \end{cases}$$

are measurable and converge to 0. By Egoroff's theorem for every $\epsilon > 0$ there is a measurable subset $\Omega_\epsilon \subset \Omega$ with $m(\Omega_\epsilon) > 1 - \epsilon$ where $\{H_T\}$ converge uniformly. It follows that $\{h_T\}$ converge uniformly in $\Omega_\epsilon \times S^1$.

Let k be a continuous solution of (13) . Then

$$p_T(\xi,\varphi) = e^{k(\xi,\varphi)}\frac{1}{2T}\int_{-T}^{T} e^{-k(\Phi_t(\xi,\varphi))dt}$$

$$q_T(\xi,\varphi) = e^{-k(\xi,\varphi)}\frac{1}{2T}\int_{-T}^{T} e^{k(\Phi_t(\xi,\varphi))dt}$$

and all the above conclusions about convergence hold.

Let μ be a normalized invariant measure on K. Note that μ is mapped into the unique normalized invariant measure on the base under the natural projection $\Pi : K \to \Omega; (\xi,\varphi) \to \xi$. Hence $\mu(\Omega_\epsilon \times S^1) > 1-\epsilon$ and the functions $\{p_T\}$, $\{q_T\}$ converge almost everywhere for every invariant measure under the flow.

Now we suppose that (Ω, T) is an almost-periodic flow. Theorem (3.1) permits an alternative proof of several results contained in [16] and [17]. If (K,Φ) is not minimal then each subset K_j is a finite cover of Ω. See [27]. The functions $\{p_T\}$, $\{q_T\}$ converge everywhere to a continuous function on each subset K_j. These limits are also continuous with respect to φ and which as shown in theorem (3.1) implies the continuity on K.

Similar conclusions can be derived when (K, Φ) is a unique ergodic flow. From proposition (3.2) we deduce that there are constants η_1, η_2 such that

$$\eta_1 = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} e^{-k(\Phi_t(\xi,\varphi))dt}$$

and

$$\eta_2 = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} e^{k(\Phi_t(\xi,\varphi))dt}$$

almost everywhere. In fact the convergence holds everywhere and moreover is uniform in K.

However a different situation occurs when the flow on K is minimal but many invariant measures exist. A slight modification of an argument of Furstenberg [12] allows to prove that if $\{p_T\}$ or $\{q_T\}$ converge everywhere their limits are continuous functions and the convergence is uniform. Otherwise $\overset{\circ}{\Omega}_{1/n} = \emptyset$ for every $n \in N$ and hence $\bigcup_{n\in N}(\Omega_{1/n} \times S^1)$ is a set of first cathegory in K whose elements are points of convergence.

4 The case of the Schrödinger equation

Let us consider in this Section the one dimensional Schrödinger equation

$$L_E x = -x'' + (\xi(t) - E)x = 0 \qquad \xi \in \Omega, \tag{17}$$

with almost-periodic potential where we are going to apply the conclusions obtained in the previous Sections. The phase of the solutions of (17) satisfy the differential equation

$$\dot{\varphi} = \cos^2\varphi - (J(\xi_t) - E)\sin^2\varphi\,. \tag{18}$$

The map $\Phi^E(\xi, \varphi) = (\xi_t, \varphi_t^E(\xi, \varphi))$ defines a continuos flow on $\Sigma = \Omega \times P^1$. This flow allows a complete description of the solutions of (17).

We shall identify P^1 with $S^1/\pi Z$ and maintain on P^1 the notation used on S^1 in the previous Sections.

Let $x(t, \xi, \varphi)$ stand for the solution of (17) verifying $x'(0) + ix(0) = e^{i\varphi}$ and $r(t, \xi, \varphi) = x^2(t, \xi, \varphi) + (x')^2(t, \xi, \varphi)$. Then one has

$$r(t, \xi, \varphi) = e^{-\int_0^t \frac{\partial f}{\partial \varphi}(\Phi_s(\xi,\varphi))ds}$$

and

$$\frac{\partial \varphi}{\partial \varphi_0}(t, \xi, \varphi_0) = \frac{1}{r(t, \xi, \varphi_0)} = e^{\int_0^t \frac{\partial f}{\partial \varphi}(\Phi_s(\xi, \varphi_0))ds} .$$

Let us fix $E \in R$. If all the solutions of the equation (17) with potential $\xi_0 \in \Omega$ are bounded, then all of them are bounded for every $\xi \in \Omega$. In this conditions we can state

Proposition 4.1 *If the solutions of (17) are bounded then (Σ, Φ^E) admits an invariant absolutely continuous measure with continuous density function.*

Proof: We recall the following inequality for the first derivative

$$\max_{t \in R} \mid x' \mid \leq \max_{t \in R} \mid x \mid + \max_{t \in R} \mid x'' \mid \ .$$

This means that the family $(F_t)_{t \in R}$ introduced in theorem (3.1) is bounded. Moreover for every $\epsilon > 0$ there is a $\delta > 0$ such that for any two elements φ_1, φ_2 of S^1 with $\mid \varphi_1 - \varphi_2 \mid < \delta$ we have

$$\mid \int_0^t \frac{\partial f}{\partial \varphi}(\Phi_s(\xi, \varphi_1)) - \int_0^t \frac{\partial f}{\partial \varphi}(\Phi_s(\xi, \varphi_2))ds \mid = \mid \ln r(t, \xi, \varphi_1) - \ln r(t, \xi, \varphi_2) \mid < \epsilon$$

for every $t \in R$. This shows that (F_t) is conditionally compact. Hence our claim is a consequence of theorem (3.1).

This is not the only case where invariant absolutely continuous measures with L^2-density function occur. Let $\gamma(E)$ be the Lyapunov exponent of L_E. Then the set $A = \{E \in R / \gamma(E) = 0\}$ is known to be the essential support with respect to the Lebesgue measure of the absolutely continuous part of the spectral measure . This is a consequence of the work by Pastur, Ishii and Kotani. The following theorem was demonstrated by Kotani [21] and Deift-Simon [8] and show the kind of almost-periodic solutions we find for these values of the energy.

Theorem 4.1 *There exist two linearly independent solutions $u_\pm(t, \xi, E)$ of $x'' + \xi(t)x = Ex$ for almost every $(\xi, E) \in \Omega \times A$ and two positive measurable functions $P_1, P_2 : \Omega \longrightarrow R$ for almost every $E \in A$ such that*
i)$u_+ = \bar{u}_-$.
ii) $\mid u_\pm(t, \xi, E) \mid = P_1(\xi_t, E), \quad \mid u'_\pm(t, \xi, E) \mid = P_2(\xi_t, E)$.
iii) $\int_\Omega [P_2^2(\xi, E) + P_2^2(\xi, E)] dm_0 = \eta_E \in (0, \infty)$.

We take $E \in A$ where the claim of the theorem holds. Our objetive is to prove that an invariant square integrable measure under Φ^E can be constructed.

It follows from Birkhoff's ergodic theorem that

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} [\mid u_{\pm}^2(t,\xi,E) \mid + \mid u_{\pm}'^2(t,\xi,E) \mid] dt = \int_{\Omega} [P_1^2(\xi,E) + P_2^2(\xi,E)] dm_0$$

for almost every $\xi \in \Omega$. Now it is easy to check that

$$\lim_{|t|\to\infty} \frac{\mid u_{\pm}^2(t,\xi,E) \mid + \mid u_{\pm}'^2(t,\xi,E) \mid}{t} = 0 \tag{19}$$

for almost every $\xi \in \Omega$. In consequence

$$\lim_{|t|\to\infty} \frac{x^2(t,\xi,E) + x'^2(t,\xi,E)}{t} = 0 \tag{20}$$

for every solution of these linear equations.

If $u_+(0,\xi,\varphi) = \alpha_1(\xi)$, $u_+'(0,\xi,E) = \alpha_2(\xi)$, it follows from the proof of the theorem that $\alpha_1, \alpha_2 \in L^2(\Omega, m_0)$ and moreover $\mid \alpha_1^2(\xi) \mid + \mid \alpha_2^2(\xi) \mid \geq 1$. Then we have

$$x(t,\xi,\varphi) = \alpha(\xi,\varphi) u_+(t,\xi,E) + \overline{\alpha(\xi,\varphi) u_+(t,\xi,E)}$$

where

$$\alpha(\xi,\varphi) = \frac{1}{2i}(\overline{\alpha_2(\xi)} \sin\varphi - \alpha_1(\xi)\cos\varphi) \ .$$

Taking $\beta(\xi) = 1/2\,(\mid \alpha_1^2(\xi) \mid + \mid \alpha_2^2(\xi) \mid)$ we see that $\beta \in L^1(\Omega, m_0)$. Moreover $1/2 \leq \beta(\xi)$ and $\mid \alpha^2(\xi,\varphi) \mid \leq \beta(\xi)$ for every $(\xi,\varphi) \in \Sigma$.

