DELOCALIZATION FOR SLOWLY DECAYING POTENTIALS RANDOMLY DEPENDING ON TIME

by

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Abstract

We consider the one-dimensional Schrodinger equation with slowly decaying potential which depends on time in a random manner.

We prove that solutions of this equation are a.s. delocalized as t tends to infinity and give the bound from below for the rate of propagation.

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1. Introduction

Theory of quantum systems with time-dependent Hamiltonians has been actively developed in the last ten years. Time-periodic interactions, in particular, were treated in many papers. A new direction in this field is studying of quantum systems with Hamiltonians depending on time in a random manner. There are here only few rigorous mathematical results.

When considering quantum systems with time-dependent Hamiltonians $H(t)$ (random or not) one can classify them according to the character of spectrum of $H(t)$ for a fixed t. Three situations are possible:

1. $H(t)$ have absolutely continuous and, possibly, discrete spectrum.
2. $H(t)$ have only discrete spectrum.
3. $H(t)$ have dense pure point spectrum.

If H does not depend on time, the spectral properties of H determine the behaviour of solutions of nonstationary equation as $t \to \infty$. For time-dependent Hamiltonians this is in general not true. The behaviour of solutions of the Schrodinger equation

$$i\frac{\partial \Psi}{\partial t} = H(t)\Psi$$

is not directly related with spectral properties of $H(t)$ for a fixed t. However, this classification turns out to be useful.

In the case where $H(t)$ has the form

$$H(t) = -\Delta_x + V(t,x),$$

the case 1 corresponds to $V(t,x)$, sufficiently fast decaying as $|x| \to \infty$; $V(t,x) \to +\infty$ in the case 2, and $V(t,x) \sim$ const or slowly decays as $|x| \to \infty$ in the case 3 (Anderson potentials).

The case 1 for $V(t,x)$ randomly depending on time by some Markov process was studied by Pillet [1,2] and Cheremshantsev [3,4] for different kinds of Markov processes and conditions on V. The main results are asymptotic completeness and absence of bound states for a.e. path of the Markov process.

The problem of second kind for random $H(t)$ was investigated by Guarneri [5], Combescure [6] and some other authors [7,8]. The typical result is a growth of energy as $t \to \infty$.

As to the case 3, it seems that so far there were no rigorous mathematical results. What we suggest below is an attempt to observe for a relatively simple model possible nontrivial effects caused by random dependence on time.

Let us consider one-dimensional Schrodinger equation with potential whose sign randomly depends on time:

$$i\frac{\partial \Psi}{\partial t} = -\frac{d^2}{dx^2}\Psi + q(x)\omega(t)\Psi, \;\; \Psi|_{t=0} = \Psi_0(x), \;\; t \geq 0 \tag{1}$$

Here $\Psi_0 \in L_2(\mathbf{R})$, q is some bounded real function, $\omega : [0, +\infty) \to \{1, -1\}$. Suppose that $\omega(t)$ is some path of Markov process with two states 1 and -1 and transition probabilities

$$p(\tau, i, j) = 1/2(1 - \exp(-2p\tau)), \;\; i \neq j, \tag{2}$$

$$p(\tau, i, i) = 1/2(1 + \exp(-2p\tau)), \tag{3}$$

where $i, j = \pm 1$, $p > 0, \tau > 0$. Let $\omega(0) = 1$ (one can make this assumption without loss of generality) and μ be the corresponding probability measure on the space of paths X. It is known [9] that μ-a.e. ω has only a finite number of jumps on $[0, t]$, so in fact

$$\Psi(t, \omega) = U_{(-1)^n}(t - t_n)U_{(-1)^{n-1}}(t_n - t_{n-1})...U_1(t_1 - t_0)\Psi_0, \tag{4}$$

where $0 = t_0 < t_1(\omega) < t_2(\omega) < ... < t_n(\omega) \leq t$, $\omega(t) = (-1)^{n+1}$ for $t \in [t_{n-1}, t_n)$, $U_1(t)$ and $U_{-1}(t)$ are propagators for operators $-\frac{d^2}{dx^2} + q(x)$ and $-\frac{d^2}{dx^2} - q(x)$ respectively. It is important that these two operators have no common eigenfunctions (one can easily show it). Therefore there are no solutions looking like $\Psi(t, x) = \exp(ih(t))\Psi_0(x)$. The main question is what is the behaviour of $\Psi(t, x)$ given by (4) as $t \to +\infty$.

Definition.(see [10]). If

$$\lim_{R\to+\infty} \sup_{t\geq 0} ||F(|x| \geq R)\Psi(t, x)||_2 = 0, \tag{5}$$

then Ψ_0 is called bound state.

If $\forall R > 0$

$$\lim_{T\to+\infty} \frac{1}{T}\int_0^T dt||F(|x| \leq R)\Psi(t, x)||_2^2 = 0 \tag{6}$$

then Ψ_0 is called scattering or propagating state.

Let $H_b = \{\Psi_0 : \Psi$ satisfies $(5)\}$. For time-independent potentials V with reasonable properties it is known [11] that (5) holds for any $\Psi_0 \in H_{pp}(-\frac{d^2}{dx^2} + V(x))$, and (6) holds for $\Psi_0 \in H_c(-\frac{d^2}{dx^2} + V(x))$ (RAGE-theorem). For time-dependent potentials, however, we cannot be sure that (6) holds for all $\Psi_0 \in H_b^{\perp}$.

