Free energy of a four-dimensional chiral bag *

M. De Francia, H. Falomir,
and E. M. Santangelo

Departamento de Física
Fac. de Ciencias Exactas, U. N. L. P.
c. c. 67, 1900 La Plata, Argentina.

June 1991

Abstract

Through the application of mathematical methods developed in a recent paper, we obtain a closed expression for the chiral correction to the free energy of a bag, due to the presence of the external pionic field in a hedgehog configuration. Low and high temperature developments are given and, in particular, the zero temperature limit is shown to be consistent with previous evaluations of the Casimir energy. The subject of renormalization is also discussed.

Pacs: 12.40.Aa, 02.90.+p

* *This work was partially supported by CONICET (Argentina).*

1 Introduction

It is by now a well-established fact that, when field configurations are subject to boundary conditions, their quantum fluctuations are modified in a fundamental way. Starting from the well-known work of Casimir [1], there has been ever growing evidence of the relevance of boundary effects on the vacuum structure of confined systems. One of the best illustrations of this fact is provided by bag models of hadrons, both in the early MIT version [2] and in the more recent chiral version [3], which provides the original formulation with the ingredient of axial current conservation.

In a recent paper [4], a simple two-dimensional chiral bag was studied, as an application of methods developed by R. Forman [5] for the evaluation of Fredholm determinants of quotients of differential operators in terms of Fredholm determinants of quotients of pseudodifferential operators. These last (Forman's) operators turn out to be entirely determined by boundary values of functions in the kernel of the original differential one, which is independent of boundary conditions. This approach proved comparatively simpler than a previous one [6], based on Seeley's developments for complex powers of elliptic boundary systems [7], which requires the knowledge of the Green's function.

However, the conditions under which Forman's approach can be applied turn out to be too restrictive when trying to use it for the more realistic case of a four dimensional chiral bag. For this reason, a generalization of the results in reference [5] has recently been presented [8], which applies to the case where these Fredholm determinants do not exist, but a regularized determinant (p-determinant) based on Hilbert's approach does.

It is the aim of this paper to apply the regularization approach provided by the results in reference [8] to evaluate the one-loop contribution to the free energy of a chiral bag, due to the presence of a hedgehog configuration of the external pionic field. Even though this approach requires the introduction of two different boundary conditions, depending on an ad-hoc parameter μ, an adequate choice of all these elements will allow us to get complete information about a unique boundary condition for the bag.

In section 2, we give a brief review of the mathematical tool developed in reference [8].

In section 3, we describe the model to be studied and discuss the definition of (the chiral correction to) the renormalized free energy in the framework of this regularization scheme.

Section 4 is devoted to the explicit evaluation of Forman's operator for the problem at hand. There it is shown that a judicious choice of a basis in the kernel of the differential operator, suggested by the symmetries of the problem, allows to reduce Forman's operator to a block-diagonal form, which greatly simplifies subsequent calculations.

In section 5, we give closed expressions for the chiral correction to the free energy at any temperature. Making use of asymptotic expansions for Bessel functions, we also get developments for its behavior at high and low temperatures. In particular, for the zero-temperature limit, our results turn out to be consistent with previous evaluations of the Casimir energy [9].

Finally, section 6 contains some comments and conclusions.

2 Brief review of the mathematical tool

In this section, we will briefly review the main result in a recent paper [8]. There, it was shown that - under certain conditions - a regularized determinant of the quotient of elliptic operators defined in a region $\Omega \subset \mathbf{R}^N$ with boundary $\partial\Omega$ is equal to the determinant of an operator acting on functions defined on $\partial\Omega$. The last one can be entirely expressed in terms of boundary values of the solutions of the original elliptic operator, as will become clear in what follows.

This approach allows one to define the (difference of) renormalized free energies in terms of such boundary values.

We will be interested in elliptic linear differential operators $L(\mu)$, depending on a parameter μ and restricted to functions defined in a region $\Omega \subset \mathbf{R}^n$ and satisfying elliptic boundary conditions corresponding to the projection $Bf = 0$ at $\partial\Omega$. We will suppose that the homogeneous problem:

$$Lf(x) = 0, \ x \in \Omega$$

$$Bf(x) = 0, \ x \in \partial\Omega \tag{1}$$

has no nontrivial solution. The Poisson map for the operator L and the boundary condition B, $P_B(\mu)$, is defined such that the unique solution of

$$Lf(x) = 0, \ x \in \Omega$$

$$Bf(x) = h(x),\ \ x \in \partial\Omega \tag{2}$$

is given by

$$f(x) = P_B(\mu)h(x). \tag{3}$$

We will denote by $L_B^{-1}(\mu)$, the Green's function of L, satisfying the boundary condition $BL_B^{-1}(\mu) = 0$ at $\partial\Omega$.

Let us consider a second elliptic boundary condition, defined by the operator A, and its corresponding Green's function $L_A^{-1}(\mu)$ $(AL_A^{-1}(\mu) = 0$ at $\partial\Omega)$.

Following reference [5], we will introduce the operator:

$$\Phi_{AB}(\mu) = AP_B(\mu). \tag{4}$$

Notice that, even though $P_B(\mu)$ is difficult to evaluate (it requires the knowledge of the Green's function $L_B^{-1}(\mu)$), $\Phi_{AB}(\mu)$ is such that:

$$A\Psi = \Phi_{AB}(\mu)B\Psi \tag{5}$$

for any solution of the homogeneous equation:

$$L\Psi = 0. \tag{6}$$

So, $\Phi_{AB}(\mu)$ can - in principle - be obtained from the boundary values of the functions in a basis of the kernel of $L(\mu)$.

In reference [8] it has been shown that:

$$det_p\left(L_B^{-1}(\mu)L_A(\mu)L_A^{-1}(0)L_B(0)\right) = det_p\left(\Phi_{AB}^{-1}(\mu=0)\Phi_{AB}(\mu)\right), \tag{7}$$

as long these p-determinants, to be defined in the following, exist.

If M is an operator such that M^p $(p \geq 1)$ is trace class, the p-determinant of $(1-M)$ is defined through [10]:

$$\ln det_p(1-M) = -\int_0^1 dz z^{p-1} Tr\left\{M^p\left(1-zM\right)^{-1}\right\}, \tag{8}$$

Notice that, if M itself is trace class, this definition reduces to:

$$\ln det_p(1-M) = \ln det(1-M) + \sum_{n=1}^{p-1}\frac{Tr(M^n)}{n} \tag{9}$$

So, one can see that this regularization amounts to the subtraction of the first (p-1) (divergent) traces in the development of $\ln det(1-M)$.

