# The Geometric Phase in Quantum Physics

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Berry's phase has been fashionable in many areas of physics and in chemistry and also among mathematicians and mathematical physicists. The mathematical people are attracted to this area because it is related to the beautiful mathematics of fibre bundles which underlay gauge theories. In fact this is the most accessible example of a gauge theory for people who know just the elementary facts of nonrelativistic quantum mechanics.

The chemists and physicists are interested because the geometric or Berry phase has observable consequences, which could not be explained before. It is this aspect which distinguishes the Berry phase from the many other fashions of mathematical physics. Though the Berry phase may turn out not to be as important as its present popularity suggests, it is a discovery which will remain forever. The surprising thing about it is that its importance has been realized 60 years too late. The reason for this was that many people (including myself) thought that phases are unimportant in quantum mechanics, because a quantum mechanical state is not described by a vector $\psi$ but by a ray or a projection operator $\mid \psi >< \psi \mid$ and that phase factors could always be removed by a suitable phase or gauge transformation.

That this is not always possible and under which conditions this is not possible was shown by Berry in his famous 1984 paper.[1)] And the Berry phase fashion started when Simon[2)] explained that Berry's phase is the holonomy (element of the holonomy group) for a fibre bundle with a particular connection, the adiabatic connection.

But the physical effect of the Berry phase had been known for quite some time. It was observed as some anomalies in the spectra of molecules[3)] and then, (1978), explained in a series of remarkable papers by C.A. Mead and Truhlar[4)] by the introduction of a gauge potential, which is identical with the one derived by Berry and which is now called Berry connection.[5)] This gauge potential emerges naturally from the Born-Oppenheimer procedure in molecular physics[6)] if one does not make the drastic Born-Oppenheimer approximation.[7)]

The Born-Oppenheimer method is concerned with the study of complicated molecules by dividing them into two parts: the electronic motion described by a set of "fast" variables, and the collective motion described by a set of "slow" variables. Berry connection and Berry phase, therefore, arise in the dissection of complicated quantum physical systems into simpler subsystems. This reduction to the simpler is the basic meaning of understanding in science. In the old, drastic approximation the dissection results in the trivial direct product of the states for the two subsystems. In the less-drastic, adiabatic approximation or in the exact theory, the motion of one subsystem (the "fast" subsystem) alters the dynamics of the "slow" subsystem by inducing in it a Berry gauge potential. Thus, the parts of a complicated quantum physical system turn out to be different from what we naively expected. Our discussions here will be mainly concerned with these aspects of the Berry connection.

The Hamiltonian for a (diatomic) molecule is,

$$H = \frac{\mathbf{P}^2}{2\mu} + \frac{\hat{\mathbf{p}}^2}{2m} + V(\mathbf{X}, \hat{\mathbf{r}}) \tag{1}$$

where $\hat{\mathbf{p}}$, $\hat{\mathbf{r}}$ stand for the observables of the fast electrons and $\mathbf{P}$, $\mathbf{X}$ stand for the observables of the slow nuclei. Since the light electrons instantaneously follow the motion of the heavy nuclei, the slow variables can also be understood as being the variables of the molecule as a whole, i.e. the collective variables. In particular, for the diatomic molecule $\mathbf{X}$ will be the vector along the internuclear axis and $\mathbf{P}$ its conjugate momentum (in addition there are the center of mass position and momenta which are, as always in a non-relativistic theory, ignored). The potential $V(\mathbf{X}, \hat{\mathbf{r}})$ is a complicated function of the operators $\mathbf{X}$,$\hat{\mathbf{r}}$ and possibly some other operators like spin.

The Hamiltonian $H$ of (1) is split into two parts,

$$H = \frac{\mathbf{P}^2}{2\mu} + h(\mathbf{X}) \tag{2}$$

$$h(\mathbf{X}) = \frac{\hat{\mathbf{p}}^2}{2m} + V(\mathbf{X}, \hat{\mathbf{r}}) \tag{3}$$

where $h(\mathbf{X})$ denotes the "fast" or electronic Hamiltonian that depends upon the "slow" operator $\mathbf{X}$. The eigenvalue problem,

$$H \mid \psi^E\rangle = E \mid \psi^E\rangle \tag{4}$$

is solved in the Born-Oppenheimer procedure by first solving the eigenvalue problem for the operator $h(\mathbf{X})$. In the drastic Born-Oppenheimer approximation $\mathbf{X}$ is considered as a classical parameter $\mathbf{x}$ which is fixed. With $\mathbf{X} = \mathbf{x} = fixed$, $h(\mathbf{X})$ commutes with $\mathbf{P}$ and thus $h(\mathbf{X})$ and $H$ can be diagonalized together (which amounts to ignoring the effect of the kinetic energy of the slow variables in (1)):