What kind of behaviour can we expect for the solutions of (1) given by (4)? If q decays sufficiently fast, then two operators $-\frac{d^2}{dx^2} \pm q(x)$ have only absolutely continuous spectrum on $(0, +\infty)$ and complete wave operators. One can prove by the same method as in [2] or [3] that solutions (4) behave like free ones as $t \to +\infty$ for μ -a.e ω for any $\Psi_0 \in L_2(\mathbf{R})$. In particular, (6) holds for any Ψ_0 and there are no bound states. Here the behaviour of solutions (4) as $t \to +\infty$ is essentially the same (except the absence of bound states) as the behaviour of $U_{\pm 1}(t)\Psi_0$.

More interesting is the situation where q decays slowly. It is known that in this case operators $-\frac{d^2}{dx^2} \pm q(x)$ may have only pure point spectrum on $(0, +\infty)$. An important example is given in the paper of Kotani and Ushiroya [12] where some class of random slowly decaying potentials q is considered. If $q(x)$ decay like $|x|^{-\nu}, \nu \in (0, 1/2)$ as $|x| \to \infty$, then a.s. both operators $-\frac{d^2}{dx^2} \pm q(x)$ have only pure point spectrum dense on $(0, +\infty)$. That means that all the states $U_1(t)\Psi_0$ and $U_{-1}(t)\Psi_0$ are localized (5) as $t \to +\infty$. One can ask whether the solutions (4) are also localized, where $U_1(t)$ and $U_{-1}(t)$ interchange with random t. The answer is negative, solutions (4) being delocalized for μ-a.e. ω for any Ψ_0. Let us formulate now the main results of the paper.

Theorem 3. Assume that for some $r \geq 1$ $q \in L_r(\mathbf{R}) \cap L_\infty(\mathbf{R})$. Set $\beta_0 = \frac{1}{2r}$ for $r \geq 2$ and $\beta_0 = 1/4$ for $1 \leq r \leq 2$. If $0 < \beta < \beta_0$ then for μ-a.e. ω for any $\Psi_0 \in L_2(\mathbf{R})$

$$\lim_{T \to +\infty} \frac{1}{T} \int_0^T dt \|F(|x| \leq t^\beta)\Psi(t, x, \omega)\|_2^2 = 0$$

In particular, (6) holds and there are no bound states.

Theorem 4. If $\nu < 2\beta_0$ then for μ-a.e. ω for any $\Psi_0 \in L_2(\mathbf{R})$, $\Psi_0 \neq 0$

$$\liminf_{T \to +\infty} \frac{1}{T} \int_1^T dt \frac{< x^2(t) >}{t^\nu} > 0$$

where

$$< x^2(t) > \stackrel{\text{def}}{=} \int dx x^2 |\Psi(t, x, \omega)|^2$$

The proof of these results consist of three steps. First of all we get formula for the random dynamics averaged over probability measure. Then we introduce some nonnegative functional Φ on the space of paths X and prove that its expectation is finite. From this fact we get finally the statements of Theorems.

In further consideration $\int dx$ denotes integrals over $(-\infty, +\infty)$, $\|f\|_p$ is the norm in $L_p(\mathbf{R})$, $(f, g)_H$ - the inner product in H (on the contrary, (f, g) is an element of $H \oplus H$). We denote by $\hat{f}$ the Fourier transform of the function f.

2. Averaging of solutions of evolution equation

Let us consider an evolution equation in Hilbert space H :

$$i\frac{d\phi}{dt} = A(\omega(t))\phi, \ \ \phi|_{t=0} = \phi_0, \qquad (7),$$

where ω is a path of Markov process introduced above, $A(1)$ and $A(-1)$ are selfadjoint operators in H. The solution of the equation (7) is given by

$$\phi(t,\omega)=\prod_{j=0}^{k} U_{(-1)^j}(t_{j+1}-t_j))\phi_0 \equiv U(t,\omega)\phi_0, \qquad (8),$$

where $0=t_0<t_1<...<t_{k-1}\le t_k=t$; $\omega(t)=(-1)^j$ for $t\in[t_j,t_{j+1})$, $U_1(t)\equiv \exp(-iA(1)t)$, $U_{-1}(t)\equiv\exp(-iA(-1)t)$. We may consider (8) as a definition of solutions of the equation (7).

Set

$$U_N(t,\omega)\stackrel{\mathrm{def}}{=}\prod_{j=1}^{N-1} U_{\omega(\tau_j)}(t/N), \qquad (9)$$

where $\tau_j=tj/N$. Both groups $U_1(t)$ and $U_{-1}(t)$ being unitary and strongly continuous, it is easy to get from (8) and (9) that

$$U(t,\omega)=s-\lim_{N\to\infty}U_N(t,\omega) \qquad (10)$$

for a.e. ω and any $t>0$.