Let us consider for a while that the (p-1)-determinants of the operators in equation (7) also exist. Then, from the following property of p-determinants[11]:

$$det_p\,(1-M) = \exp\left(\frac{1}{p-1}TrM^{p-1}\right) det_{p-1}(1-M), \tag{10}$$

and equation (7), we conclude that

$$Tr\left\{\left(1-L_B^{-1}(\mu)L_A(\mu)L_A^{-1}(0)L_B(0)\right)^{p-1}\right\} =$$

$$Tr\left\{\left(1-\Phi_{AB}^{-1}(\mu=0)\Phi_{AB}(\mu)\right)^{p-1}\right\} \tag{11}$$

When the (p-1)-determinant does not exist, equation (11) can be considered to be a formal identity, which can be extended to all integer powers from 1 to (p-1). In such a way, the (divergent) subtractions in the present regularization scheme (and, therefore, the finite counterterms required) can be expressed entirely in terms of the operator $\Phi_{AB}(\mu)$.

We will be interested in a situation in which boundary conditions, rather than operators, depend on μ through a regular transformation $\mathcal{U}(\mu)$ such that the relevant boundary values of a solution of L, f, are given by $A\mathcal{U}^{-1}(\mu)f$ ($B\mathcal{U}^{-1}(\mu)f$). Again, we will suppose that the boundary value problem is nonsingular.

We can translate the μ-dependence to the differential operator by noticing that:

$$\left(\begin{array}{c} L\Psi = 0 \\ A\mathcal{U}^{-1}\Psi = h \end{array}\right) \Rightarrow \left(\begin{array}{c} \mathcal{U}^{-1}L\mathcal{U}\chi = 0 \\ A\chi = h \end{array}\right), for\ \chi = \mathcal{U}^{-1}\Psi, \tag{12}$$

and, therefore, apply our previous result to $L_A(\mu) \equiv \left(\mathcal{U}^{-1}L\mathcal{U}\right)_A$.

Since

$$\left(\mathcal{U}^{-1}L\mathcal{U}\right)_A^{-1} = \mathcal{U}^{-1}L_{A\mathcal{U}^{-1}}^{-1}\mathcal{U}, \tag{13}$$

equation (7) implies:

$$det_p\left\{\left(\mathcal{U}^{-1}L\mathcal{U}\right)_B^{-1}\left(\mathcal{U}^{-1}L\mathcal{U}\right)_A L_A^{-1}L_B\right\} =$$

$$det_p \left\{ \mathcal{U}^{-1} L^{-1}_{B\mathcal{U}^{-1}} L_{A\mathcal{U}^{-1}} \mathcal{U} L_A^{-1} L_B \right\} =$$
$$det_p \left\{ \Phi^{-1}_{AB}(\mu = 0) \Phi_{AB}(\mu) \right\}, \tag{14}$$
where
$$\Phi_{AB}(\mu) B \mathcal{U}^{-1} \Psi = A \mathcal{U}^{-1} \Psi, \tag{15}$$
for any solution of $L\Psi = 0$.

Finally, we wish to stress that, the boundary-value problem of equation (2) being nonsingular, the projected values of a complete system of functions in the kernel of L constitute a complete system in the space of boundary values.

We will use this fact to explicitly evaluate $\Phi_{AB}(\mu)$ in following sections.

3 Definition of the free energy of a chiral bag

In what follows, we will consider the free energy for a four-dimensional static chiral bag at temperature $T = 1/\beta$. This amounts to consider a theory of free massless fermions confined to a spherical cavity of fixed radius R and interacting at the boundary with a hedgehog configuration of an external pionic field.

The corresponding partition function is given by:
$$Z = \int \mathcal{D}\bar{\psi} \mathcal{D}\psi e^{-S_E}, \tag{16}$$
where the functional integral is performed over functions satisfying antiperiodic boundary conditions in the "time" direction, and the Euclidean action is:
$$S_E = \int_\Omega d^4x \bar{\psi} i \not{\partial} \psi - \frac{1}{2} \int_{r=R} d^3x \bar{\psi} (e^{-i\theta \vec{\tau}.\mathbf{n}\gamma_5} + i \not{n}) \psi, \tag{17}$$
where Ω is the region corresponding to $r \leq R$, and $t \in [0, \beta]$. Here, $\mathbf{n}$ is the outward normal to the bag surface and τ^i $(i = 1, 2, 3)$ are the Pauli matrices[1].

---

[1]Our convention for gamma matrices is:
$$\gamma_0 = i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}; \gamma_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
where σ^i are the Pauli matrices.

This means that

$$Z = e^{-\beta F} \sim Det(i\not{\partial})_A \tag{18}$$

where $(i\not{\partial})_A$ acts as the differential Dirac operator $i\not{\partial}$ on functions satisfying the chiral boundary condition A

$$A\psi = \frac{1}{2}(1 + i\not{n}e^{-i\theta\vec{\tau}.\mathbf{n}\gamma_5})\psi = 0 \ at \ r = R, \tag{19}$$

and being antiperiodic in the temperature interval $[0, \beta]$.

Equation (18) is just a formal relation, since the functional determinant must be defined through a suitable regularization. To do so, our aim is to employ the scheme developed in reference [8] and reviewed in the previous section.

In order to apply equation (14), we introduce a second boundary operator:

$$B = \frac{1}{2}(1 + i\not{n}) = A(\theta = 0) \tag{20}$$

and a regular transformation which, at the boundary, takes the value:

$$\mathcal{U}(\mu) = e^{-i\mu\vec{\tau}.\mathbf{n}\gamma_5}. \tag{21}$$

Thus,

$$det_p\left(\mathcal{U}^{-1}(i\not{\partial})^{-1}_{B\mathcal{U}^{-1}}(i\not{\partial})_{A\mathcal{U}^{-1}}\mathcal{U}(i\not{\partial})^{-1}_{A}(i\not{\partial})_B\right) = det_p\left(\Phi^{-1}_{AB}(\mu = 0)\Phi_{AB}(\mu)\right), \tag{22}$$

where $\Phi_{AB}(\mu)$ satisfies equation (15) for any solution of:

$$(i\not{\partial})\psi = 0. \tag{23}$$

Notice that

$$A(\theta)\mathcal{U}^{-1}(\mu)\psi = 0 \Rightarrow A(\theta - 2\mu)\psi = 0. \tag{24}$$

Then, the spectrum of the Dirac Hamiltonian for this modified boundary condition will depend on $(\theta - 2\mu)$ and the same will be true for the corresponding free energy $F = F(\theta - 2\mu)$. Consequently, we will adopt the following definition:

$$\beta\left\{F(-2\mu) - F(\theta - 2\mu) + F(\theta) - F(0)\right\} \equiv \ln det_p\left(\Phi^{-1}_{AB}(\mu = 0)\Phi_{AB}(\mu)\right) -$$

$$\left[\sum_{q=1}^{p-1} \frac{1}{q} \, Tr \left\{ [1 - \Phi_{AB}^{-1}(\mu = 0)\Phi_{AB}(\mu)]^q \right\} \right]_{renormalized} \tag{25}$$

where the least integer p for which the p-determinant exists will be determined later in this paper. In this expression we have used the formal identity established in equation (11).