In order to achieve an invariant absolutely continuous measure whose density function belongs to $L^2(\Sigma, m)$, we consider the family

$$q_T(\xi,\varphi) = \frac{1}{2T}\int_{-T}^{T} [x^2(t,\xi,\varphi) + (x')^2(t,\xi,\varphi)] dt =$$

$$\frac{1}{2T}\int_{-T}^{T} 2\Re\{\alpha^2(\xi,\varphi)(u_+^2(t,\xi,E) + u_+'^2(t,\xi,E))\} dt +$$

$$\frac{1}{2T}\int_{-T}^{T} 2 \mid \alpha^2(\xi,\varphi) \mid (\mid u_+^2(t,\xi,E) \mid + \mid u_+'^2(t,\xi,E) \mid)\} dt \ .$$

We cannot prove the pointwise convergence of $\{q_T\}$ directly from theorem (4.1). In consequence we will establish the convergence of $\sqrt{q_T}$ in a weak-topology whose limit allows us to obtain a positive solution of the functional equation (12). Of course this finally implies the pointwise convergence of $\{q_T\}$ to a finite limit.

We will always refer through the proof to the measures $\nu_0 = 1/\beta(\xi)\, dm_0$ on Ω and $\nu = \nu_0 \times d\rho$ on Σ. Notice that

$$\frac{q_T(\xi,\varphi)}{\beta(\xi)} \le \frac{1}{2T}\int_{-T}^{T} 4(\mid u_+^2(t,\xi,E)\mid + \mid u_+'^2(t,\xi,E)\mid)dt$$

and

$$\int_\Sigma q_T(\xi,\varphi)d\nu \le \frac{1}{2T}\int_{-T}^{T}\int_\Omega 4(P_1^2(\xi_t,E)+P_2^2(\xi_t,E))dm_0 dt = 4\eta_E < \infty\ .$$

Then $\{\sqrt{q_T}\}$ is a bounded subset of $L^2_\nu = L^2(\Sigma,\nu)$ which possesses a subsequence $\{\sqrt{q_{T_j}}\}$, with $\lim_{j\to\infty} T_j = \infty$, satisfying:

$$\begin{cases} i)\sqrt{q_{T_j}} & \text{converges in the weak topology } \sigma(L^2_\nu, L^2_\nu) \text{ to a limit } \sqrt{q}. \\ ii)\dfrac{q_{T_j}}{T_j} & \text{converges to zero almost everywhere.} \end{cases}$$

The next lemmas show certain important properties of this sequence.

Lemma 4.1 *Let $l \in Z$ be fixed. Then*
i) $\{q_{T_j+l} - q_{T_j}\}$ converges to zero almost everywhere.
ii) $\{\sqrt{q_{T_j+l}}\}$ converges to $\sqrt{q}$ in the $\sigma(L^2_\nu, L^2_\nu)$-topology.

Proof: i) We suppose that l is positive. The other case is completely analogous

$$q_{T_j+l}(\xi,\varphi) - q_{T_j}(\xi,\varphi) =$$
$$\frac{1}{2(T_j+l)}\int_{-T_j-l}^{T_j+l} e^{-\int_0^t \frac{\partial f}{\partial\varphi}(\Phi_s(\xi,\varphi))ds}dt - \frac{1}{2T_j}\int_{-T_j}^{T_j} e^{-\int_0^t \frac{\partial f}{\partial\varphi}(\Phi_s(\xi,\varphi)ds}dt =$$
$$S_{1,j}(\xi,\varphi) + S_{2,j}(\xi,\varphi) + S_{3,j}(\xi,\varphi).$$

where

$$S_{1,j}(\xi,\varphi) = \frac{-l}{2T_j(T_j+l)}\int_{-T_j}^{T_j} e^{-\int_0^t \frac{\partial f}{\partial\varphi}(\Phi_s(\xi,\varphi))ds}dt$$

$$S_{2,j}(\xi,\varphi)=\frac{1}{2(T_j+l)}\int_{T_j}^{T_j+l}e^{-\int_0^t\frac{\partial f}{\partial\varphi}(\Phi_s(\xi,\varphi))ds}dt$$

$$S_{3,j}(\xi,\varphi)=\frac{1}{2(T_j+l)}\int_{-T_j-l}^{-T_j}e^{-\int_0^t\frac{\partial f}{\partial\varphi}(\Phi_s(\xi,\varphi))ds}dt\ .$$

First we know that

$$\lim_{j\to\infty}S_{1,j}(\xi,\varphi)=\lim_{j\to\infty}\frac{-l}{T_j+l}q_{T_j}(\xi,\varphi)=0$$

almost everywhere. Besides

$$S_{2,j}(\xi,\varphi)=\frac{1}{2(T_j+l)}e^{-\int_0^{T_j}\frac{\partial f}{\partial\varphi}(\Phi_s(\xi,\varphi)ds}\int_{T_j}^{T_j+l}e^{-\int_{T_j}^{t}\frac{\partial f}{\partial\varphi}(\Phi_s(\xi,\varphi)ds}dt\ .$$

If $M=max\left\{\mid\frac{\partial f}{\partial\varphi}(\xi,\varphi)\mid/(\xi,\varphi)\in\Sigma\right\}$ we obtain using (20) that

$$\lim_{j\to\infty}S_{2,j}(\xi,\varphi)=le^{M.l}\lim_{j\to\infty}\frac{2r(T_j,\xi,\varphi)}{2(T_j+l)}=0\ .$$

Similarly $\lim_{j\to\infty}S_{3,j}(\xi,\varphi)=0$. This completes the proof.

ii) Since $q_{T_j+l}-q_{T_j}=\left(\sqrt{q_{T_j+l}}-\sqrt{q_T}\right)\left(\sqrt{q_{T_j+l}}+\sqrt{q_T}\right)$ we deduce that $\sqrt{q_{T_{j+l}}}-\sqrt{q_T}$ converges to zero almost everywhere.
Moreover the family $\{\sqrt{q_T}\}$ is uniformly integrable . Indeed, if ϵ is a positive constant, $\delta_\epsilon=\epsilon^2/4\eta_E$ and A is a measurable set with $\nu(A)\le\delta_\epsilon$ then

$$\int_A\sqrt{q_T}d\nu=\int_\Sigma\sqrt{q_T}.\chi_A d\nu\le\left(\int_\Sigma q_T d\nu\right)^{1/2}\nu(A)^{1/2}\le\epsilon\ .$$

Let $B\subset L^2(\Sigma,\nu)$ be the unit ball and $B'=4\eta_E^{1/2}B$. The topologies $\sigma(L^2_\nu,L^2_\nu)$ and $\sigma(L^2_\nu,C)$ where $C=C(\Sigma)$, are identical on B', so we will prove the convergence of $\sqrt{q_{T_j+l}}-\sqrt{q_{T_j}}$ in the last topology.

If $h\in C(\Sigma)$ the sequence $\{(\sqrt{q_{T_j+l}}-\sqrt{q_{T_j}})h\}$ is uniformly integrable and converges to zero almost evertywhere. According to Vitali's theorem we find that $\lim_{j\to\infty}\int_\Sigma\mid(\sqrt{q_{T_j+l}}-\sqrt{q_{T_j}})h\mid d\nu=0$.

This confirms that $\{\sqrt{q_{T_j+l}}\}$ converges to $\sqrt{q}$ in the weak topology.