Consider the complex-valued function

$$F_N(\mathbf{i})\stackrel{\mathrm{def}}{=}F_N(i_1,i_2,...,i_{N-1})=(\prod_{j=1}^{N-1}U_{i_j}(t/N)\phi_0,g)_H,$$

where $i_j=\pm1$, $\phi_0,g\in H$. Obviously,

$$h_N(t,\omega)\equiv(U_N(t,\omega)\phi_0,g)_H=F_N(\omega(\tau_1),\omega(\tau_2),...,\omega(\tau_{N-1}))$$

The functional $h_N(t,\cdot)$ is peacewise constant on probability space. Therefore $h_N(t,\cdot)$ is measurable with respect to probability measure μ and for any measurable set Ω we have

$$J_{N,\Omega}(t)\equiv\int_\Omega d\mu(\omega)h_N(t,\omega)=\sum_{\mathbf{i}}F_N(\mathbf{i})\mu(\Omega\cap I_N(\mathbf{i})) \qquad (11)$$

Here $I_N(\mathbf{i})=\{\omega:(\omega(\tau_1),\omega(\tau_2),...,\omega(\tau_{N-1}))=\mathbf{i}\}$, $I_N(\mathbf{i})\cap I_N(\mathbf{j})=\emptyset$ for $\mathbf{i}\ne\mathbf{j}$ and the summation is carried over all possible $\mathbf{i}\in\{1,-1\}^{N-1}$.

It follows from (10) that $(U(t,\omega)\phi_0,g)_H$ is μ- measurable for any t,ϕ_0,g and

$$\int_\Omega d\mu(\omega)(U(t,\omega)\phi_0,g)_H=\lim_{N\to\infty}J_{N,\Omega}(t) \qquad (12)$$

Define the following two operators in $H\oplus H$:

$$A(\phi_1,\phi_2)\stackrel{\mathrm{def}}{=}(A(1)\phi_1,A(-1)\phi_2),$$

$$M(\phi_1,\phi_2) \stackrel{\text{def}}{=} (-p\phi_1+p\phi_2, -p\phi_2+p\phi_1),$$

where $\phi_1,\phi_2 \in H,\ (\phi_1,\phi_2)\in H\oplus H$. One can easily show that A and M are selfadjoint in $H\oplus H$, M is bounded and Re $M\le 0$.

Theorem 1. Let $\Omega_l=\{\omega:\omega(t)=l\},\ \ l=\pm 1$. Then for any $\phi_0, g\in H$

$$\int_{\Omega_l} d\mu(\omega)(U(t,\omega)\phi_0,g)_H = (P_l \exp(t(-iA+M))\vec{\phi}_0, g)_H \tag{13}$$

where $\vec{\phi}_0=(\phi_0,0)\in H\oplus H,\ \ P_1(f,h)=f,\ \ P_2(f,h)=h.$

Proof. It is essentially the same as in [2] or [3]. Let $p(\tau,i,j)$ be the transition probabilities of considered Markov process given by (2),(3). The measure can be easily calculated:

$$\mu(\Omega_l\cap I_N(\mathbf{i}))=\prod_{j=1}^{N} p(t/N,i_j,i_{j-1}), \tag{14}$$

where $i_0=1,\ \ i_N=l$. It follows from (11) and (14) that

$$J_{N,l}(t)\stackrel{\text{def}}{=}\int_{\Omega_l} d\mu(\omega)h_N(t,\omega)=\Bigg(U_l(t/N)\sum_{i_{N-1}=\pm1} p(t/N,l,i_{N-1})$$

$$U_{i_{N-1}}(t/N)\sum_{i_{N-2}=\pm1} p(t/N,i_{N-1},i_{N-2})...$$

$$\sum_{i_1=\pm1} p(t/N,i_2,i_1)U_{i_1}(t/N)p(t/N,i_1,1)\phi_0,\ \ g\Bigg)_H \tag{15}$$

Let the operator L_N in $H\oplus H$ be given by

$$P_l(L_N\vec{h})=U_l(t/N)(p(t/N,l,1)h_1+p(t/N,l,-1)h_{-1}),$$

where $\vec{h}=(h_1,h_{-1}),\ \ l=\pm1$. It follows from the definition of A and M that

$$L_N=\exp(-i(t/N)A)\exp((t/N)M)$$

Hence one can rewrite (15) as follows:

$$J_{N,l}(t)=(P_l(L_N^N\vec{\phi}_0),\ g)_H, \tag{16}$$

By Trotter formula for dissipative operators, (12) and (16) we get (13). The proof is completed.

Corollary. The average over all ω is given by

$$\int_X d\mu(\omega)(U(t,\omega)\phi_0,g)_H=(\exp(t(-iA+M))\vec{\phi}_0,\vec{g})_{H\oplus H},$$

where $\vec{g} = (g, g)$**.**

In our further consideration we shall estimate the integral

$$I_\delta = \int_X d\mu(\omega) K_\delta(\omega),$$

where

$$K_\delta(\omega) = \int_\delta^1 d\epsilon \epsilon^{-\gamma} \int_0^{+\infty} dt \exp(-2\epsilon t)(U(t,\omega)\phi_0, g(t))_H,$$

$0 < \delta < 1,\ \gamma > 0,\ g$ is some H-valued function.

Lemma 1. Suppose that g **is strongly continuous and**

$$\|g(t)\|_H \leq C \exp(\nu t),\ \ \nu < 2\delta$$

Then $K_\delta(\omega)$ **is** μ**-integrable and**

$$I_\delta = \int_\delta^1 d\epsilon \epsilon^{-\gamma} \int_0^{+\infty} dt \exp(-2\epsilon t)(\exp(t(-iA + M))\vec{\phi}_0, \vec{g}(t))_{H \otimes H},$$

where $\vec{g}(t) = (g(t), g(t))$**.**

The result follows from the statement of Theorem 1, continuity of the function $\epsilon^{-\gamma} \exp(-2\epsilon t)(U(t,\omega)\phi_0, g(t))_H$ with respect to ϵ, t for a.e. ω and the proof of Lemma 1.2 in [3].