Now, the parameter μ is arbitrary and can be chosen so that one gets the chiral correction to the free energy. Indeed, for $\mu = \theta/2$ and taking into account that $F(-\theta) = F(\theta)$ (since multiplication by $\gamma_0\gamma_5$ changes θ into $-\theta$ in the boundary condition and E_n into $-E_n$ in the Dirac equation, which does not affect the fermionic partition function), one obtains:

$$2\beta[F(\theta) - F(0)] = lndet_p \left(\Phi_{AB}^{-1}(\mu = 0)\Phi_{AB}(\theta/2) \right) -$$

$$\left[\sum_{q=1}^{p-1} \frac{1}{q} Tr \left\{ \left(1 - \Phi_{AB}^{-1}(\mu = 0)\Phi_{AB}(\theta/2) \right)^q \right\} \right]_{renormalized} \tag{26}$$

In the next section we will explicitly evaluate the operator $\Phi_{AB}(\mu)$ and in section 5 an expression for the chiral free energy will be given.

4 The operator $\Phi_{AB}(\mu)$

As remarked in Section 2, $\Phi_{AB}(\mu)$ can be obtained from the boundary values of a complete system of functions in the kernel of $i \not{\partial}$. This task is made easier by first studying the symmetries of the homogeneous Dirac equation and boundary operators [12, 13].

While the homogeneous differential equation (23) is invariant under $SU(2)_{rot.} \otimes SU(2)_{isospin}$, the boundary operator $A(\theta)\mathcal{U}^{-1}(\mu)$ is invariant only under the diagonal $SU(2)$ subgroup corresponding to simultaneous rotation in space and isospace. So, following reference [13], we introduce $\mathbf{K} = \mathbf{J} + \mathbf{I}$, where $\mathbf{J} = \mathbf{L} + \mathbf{S}$ and $\mathbf{I} = \vec{\tau}/2$.

Moreover, $i \not{\partial}$ and $A(\theta)\mathcal{U}(\mu)$ also commute with the parity operator $i\gamma_0 P$ (where P is spatial parity), so we can classify the solutions of equation (23) through the eigenvalues of K^2, K_3, J^2 and parity. The corresponding eigenvectors are the following spin-isospin spherical harmonics:

$$\begin{array}{l} |1>=|k,j=k+1/2,l=k,m> \\ |2>=|k,j=k-1/2,l=k,m> \\ |3>=|k,j=k+1/2,l=k+1,m> \\ |4>=|k,j=k-1/2,l=k-1,m>, \end{array} \tag{27}$$

where the parity of each state is given by $(-)^l$. Now, factoring out the "temporal" dependence and solving the radial equation, it can be easily seen that the solutions are expressed in terms of modified Bessel functions.

Throughout our calculation, we will only need their boundary values (conveniently normalized).

For $k > 0$, they can be written as the Dirac spinors [13]:

$$\begin{aligned} \Psi_1^{n,k,m}(r=R) &= \begin{pmatrix} |1> \\ \\ +iS_n W^+_{k,n}|3> \end{pmatrix} e^{i\omega_n t}, \\ \Psi_2^{n,k,m}(r=R) &= \begin{pmatrix} |2> \\ \\ -iS_n W^-_{k,n}|4> \end{pmatrix} e^{i\omega_n t}, \\ \Psi_3^{n,k,m}(r=R) &= \begin{pmatrix} -iS_n W^+_{k,n}|3> \\ \\ -|1> \end{pmatrix} e^{i\omega_n t}, \\ \Psi_4^{n,k,m}(r=R) &= \begin{pmatrix} +iS_n W^-_{k,n}|4> \\ \\ -|2> \end{pmatrix} e^{i\omega_n t}, \end{aligned} \tag{28}$$

for $n, k \in \mathbf{Z},\ k \geq 1$ and $-k \leq m \leq k$. In equations (28) $\omega_n = (2n+1)\pi/\beta$, $S_n = sgn(n+1/2)$, and

$$W^{\pm}_{k,n} = \frac{I_{k\pm 1+1/2}(|\omega_n|R)}{I_{k+1/2}(|\omega_n|R)} \ . \tag{29}$$

In the $k=0$ case, only the solutions $\Psi_1^{n,0}$ and $\Psi_3^{n,0}$ survive.

For this choice of basis in the kernel of $i \not\partial$, the boundary operators $A(\theta)\mathcal{U}^{-1}(\mu)$ and $B\mathcal{U}^{-1}(\mu)$ leave the subspaces characterized by given k,m, $(-)^l$ invariant. Thus, the operator $\Phi_{AB}(\mu)$ turns out to be block diagonal and it can be evaluated in each invariant two-dimensional subspace (their directions corresponding to the values of $J = k \pm 1/2$).

To obtain the projected boundary values, we need the explicit form of the operator $A(\theta)$, which is given by[2]:

$$A(\theta) = \begin{pmatrix} \mathcal{A}(\theta) \\ \frac{i(\vec{\sigma}.\mathbf{n})}{\cos\theta}[1 + i(\vec{\sigma}.\mathbf{n})i(\vec{\tau}.\mathbf{n})\sin\theta]\mathcal{A}(\theta) \end{pmatrix}, \tag{30}$$

where

$$\mathcal{A}(\theta) = \frac{1}{2}\left[1 + i(\vec{\sigma}.\mathbf{n})i(\vec{\tau}.\mathbf{n})\sin\theta; i(\vec{\sigma}.\mathbf{n})\cos\theta\right]. \tag{31}$$

In order to discard redundant information, we will only retain the upper spin-isospin component in the projected value. So, we introduce the matrices:

$$H^{I}_{n,k,m}(\mu,\theta) = \mathcal{A}(\theta)\mathcal{U}^{-1}(\mu)(\Psi_1,\Psi_2)^{n,k,m}, \tag{32}$$

$$H^{II}_{n,k,m}(\mu,\theta) = \mathcal{A}(\theta)\mathcal{U}^{-1}(\mu)(\Psi_3,\Psi_4)^{n,k,m}, \tag{33}$$

where I stands for $(-)^l = (-)^k$, and II for $(-)^l = (-)^{k+1}$. It is straightforward to obtain their matrix elements from equations (28), (32) and (33).They are explicitly given in Appendix 1.