Lemma 4.2 *Let us fix $l \in Z$. Then*
i) $\{\sqrt{q_{T_j}(\Phi_l(\xi,\varphi))}\}$ *converges to* $\sqrt{q(\Phi_l(\xi,\varphi))}$ *in the* $\sigma(L^2_\nu, L^2_\nu)$ *-topology.*
ii) q satisfies the functional equation

$$q(\Phi_l(\xi,\varphi)) = q(\xi,\varphi).e^{\int_0^l \frac{\partial f}{\partial \varphi}(\Phi(\xi,\varphi))ds} .$$

Proof: i) Again we suppose that l is positive. We will prove the convergence in the $\sigma(L^2_\nu, C)$-topology. If $h \in C(\Sigma)$, and

$$\upsilon(\xi,\varphi) = \frac{h(\Phi_{-l}(\xi,\varphi))}{\beta(\xi_{-l})} e^{\int_0^{-l} \frac{\partial f}{\partial \varphi}(\Phi_s(\xi,\varphi))ds} \beta(\xi)$$

then

$$\int_\Sigma \left[\sqrt{q_{T_j}(\Phi_l(\xi,\varphi))} - \sqrt{q(\Phi_l(\xi,\varphi))}\right] h(\xi,\varphi)d\nu =$$
$$\int_\Sigma \left[\sqrt{q_{T_j}(\xi,\varphi)} - \sqrt{q(\xi,\varphi)}\right] \upsilon(\xi,\varphi)d\nu .$$

We see that there exists a constant M such that $\upsilon(\xi,\varphi) \leq M\beta(\xi)$ and $\int_\Sigma \beta^2(\xi)d\nu = \int_\Omega \beta(\xi)dm_0 < \infty$. Hence $\upsilon \in L^2(\Sigma, d\nu)$. Now using lemma (4.1), we get

$$\lim_{j\to\infty} \int_\Sigma \left[\sqrt{q_{T_j}(\Phi_l(\xi,\varphi))} - \sqrt{q(\Phi_l(\xi,\varphi))}\right] h(\xi,\varphi)d\nu = 0 .$$

ii) The argument used in Section (1) yields

$$\sqrt{\frac{T_j - l}{T_j}} q^{\frac{1}{2}}_{T_j - l}(\xi,\varphi) \leq q^{\frac{1}{2}}_{T_j}(\Phi_l(\xi,\varphi)) e^{-\frac{1}{2}\int_0^l \frac{\partial f}{\partial \varphi}(\Phi_s(\xi,\varphi))ds} \leq \sqrt{\frac{T_j + l}{T_j}} q^{\frac{1}{2}}_{T_j + l}(\xi,\varphi) .$$

Taking limits in the $\sigma(L^2_\nu, L^2_\nu)$-topology as $j \to \infty$ we obtain

$$\sqrt{q(\Phi_l(\xi,\varphi))} = \sqrt{q(\xi,\varphi)} e^{\frac{1}{2}\int_0^l \frac{\partial f}{\partial \varphi}(\Phi(\xi,\varphi))ds}$$

almost everywhere. Taking squares we prove (*ii*). We can assume that q satisfies (12) for every $t \in R$ and $(\xi,\varphi) \in \Sigma$.

Finally we verify that q is different from zero almost everywhere. The set $A = \{(\xi,\varphi) \in \Sigma \ / \ q(\xi,\varphi) = 0\}$ is invariant and it follows from (11) that

$$m(A) \leq (\int_A \sqrt{q_{T_j}} dm)^{\frac{1}{2}} (\int_\Sigma \frac{1}{\sqrt{q_{T_j}}} dm)^{\frac{1}{2}} \leq$$

$$(\int_A \sqrt{q_{T_j}} dm)^{\frac{1}{2}} (\int_\Sigma \frac{1}{q_{T_j}} dm)^{\frac{1}{4}} \leq (\int_A \sqrt{q_{T_j}} dm)^{\frac{1}{2}}$$

then

$$m(A) = \lim_{j\to\infty} \int_A \sqrt{q_{T_j}} dm = \int_A \sqrt{q} dm = 0 \ .$$

This confirms the pointwise convergence of $\{q_T\}$. We may apply proposition (2.3) to get

Theorem 4.2 *The flow (Σ, Φ^E) admits an invariant measure, equivalent to m whose density function belongs to $L^2(\Sigma, m)$ for almost every $E \in A$.*

Let q be the limit of the above family of functions $\{q_T\}$. Note that

$$(i)\ q \in L^1(\Sigma, m) \quad , \quad (ii)\ \frac{1}{q} \in L^2(\Sigma, m) \ . \tag{21}$$

These conditions on convergence will be essential in the next Sections. From now on we denote by $A_0 \subset A$ the set of energies where they are satisfied.

If (Σ, Φ^E) is uniquely ergodic then $\gamma(E) = 0$. Let us assume that $\gamma(E) > 0$. In this conditions the function $r(t, \xi, \varphi)$ increases exponentially for almost every $(\xi, \varphi) \in \Sigma$ when $t \to \pm\infty$ and hence

$$\lim_{T\to\infty} p_T(\xi, \varphi) = \lim_{T\to\infty} \frac{1}{2T} \int_T^T \frac{1}{r(t, \xi, \varphi)} dt = 0$$

almost everywhere. ¿From proposition (2.4) we deduce that (Σ, Φ^E) does not admit an invariant absolutely continuous measure. In fact invariant singular measures which induce purely discontinuous measures on the fibres are known in this case. See [12], [22].

5 The admissible directions of differentiability

We consider again the subset A_0 where the functions $\{q_T\}$ converge as shown in (21). Let us fix $E \in A_0$ and $\Gamma \in C(\Omega)$. If $f_E(\xi, \varphi)$ defines the differential equation (18) then $\partial f / \partial E = \sin^2 \varphi$. The family

$$\tilde{q}_{T,\Gamma}(\xi, \varphi) = \frac{1}{2T} \int_{-T}^T e^{-\int_0^t \frac{\partial f}{\partial \varphi}(\Phi_s(\xi,\varphi))ds} \Gamma(\xi_t) \sin^2 \varphi_t(\xi, \varphi) dt = \tag{22}$$

$$\frac{1}{2T}\int_{-T}^{T}\frac{q(\xi,\varphi)}{q(\Phi_t(\xi,\varphi))}\Gamma(\xi_t)\sin^2\varphi_t(\xi,\varphi)dt = \frac{1}{2T}\int_{-T}^{T}x^2(t,\xi,\varphi)\Gamma(\xi_t)dt$$

converges to a limit $\tilde{q}_\Gamma(\xi,\varphi)$. Moreover $\tilde{q}_\Gamma(\xi,\varphi) = q(\xi,\varphi)q^\star_\Gamma(\xi,\varphi)$ where $q^\star_\Gamma(\xi,\varphi)$ is an invariant function. This also shows that $q_\Gamma(\xi,\varphi)$ satisfies the functional equation (12).

We introduce the following set of directions:

Definition 5.1 *We say that* Γ *is an admissible direction of differentiability if*

$$(i)\ \tilde{q}_\Gamma \in L^1(\Sigma,m) \quad , \quad (ii)\ \tilde{q}_{1,\Gamma} = \frac{1}{\tilde{q}_\Gamma} \in L^2(\Sigma,m) \tag{23}$$

The set $C_{ad}(\Omega)$ contains the admissible directions of differentiability. This Section is transitional. We are going to show the basic properties of the measure $\mu_\Gamma = \tilde{q}_{1,\Gamma}dm$ for $\Gamma \in C_{ad}(\Omega)$. Our conclusions will be essential in Section 6.

The condition (i) of the definition always holds, so only (ii) has to be verified. We represent

$$\tilde{q}_T(\xi,\varphi) = \frac{1}{2T}\int_{-T}^{T}e^{-\int_0^t \frac{\partial f}{\partial\varphi}(\Phi_s(\xi,\varphi))ds}\sin^2\varphi_t(\xi,\varphi)dt = \frac{1}{2T}\int_{-T}^{T}x^2(t,\xi,\varphi)\ .dt$$

Note that for almost $\xi \in \Omega$ one has

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}x'^2(t,\xi,\varphi)dt = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}x^2(t,\xi,\varphi)J(\xi_t)dt\ .$$

Thus, there is a constant M such that $\tilde{q} \le q \le M\tilde{q}$. Hence $\tilde{q}_1 \in L^2(\Sigma,m)$ and $\tilde{q}_1 dm$ is an invariant measure. This shows that $\Gamma \equiv 1 \in C_{ad}(\Omega)$.