3. Finiteness of the main functional.

Define the following nonnegative functional on the paths:

$$\Phi(\omega) = \int_0^1 d\epsilon \epsilon^{-\gamma} \int_0^{+\infty} dt \exp(-2\epsilon t) f(t,\omega), \tag{17}$$

where

$$f(t,\omega) = \int dx F(|x| \leq t^\beta) |\Psi(t,x,\omega)|^2,$$

$\gamma, \beta \in (0,1),\ \ 0 \leq f \leq \|\Psi_0\|_2^2,\ \ \Psi$ is a solution of the equation (1). If the integral in (17) diverges, we set $\Phi = +\infty$. Our aim is to prove that for some set of Ψ_0

$$J \stackrel{\text{def}}{=} \int_X d\mu(\omega) \Phi(\omega) < +\infty$$

To do it, we first consider the following functionals:

$$\Phi_\delta(\omega) \stackrel{\text{def}}{=} \int_\delta^1 d\epsilon \epsilon^{-\gamma} \int_0^{+\infty} dt \exp(-2\epsilon t) f(t,\omega),$$

where $0 < \delta < 1$. It is obvious that

$$\Phi(\omega) = \lim_{\delta \to +0} \Phi_\delta(\omega) \tag{18}$$

for μ-a.e. ω. Let us introduce the function

$$N(t,x,y,\omega) \stackrel{\text{def}}{=} \Psi(t,x,\omega) \cdot \overline{\Psi(t,y,\omega)}$$

According to Lemma 1.1 of [3] we can write the equality

$$f(t,\omega) = \lim_{n\to\infty} (N(t,\omega), R_n(t))_{L_2(\mathbf{R}^2)} \equiv \lim_{n\to\infty} f_n(t,\omega), \tag{19}$$

where $R_n(t,x,y) = F(|x| \le t^\beta) l_n(x-y)$, $l_n \in C_0^\infty(\mathbf{R}), l_n(x) \ge 0, \int l_n(x)dx = 1$ and

$$\lim_{n\to\infty} \int dx F(|x| \ge \nu) l_n(x) = 0$$

for any $\nu > 0$.One can see [3] that

$$|f_n(t,\omega)| \le C \tag{20}$$

uniformly on n, t, ω. From (19)-(20) for any $\delta > 0, \omega \in X$

$$\begin{aligned}\Phi_\delta(\omega) = \lim_{n\to\infty} \int_\delta^1 d\epsilon \epsilon^{-\gamma} \int_0^{+\infty} dt \exp(-2\epsilon t) f_n(t,\omega) \\ \equiv \lim_{n\to\infty} \Phi_{\delta,n}(\omega) \end{aligned} \tag{21}$$

by dominated convergence theorem.

Consider two operators in $L_2(\mathbf{R}^2)$:

$$A(j) = -\partial_{xx} + \partial_{yy} + (q(x) - q(y))j, \;\; j = \pm 1$$

It is not hard to show (see [3]) that these operators are selfadjoint on the domain

$$D = \{f : f \in L_2(\mathbf{R}^2), \int ds dk |\hat{f}(s,k)|^2 (|k^2 - s^2|^2 + 1) < +\infty\}$$

for bounded q. One can see also that N satisfies to the equation

$$i\frac{\partial N}{\partial t} = A(\omega(t))N, \;\; N|_{t=0} = \Psi_0(x) \cdot \overline{\Psi_0(y)}$$

and all the conditions of Theorem 1 and Lemma 1 are satisfied. Hence, functionals $\Phi_{\delta,n}$ are μ-integrable for all δ, n and

$$J(\delta,n) \stackrel{\text{def}}{=} \int_X d\mu(\omega) \Phi_{\delta,n}(\omega) =$$

$$\int_\delta^1 d\epsilon\epsilon^{-\gamma}\int_0^{+\infty} dt\exp(-2\epsilon t)(\exp(t(-iA+M))\vec{\phi}_0,\vec{R}_n(t))_{H\oplus H}, \qquad (22)$$

where $\vec{\phi}_0=(\Psi_0(x)\cdot\overline{\Psi_0(y)},\ 0)$, $\vec{R}_n(t)=(R_n(t),R_n(t))$, $H=L_2(\mathbf{R}^2)$. From (20)-(22) by dominated convergence theorem we get for all $\delta\in(0,1)$

$$J(\delta)\stackrel{\text{def}}{=}\int_X d\mu(\omega)\Phi_\delta(\omega)=\lim_{n\to\infty}J(\delta,n) \qquad (23)$$

Let us estimate the function

$$Q_n(\epsilon)\stackrel{\text{def}}{=}\int_0^{+\infty} dt\exp(-2\epsilon t)(\exp(t(-iA+M))\vec{\phi}_0,\vec{R}_n(t))_{H\oplus H}, \qquad (24)$$