Following equation (15), and taking into account equation (20), we see the operator $\Phi_{AB}(\mu)$ is such that:

$$H^{(i)}_{n,k,m}(\mu,\theta) = \Phi_{AB}(\mu)H^{(i)}_{n,k,m}(\mu,\theta=0),\ i = I, II. \tag{34}$$

[2]Referred to the basis of the k,m subspace in equation (27):

$$(i\vec{\sigma}.\mathbf{n}) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$(i\vec{\tau}.\mathbf{n}) = \frac{1}{2k+1}\begin{pmatrix} 0 & 0 & 1 & -2\sqrt{k(k+1)} \\ 0 & 0 & 2\sqrt{k(k+1)} & 1 \\ -1 & -2\sqrt{k(k+1)} & 0 & 0 \\ 2\sqrt{k(k+1)} & -1 & 0 & 0 \end{pmatrix}$$

(see reference [13])

Therefore, in the $n, k, (-)^l$ subspace:

$$[\Phi_{AB}(\mu)]^{(i)}_{n,k,m} = [H(\mu,\theta)H^{-1}(\mu,\theta=0)]^{(i)}_{n,k,m}, \tag{35}$$

with $n, k \in \mathbf{Z},\ k \geq 1, -k \leq m \leq k$, and $i = I, II$.

For $k = 0$, the invariant subspaces are one-dimensional and $H^{(i)}_{n,k=0}$ are similarly obtained from $\Psi_1^{n,0}$ and $\Psi_3^{n,0}$:

$$H^{I}_{n,k=0}(\mu,\theta) = \mathcal{A}(\theta)\mathcal{U}^{-1}(\mu)\Psi_1^{n,0} \tag{36}$$

and

$$H^{II}_{n,k=0}(\mu,\theta) = \mathcal{A}(\theta)\mathcal{U}^{-1}(\mu)\Psi_3^{n,0}, \tag{37}$$

for $n \in \mathbf{Z}$. Their expressions are also given in Appendix 1.

5 Evaluation of the chiral correction to the free energy

In section 3, we have proposed a definition for the chiral correction to the free energy in terms of the p-determinant of $[\Phi^{-1}_{AB}(\mu=0)\ \Phi_{AB}(\theta/2)]$. Moreover, in the previous section, we have shown that this operator is block-diagonal. So, its p-determinant reduces to:

$$\ln det_p\left[\Phi^{-1}_{AB}(\mu=0)\Phi_{AB}(\theta/2)\right] =$$

$$\sum_{n=-\infty}^{\infty}\sum_{k=0}^{\infty}\sum_{m=-k}^{m=k}\sum_{i=I,II} \ln det_p[\Phi^{-1}_{AB}(\mu=0)\Phi_{AB}(\theta/2)]^{(i)}_{n,k,m}\ , \tag{38}$$

where each term in the series is given by equation (9).

The least p for which the right-hand side in equation(38) exists can be determined by studying the behavior of $\left[1-\Phi^{-1}_{AB}(\mu=0)\Phi_{AB}(\theta/2)\right]^{(i)}_{n,k,m}$, for large n and/or k.

In what follows, we will - for convenience - study in a separate way the $k > 0$ and $k = 0$ cases.

5.1 $k \geq 1$ contribution

In this subspace one obtains, from equation (35) and the results quoted in Appendix 1:

$$\left[1-\Phi_{AB}^{-1}(\mu=0)\Phi_{AB}(\theta/2)\right]_{n,k,m}^{(I)}=\frac{1}{d}\left(\begin{array}{cc} a+b & c-e \\ c+e & a-b \end{array}\right)$$

$$\left[1-\Phi_{AB}^{-1}(\mu=0)\Phi_{AB}(\theta/2)\right]_{-n-1,k,m}^{(II)}=\frac{1}{d}\left(\begin{array}{cc} a+b & -c+e \\ -c-e & a-b \end{array}\right), \qquad (39)$$

where:

$$a = iS_n X_{k,n}(1-\cos\theta)+[X_{k,n}^2-Y_{k,n}^2]\cos\theta(1-\cos\theta)$$
$$b = -iS_n Y_{k,n} Z_{k,n}(1-\cos\theta)$$
$$c = -2i\sqrt{k(k+1)}S_n Y_{k,n} Z_{k,n}(1-\cos\theta)$$
$$d = 1-2iS_n X_{k,n}\cos\theta - X_{k,n}^2\cos^2\theta - Y_{k,n}^2\sin^2\theta$$
$$e = -2\sqrt{k(k+1)}Y_{k,n}^2(1-\cos\theta). \qquad (40)$$

Here we have defined:

$$X_{k,n}=\frac{1-W_{k,n}^+W_{k,n}^-}{W_{k,n}^++W_{k,n}^-}=$$

$$\frac{1}{2}\frac{z_n}{\rho^2}+\frac{1}{2}\frac{z_n}{\rho^3}\left(\frac{z_n^2-2\nu^2}{2\rho^2}\right)+\frac{1}{2}\frac{z_n}{\rho^4}\left(\frac{(z_n^2-2\nu^2)^2-2\nu^2\rho^2}{2\rho^4}\right)+O(1/\rho^4) \qquad (41)$$

$$Y_{k,n}=\frac{W_{k,n}^+-W_{k,n}^-}{(2k+1)(W_{k,n}^++W_{k,n}^-)}=$$

$$-\frac{1}{2}\frac{1}{\rho}-\frac{1}{4}\frac{z_n^2}{\rho^4}+\frac{1}{4}\frac{z_n^2}{\rho^5}\left(-\frac{1}{4}+\frac{5}{4}\frac{\nu^2}{\rho^2}\right)-\frac{1}{8}\frac{z_n^4}{\rho^7}+O(1/\rho^4), \qquad (42)$$

$$Z_{k,n}=\frac{1+W_{k,n}^+W_{k,n}^-}{(2k+1)(W_{k,n}^++W_{k,n}^-)}=$$

$$\frac{z_n}{\nu\rho}-\frac{1}{2}\frac{z_n}{\nu\rho^2}+\frac{1}{2}\frac{z_n^3}{\nu\rho^4}+O(1/\rho^3) \qquad (43)$$

where we have used the Debye expansion [9, 15] of $I_\nu(z_n)$, for $\nu = k + 1/2$, and $z_n = |\omega_n| R$, and called $\rho = \sqrt{\nu^2 + z_n^2}$.