$$\mid N, n; \mathbf{x}\,\rangle = \mid N\rangle \otimes \mid n(\mathbf{x})\,\rangle \tag{5}$$

$$h(\mathbf{x}) \mid n(\mathbf{x})\rangle = \varepsilon_n(x) \mid n(\mathbf{x})\,\rangle \tag{6}$$

$$H \mid N, n; \mathbf{x}\rangle = \Big(\frac{\mathbf{P}^2}{2\mu} + \varepsilon_n(x)\Big) \mid N, n; \mathbf{x}\rangle = E_{N,n} \mid N, n; \mathbf{x}\rangle \tag{7}$$

One first solves (6) for every value of the *fixed* parameter $\mathbf{x}$, and obtains the electronic energy values $\varepsilon_n(x_e)$ where $x_e$ is the minimum (equilibrium) value of the "potential curve" $\varepsilon_n(x)$. The eigenvectors $\{| n(\mathbf{x})\rangle \mid n = 1, 2, \ldots\}$ for every value of $\mathbf{x}$ form a complete system of basis vectors for the space of physical states $\mathcal{H}^{fast}$ for the fast subsystem. After $\varepsilon_n(x)$ has been obtained from (6) for every value of the fixed parameter $\mathbf{x}$, one inserts it into the right-hand side of (7) as an "induced scalar potential" and solves (7) for a given value of the electronic quantum number $n$. As $\varepsilon_n(x)$ is (often) approximately an oscillator potential, $\varepsilon_n(x_e)$ splits into vibrational excitations with quantum number $\nu$. And as the diatomic molecule (dumbbell) also rotates about its center of mass, each vibrational excitation splits into rotational bands with quantum number $j$. The collective quantum numbers $N$ are thus the vibrational quantum number $\nu$ and the angular momentum $j : N = \nu, j$; and one obtains the typical spectrum of molecules, Fig. 1.

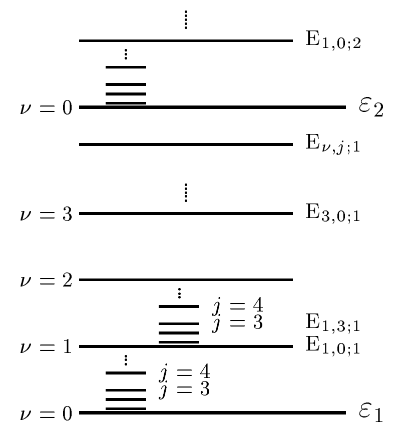

Fig.1 Schematics of typical molecular spectra

The time evolution of the fast system is described by the Schrödinger equation

$$i\hbar \frac{d \mid \psi(t)\rangle}{dt} = h(X) \mid \psi(t)\rangle \tag{8}$$

and if initially the state vector is an electronic energy eigenstate,

$$\psi(0) = \mid n(\mathbf{x})\rangle, \tag{9}$$

then the solution of (8) is

$$\psi(t) = e^{-\frac{i}{\hbar}\varepsilon_n(x)t} \mid n(\mathbf{x})\rangle = e^{-\frac{i}{\hbar}\int_0^t dt' \varepsilon_n(x(t'))} \mid n(\mathbf{x})\rangle \tag{10}$$

iff $\mathbf{x}$="fixed" parameter.

We will now consider the less drastic, adiabatic approximation.[8)] If the internuclear distance and direction $\mathbf{X}$ is considered a classical parameter $\mathbf{x}(t)$ which changes slowly in time (fast quantum system in a slowly changing classical environment) then an initial eigenstate of $h(\mathbf{X}(t))$ ; $\psi(0) = \mid n(\mathbf{x}(0))\rangle$ may "jump" into a state which also has different electronic quantum numbers $n' \neq n$. The adiabatic approximation is an evolution in which $\mathbf{x}(t)$ changes so slowly that an eigenstate of $h(\mathbf{X}(t))$ always remains in the same eigenstate:[9)]

$$\mid \psi(t)\rangle\langle\psi(t) \mid = \mid n(\mathbf{x}(t))\rangle\langle n(\mathbf{x}(t)) \mid \tag{11}$$