By Plancherel Theorem

$$Q_n(\epsilon)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}(\vec{L}_1(\lambda,\epsilon),\vec{L}_2(\lambda,\epsilon))_{H\oplus H}, \qquad (25)$$

where

$$\vec{L}_1(\lambda,\epsilon)=\int_0^{+\infty} dt\exp(-\epsilon t-it\lambda)(\exp(t(-iA+M))\vec{\phi}_0=$$

$$-i(A+iM+\lambda-i\epsilon)^{-1}\vec{\phi}_0,\ \ \vec{L}_2(\lambda,\epsilon)=(L_2(\lambda,\epsilon),L_2(\lambda,\epsilon)),$$

$$L_2(x,y,\lambda,\epsilon)=l_n(x-y)\int_0^{+\infty} dt\exp(-\epsilon t-it\lambda)F(|x|\le t^\beta)=$$

$$l_n(x-y)(\epsilon+i\lambda)^{-1}\exp(-i|x|^{1/\beta}\lambda-\epsilon|x|^{1/\beta}) \qquad (26)$$

It follows from (26) and $\int dx l_n(x)=1$ that

$$\hat{L}_2(s,k,\lambda,\epsilon)\stackrel{\text{def}}{=}\int dxdy\exp(-isx-iky)L_2(x,y,\lambda,\epsilon)$$

can be estimated as

$$|\hat{L}_2(s,k,\lambda,\epsilon)|\le|\epsilon+i\lambda|^{-1}\int dxdy l_n(x-y)\exp(-\epsilon|x|^{1/\beta})$$

$$\le C(\beta)\epsilon^{-\beta}|\epsilon+i\lambda|^{-1} \qquad (27)$$

uniformly in s,k,λ,ϵ, where $C(\beta)=\int dx\exp(-|x|^{1/\beta})$.

Consider two functions:

$$\eta(\lambda,\epsilon)=(P_1+P_{-1})(A+iM+\lambda-i\epsilon)^{-1}\vec{\phi}_0,$$

$$\theta(\lambda,\epsilon)=(P_1-P_{-1})(A+iM+\lambda-i\epsilon)^{-1}\vec{\phi}_0,$$

$\eta(\cdot,\cdot,\lambda,\epsilon),\ \theta(\cdot,\cdot,\lambda,\epsilon)\in L_2(\mathbf{R}^2)$. It follows from (25)-(27) and Plancherel Theorem that

$$|Q_n(\epsilon)|\le C(\beta)\epsilon^{-\beta}\int_{-\infty}^{+\infty}d\lambda|\epsilon+i\lambda|^{-1}\int\int dsdk|\hat{\eta}(s,k,\lambda,\epsilon)|, \tag{28}$$

where η is taken in Fourier represantation. One can see from the definition of A and M that η,θ satisfy in space represantation the following system of equations:

$$(-\partial_{xx}+\partial_{yy}+\lambda-i\epsilon)\eta+(q(x)-q(y))\theta=\Psi_0(x)\cdot\overline{\Psi_0(y)} \tag{29}$$

$$(-\partial_{xx}+\partial_{yy}+\lambda-i\epsilon-2ip)\theta+(q(x)-q(y))\eta=\Psi_0(x)\cdot\overline{\Psi_0(y)} \tag{30}$$

The following Lemma plays very important role in further consideration. Let $\lambda\in\mathbf{R},\ \epsilon\in(0,1),\ p>0$.

Lemma 2. The following uniform estimate holds

$$\|\theta(\cdot,\cdot,\lambda,\epsilon)\|_{L_2(\mathbf{R}^2)}\le C(\epsilon p)^{-1/2}\|\Psi_0\|_2^2$$

Proof. One can rewrite (29)-(30) as

$$(B_\lambda-i\epsilon+iM)\vec{\phi}=\vec{\phi}_0, \tag{31}$$

where $\vec{\phi}=(\eta,\theta),\ \ \vec{\phi}_0=(\Psi_0(x)\cdot\overline{\Psi_0(y)},\ \ \Psi_0(x)\cdot\overline{\Psi_0(y)})$,

$$B_\lambda\vec{\phi}=((-\partial_{xx}+\partial_{yy}+\lambda)\eta+(q(x)-q(y))\theta,\ \ (-\partial_{xx}+\partial_{yy}+\lambda)\theta+(q(x)-q(y))\eta),$$

$M\vec{\phi}=(0,\ -2p\theta)$. Both operators B_λ and M are selfadjoint in $L_2(\mathbf{R}^2)\oplus L_2(\mathbf{R}^2)$. From (31) we have

$$\|\vec{\phi}_0\|^2=((B_\lambda-i\epsilon+iM)\vec{\phi},(B_\lambda-i\epsilon+iM)\vec{\phi})_{H\oplus H}=$$
$$\|(B_\lambda+iM)\vec{\phi}\|^2_{H\oplus H}+\epsilon^2\|\vec{\phi}\|^2_{H\oplus H}-2\epsilon(M\vec{\phi},\vec{\phi})_{H\oplus H}\ge$$
$$-2\epsilon(M\vec{\phi},\vec{\phi})_{H\oplus H}=4p\epsilon\|\theta\|^2_H.$$

The proof is completed.

We shall need also the following result of technical caracter.

Lemma 3. Let $1<\alpha<2,\ \ 0<\epsilon<1,\ \ \lambda\in\mathbf{R}$. **Consider the integral**

$$I(s,\lambda,\epsilon)=\int_{-\infty}^{+\infty}\frac{dk}{|s^2-k^2-\lambda|^\alpha+\epsilon^\alpha}$$

The estimate holds

$$\int ds(I(s,\lambda,\epsilon))^{2/\alpha}\le C(\alpha)\epsilon^{2/\alpha-2}|\lambda|^{1/2-1/\alpha}$$

uniform in λ,ϵ.