It can be easily seen that the matrices in equation (39) are $O(1/\rho)$. Thus, at least three traces must be subtracted in equation (9). So, from now on, we will take $p = 4$, thus getting for each term in the right hand side of equation (38):

$$\ln det_4[\Phi_{AB}^{-1}(\mu = 0)\Phi_{AB}(\theta/2)]_{n,k,m}^{(i)} =$$
$$\ln det[\Phi_{AB}^{-1}(\mu = 0)\Phi_{AB}(\theta/2)]_{n,k,m}^{(i)} +$$
$$\sum_{q=1}^{3} \frac{1}{q} Tr[(1 - \Phi_{AB}^{-1}(\mu = 0)\Phi_{AB}(\theta/2))^q]_{n,k,m}^{(i)}, \tag{44}$$

where

$$\ln det[\Phi_{AB}^{-1}(\mu = 0)\Phi_{AB}(\theta/2)]_{n,k,m}^{I} = \ln det[\Phi_{AB}^{-1}(\mu = 0)\Phi_{AB}(\theta/2)]_{-n-1,k,m}^{II}$$
$$= \frac{(1 - iS_n X_{k,n})^2}{(1 - iS_n \cos\theta X_{k,n})^2 - Y_{k,n}\sin^2\theta}, \tag{45}$$

and

$$\sum_{q=1}^{3} \frac{1}{q}\, Tr\left\{\left(\left[1 - \Phi_{AB}^{-1}(\mu = 0)\Phi_{AB}(\theta/2)\right]_{n,k,m}^{(I)}\right)^q\right\} =$$
$$\sum_{q=1}^{3} \frac{1}{q} Tr\left\{\left(\left[1 - \Phi_{AB}^{-1}(\mu = 0)\Phi_{AB}(\theta/2)\right]_{-n-1,k,m}^{(II)}\right)^q\right\} =$$
$$2iS_n(X_{k,n} - \frac{1}{3}X_{k,n}^3)(1 - \cos\theta) +$$
$$-2iS_n(X_{k,n}Y_{k,n}^2 + \frac{1}{3}X_{k,n}^3)\cos\theta\sin^2\theta +$$
$$-(X_{k,n}^2 + Y_{k,n}^2)\sin^2\theta + O(1/\rho^4). \tag{46}$$

Notice that the terms up to order $O(1/\rho^3)$ in the Debye expansion of the right hand side of equation (46) exactly cancel the bad behavior of $\ln det[\Phi_{AB}^{-1}(\mu = 0)\Phi_{AB}(\theta/2)]$ in equation (45). The rest of the terms contribute to equation (38) with an absolutely convergent series and will cancell against their contribution to the renormalized traces in equation (26); so, they can be ignored from now on[3].

Consequently, we can write for the $k>0$ contribution to the chiral correction of the free energy

$$
\begin{aligned}
&[F(\theta)-F(0)]_{k>0} = \\
&= \frac{1}{\beta}\sum_{n=-\infty}^{\infty}\sum_{k=1}^{\infty}\nu\left\{\ln\left|\frac{(1-iS_nX_{k,n})^2}{(1-iS_nX_{k,n}\cos\theta)^2-Y_{k,n}^2\sin^2\theta}\right|^2\right. \\
&\left.-\frac{1}{2}\left[\frac{2z^2+\nu^2}{\rho^4}+\frac{z^2}{\rho^5}\frac{2z^2-\nu^2}{\rho^2}\right]\sin^2\theta\right\}+K\sin^2\theta \\
&+\sin^2\theta\ F.P._{s=0}\left\{\frac{1}{\beta}\sum_{k=1}^{\infty}\sum_{n=-\infty}^{\infty}\nu\rho^{-s}\frac{1}{2}\left[\frac{2z^2+\nu^2}{\rho^4}+\frac{z^2}{\rho^5}\frac{2z^2-\nu^2}{\rho^2}\right]\right\}, \qquad (47)
\end{aligned}
$$

where we have first summed over the two parities, taken into account that the terms do not depend on m and introduced a finite counterterm proportional to $\sin^2\theta$, with an undetermined constant K.[4]

In order to define the renormalized piece, we have chosen an analytic regularization in the last term of equation (47) by introducing a factor ρ^{-s}, and taken its finite part at $s=0$. It can be verified that the residue at $s=0$ is temperature-independent, so that only temperature independent divergencies must be removed from the free energy. This can be seen in the following expansions at low and high temperatures, which are obtained by

[3]At this point, we should stress that in the present case - where we were able to reduce $\Phi_{AB}(\mu)$ to a block-diagonal form - we could equally well have studied the pieces leading to divergencies in equation (45), avoiding the explicit construction of the traces in equation (46). However, the approach presented in section 2 is valid in the general case.

[4]Expression (46) suggests that, in order to compensate the arbitrariness in the choice of this regularization scheme, we should also add finite counterterms proportional to $(1-\cos\theta)$ and $\cos\theta\sin^2\theta$. However, since a discrete chiral γ_5 transformation [9] leaves the Dirac spectrum invariant, while it turns θ into $(\theta+\pi)$ (i.e., $F(\theta+\pi)=F(\theta)$), these two posibilities are ruled out.

means of the Euler-Mac Laurin formula [15] [5]:

$$C(T) = F.P._{s=0}\left\{\frac{1}{2\beta}\sum_{k=1}^{\infty}\sum_{n=-\infty}^{\infty}\nu\rho^{-s}\left[\frac{2z^2+\nu^2}{\rho^4}+\frac{z^2}{\rho^5}\frac{2z^2-\nu^2}{\rho^2}\right]\right\} =$$

$$\frac{1}{4\pi R}\ F.P._{s=0}\left[\frac{8}{15\ s} - 5.313 + O\left(\left(\frac{2\pi R}{\beta}\right)^5\right)\right],\quad for\ \frac{\beta}{2\pi R} >> 1, \tag{48}$$

$$= \frac{1}{4\pi R}\ F.P._{s=0}\left\{\frac{8}{15\ s} + 1.012 + 2\ln 2\ \left(\frac{2\pi R}{\beta}\right) + \right.$$

$$\left.\frac{8}{15}\ln\left(\frac{\beta}{2\pi R}\right) + O\left(\frac{\beta}{2\pi R}\right)\right\},\quad for\ \frac{\beta}{2\pi R} << 1. \tag{49}$$

The first term in the right hand side of equation (47) can be approximated by means of the Debye asymptotic expansion in equations (41) and (42). Up to the first nonvanishing order in this expansion, we get for the $k \geq 1$ contribution:

$$[F(\theta) - F(0)]_{k>0} = A(T)\sin^2\theta + B(T)\sin^4\theta +$$

$$C(T)\sin^2\theta + K\sin^2\theta, \tag{50}$$

where:

$$A(T) = \frac{1}{\beta}\sum_{k=1}^{\infty}\nu\sum_{n=-\infty}^{\infty}\frac{5}{8}\left[\frac{z^2}{\rho^6} - 5\frac{\nu^2 z^4}{\rho^{10}}\right] \tag{51}$$

$$B(T) = \frac{1}{\beta}\sum_{k=1}^{\infty}\nu\sum_{n=-\infty}^{\infty}\frac{1}{16}\left[\frac{8z^4+8\nu^2z^2+\nu^4}{\rho^8}\right]. \tag{52}$$