The solution of (8) for an initial eigenstate (9) with time-dependent $\mathbf{x}(t)$ is then given by

$$\psi(t) = e^{-\frac{i}{\hbar}\int_0^t dt' \varepsilon_n(x(t'))} e^{i\gamma_n(t)} \mid n(\mathbf{x}(t))\rangle \tag{12}$$

In addition to the dynamical phase factor of (10), there appears another phase factor $e^{i\gamma_n(t)}$. This phase factor had always been omitted in the old adiabatic approximation because it was believed that it can always be absorbed into the eigenvector $\mid n(\mathbf{x}) >$ by a phase (gauge) transformation:

$$\mid n(\mathbf{x}) >\to\mid n(\mathbf{x}) >'= e^{i\zeta_n(\mathbf{x})} \mid n(\mathbf{x}) > \tag{13}$$

where $\mid n(\mathbf{x}) >'$ is again a normalized eigenvector in (6).

For cyclic time evolution

$$\mathcal{C} : \mathbf{x}(0) \to \mathbf{x}(t) \to \mathbf{x}(T) = \mathbf{x}(0) \ , \tag{14}$$

when the internuclear axis returns to its original position after a period $T$, the solution of the Schrödinger equation (8) for the vector of the state $\mid \psi(T) >< (\psi(T) \mid=$
$\mid n(\mathbf{x}(T)) >< n(\mathbf{x}(T)) \mid$ is

$$\psi(T) \ = \ e^{-\frac{i}{\hbar}\int_0^T dt'\varepsilon_n(\mathbf{x}(t'))} e^{i\gamma_n(T)} \mid n(\mathbf{x}(0))\rangle \tag{15}$$

where

$$\gamma_n(T) \ = \ \int_{\mathcal{C}} d\mathbf{x}\mathbf{A}_n(\mathbf{x}) = \int_S \int d\mathbf{S}\cdot\mathbf{B}_n \tag{16}$$

with

$$\mathbf{A}_n \ \equiv \ i\langle n(\mathbf{x}) \mid \nabla_x \mid n(\mathbf{x})\rangle \quad ; \quad \mathbf{B}_n = \nabla_x \wedge \mathbf{A}_n \tag{17}$$

and where $\mathcal{C}$ is the closed path in the parameter space (14) and $S$ is a surface spanned by $\mathcal{C}$. It is convenient and a standard convention to choose eigenvectors which are single valued functions of the parameter $\mathbf{x}$ in the region that contains $\mathcal{C}$:

$$\mid n(\mathbf{x}(T))\rangle =\mid n(\mathbf{x}(0)\rangle \tag{18}$$

Under this convention $\mathbf{A}_n(\mathbf{x})$ is called Berry connection or Berry gauge potential, $\mathbf{B}_n$ is called Berry curvature and $\gamma_n(T)$ is called the Berry phase angle.

It can be shown that $e^{i\gamma_n(T)}$ is an invariant with respect to the transformation (13) (gauge invariant) which may be different from unity.[1)] It can therefore not be transformed away by (13).

An example of a system where the Berry phase $\gamma_n(T)$ and Berry connection are non-trivial (*i.e.* not removable by a gauge transformation) is the quantum magnetic moment $\mathbf{m} = -\frac{e}{2mc}g\mathbf{j}$ (of the electrons with spin $\mathbf{j}$) in a slowly rotating magnetic field $\mathbf{B}^{mag} = B\hat{\mathbf{X}}(t)$ (along the internuclear axis of the molecule caused by the rapidly orbiting electrons). The fast Hamiltonian for this case is

$$h(t) = h(\mathbf{X}(t)) = -\mathbf{m}\cdot\mathbf{B}^{mag}(t) = -\frac{e}{2mc}gB\hat{\mathbf{X}}(t)\cdot\mathbf{j} = b\hat{\mathbf{X}}(t)\cdot\mathbf{j} \tag{19}$$

The eigenvectors $\mid n(\mathbf{x})\rangle =\mid k(x,\theta(t),\varphi(t))\rangle$ depend upon the polar coordinates ($\theta(t)$, $\varphi(t)$) of the unit vector $\hat{\mathbf{X}}(t)$, and $k$ denotes the component of angular momentum of the fast system along the internuclear axis

$$\mathbf{x}(t)\cdot\mathbf{j} \mid k(\theta,\varphi)\rangle = k \mid k(\theta,\varphi)\rangle \quad ; \quad \varepsilon_k = \hbar bk \tag{20}$$