Proof. From $|k^2 - s^2 + \lambda|^\alpha \geq |k^2 - |s^2 - \lambda||^\alpha$ after the change of variable $t = k|s^2 - \lambda|^{-1/2}$ we have

$$I(s, \lambda, \epsilon) \leq C|s^2 - \lambda|^{1/2-\alpha} \int_0^{+\infty} \frac{dt}{|t^2 - 1|^\alpha + \nu^\alpha}\,, \tag{32}$$

where $\nu = \epsilon|s^2 - \lambda|^{-1}$. Consider two cases.

1. $\nu \geq 1$. From

$$\sup_{\nu \geq 1} \sup_{t \geq 0} \frac{|t^2 - 1|^\alpha + \nu^\alpha}{|t - 1|^{2\alpha} + \nu^\alpha} < +\infty$$

we obtain

$$I(s, \lambda, \epsilon) \leq C|s^2 - \lambda|^{1/2-\alpha} \int_{-\infty}^{+\infty} d\tau(|\tau|^{2\alpha} + \nu^\alpha)^{-1} \leq$$

$$C|s^2 - \lambda|^{1/2-\alpha}\nu^{1/2-\alpha} = C\epsilon^{1/2-\alpha} \tag{33}$$

2. $\nu \leq 1$. Then

$$T(\nu) \stackrel{\text{def}}{=} \int_0^{+\infty} dt(|t^2 - 1|^\alpha + \nu^\alpha)^{-1} = \int_0^{+\infty} dt(|t^2 - 1|^\alpha + \nu^\alpha)^{-1} \cdot$$

$$(F(|t - 1| \geq 1/2) + (F(|t - 1| \leq 1/2)) \equiv T_1(\nu) + T_2(\nu)$$

Further,

$$T_1(\nu) \leq \int_0^{+\infty} dtF(|t - 1| \geq 1/2)|t^2 - 1|^{-\alpha} = C < +\infty, \tag{34}$$

since $\alpha > 1$. Finally,

$$T_2(\nu) \leq C \int_{1/2}^{3/2} dt(|t^2 - 1|^\alpha + \nu^\alpha)^{-1} \leq C \int_{-1/2}^{1/2} d\tau(|\tau|^\alpha + \nu^\alpha)^{-1} \leq C\nu^{1-\alpha}, \tag{35}$$

since $1 < \alpha < 2$. From (32),(34)-(35) we have for $\nu \leq 1$

$$I(s, \lambda, \epsilon) \leq C\epsilon^{1-\alpha}|s^2 - \lambda|^{-1/2} \tag{36}$$

Now using (33) and (36) estimate the integral

$$L \stackrel{\text{def}}{=} \int ds(I(s, \lambda, \epsilon))^{2/\alpha} \leq C \int ds(F(|s^2 - \lambda| \leq \epsilon)\epsilon^{2/\alpha(1/2-\alpha)} +$$

$$F(|s^2 - \lambda| \geq \epsilon)(\epsilon^{1-\alpha}|s^2 - \lambda|^{-1/2})^{2/\alpha}) \equiv L_1 + L_2$$

From $|s^2 - |\lambda|| \leq |s^2 - \lambda|$ we have

$$L_1 \leq C \int dsF(|s^2 - |\lambda|| \leq \epsilon)\epsilon^{1/\alpha-2}$$

If $|\lambda| \le \epsilon$ then

$$L_1 \le C\epsilon^{1/2}\epsilon^{1/\alpha-2} \le C\epsilon^{2/\alpha-2}|\lambda|^{1/2-1/\alpha} \tag{37}$$

because $1/2 - 1/\alpha < 0$.

If $|\lambda| \ge \epsilon$ then

$$L_1 \le C\epsilon^{1/\alpha-2}((|\lambda|+\epsilon)^{1/2} - (|\lambda|-\epsilon)^{1/2}) \le$$
$$C\epsilon^{1/\alpha-1}(|\lambda|+\epsilon)^{-1/2} \le C\epsilon^{2/\alpha-2}|\lambda|^{1/2-1/\alpha} \tag{38}$$

Finally, for all λ

$$L_2 \le C\epsilon^{2/\alpha-2}\int ds|s^2-\lambda|^{-1/\alpha} \le$$
$$C\epsilon^{2/\alpha-2}\int ds|s^2-|\lambda||^{-1/\alpha} \le C\epsilon^{2/\alpha-2}|\lambda|^{1/2-1/\alpha} \tag{39}$$

From (37)-(39) follows the statement of Lemma.