As regards the coefficients $A(T)$ and $B(T)$, the sum over n can be exactly evaluated and, as shown in Appendix 2, they are given by:

$$-4\pi R\ A(T) = \frac{5}{8}\pi\beta^3\left(\frac{1}{\beta}\frac{d}{d\beta}\right)S(\beta) + \frac{15}{16}\pi\beta^5\left(\frac{1}{\beta}\frac{d}{d\beta}\right)^2 S(\beta) +$$

[5]Notice that, in the high temperature development, the Euler-Mac Laurin formula was used to evaluate the sum over k, which was the first to performed. The order of the infinite sums can be freely interchanged for $Re(s)$ large enough, where the double series is absolutely convergent.

$$\frac{25}{96}\pi\beta^7\left(\frac{1}{\beta}\frac{d}{d\beta}\right)^3 S(\beta) + \frac{25}{1536}\pi\beta^9\left(\frac{1}{\beta}\frac{d}{d\beta}\right)^4 S(\beta) \tag{53}$$

$$-4\pi R\ B(T) = \frac{\pi}{2}\beta^3\left(\frac{1}{\beta}\frac{d}{d\beta}\right) S(\beta) + \frac{\pi}{8}\beta^5\left(\frac{1}{\beta}\frac{d}{d\beta}\right)^2 S(\beta) +$$

$$\frac{\pi}{384}\beta^7\left(\frac{1}{\beta}\frac{d}{d\beta}\right)^3 S(\beta), \tag{54}$$

with

$$S(\beta) = \frac{1}{\beta}\sum_{k=1}^{\infty}\nu^{-2}\tanh\left(\frac{\beta\nu}{2R}\right). \tag{55}$$

Their high and low temperature limits, also derived in Appendix 2, are:

$$A(T) = \left[\frac{1}{4\pi R}0.029 + O(e^{-\beta/2R})\right], \quad for\ \frac{\beta}{2\pi R} >> 1,$$

$$= \left[\frac{1}{4\pi R}0.514\left(\frac{\beta}{2\pi R}\right) + O\left(\left(\frac{\beta}{2\pi R}\right)^3\right)\right], \quad for\ \frac{\beta}{2\pi R} << 1. \tag{56}$$

$$B(T) = \left[\frac{1}{4\pi R}0.482 + O(e^{-\beta/2R})\right], \quad for\ \frac{\beta}{2\pi R} >> 1,$$

$$= \left[\frac{1}{4\pi R}2,673\left(\frac{\beta}{2\pi R}\right) + O\left(\left(\frac{\beta}{2\pi R}\right)^3\right)\right], \quad for\ \frac{\beta}{2\pi R} << 1. \tag{57}$$

In particular, when $\beta \to \infty$ we obtain for the chiral correction to the Casimir energy of the bag, up to this order in the Debye expansion:

$$[E(\theta) - E(0)]_{k>0} = \lim_{\beta\to\infty}[F(\theta) - F(0)]_{k>0} =$$

$$= \frac{1}{4\pi R}(0.029 - 5.313 + 4\pi RK)\sin^2\theta + \frac{1}{4\pi R}0.482\sin^4\theta. \tag{58}$$

Notice that the coefficient of $\sin^4\theta$, which is renormalization independent, coincides with the one derived in reference [9] through the propagator approach. However, our result is finite, the renormalization prescription has completely removed the divergencies and only an undetermined constant remains. The corresponding (finite) counterterm can be understood as a surface integral of a local density in the external pionic field. Indeed, if we call

$$L_\alpha = e^{-i\theta\vec{\tau}.\mathbf{n}}\left(i\partial_\alpha e^{i\theta\vec{\tau}.\mathbf{n}}\right), \tag{59}$$

we can write:

$$K\sin^2\theta = \frac{K}{16\pi}\int_{r=R} d^2x\ tr\left\{L_\alpha L_\alpha - (n_\alpha L_\alpha)^2\right\}, \tag{60}$$

showing that KR can be identified with a "surface tension", as discussed in reference [9].

5.2 $k = 0$ contribution

For $k = 0$, from equations (36) and (37) and Appendix 1, we get

$$[\Phi^{-1}_{AB}(\mu=0)\Phi_{AB}(\theta/2)]^{(I)}_{n,k=0} = [\Phi^{-1}_{AB}(\mu=0)\Phi_{AB}(\theta/2)]^{(II)}_{-n-1,k=0} =$$

$$= \frac{(1+iS_nW^+_{0,n})^2}{\cos\theta[1-(W^+_{0,n})^2]+2iS_nW^+_{0,n}}, \tag{61}$$

(since the corresponding subspaces are one-dimensional).

Taking into account that

$$W^+_{0,n} = \frac{\cosh z_n}{\sinh z_n} - \frac{1}{z_n}, \tag{62}$$

with $z_n = |\omega_n|R$, we get the asymptotic behavior

$$\ln[\Phi^{-1}_{AB}(\mu=0)\Phi_{AB}(\theta/2)]^{(I)}_{n,k=0} = \ln[\Phi^{-1}_{AB}(\mu=0)\Phi_{AB}(\theta/2)]^{(II)}_{-n-1,k=0} =$$

$$= -\frac{iS_n}{z_n}(1-\cos\theta) + O(1/z_n^2), \tag{63}$$

while

$$[1-\Phi^{-1}_{AB}(\mu=0)\Phi_{AB}(\theta/2)]^{(I)}_{n,k=0} = [1-\Phi^{-1}_{AB}(\mu=0)\Phi_{AB}(\theta/2)]^{(II)}_{-n-1,k=0} =$$

$$= \frac{iS_n}{z_n}(1-\cos\theta) + O(1/z_n^2). \tag{64}$$

So, one can see that only the $O(1/z_n)$ term of the first subtracted trace in equation (44) (for $k = 0$) is needed to cancel the bad behavior of the right hand side in equation (63). The remaining ones, leading to convergent series, cancel against their contribution to the renormalized terms in equation (26) and will be disregarded from now on.

Finally, noticing that terms of $O(1/z_n)$ cancel when summing over the two parities, we simply get for the $k=0$ contribution to the chiral correction of the free energy:

$$[F(\theta)-F(0)]_{k=0}=\frac{1}{2\beta}\sum_{n=-\infty}^{\infty}\ln\left|\frac{(1+iS_nW_{0,n}^{+})^2}{\cos\theta[1-(W_{0,n}^{+})^2]+2iS_nW_{0,n}^{+}}\right|^2 . \tag{65}$$

In this expression we have again discarded a finite counterterm proportional to $(1-\cos\theta)$ on physical grounds (see footnote 4).