By a straightforward calculation using

$$| k(\theta,\varphi)\rangle = e^{-i\varphi j_3} e^{-i\theta j_2} e^{i\varphi j_3} \mid k(\theta=0,\varphi=0)\rangle \qquad \text{for} \qquad \theta<\pi \tag{21}$$

and

$$| k(\theta,\varphi)\rangle' = e^{-i\varphi j_3} e^{-i\theta j_2} e^{-i\varphi j_3} \mid k(\theta=0,\varphi=0)\rangle \qquad \text{for} \qquad \theta>0 \tag{22}$$

one obtains for the Berry connection

$$\mathbf{A}^{k'k} = i\langle k^{'}(\theta,\varphi) \mid \nabla \mid k(\theta,\varphi)\rangle = \mathbf{e}_x \hat{A}_x^{k'k} + \mathbf{e}_\theta \hat{A}_\theta^{k'k} + \mathbf{e}_e \hat{A}_\varphi^{k'k} : \tag{23}$$

the following results

$$\mathbf{A}_x = 0 \qquad A_\theta^{k'k} = 0\ ,\ A_\varphi^{k'k} = -\frac{k(1-\cos\theta)}{x\sin\theta} \qquad \theta<\pi \tag{24}$$

$$A_\varphi^{\prime k'k} = \frac{k(1+\cos\theta)}{x\sin\theta} \qquad \theta>0 \tag{24'}$$

The two vectors $\mid k(\theta,\varphi)\rangle$ and $\mid k(\theta,\varphi)\rangle'$ (and the corresponding two vector potentials $\mathbf{A}$ and $\mathbf{A}'$) had to be chosen differently for the domain $(\theta<\pi)$ and the domain $(\theta>0)$ so that they are single-valued in each domain.

The Berry curvature (17) calculated from (24) and (24′) is

$$\mathbf{B}^k = -\frac{k}{x^2}\hat{\mathbf{X}} \tag{25}$$

And the Berry phase (16) calculated from (24) (or (24′)) is given by the standard result

$$\gamma_k(\mathcal{C}) = -k(\text{solid angle subtended by } \mathcal{C}) = -k\Omega. \tag{26}$$

The result (25) is identical with the field strength of Dirac's magnetic monopoles[10] $e\mathbf{B} = \frac{eg}{4\pi}\frac{\hat{\mathbf{X}}}{X^2}$, except that the electromagnetic constant $\frac{eg}{4\pi}$ is replaced by the motion constant $k$, the component of angular momentum along the internuclear axis, which in this approximation is a fixed number. ¿From this result we already suspect that something like a magnetic monopole must be part of the diatomic molecule (except if $k=0$). This motional or mechanical "monopole" will remain uncovered if one uses the drastic Born-Oppenheimer approximation.[11]

If the eigenvalues of the fast Hamiltonian $\varepsilon_n(x_e)$ are degenerate or close to each other (compared with the splitting between $E_{N',n}$ and $E_{N'',n}$) then the adiabatic approximation (11) is apparently not good and one may consider in place of the $U(1)$ gauge transformation (13) an $U(\mathcal{N})$ gauge transformation (where $\mathcal{N}$ is the degeneracy of $\varepsilon_n$) and obtain a non-abelian Berry connection[12] $\mathbf{A}^{mn}(\mathbf{x})$ in place of the abelian $\mathbf{A}_n(\mathbf{x}) = \mathbf{A}^n(\mathbf{x})$ of (17). Alternatively one can treat the slow variables $\mathbf{P}$ and $\mathbf{X}$ as operators and solve the problem quantum mechanically by the Born-Oppenheimer method.[13] Then these non-abelian Berry connections will emerge naturally. This we will discuss now:

The space of physical states is, according to the basic principles of quantum mechanics, the direct product of the space for the fast motion $\mathcal{H}^{fast}$ and the space for the slow motion $\mathcal{H}^{slow}$: $\mathcal{H} = \mathcal{H}^{slow} \otimes \mathcal{H}^{fast}$. The slow variables $\mathbf{P}$ and $\mathbf{X}$ are the operators $\mathbf{P} = \mathbf{P} \otimes 1^{fast}$ and $\mathbf{X} = \mathbf{X} \otimes 1^{fast}$ and the operator $h(\mathbf{X}) = h(\hat{\mathbf{p}}, \hat{\mathbf{r}}; \mathbf{X})$ acts in both factors of $\mathcal{H}$. As basis of $\mathcal{H}^{fast}$ one takes the $\mid \mathbf{n}(\mathbf{x})\rangle$, as basis of $\mathcal{H}^{slow}$ one takes a basis of generalized eigenvectors of $\mathbf{X} :\mid \mathbf{x} \ldots\rangle$ and the basis system of $\mathcal{H}$ is something like the direct product basis $\mid \mathbf{x} \ldots n\rangle = \mid \mathbf{x} \ldots\rangle \tilde{\otimes} \mid n(\mathbf{x})\rangle$.