Proposition 5.1 *There exists an invariant set* $\Omega_0 \subset \Omega$ *with* $m_0(\Omega_0) = 1$ *such that if* $\xi \in \Omega_0$ *then*
i) $\{\tilde{q}_{T,\Gamma}(\xi,\varphi)\}$ *converges to* $\tilde{q}_\Gamma(\xi,\varphi)$ *for every* $\varphi \in P^1$.
ii) $\tilde{q}_{1,\Gamma}(\xi,\varphi) \in C^\infty(P^1)$ *and* $\int_{P^1}\tilde{q}_{1,\Gamma}(\xi,\varphi)d\rho_0 = 1/\lambda_\Gamma$ *(constant).*
iii) $\int_\Sigma \tilde{q}_{1,\Gamma}(\xi,\varphi)\Gamma(\xi)\sin^2\varphi d\mu = 1$ *for every invariant measure under the flow.*

Proof: If φ_1,φ_2 are different elements of Σ we represent

$$x(t,\xi,\varphi) = c_1(\varphi)x(t,\xi,\varphi_1) + c_2(\varphi)x(t,\xi,\varphi_2)$$

where $c_1, c_2 : P^1 \to R$ are smooth functions. We have

$$\tilde{q}_\Gamma(\xi,\varphi) = 2c_1(\varphi)c_2(\varphi) \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t,\xi,\varphi_1)x(t,\xi,\varphi_2)\Gamma(\xi_t)dt \; + \tag{24}$$

$$c_1^2(\varphi) \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x^2(t,\xi,\varphi_1)\Gamma(\xi_t)dt + c_2^2(\varphi) \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x^2(t,\xi,\varphi_2)\Gamma(\xi_t)dt$$

when these limits exist. If we choose ξ of Ω and $\varphi_1, \varphi_2, \varphi = \varphi_3$ of Σ such that $\{\tilde{q}_T(\xi,\varphi_i)\}$ converges for every $i = 1, 2, 3$ then it is obvious that there exists $\lim_{T\to\infty} 1/2T \int_{-T}^{T} x(t,\xi,\varphi_1)x(t,\xi,\varphi_2)\Gamma(\xi_t)dt$ and in consequence $\{\tilde{q}_{T,\Gamma}(\xi,\varphi)\}$ converges for every $\varphi \in P^1$.

The map $I : \Omega \to R$, $\xi \to \int_{P^1} \tilde{q}_{1,\Gamma}(\xi,\varphi)d\rho_0$ is invariant. We can find an invariant subset $\Omega_0 \subset \Omega$ with $m_0(\Omega_0) = 1$ and a constant λ_Γ such that $\{\tilde{q}_{T,\Gamma}(\xi,\varphi)\}$ converge, $\tilde{q}_{1,\Gamma}(\xi,\varphi) \in L^2(P^1,\rho_0)$ and $\int_{P^1} \tilde{q}_{1,\Gamma}(\xi,\varphi)d\rho_0 = 1/\lambda_\Gamma$ for every $\xi \in \Omega_0$.

Let us fix $\varphi_0 \in P^1$. Choosing $\varphi_1 = \varphi_0$, $\varphi_2 = \varphi_0 + \pi/2$ we obtain

$$x(t,\xi,\varphi+\varphi_0) = \cos\varphi\, x(t,\xi,\varphi_0) + \sin\varphi\, x(t,\xi,\varphi_0 + \frac{\pi}{2}) \,.$$

Denoting $\kappa_1(\xi) = \tilde{q}_\Gamma(\xi,\varphi_0)$, $\kappa_2(\xi) = \tilde{q}_\Gamma(\xi,\varphi_0 + \frac{\pi}{2})$ and

$$\kappa_3(\xi) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t,\xi,\varphi_0)x(t,\xi,\varphi_0 + \frac{\pi}{2})\Gamma(\xi_t)dt$$

we find

$$\tilde{q}_\Gamma(\xi,\varphi+\varphi_0) = \cos^2\varphi\,\kappa_1(\xi) + \sin^2\varphi\,\kappa_2(\xi) + 2\sin\varphi\,\cos\varphi\,\kappa_3(\xi) \tag{25}$$

and

$$\tilde{q}_{1,\Gamma}(\xi,\varphi+\varphi_0) = \frac{1}{\cos^2\varphi\,\kappa_1(\xi) + \sin^2\varphi\,\kappa_2(\xi) + 2\sin\varphi\,\cos\varphi\,\kappa_3(\xi)} \,. \tag{26}$$

If $\xi \in \Omega_0$ it is obvious that $\tilde{q}_\Gamma(\xi,\varphi)$ is a smooth function in φ. Since $\tilde{q}_{1,\Gamma} \in L^2(P^1,\rho_0)$ then $\tilde{q}_\Gamma(\xi,\varphi) \neq 0$ for every $\varphi \in P^1$. This shows that $\tilde{q}_{1,\Gamma} \in C^\infty(P^1)$. We assume that $\tilde{q}_\Gamma$ is always positive on $\Omega_0 \times P^1$. Then we have

$$\frac{1}{\lambda_\Gamma} = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{\cos^2\varphi\,\kappa_1(\xi) + \sin^2\varphi\,\kappa_2(\xi) + 2\sin(\varphi)\,\cos(\varphi)\,\kappa_3(\xi)} d\varphi \,.$$

Taking $s = \tan\varphi$ and $\kappa_0 = \sqrt{\kappa_1\kappa_2 - \kappa_3^2}$ we obtain

$$\frac{1}{\lambda_\Gamma} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1}{\kappa_1(\xi) + s^2\kappa_2(\xi) + 2s\kappa_3(\xi)} ds = \tag{27}$$

$$\frac{2}{\pi\kappa_0(\xi)}\left[\tan^{-1}\left(\frac{2\kappa_3(\xi) + 2\kappa_2(\xi)s}{\kappa_0(\xi)}\right)\right]_{-\infty}^{\infty} = \frac{2}{\kappa_0(\xi)} .$$

Moreover since

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\tilde{q}_{1,\Gamma}(\Phi_t(\xi,\varphi)\Gamma(\xi_t)\sin^2\varphi_t(\xi,\varphi)dt = 1$$

at every $(\xi,\varphi) \in \Omega_0 \times P^1$ we conclude that $\int_\Sigma \tilde{q}_{1,\Gamma}(\xi,\varphi)\Gamma(\xi)\sin^2\varphi d\mu = 1$ for every normalized invariant measure under the flow. This completes the proof.

The sequence $\{\tilde{q}_{T,\Gamma}\}$ converges to a limit $\tilde{q}_\Gamma \in L^1(\Sigma, m)$. We can assume that $\tilde{q}_\Gamma > 0$ on $\Omega_0 \times P^1$, otherwise we would change Γ to $-\Gamma$. Using the relation (25) we define

$$\beta(\xi) = \frac{1}{2}(\tilde{q}_\Gamma(\xi,\varphi) + \tilde{q}_\Gamma(\xi,\varphi + \frac{\pi}{2})) = \frac{1}{2}(\kappa_1(\xi) + \kappa_2(\xi)) . \tag{28}$$

Then it is obvious that $\beta \in L^1(\Omega, m_0)$ and $\tilde{q}_1\beta \in L^1(\Sigma, m_0)$.

We take $\varphi_0 \in P^1$ such that $\tilde{q}_{1,\Gamma}(\xi,\varphi_0), \tilde{q}_{1,\Gamma}(\xi,\varphi_0 + \pi/2) \in L^2(\Omega_0, m_0)$ and $\beta(\xi)\tilde{q}_{1,\Gamma}(\xi,\varphi_0),\ \beta(\xi)\tilde{q}_{1,\Gamma}(\xi,\varphi_0 + \frac{\pi}{2}) \in L^1(\Omega, m_0)$. We assume that

$$\int_\Omega (\beta(\xi) + \tilde{q}_{1,\Gamma}(\xi,\varphi_0))\tilde{q}_{1,\Gamma}(\xi,\varphi_0)dm_0 \leq \| \tilde{q}_{1,\Gamma} \|_2^2 + \frac{1}{\lambda_\Gamma} \| \beta \|_1 \ .$$