A straightforward algebra allows one to write equation (65) in the form:

$$[F(\theta)-F(0)]_{k=0}=A'+B' , \tag{66}$$

with:

$$A'=\frac{1}{2\beta}\sum_{n=-\infty}^{\infty}\ln\frac{1+2e^{-4z_n}+e^{-8z_n}}{1+2e^{-4z_n}\cos(2\theta)+e^{-8z_n}}=\frac{1}{\beta}\ln\frac{\theta_3(0)}{\theta_3(\theta)}$$

$$B'=\frac{1}{2\beta}\sum_{n=-\infty}^{\infty}\left\{\ln\left[1+\frac{-\frac{2a_n}{z_n}+\frac{1}{z_n^2}}{1+a_n^2}\right]^2-\right.$$

$$\left.\ln\left[1+\frac{4\left(-\frac{2a_n}{z_n}+\frac{1}{z_n^2}\right)+\left[2(1-a_n^2)+\left(\frac{2a_n}{z_n}-\frac{1}{z_n^2}\right)\right]\left(\frac{2a_n}{z_n}-\frac{1}{z_n^2}\right)\cos^2\theta}{4a_n^2+(1-a_n^2)^2\cos^2\theta}\right]\right\}, \tag{67}$$

where θ_3 is the Jacobi's theta function [14], and a_n stands for $\coth z_n$. These expressions can be studied in the low temperature limit by making use of the Euler-Mac Laurin summation formula [15]. In particular, B' can be estimated in this limit from the asymptotic behavior of the integrand. As concerns the high temperature limit, it can be obtain from suitable expansions of the general terms in equations (67). We thus get:

$$A'=\frac{1}{4\pi R}\theta^2+O\left(\frac{2\pi R}{\beta}\right) \ , \ for \ \frac{\beta}{2\pi R}>>1$$

$$= \frac{1}{4\pi R} 8 \left(\frac{2\pi R}{\beta} \right) e^{-4\pi R/\beta} + O(e^{-8\pi R/\beta}), \quad for \ \frac{\beta}{2\pi R} << 1 \tag{68}$$

and

$$B' \simeq \frac{1}{4\pi R} \left(2.587 \sin^2 \theta + 0.922 \sin^4 \theta \right) + O\left(\left(\frac{2\pi R}{\beta} \right)^3 \right), \ for \ \frac{\beta}{2\pi R} >> 1$$

$$= \frac{1}{4\pi R} \pi^2 \left(\frac{\beta}{2\pi R} \right) \sin^2 \theta + O\left(\left(\frac{\beta}{2\pi R} \right)^2 \right), \quad for \ \frac{\beta}{2\pi R} << 1 \tag{69}$$

In particular, when $\beta \to \infty$, we obtain for the $k = 0$ contribution to the chiral correction to the Casimir energy:

$$[E(\theta) - E(0)]_{k=0} \simeq \frac{1}{4\pi R} \theta^2 + \frac{1}{4\pi R} \left(2.587 \sin^2 \theta + 0.922 \sin^4 \theta \right), \tag{70}$$

which is to be compared with the corresponding result in reference [9], where the θ^2 term is obtained from the asymptotic behavior of the Dirac spectrum.

6 Comments and conclusions

In this paper, we have applied the results in reference [8] to evaluate the one-loop contribution to the free energy of a four-dimensional chiral bag due to the presence of an external pionic field in a hedgehog configuration. In that reference, it has been established that the p-determinant of a quotient of elliptic differential operators, defined on a space of functions living in a region Ω and subject to given elliptic boundary conditions at $\partial\Omega$, equals the p-determinant of a quotient of pseudodifferential (Forman's) operators, acting on functions defined just on $\partial\Omega$, the later being entirely expressible in terms of boundary values of functions in the kernel of the original differential operator.

In order to define renormalized free energies on the basis of this regularized functional determinants, one must readd the subtracted traces in the p-determinant, duely renormalized. As discussed in reference [8], these can be also be expressed in terms of Forman's operators.

Even though this approach relies on the simultaneous consideration of two distinct boundary conditions (each depending on an ad-hoc parameter

μ), a choice of these conditions and of μ adequate to the symmetries of the problem, allowed us to get the difference between the (one-loop) renormalized free energies of the bag with and without the external hedgehog pionic field, $\Delta F(\theta) = F(\theta) - F(0)$.

A convenient selection of a basis in the kernel of the differential operator allowed us to show that the relevant Forman's operator is block-diagonal. In this way, the evaluation of its p-determinant was performed in each finite-dimensional invariant subspace, thus greatly simplifying the task.

Through an analytic regularization of the subtracted traces, together with a finite part prescription, we got a closed expression, in the form of a double series, for $\Delta F(\theta)$ at any finite temperature. We have verified that the residue at the discarded pole is temperature-independent, so that only temperature-independent divergencies were removed.

Notice that the only term affected by the renormalization procedure, i.e., that proportional to $\sin^2\theta$, can be expressed as a boundary integral of a local density in the pionic field. Thus, its undetermined coefficient can be understood as a phenomenological "surface tension" (as discussed in reference [9]).

In the $k > 0$ sector, a Debye expansion of Bessel functions was employed; up to the first nonvanishing order, the sum over n was explicitly performed and high and low temperature expansions of the remaining sum were given. In particular, in the zero-temperature limit, our results are consistent with the Casimir energy in reference [9]. In fact, the coefficient of $\sin^4\theta$ (which is renormalization independent) exactly coincides with the one obtained in that reference. But our result is free of divergencies and $\sin^2\theta$ is affected by an undetermined factor as a consequence of renormalization.

For the $k = 0$ sector, the asymptotic expansion of Bessel functions for large arguments was used to again obtain high and low temperature developments. In particular, for $T = 0$, we obtained a contribution proportional to θ^2, which coincides with the one derived in reference [9] from the asymptotic behaviour of the Dirac spectrum.

Finally, we wish to stress that those quantities which can be derived from the renormalized energy turn out to be automatically finite in this approach. For example, the flux of the vacuum axial current, which in reference [9] is related to the spectral asymmetry of the energy, can be defined from equations

(16) and (17) as
$$\Phi(\theta) = \frac{d}{d\theta}[E(\theta) - E(0)].$$
So, from equations (58) and (70), we get:
$$\Phi(\theta) = \frac{\theta}{2\pi R} + \frac{constant}{4\pi R}\sin 2\theta + \frac{(0.482 + 0.922)}{2\pi R}\sin^2\theta \sin 2\theta + ...,$$
which depends on the undetermined constant K only through the coefficient of $\sin 2\theta$ in the second term on the right hand side of this expression.