A straightforward[6,13] but lengthy calculation shows that the effective Hamiltonian $H^{eff}$ for the slow motion in $\mathcal{H}^{slow}$ is not given as in the drastic approximation by (7) but by

$$H^{eff} = \frac{1}{2\mu}\mathbf{\Pi}^2 + \varepsilon_n(X) \tag{27}$$

where

$$\mathbf{\Pi}^{mn}(\mathbf{x}) = \mathbf{P}\delta^{mn} - \mathbf{A}^{mn}(\mathbf{X}) \tag{28}$$

and where

$$\mathbf{A}^{mn}(\mathbf{x}) = i\langle m(\mathbf{x}) \mid \nabla \mid n(\mathbf{x})\rangle = (\mathbf{A}^{mn})^\dagger \tag{29}$$

$H^{eff}$, $\mathbf{\Pi}$ and $\mathbf{A}$ are $\mathcal{N} \times \mathcal{N}$ matrices and $\mathbf{\Pi}^2$ in (27) means matrix multiplication $\Sigma_m \mathbf{\Pi}^{n'm} \cdot \mathbf{\Pi}^{mn}$. Thus the fast motion induces in the dynamics of the slow motion not only a scalar potential $\varepsilon_n(X)$ as in the drastic approximation (7) but also a non-abelian vector potential $\mathbf{A}^{mn}(\mathbf{X})$ and in place of the canonical momentum $\mathbf{P}$ of (7) one has the gauge-covariant momentum (28).

If one makes no approximation then $m, n$ in (29) range over all values of the electronic quantum numbers ($\mathcal{N} = \infty$) and the Berry connection $\mathbf{A}^{mn}$ is an infinite matrix. But this $\mathbf{A}^{mn}$ is trivial (can be gauged away by a $U(\infty)$ gauge transformation). The eigenvalue equation of $H^{eff}$ is then an infinite system of coupled differential equations, which is useless for practical calculations. One obtains a workable eigenvalue equation only if one can restrict oneself to a small number $\mathcal{N}$ of eigenvalues $\varepsilon_n$ and eigenvectors $\mid n(\mathbf{x})\rangle$ $n = 1, 2, \ldots \mathcal{N}$ (Born-Huang approximation).[14] $\mathbf{A}^{mn}(\mathbf{x})$ of (29) is then a connection of a $U(\mathcal{N})$ gauge theory, which is in general non-trivial.

As a special case we consider the doubly degenerate $\Lambda$-levels of a diatomic molecule for which $\mathcal{N} = 2$ and $m, n$ takes the two values $k = \pm\Lambda$ where $k$ is again the component of angular momentum along the internuclear axis and the $\mid \mathbf{n}(\mathbf{x})\rangle$ in (29) are given by (21) (or (22)).

The space of physical states of the slow system is

$$\mathcal{H}^{slow} = \mathcal{H}^{k=1} \oplus \mathcal{H}^{k=-1} \tag{30}$$

and (27), (28), (29) are $2 \times 2$ operator matrices:

$$H^{k'k} = \frac{1}{2\mu} \sum_{k''=-1}^{+1} \mathbf{\Pi}^{k'k''} \cdot \mathbf{\Pi}^{k''k} + \varepsilon(x) \tag{31}$$

$$\mathbf{\Pi}^{k'k} = \delta^{k'k}\mathbf{P} - \mathbf{A}^{k'k}. \tag{32}$$

By a straightforward calculation one obtains as in (24) for the spherical components of the Berry connection $\mathbf{A}^{k'k}$:

$$A_\theta^{k'k} = \frac{\Lambda}{x\sin\theta}\begin{pmatrix} -(1-\cos\theta) & 0 \\ 0 & (1-\cos\theta) \end{pmatrix} \quad ; \quad A_\varphi^{k'k} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} . \tag{33}$$

The rapidly orbiting and spinning motion of the electrons about the internuclear axis of a diatomic molecule thus leads to an induced vector potential in the dynamics of the slow (collective) motion, which is the same as that of a pair of magnetic monopoles with the monopole strength $g$ given by $\frac{eg}{4\pi} = \pm\Lambda$. The components of the gauge covariant momentum operator do no more commute but fulfill the commutation relation

$$\left[\Pi_i^{k'k''} \ , \ \Pi_j^{k''k}\right] = -i\varepsilon_{ij\ell}\frac{X_\ell}{X^3}\Lambda\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = iB_{ij}^{k'k} \ , \tag{34}$$

where $B_{ij}^{k'k} = \varepsilon_{ij\ell}B_\ell^{k'k}$, is the Berry curvature (17) for the $\mathbf{A}^{k'k}$ given by (33).