Now we are able to estimate $\|\hat\eta(\cdot,\cdot,\lambda,\epsilon)\|_{L_1(\mathbf{R}^2)}$. It follows from (29) that

$$\hat\eta(s,k,\lambda,\epsilon) = (s^2-k^2+\lambda-i\epsilon)^{-1}(\hat\Psi_0(s)\cdot\overline{\hat\Psi_0(-k)} - U_1 + U_2) \stackrel{\text{def}}{=} \hat\eta_0 - \hat\eta_1 + \hat\eta_2 \tag{40}$$

where

$$U_1(s,k,\lambda,\epsilon) = \int dx\exp(-isx)q(x)\theta_1(x,k,\lambda,\epsilon),$$

$$U_2(s,k,\lambda,\epsilon) = \int dy\exp(-iky)q(y)\theta_2(s,y,\lambda,\epsilon),$$

$$\theta_1(x,k) = \int dy\exp(-iky)\theta(x,y),\ \ \theta_2(s,y) = \int dx\exp(-isx)\theta(x,y).$$

Set $\alpha = (1/r+1/2)^{-1}$ for $r>2$ and $\alpha = 1+\nu, \nu\in(0,1/3)$ for $r\in[1,2]$, where $q\in L_r(\mathbf{R})\cap L_\infty(\mathbf{R})$. Suppose that $\hat\Psi_0\in L_2(\mathbf{R})\cap L_\mu(\mathbf{R}),\ \ \alpha^{-1}+\mu^{-1}=1$. By Lemma 3 and Holder inequality

$$\|\hat\eta_0\|_{L_1(\mathbf{R}^2)} \le C\|\hat\Psi_0\|_\mu\|\hat\Psi_0\|_2\,\epsilon^{1/\alpha-1}|\lambda|^{1/4-1/2\alpha} \tag{41}$$

and

$$\|\hat\eta_2\|_{L_1(\mathbf{R}^2)} \le CI(s,\lambda,\epsilon)^{1/\alpha}\|U_2(s,\cdot,\lambda,\epsilon)\|_\mu, \tag{42}$$

where $I(s,\lambda,\epsilon)$ is integral introduced in Lemma 3. Further, by Hausdorff-Young inequality and Holder inequality

$$\|U_2(s,\cdot,\lambda,\epsilon)\|_\mu \le C\|q(\cdot)\theta_2(s,\cdot,\lambda,\epsilon)\|_\alpha \le C\|q\|_\gamma\|\theta_2(s,\cdot,\lambda,\epsilon)\|_2, \tag{43}$$

where $\gamma^{-1} = -1/2+\alpha^{-1}, \gamma\in(2,+\infty)$ One can easily see that $\|q\|_\gamma < +\infty$. On the basis of Lemma 3, Lemma 2, Plancherel Theorem and (42)-(43) one can write the estimate

$$\|\hat\eta_2\|_{L_1(\mathbf{R}^2)} \le C\epsilon^{1/\alpha-1}|\lambda|^{1/4-1/(2\alpha)}\|\theta_2(\cdot,\cdot,\lambda,\epsilon)\|_{L_2(\mathbf{R}^2)} \le C\epsilon^{1/\alpha-3/2}|\lambda|^{1/4-1/(2\alpha)} \tag{44}$$

The norm $\|\hat{\eta}_1\|$ can be estimated as (44) in the same manner. From (41) and (44) we obtain the following estimate for $Q_n(\epsilon)$:

$$|Q_n(\epsilon)| \leq C(\beta)\epsilon^{-\beta+1/\alpha-3/2}\int d\lambda(|\lambda|+\epsilon)^{-1}|\lambda|^{1/4-1/(2\alpha)} =$$

$$C_1\epsilon^{-\beta+1/(2\alpha)-5/4}, \tag{45}$$

because $-1 < 1/4 - 1/(2\alpha) < 0$. This inequality is uniform in n, ϵ.

Substituting (45) in (22) we get the estimate for $J(\delta, n)$:

$$|J(\delta, n)| \leq C\int_\delta^1 d\epsilon\epsilon^{-\gamma-\beta+1/(2\alpha)-5/4} \tag{46}$$

Let $\gamma + \beta < 1/(2\alpha) - 1/4$. It follows from (46) that

$$|J(\delta, n)| \leq C < +\infty \tag{47}$$

uniformly on $\delta \in (0,1)$, $n \in \mathbf{N}$. From (23) and (47) we have

$$|J(\delta)| \leq C < +\infty \tag{48}$$

uniformly in δ. Functionals $\Phi_\delta(\omega)$ being nonnegative, by Fatou Lemma we get from (18) and (48):

$$J = \int_X d\mu(\omega)\Phi(\omega) < +\infty \tag{49}$$

4. Proof of Theorems.

Now we can prove the results formulated in the introduction. First of all, one can easily see from the definition of $\Phi(\omega)$ that

$$\Phi(\omega) \geq C\int_1^{+\infty} dt t^{\gamma-1}\int dx F(|x| \leq t^\beta)|\Psi(t,x,\omega)|^2 \equiv \Xi(\omega), \tag{50}$$

where $C > 0$. From (49) and (50) we obtain the following statement.

Theorem 2. Let $q \in L_r(\mathbf{R}) \cap L_\infty(\mathbf{R}), r \geq 1$. Set $\alpha = (r^{-1} + 1/2)^{-1}$ for $r > 2$ and $1 < \alpha < 4/3$ for $r \in [1,2]$. If $\gamma, \beta \in (0,1)$, $0 < \gamma + \beta < 1/(2\alpha) - 1/4$ and $\Psi_0 \in L_2(\mathbf{R}) \cap L_\alpha(\mathbf{R})$, then

$$\int_X d\mu(\omega)\Xi(\omega) < +\infty$$

Corollary 1. For any Ψ_0 satisfying condition of the Theorem, for a.e. ω $\Xi(\omega) < +\infty$.

Let $\beta_0 = \frac{1}{2r}$ for $r \geq 2$ and $\beta_0 = 1/4$ for $r \leq 2$.