Lemma 5.1 *The functions κ_3/κ_1^2 and κ_3/κ_2^2 belong to $L^1(\Omega, m_0)$.*

Proof: We start by showing that $\kappa_3/\kappa_1^2 \in L^1(\Omega, m_0)$. We have proved that $| \kappa_3(\xi) | \leq \sqrt{\kappa_1(\xi)\kappa_2(\xi)}$. ¿From (28) we deduce

$$1 + \frac{\tilde{q}_\Gamma(\xi,\varphi + \frac{\pi}{2})}{\tilde{q}_\Gamma(\xi,\varphi)} \leq 2\beta(\xi)\tilde{q}_{1,\Gamma}(\xi,\varphi)$$

and hence

$$\frac{\kappa_2(\xi)}{\kappa_1(\xi)} = \frac{\tilde{q}_\Gamma(\xi,\varphi_0 + \frac{\pi}{2})}{\tilde{q}_\Gamma(\xi,\varphi_0)} \leq 2\beta(\xi)\tilde{q}_{1,\Gamma}(\xi,\varphi_0) \in L^1(\Omega, m_0) .$$

Combining these inequalities, we obtain

$$\int_\Omega \frac{\mid \kappa_3 \mid}{\kappa_1^2} dm_0 \leq \int_\Omega \sqrt{\frac{\kappa_2}{\kappa_1}} \frac{1}{\kappa_1} dm_0 \leq \left[\int_\Omega \frac{\kappa_2}{\kappa_1} dm_0\right]^{\frac{1}{2}} \left[\int_\Omega \frac{1}{\kappa_1^2} dm_0\right]^{\frac{1}{2}} \leq \infty \ .$$

The other case is completely analogous. Moreover

$$\|\frac{\kappa_3}{\kappa_1^2}\|_1 \leq \| \tilde{q}_{1,\Gamma} \|_2^2 + \| \tilde{q}_\Gamma \|_1 \| \tilde{q}_{1,\Gamma} \|_1 \ . \tag{29}$$

Let μ be an invariant measure on Σ which is mapped into m_0 under the natural projection on Ω. Then $\tilde{q}_{1,\Gamma} \in L^1(\Sigma, \mu)$. This is the main consequence of the next

Proposition 5.2 *There exist $\kappa \in L^1(\Omega, m_0)$, $\kappa' \in L^2(\Omega, m_0)$ such that $0 < \kappa'(\xi) \leq \tilde{q}_{1,\Gamma}(\xi, \varphi) \leq \kappa(\xi)$ for every $(\xi, \varphi) \in \Omega_0 \times P^1$. Moreover there exists a constant M such that $\|\kappa\|_1 \leq M(\| \tilde{q}_{1,\Gamma} \|_2^2 + \| \tilde{q}_\Gamma \|_1 \| \tilde{q}_{1,\Gamma} \|_1)$.*

Proof: It is known that $\frac{1}{2\beta} \leq \tilde{q}_{1,\Gamma}(\xi, \varphi)$ for every $(\xi, \varphi) \in \Omega_0 \times P^1$. Therefore the only problem is to obtain the function κ of our claim.

Let us identify $\Omega \times \{\varphi_0\}$ with Ω. We are going to prove that $\tilde{q}_{1,\Gamma}$ possesses continuous derivative along the trajectories lying on Ω_0.

Let us fix $\xi \in \Omega_0$. We know that

$$\tilde{q}_{1,\Gamma}(\xi_t, \varphi_0) = \tilde{q}_{1,\Gamma}(\xi, \varphi_{-t}(\xi_t, \varphi_0)) e^{-\int_0^t \frac{\partial f}{\partial \varphi}(\Phi_s(\xi, \varphi_{-t}(\xi_t, \varphi_0)))ds}$$

and $\tilde{q}_{1,\Gamma}$ is smooth with respect to φ. Thus taking limits as $t \to 0$ it follows that $\lim_{t\to 0} \tilde{q}_{1,\Gamma}(\xi_t, \varphi_0) = \tilde{q}_{1,\Gamma}(\xi, \varphi_0)$. The same conclusion can be derived for any φ of S^1.

In this way we conclude that $\lim_{t\to 0} \kappa_i(\xi_t) = \kappa_i(\xi)$ for $i = 1, 2, 3$. We decompose

$$\tilde{q}_{1,\Gamma}(\xi_t, \varphi_0) - \tilde{q}_{1,\Gamma}(\xi, \varphi_0) =$$
$$[\tilde{q}_{1,\Gamma}(\xi_t, \varphi_0) - \tilde{q}_{1,\Gamma}(\xi_t, \varphi_t(\xi, \varphi_0))] + [\tilde{q}_{1,\Gamma}(\xi_t, \varphi_t(\xi, \varphi_0)) - \tilde{q}_{1,\Gamma}(\xi, \varphi_0)] \ .$$

First we have

$$\tilde{q}_{1,\Gamma}(\xi_t, \varphi_0) - \tilde{q}_{1,\Gamma}(\xi_t, \varphi_t(\xi, \varphi_0)) = -\frac{\partial \tilde{q}_{1,\Gamma}}{\partial \varphi}(\xi_t, \varphi_t')(\varphi_t(\xi, \varphi_0) - \varphi_0)$$

where φ'_t is an intermediate point between φ_0 and $\varphi_t(\xi,\varphi_0)$ and

$$\frac{\partial \tilde{q}_{1,\Gamma}}{\partial \varphi}(\xi_t,\varphi'_t) = \frac{2\sin(\varphi'_t-\varphi_0)\cos(\varphi'_t-\varphi_0)\,[\kappa_1(\xi_t)-\kappa_2(\xi_t)]}{\tilde{q}^2_{1,\Gamma}(\xi_t,\varphi'_t)} +$$

$$\frac{2\left[\sin^2(\varphi'_t-\varphi_0)-\cos^2(\varphi'_t-\varphi_0)\right]\kappa_3(\xi_t)}{\tilde{q}^2_{1,\Gamma}(\xi_t,\varphi'_t)}$$

thus

$$\lim_{t\to 0}\frac{\tilde{q}_{1,\Gamma}(\xi_t,\varphi_0)-\tilde{q}_{1,\Gamma}(\xi_t,\varphi_t(\xi,\varphi_0))}{t} = \frac{2\kappa_3(\xi)}{\kappa_1^2(\xi)}f_E(\xi,\varphi_0)\ .$$

We also see that

$$\lim_{t\to 0}\frac{\tilde{q}_{1,\Gamma}(\xi_t,\varphi_t(\xi,\varphi_0))-\tilde{q}_{1,\Gamma}(\xi,\varphi_0)}{t} = -\tilde{q}_{1,\Gamma}(\xi,\varphi_0)\frac{\partial f_E}{\partial\varphi}(\xi,\varphi_0)\ .$$

Therefore, from lemma (5.1) we obtain

$$\frac{d\tilde{q}_{1,\Gamma}(\xi_t,\varphi_0)}{dt}|_{t=0} = \frac{2\kappa_3(\xi)}{\kappa_1^2(\xi)}f_E(\xi,\varphi_0)-\tilde{q}_{1,\Gamma}(\xi,\varphi_0)\frac{\partial f_E}{\partial\varphi}(\xi,\varphi_0)\in L^1(\Omega,m_0)\ .$$

On the other hand there exists $T>0$ such that for every $(\xi,\varphi)\in\Sigma$ we can find $t=t(\xi,\varphi)\in[-T,T]$ with $\varphi_t(\xi,\varphi)=\varphi_0$.

If $M_0 = max\{\mid \partial f/\partial\varphi(\xi,\varphi)\mid /(\xi,\varphi)\in\Sigma\}$ and we define

$$\kappa(\xi) = e^{M_0T}\left[\tilde{q}_{1,\Gamma}(\xi,\varphi_0)+\int_{-T}^{T}\mid\frac{d\tilde{q}_{1,\Gamma}}{ds}(\xi_s,\varphi_0)\mid dt\right]$$

then $\kappa\in L^1(\Omega,m_0)$ and finally

$$\tilde{q}_{1,\Gamma}(\xi,\varphi) = \tilde{q}_{1,\Gamma}(\xi_t,\varphi_0)e^{-\int_0^{-t}\frac{\partial f}{\partial\varphi}(\Phi_s(\xi_t,\varphi_0)ds} =$$

$$\left[\tilde{q}_{1,\Gamma}(\xi,\varphi_0)+\int_0^t\frac{d\tilde{q}_{1,\Gamma}}{ds}(\xi_s,\varphi_0)ds\right]e^{-\int_0^{-t}\frac{\partial f}{\partial\varphi}(\Phi_s(\xi_t,\varphi_0)dt}\leq\kappa(\xi)\ .$$

This proves the first part of the statement. The bound of $\|\kappa\|_1$ comes from inequality (29). Thus the proposition is established.