The commutation relations (34) are the well known *c.r.* for a charge-monopole system and the first term in the Hamiltonian (31) is the monopole Hamiltonian with free radial motion, $H_{monopole} = \frac{1}{2\mu}\mathbf{\Pi}^2$. Due to the induced scalar potential, which is approximately a radial oscillator potential $\varepsilon(x) \approx \frac{f}{2}(x-x_e)^2$, the radial motion of (31) is not free. To dissect the system described by (31) into a radial part and an angular part, we use the radial momentum operator[15)]

$$P_{rad} = \frac{1}{2}\left\{\frac{X_i}{X} \ , P_i\right\} \tag{35}$$

and the angular momentum operator of a monopole:[16)]

$$J_i = \varepsilon_{ij\ell}X_j\Pi_\ell - X_i\frac{1}{2}\varepsilon_{mn\ell}X_m B_{n\ell} = \varepsilon_{ij\ell}X_j\Pi_\ell + k\frac{X_i}{X} \ . \tag{36}$$

Then we obtain after a straightforward calculation for $H$ of (31):

$$H = \frac{1}{2\mu}\vec{\Pi}^2 + \varepsilon(X) = \underbrace{\frac{1}{2\mu X^2}(\mathbf{J}^2 - k^2)}_{} + \underbrace{\frac{1}{2\mu}P_{rad}^2 + \varepsilon(X)}_{}$$

or

$$H = H_{monopole} + \varepsilon(X) = \quad H_{rot} \quad + \quad H_{radial\ oscillator} \ . \tag{37}$$

$H_{rot}$ is the Hamiltonian of a rotating dumbbell with flywheel on its axis[17)] whose doubly degenerate rotator spectrum $\frac{1}{2\mu x_e^2}j(j+1)$ starts at $j = k$. Thus the spectrum is again something like shown in Fig. 1 except that the rotational levels do not start at $j = 0$ but, as observed for diatomic molecules, at $j = k$ (because from (36) follows that $\frac{1}{X}X_iJ_i = k$). Therewith we have obtained the standard result in a way, which shows that it is caused by the monopole dynamics induced by the fast motion in the slow collective motion.

Although we used the diatomic molecule as an example, the same arguments should hold for all kinds of quantum systems in which the fast subsystem is a rapid rotation about a slowly moving axis like a spinning quark about the axis of a thin flux tube.

Fifty years after Dirac conceived the idea of magnetic monopoles he wrote (in a letter to Abdus Salam): "I am inclined now to believe that monopoles do not exist." Though magnetic monopoles of the electromagnetic kind may not exist, physical systems with the same dynamics as that of monopoles do exist. These monopoles are "parts" of complicated physical systems, like the "part" which performs the collective motion of molecules or the collective motion of a flux tube.

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[9] Note that (8) and (11) are two conditions which are in general not compatible. This means that there does not exist a solution of (8) with the initial condition $\psi(0) =\mid n(\mathbf{x}(0)) >$ such that (11) is fulfilled. However there exists a solution of (8) with the initial condition $\psi(0) =\mid n(\tilde{\mathbf{x}}(0))\rangle =\mid n(\tilde{\mathbf{x}}(T))\rangle$ such that $\mid \psi(t) >< \psi(t) \mid=\mid n(\mathbf{x}(t) >< n(\mathbf{x}(t)) \mid$. The section $\mid n(\tilde{\mathbf{x}}(t)) >$ is not an eigenvector of $h(t)$ but of some other observables. With these sections one can define a geometric phase and a connection which describe a realistic time evolution and which go in the adiabatic limit into the Berry phase and the adiabatic connection. Y. Aharonov and J. Anandan, *Phys. Rev. Lett.* **58**, 1593 (1987); J. Anandan and Y. Aharonov, *Phys. Rev. D*38, 1863 (1988).

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[17] Section V.4 of reference 15 or G. Herzberg, "Molecular Spectra and Molecular Structure," Vol. I, D. Van Nostrand Publishers, NY (1955).