Corollary 2. If $0<\beta<\beta_0$, then for any $\Psi_0 \in L_1(\mathbf{R}) \cap L_2(\mathbf{R})$ for a.e. ω

$$\lim_{T\to+\infty} \frac{1}{T}\int_0^T dt \int dx F(|x| \leq t^{\beta})|\Psi(t,x,\omega)|^2 = 0 \tag{51}$$

The result follows from the estimate

$$\frac{1}{T}\int_0^T dt f(t) \leq \frac{1}{T}\left(\int_0^T dt t^{\gamma-1} f^2(t)\right)^{1/2}\left(\int_0^T dt t^{1-\gamma}\right)^{1/2} \leq$$

$$\frac{C}{T}T^{1-\gamma/2}\left(\int_0^T dt t^{\gamma-1} f(t)\right)^{1/2} \to 0$$

as $T \to +\infty$, where $0 \leq f(t) \leq \|\Psi_0\|^2$.

Theorem 3. There exists a set of full measure $\Omega \subset X$ such that (51) holds for any $\omega \in \Omega$ for all $\Psi_0 \in L_2(\mathbf{R})$.

Proof. It follows from Corollary 2 that there exists a dense in $L_2(\mathbf{R})$ countable set M such that (51) holds for $\Psi_{0,n} \in M$ if $\omega \in X\backslash W_n,\ \mu(W_n)=0,\ n=1,2,...$ Define $W=\cup W_n,\ \mu(W)=0$. Then (51) holds for any $\Psi_0 \in M$ and any $\omega \in X\backslash W$. From the estimate

$$\frac{1}{T}\int_0^T dt \int dx F(|x| \leq t^{\beta})|\Psi(t,x,\omega)|^2 \leq \|\Psi_0\|_2^2$$

we get (51) for any $\Psi_0 \in L_2(\mathbf{R}),\ \omega \in X\backslash W$. The proof is completed.

Lemma 4. If $0<\nu<1/\alpha-1/2$ then for any $\Psi_0 \in L_2(\mathbf{R}) \cap L_\alpha(\mathbf{R}),\ \Psi_0 \neq 0$,

$$\int_1^{+\infty} dt \frac{< x^2(t) >}{t^{1+\nu}} = +\infty$$

for a.e. ω.

Proof. From the definition of $< x^2(t) >$ we have

$$< x^2(t) >= \int dx x^2|\Psi(t,x,\omega)|^2 \geq t^{2\beta}\int dx F(|x| \geq t^{\beta})|\Psi(t,x,\omega)|^2 =$$

$$t^{2\beta}(\|\Psi_0\|_2^2 - \int dx F(|x| \leq t^{\beta})|\Psi(t,x,\omega)|^2) \tag{52}$$

By Corollary 1

$$\int_1^T dt < x^2(t) > t^{\gamma-1-2\beta} \geq \|\Psi_0\|_2^2 \int_1^T dt t^{\gamma-1} -$$

$$\int_1^T dt t^{\gamma-1}\int dx F(|x| \leq t^{\beta}|\Psi(t,x,\omega)|^2 \geq C_1 T^{\gamma} - C_2 \to +\infty$$

as $T \to +\infty$. Taking γ sufficiently small, we get the statement of the Lemma.

Theorem 4. **If** $0 < \nu < 2\beta_0$ **then for a.e.** ω **for any** $\Psi_0 \in L_2(\mathbf{R})$, $\Psi_0 \neq 0$,

$$\liminf_{T\to+\infty} \frac{1}{T}\int_1^T dt < x^2(t) > t^{-\nu} > 0 \tag{53}$$

Proof. From (52) and Theorem 3

$$\frac{1}{T}\int_1^T dt < x^2(t) > t^{-2\beta} \geq \frac{T-1}{T}\|\Psi_0\|_2^2 - \frac{1}{T}\int_1^T dt \int dx F(|x| \leq t^\beta)|\Psi(t,x,\omega)|^2 \to \|\Psi_0\|_2^2$$

as $T \to +\infty$ that proves (53).

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REFERENCIES

1. C.-A. Pillet, Commun. Math. Phys. **102** (1985) 237.
2. C.-A. Pillet, Commun. Math. Phys. **105** (1985) 259.
3. S.E. Cheremshantsev, Mat. Sbornik **137(179)**, No 4 (1988), 526. English transl. in Math. USSR Sbornik **65** No 2 (1990) 531
4. S.E. Cheremshantsev, Mat. Zametki **46** No 4 (1989) 86. English transl. in Math. Notes **46** No 3-4 (1989) 811.
5. I. Guarneri, Lett. Nuovo Cimento **40** (1984) 171.
6. M. Combescure, J. Stat. Phys. **62** (1991), 779.
7. G.A. Hagedorn, M. Loss, J. Slawny, J. Phys. **A19** (1986) 521.
8. L. Bunimovich, H.R. Jauslin, J.J. Lebowitz, A. Pellegrinotti, P. Nielaba, J. Stat. Phys. **62** (1991) 793.
9. E.B. Dynkin.: Markov Processes. Springer-Verlag, Berlin - Gottingen - Heidelberg, 1965.
10. V. Enss, K. Veselic, Ann. Inst. Henri Poincare **A39** (1983) 159.
11. M. Reed, B. Simon.: Methods of modern Mathematical Physics, v.3. Academic Press, New York, 1979.
12. S. Kotani, N. Ushiroya, Commun. Math. Phys. **115** (1988) 247.