6 The differentiability of the rotation number

We shall describe the behaviour of the rotation number α as function of the potential g. We recall the equation (1)

$$-x'' + (g_0(t) - E)x = 0 \ : . \tag{30}$$

If $E_0 \in A_0$ then (Σ, Φ^{E_0}) admits an invariant measure equivalent to m whose density function belongs to $L^2(\Sigma, m)$, we may define a fibered map which transforms Φ^{E_0} into a skew-translation and preserves the rotation number. This is a variant of an old idea used by Brunowski [3] and Hermann [15] for families of diffeomorphims of the circle which give us an effective tool to analyze the variation of the rotation number across $g = g_0 - E_0$. In this paragraph we will verify the Lipschitz character of the rotation number on $g_0 - E_0$ and will deduce its differentiability properties.

In order to formulate properly the problem we are going to deal we remember some known facts. If $\mathcal{M}(g_0)$ stands for the frecuency module of g_0 and $A(\mathcal{M})$ for the set of all almost-periodic functions with frequency module contained in $\mathcal{M}$, then it is possible to identify $A(\mathcal{M})$ with $C(\Omega)$. In fact they are isomorphic as Banach algebras.

Now if g' is a second almost-periodic function with $\mathcal{M}(g_0) \subset \mathcal{M}(g')$ then for any sequence $(t_n)_{n=1}^{\infty}$ for which $\{g'(t+t_n)\}_{n\in N}$ converges uniformly to a limit ξ' , also the family $\{g_0(t+t_n)\}_{n\in N}$ converges uniformly ; we denote its limit by ξ . The map $\Pi : \Omega' \to \Omega, \xi' \to \xi$ is a continuous and surjective homomorphism. The Haar measure on Ω' is the image under Π of the Haar measure on $\tilde{\Omega}$. Thus if $p \in L^1(\Omega, m_0)$ then $p \circ \Pi \in L^1(\Omega', m_0')$ and $\int_\Omega p dm_0 = \int_{\Omega'} p \circ \Pi dm_0'$.

The solutions of (30) can be described by the trajectories of the flow defined on $\Omega' \times P^1$ by

$$\dot{\varphi} = \cos^2 \varphi - (J \circ \Pi(\xi_t') - E) \sin^2 \varphi \tag{31}$$

Let pdm be an square integrable invariant measure under Φ on Σ. If we define $p'(\xi', \varphi) = p(\Pi(\xi'), \varphi)$ then $p'dm'$ is an invariant measure under Φ' on Σ'. Moreover $p' \in L^2(\Sigma', dm')$.

Let us consider a second almost periodic potential g. We can find a constant ϵ such that if $g' = g_0 + \epsilon g$ then $\mathcal{M}(g_0) \cup \mathcal{M}(g) \subset \mathcal{M}(g')$. The solutions of the Schrödinger equations with potentials g and $g_0 - E_0$ can be described by flows of type (31) defined on a common compact metric space.

We state the variation of the rotation number under the following conditions:

i) Ω *stands by the hull of an adequate almost periodic function.*
Γ_0, Γ *are continuous functions on* $C(\Omega)$

ii) $\Phi_t^\eta(\xi,\varphi) = (\xi_t, \varphi_t^\eta(\xi,\varphi))$ *denote the flow defined on* Σ *by the differential equation*

$$\dot{\varphi} = \cos^2\varphi - [(1-\eta)\Gamma_0(\xi_t) + \eta\Gamma(\xi_t)]\sin^2\varphi = f_\eta(\xi,\varphi) \ . \tag{32}$$

We assume that for $\eta = 0$ *the flow* (Σ, Φ) *admits invariant measures with square integrable density function.*
We will consider α as a continuous function from $C(\Omega)$ to R and we analyze the variation of the rotation number on Γ_0 in the direction Γ.

Let B' the set of the functions $p \in L^2(\Sigma, m)$ such that
i) $\mu = pdm$ is a normalized invariant measure under Φ.
ii) $p(\xi,\varphi) \in C^\infty(P^1)$ for almost every $\xi \in \Omega$
iii) There exists $\kappa_p \in L^1(\Omega, m_0)$ such that $p(\xi,\varphi) \le \kappa_p(\xi)$ for almost every $\xi \in \Omega$.

We take $p \in B'$. Arguing as in the previous Section we know that there exists $\Omega_0 \subset \Omega$ with $m_0(\Omega_0) = 1$ such that $\int_{P1} p(\xi,\varphi)dm_0 = 1$ for every $\xi \in \Omega_0$. Actually $p(\xi,\varphi)dm_0$ is nothing but the conditional measure induced by μ on $\{\xi\} \times P^1$.

We introduce the transformation given by

$$\begin{array}{lcl} H : \Sigma & \longrightarrow & \Sigma \\ (\xi,\varphi) & \longrightarrow & \left\{ \begin{array}{ll} (\xi, \int_0^\varphi p(\xi,s)ds & \text{if } \xi \in \Omega_0 \\ (\xi,\varphi) & \text{if } \xi \in \Omega - \Omega_0 \end{array} \right. . \end{array}$$

The same formula extends the definition of H to $L = \Omega \times R$. If $\xi \in \Omega_0$ and $\varphi_2 = n\pi + \varphi_1$ then

$$\int_0^{\varphi_2} p(\xi,s)ds = n\pi + \int_0^{\varphi_1} [p(\xi,s) - 1]ds \ .$$

This shows that if $(\xi,\varphi)\in L$ and $(\xi,\psi)=H(\xi,\varphi)$ then $\psi=\varphi+h(\xi,\varphi)$ where the function h is measurable, bounded and periodic in φ .

If $\Psi_t^\eta = H.\Phi_t^\eta.H^{-1}$ and $\psi_t^\eta=\varphi_t^\eta+h(\xi_t,\varphi_t^\eta)$ then

$$\lim_{t\to\infty}\frac{\psi_t^\eta}{t}=\lim_{t\to\infty}\frac{\varphi_t^\eta+h(\xi_t,\varphi_t^\eta)}{t}=\lim_{t\to\infty}\frac{\varphi_t^\eta}{t} \tag{33}$$

when both limits exist. Hence H preserves the rotation number.

Note that H is a bijective map and setting $r(\xi,\psi)=\dfrac{1}{p(\xi,\varphi(\xi,\psi))}$ then

$$\begin{array}{rcl} H^{-1}:\Sigma & \longrightarrow & \Sigma \\ (\xi,\psi) & \longrightarrow & \left\{\begin{array}{ll}(\xi,\int_0^\psi r(\xi,s)ds) & \text{if}\xi\in\Omega_0\\ (\xi,\psi) & \text{if}\xi\in\Omega-\Omega_0\end{array}\right.\end{array}$$

defines its inverse. We search for the transformation of the flow under H.

Proposition 6.1 *If $(\xi,\varphi)\in\Omega_0\times P^1$ and $(\xi_t,\psi_t^\eta(\xi,\psi))=H(\xi_t,\varphi_t^\eta(\xi,\varphi))$ then*

$$\dot{\psi}_t=p(\xi_t,0)+[f_\eta(\xi_t,\varphi_t^\eta)-f_0(\xi_t,\varphi_t^\eta)]\,p(\xi_t,\varphi_t^\eta)=f'_\eta(\xi_t,\psi_t^\eta)\,. \tag{34}$$

Proof: First we consider the case $\eta=0$. The symbol $\varphi_t^\star(\xi)$ stands by the unique solution on L of the implicit equation $\varphi_t(\xi,\varphi_t^\star(\xi))=0$. Taking a derivative with respect to t, we obtain

$$\frac{d}{dt}\varphi_t(\xi,\varphi_t^\star(\xi))=f_0(\xi_t,0)+\frac{\partial\varphi_t}{\partial\varphi}(\xi,\varphi_t^\star(\xi))\frac{d\varphi_t^\star}{dt}=0$$

which yields

$$\frac{d\varphi_t^\star(\xi)}{dt}=-f_0(\xi_t,0)e^{-\int_0^t\frac{\partial\varphi}{\partial\varphi}(\Phi_s(\xi,\varphi_t^\star(\xi))ds}\,.$$

On the other hand

$$\psi_t(\xi,\psi)=\int_0^{\varphi_t(\xi,\varphi)}p(\xi_t,s)ds=\int_{\varphi_t^\star(\xi)}^{\varphi}p(\xi,s)ds\,.$$

Since $p(\xi,\varphi)\in C^\infty(P^1)$ then the above transformation admits the derivative with respect to t and

$$\frac{d\psi_t(\xi,\varphi)}{dt}=f_0(\xi_t,0)p(\xi_t,\varphi_t^\star(\xi))e^{-\int_0^t\frac{\partial\varphi}{\partial\varphi}(\Phi_s(\xi,\varphi_t^\star(\xi))ds}=p(\xi_t,0)\,.$$

The case $\eta \neq 0$ can be verified in an analogous way. Let $\varphi_t^\star(\xi,\varphi)$ be the unique solution on L of the equation $\varphi_t(\xi,\varphi_t^\star(\xi,\varphi)) = \varphi_t^\eta(\xi,\varphi)$. Differentiating with respect to t we obtain

$$\frac{d\varphi_t}{dt}(\xi,\varphi_t^\star(\xi,\varphi)) = f_0(\xi,\varphi_t^\eta(\xi,\varphi)) + \frac{\partial\varphi_t}{\partial\varphi}(\xi,\varphi_t^\star(\xi))\frac{d\varphi_t^\star}{dt} = f_\eta(\xi_t,\varphi_t^\eta(\xi,\varphi))$$

which leads us to

$$\frac{d\varphi_t^\star}{dt} = \left[f_\eta(\xi_t,\varphi_t^\eta(\xi,\varphi)) - f_0(\xi_t,\varphi_t^\eta(\xi,\varphi))\right] e^{-\int_0^t \frac{\partial\varphi}{\partial\varphi}(\Phi_s(\xi,\varphi_t^\star(\xi))ds} .$$

Now we can write

$$\psi_t^\eta(\xi,\psi) = \psi_t^\eta(\xi,\psi) - \psi_t(\xi,\psi) + \psi_t(\xi,\psi)$$

where the last term has already been considered. Then

$$\psi_t^\eta(\xi,\psi) - \psi_t(\xi,\psi) = \int_{\varphi_t(\xi,\varphi)}^{\varphi_t^\eta(\xi,\varphi)} p(\xi_t,s)ds = \int_{\varphi}^{\varphi_t^\star(\xi,\varphi)} p(\xi,s)ds .$$

Taking the derivative with respect to t, we get

$$\frac{d}{dt}\left[\psi_t^\eta(\xi,\psi) - \psi_t(\xi,\psi)\right] = p(\xi_t,\varphi)\frac{d\varphi_t^\star}{dt} =$$

$$\left[f_\eta(\xi_t,\varphi_t^\eta) - f_0(\xi_t,\varphi_t^\eta(\xi,\varphi))\right] p(\xi_t,\varphi_t^\eta(\xi,\varphi))$$

which yields

$$\frac{d}{dt}\psi_t^\eta(\xi,\psi) = p(\xi_t,0) + \left[f_\eta(\xi_t,\varphi_t^\eta) - f_0(\xi_t,\varphi_t^\eta(\xi,\varphi))\right] p(\xi_t,\varphi_t^\eta) .$$

This completes the proof.

The following statement shows the Lipschitz variation of α throught Γ_0.

Proposition 6.2 *Let* $C_{\Gamma_0} = \min\{\|\kappa_p\|_1 / \, p \in B'\}$. *Then*

$$| \, \alpha(\Gamma) - \alpha(\Gamma_0) \, | \leq C_{\Gamma_0}\|\Gamma - \Gamma_0\|_\infty \tag{35}$$

for every $\Gamma \in C(\Omega)$.

Proof: Let $p \in B'$. Using the proposition (6.1) we obtain

$$\alpha(\Gamma_0) = \lim_{t\to\infty} \frac{\psi_t(\xi,\psi)}{t} = \lim_{t\to\infty} \frac{1}{t}\int_0^t p(\xi_s, 0)ds = \int_\Omega p(\xi, 0)dm_0\,. \tag{36}$$

Similarly, we find that

$$\psi_t^\eta(\xi,\psi) - \psi = \int_0^t \{p(\xi_s, 0) + [f_\eta(\xi_s, \varphi_s^\eta) - f_0(\xi_s, \varphi_s^\eta)]p(\xi_s, \varphi_s^\eta)\}ds$$

and in consequence we get

$$\alpha(\Gamma) - \alpha(\Gamma_0) = \int_\Sigma [f_1(\xi,\varphi) - f_0(\xi,\varphi)]p(\xi,\varphi)d\mu_1 \tag{37}$$

for every normalized invariant measure under Φ^1. Then one has

$$\mid \alpha(\Gamma) - \alpha(\Gamma_0) \mid \leq \int_\Sigma \mid \Gamma(\xi) - \Gamma_0(\xi) \mid p(\xi,\varphi)d\mu \leq \|\Gamma - \Gamma_0\| \int_\Sigma \kappa_p(\xi)dm_0$$

and hence

$$\mid \alpha(\Gamma) - \alpha(\Gamma_0) \mid \leq C_{\Gamma_0}\|\Gamma - \Gamma_0\|_\infty\,.$$

Let $\Gamma \in C(\Omega)$ and assume that it exists $p \in B'$ such that $\int_\Sigma p(\xi,\varphi)\Gamma(\xi)d\mu$ takes the same value for every normalized invariant measure. This condition avoids considering in (37) the variation of the invariant measures (μ_η) with respect to η. The theorem (6.1) assures the differentiability of α at Γ_0 in the direction Γ.

That above property suggests the definition of $C_{ad}(\Omega)$. There are functions which satisfy it but do not belong to $C_{ad}(\Omega)$. For instance if (Σ,Φ) is a unique ergodic flow every continuous function satisfy that property but $C_{ad}(\Omega)$ is a dense subset of $C(\Omega)$. However the definition (5.1) propose a practical verification of that condition.

Theorem 6.1 *Let $\Gamma \in C_{ad}(\Omega)$. The rotation number has a derivative at Γ_0 in the direction Γ and*

$$d\alpha(\Gamma_0,\Gamma) = \lambda_\Gamma^2 \int_\Sigma \tilde{q}_{1,\Gamma}^2(\xi,\varphi)\Gamma(\xi)\sin^2\varphi dm = \lambda_\Gamma\,. \tag{38}$$

Proof: We assume that $\mid \Gamma \mid_\infty \leq 1$. The variation of α in the direction Γ can be analyzed in the differential equation

$$\dot{\varphi} = \cos^2 \varphi - (\Gamma_0(\xi_t) + \eta\Gamma(\xi_t)) \sin^2 \varphi \, .$$

We denote $I_\eta = \int_\Sigma \tilde{q}_{1,\Gamma}(\xi,\varphi)\Gamma(\xi) \sin^2 \varphi d\mu_\eta$. Then

$$\frac{\alpha(\Gamma_0 + \eta\Gamma) - \alpha(\Gamma_0)}{\eta} = \lambda_\Gamma I_\eta \, .$$

There exists a function $\kappa \in L^1(\Omega, m_0)$ satisfying $\tilde{q}_{1,\Gamma}(\xi,\varphi) \leq \kappa(\xi)$ for every $(\xi,\varphi) \in \Omega_0 \times P^1$. For all $\epsilon \in [0,1]$ there is $\delta_\epsilon > 0$ such that if D is a measurable subset of Ω with $m_0(D) < \delta_\epsilon$ then $\int_{P^1} \kappa(\xi) dm_0 < \epsilon/4$.

We set $D_T = \{\xi \in \Omega \,/\, \kappa(\xi) < T - 1\}$ and we select a constant $T > 0$ with $m_0(D_T) > 1 - \delta_\epsilon$. By virtue of Egoroff theorem we can aproximate $\kappa_i(\xi)$ by continuous functions in a measurable compact set $C_0 \subset \Omega$ with $C_0 \subset D_T$ and $m_0(C_0) > 1 - \delta_\epsilon$ for $i = 1, 2, 3$. Let $C = C_0 \times P^1$. Repeating the combination (25) we find a function $h(\xi,\varphi)$ continuous on Σ such that $\mid h(\xi,\varphi) - \tilde{q}_{1,\Gamma}(\xi,\varphi) \mid < \epsilon/4$ at every $(\xi,\varphi) \in C$. Thus we have

$$I_\eta = \int_{\Sigma - C} \tilde{q}_{1,\Gamma}(\xi,\varphi)\Gamma(\xi) \sin^2 \varphi d\mu_\eta + \int_C \tilde{q}_{1,\Gamma}(\xi,\varphi)\Gamma(\xi) \sin^2 \varphi d\mu_\eta \leq$$

$$\frac{\epsilon}{2} + \int_C h(\xi,\varphi)\Gamma(\xi) \sin^2 \varphi d\mu_\eta \, .$$

Moreover $h(\xi,\varphi) < T$ for every $(\xi,\varphi) \in C$. Now we consider an open set U_0 such that $C_0 \subset U_0$, $h(\xi,\varphi) < T$ for every $(\xi,\varphi) \in U_0 \times P^1$ and $m_0(U_0 - C_0) < \epsilon/4T$. Let $U = U_0 \times P^1$. If ϱ is a continuous function on Σ with $\chi_C \leq \varrho \leq \chi_U$ then

$$I_\mu \leq \frac{\epsilon}{2} + \int_\Sigma h(\xi,\varphi)\Gamma(\xi)\varrho(\xi,\varphi) \sin^2 \varphi d\mu_\eta \, . \tag{39}$$

We take a sequence $(\eta_n)_{n=1}^\infty$ which tends to 0. We assume that (μ_{η_n}) converges in the weak topology to μ_0. This limit is a normalized invariant measure under Φ. In this way we derive

$$\lim_{n\to\infty} \sup I_{\mu_n} \leq \frac{\epsilon}{2} + \int_\Sigma h(\xi,\varphi)\Gamma(\xi)\varrho(\xi,\varphi) \sin^2 \varphi d\mu_0 \leq$$

$$\frac{\epsilon}{2} + \int_C h(\xi,\varphi)\Gamma(\xi)\varrho(\xi,\varphi) \sin^2 \varphi d\mu_0 + \int_{U-C} h(\xi,\varphi)\Gamma(\xi)\varrho(\xi,\varphi) \sin^2 \varphi d\mu_0 \leq$$

$$\frac{3\epsilon}{4} + \int_C h(\xi,\varphi)\Gamma(\xi)\sin^2\varphi d\mu_0 \leq \epsilon + \int_\Sigma \tilde{q}_{1,\Gamma}(\xi,\varphi)\Gamma(\xi)\sin^2\varphi d\mu_0 \, .$$

Similarly we can show that

$$\lim_{n\to\infty}\inf I_{\mu_n} \geq -\epsilon + \int_\Sigma \tilde{q}_{1,\Gamma}(\xi,\varphi)\Gamma(\xi)\sin^2\varphi d\mu_0 \, .$$

But ϵ is an arbitrary constant and $\int_\Sigma \tilde{q}_{1,\Gamma}(\xi,\varphi)\Gamma(\xi)\sin^2\varphi d\mu_0$ is independent on the invariant measure μ_0 we use. We conclude that

$$\lim_{\mu\to 0} I_\mu = \lambda_\Gamma^2 \int_\Sigma \tilde{q}_{1,\Gamma}(\xi,\varphi)\Gamma(\xi)\sin^2\varphi d\mu = \lambda_\Gamma \, .$$

Note that (27) gives an alternative representation of λ_Γ involving a basis of solutions of the Schrödinger equation.

References

[1] Avron, J. and Simon, B.: *Almost periodic Schrödinger operators II. The integrated density of states.* Duke Math. J., **50** (1983) 369-391.

[2] Bellisard J. and Simon B.: *Cantor spectrum for the almost Mathieu equation.* J. Func. Anal. **48** (1982) 408-419.

[3] Brunovsky, P.: *Generic properties of the rotation number of one-parameter diffeomorphisms of the circle.* Czech. Math. J. **24** (99), (1974) 74-90.

[4] Cameron, R. H.: *Almost periodic properties of bounded solutions of linear differential equations with almost periodic coefficients.* J. Math. Phys. **15** (1936) 73-81.

[5] Cornfeld I. P., Fomin S. V. and Sinai Y.G.: *Ergodic Theory.* Springer (1982).

[6] Chierchia, L.: *Absolutely continuous spectra of quasiperiodic Schrödinger operators.* J. Math. Phys. **28** (1987) 2891-2898.

[7] Chulaevsky V. A.: *Almost periodic operators and related nonlinear integrable systems.* Manchester Univ. Press (1989).

[8] Deift, P. and Simon, B.: *Almost-periodic Schrödinger operators III. The absolutely continuous spectrum in one dimension.* Comm. Math. Phys. **90** (1983) 389-411.

[9] Dinaburg, E. and Sinai, Y.: *The one dimensional Schrödinger equation with a quasiperiodic potential.* Functional Anal. Appl. **9** (1975) 275-290.

[10] Ellis, R.: *Lectures on topological dynamics.* Benjamin. Mass 1969.

[11] Fink, A. M.: *Almost periodic differential equations.* Lecture Notes in Math. Springer **377** (1974).

[12] Furstenberg, H.: *Strict ergodicity and transformations of the torus.* Amer. J. Math. **85** (1963) 477-515.

[13] Furstenberg, H. Keines, H. K. and Saphiro, L.: *Prime flows in topological dynamics.* Israel J. Math. **14** (1973) 26-38.

[14] Gottschalk, W. and Hedlund, G.: *Topological dynamics.* Amer. Math. Soc. Colloquium **36** Providence, R. I. (1955).

[15] Herman, R.: *Mesure de Lebesgue et nombre de rotation* Lecture Notes in Math. Springer **597** (1977).

[16] Johnson, R.: *Minimal functions with unbounded integral.* Israel Journal of Math. **36, 2** (1978) 133-141.

[17] Johnson, R.: *On a Floquet theoy for almost-periodic, two-dimensional linear systems.* J. Differential Equations. **37** (1980) 184-205 .

[18] Johnson, R.: *Cantor spectrum for the quasi-periodic Schrödinger equation.* J. Differential Equations **91** (1991) 88-110.

[19] Johnson, R. and Giachetti, R.:*The Floquet exponent for two-dimensional linear systems with bounded coefficients.* J. Math. Pures Appl. **65** (1986) 93-117.

[20] Johnson, R. and Moser, J.: *The rotation number for almost- periodic potentials.* Comm. Math. Phys. **84** (1982) 403-438.

[21] Kotani, S.: *One dimensional random Schrödinger operators and Herglotz functions.* Preprint. Univ. Kyoto (1986).

[22] Molčanov, S. A.: *The structure of eigenfunctions of one dimensional unordered structures.* Math. U.S.S.R. Izvestija **12-1** (1978) 69-101.

[23] Moser, J.: *An example of a Schrödinger equation with almost periodic potential and nowhere dense spectrum.* Comment. Helv. Math. **56** (1981) 198,224.

[24] Moser, J. and Pöschel, J.: *An extension of a result by Dinaburg and Sinai on quasi-periodic potentials.* Comment. Helv. Math. **59** (1984) 39-85.

[25] Pastur, L.: *Spectral properties of disordered in one-body approximation. Comm. Math. Phys.* **75** (1980) 167-196.

[26] Rüssmann, H.: *On the one-dimensional Schrödinger equation with a quasi-periodic potential.* Annals of the New York Acad. Sci. **357** (1980) 90-107.

[27] Sacker, R. J. and Sell, G. R.: *Lifting properties in skew-product flows with applications to differential equations.* Mem. Amer. Math. Soc. No.**190** (1977).

[28] Sinai, Y. G.: *Anderson localization for one-dimensional difference Schrödinger operator with quasiperiodic potential.* J. Stat. Phys. **46** (1987) 861-909.

[29] Süto, A.: *The spectrum of a quasi-periodic Schrödinger operator.* Comm. Math. Phys. **111** (1987) 409-415.

[30] Xia, J.: *The Floquet theory and the state density of quasiperiodic Schrödinger operators.* J. Differential Equations **86** (1990) 243-259.

OBAYA: DEPARTAMENTO DE MATEMATICA APLICADA A LA INGENIERIA. E.T.S. DE INGENIEROS INDUSTRIALES. UNIVERSIDAD DE VALLADOLID. 47011 VALLADOLID. SPAIN

PARAMIO: DEPARTAMENTO DE MATEMATICAS. FACULTAD DE VETERINARIA. UNIVERSIDAD DE LEON. 24071 LEON. SPAIN