Appendix 1: Matrix elements of $H^{(i)}_{n,k,m}(\mu,\theta)$

For $k \geq 1$, the elements of the boundary values matrices deffined in equations (32) and (33) are given by:

$$\left(H^{I}_{n,k,m}(\mu,\theta)\right)_{1,1} =$$

$$\left[\cos(\mu) + iS_n \cos(\theta - 2\mu) W^{+}_{k,n}\right] - \frac{1}{2k+1}\left[\sin(\theta - 2\mu) - iS_n \sin(\mu) W^{+}_{k,n}\right],$$

$$\left(H^{I}_{n,k,m}(\mu,\theta)\right)_{1,2} = -\frac{2\sqrt{k(k+1)}}{2k+1}\left[\sin(\theta - 2\mu) - iS_n \sin(\mu) W^{-}_{k,n}\right],$$

$$\left(H^{I}_{n,k,m}(\mu,\theta)\right)_{2,1} = -\frac{2\sqrt{k(k+1)}}{2k+1}\left[\sin(\theta - 2\mu) - iS_n \sin(\mu) W^{+}_{k,n}\right],$$

$$\left(H^{I}_{n,k,m}(\mu,\theta)\right)_{2,2} =$$

$$\left[\cos(\mu) + iS_n \cos(\theta - 2\mu) W^{-}_{k,n}\right] + \frac{1}{2k+1}\left[\sin(\theta - 2\mu) - iS_n \sin(\mu) W^{-}_{k,n}\right], \tag{71}$$

$$\left(H^{II}_{n,k,m}(\mu,\theta)\right)_{1,1} =$$

$$\left[\cos(\theta - 2\mu) - iS_n \cos(\mu) W^{+}_{k,n}\right] + \frac{1}{2k+1}\left[\sin(\mu) + iS_n \sin(\theta - 2\mu) W^{+}_{k,n}\right],$$

$$\left(H^{II}_{n,k,m}(\mu,\theta)\right)_{1,2} = -\frac{2\sqrt{k(k+1)}}{2k+1}\left[\sin(\mu) + iS_n \sin(\theta - 2\mu) W^{-}_{k,n}\right],$$

$$\left(H^{II}_{n,k,m}(\mu,\theta)\right)_{2,1} = -\frac{2\sqrt{k(k+1)}}{2k+1}\left[\sin(\mu) + iS_n \sin(\theta - 2\mu) W^{+}_{k,n}\right],$$

$$\left(H^{II}_{n,k,m}(\mu,\theta)\right)_{2,2} =$$

$$\left[\cos(\theta-2\mu)-iS_n\cos(\mu)W^-_{k,n}\right]) - \frac{1}{2k+1}\left[\sin(\mu)+iS_n\sin(\theta-2\mu)W^-_{k,n}\right].$$

For $k=0$, equations (36) and (37) give:

$$H^I_{n,k=0}(\mu,\theta) =$$

$$= \left[\cos(\mu)+iS_n\cos(\theta-2\mu)W^+_{0,n}\right] - \left[\sin(\theta-2\mu)-iS_n\sin(\mu)W^+_{0,n}\right]$$

and

$$H^{II}_{n,k=0}(\mu,\theta) =$$

$$= \left[\cos(\theta-2\mu)-iS_n\cos(\mu)W^+_{0,n}\right] + \left[\sin(\mu)+iS_n\sin(\theta-2\mu)W^+_{0,n}\right].$$

Appendix 2: Evaluation of $A(T)$ and $B(T)$

When evaluating $A(T)$ and $B(T)$ one must consider sums of the form:

$$F(\beta'\nu;n) = \sum_{n'=-\infty}^{\infty} \frac{1}{[(\beta'\nu)^2+(n'+1/2)]^n} \tag{72}$$

with $n\in\mathbf{N}$. This can be generated from $F(\beta'\nu;1)$ as:

$$F(\beta'\nu;n+1) = \frac{1}{n!}(-1)^n\frac{1}{(2\nu^2)^n}\left[\frac{1}{\beta'}\frac{d}{d\beta'}\right]^n F(\beta'\nu;1). \tag{73}$$

The generating series can be exactly evaluated through the Poisson summation formula [15] and it is easily seen to be given by:

$$F(\beta'\nu;1) = \frac{\pi}{\beta'\nu}\tanh(\pi\beta'\nu) \tag{74}$$

A straightforward algebra leads from equations (51) and (52) in Section 5 to

$$-4\pi R\ A(T) = \frac{5}{8}\pi\beta^3\left(\frac{1}{\beta}\frac{d}{d\beta}\right)S(\beta) + \frac{15}{16}\pi\beta^5\left(\frac{1}{\beta}\frac{d}{d\beta}\right)^2 S(\beta) +$$

$$\frac{25}{96}\pi\beta^7\left(\frac{1}{\beta}\frac{d}{d\beta}\right)^3 S(\beta) + \frac{25}{1536}\pi\beta^9\left(\frac{1}{\beta}\frac{d}{d\beta}\right)^4 S(\beta) \tag{75}$$

$$-4\pi R\ B(T) = \frac{\pi}{2}\beta^3\left(\frac{1}{\beta}\frac{d}{d\beta}\right)S(\beta) + \frac{\pi}{8}\beta^5\left(\frac{1}{\beta}\frac{d}{d\beta}\right)^2 S(\beta) +$$
$$\frac{\pi}{384}\beta^7\left(\frac{1}{\beta}\frac{d}{d\beta}\right)^3 S(\beta), \tag{76}$$

with

$$S(\beta) = \frac{1}{\beta}\sum_{k=1}^{\infty}\nu^{-2}\tanh\left(\frac{\beta\nu}{2R}\right). \tag{77}$$

In the low temperature limit ($\beta/2\pi R >> 1$) one has:

$$\tanh\left(\frac{\beta\nu}{2R}\right) = 1 + O(e^{-\beta\nu/R}), \tag{78}$$

which gives for $S(\beta)$:

$$S(\beta) = \frac{4}{\beta}\left[\frac{\pi^2}{8} - 1\right] + O\left(\frac{e^{-\beta/2R}}{\beta}\right). \tag{79}$$

When replaced in equations (75) and (76) this gives:

$$A(T) = \left[\frac{1}{4\pi R}0.029 + O(e^{-\beta/2R})\right], \tag{80}$$

and

$$B(T) = \left[\frac{1}{4\pi R}0.482 + O(e^{-\beta/2R})\right]. \tag{81}$$

For the high temperature limit ($\beta/2\pi R << 1$), $S(\beta)$ can be evaluated through the Euler-Mac Laurin formula, which gives:

$$S(\beta) = -\frac{1}{2R}\log\left(\frac{\beta}{4R}\right) + constant + O\left(\left(\frac{\beta}{2\pi R}\right)^2\right). \tag{82}$$

So, in this limit, we obtain:

$$A(T) = \left[\frac{1}{4\pi R}0.514\left(\frac{\beta}{2\pi R}\right) + O\left(\left(\frac{\beta}{2\pi R}\right)^3\right)\right], \tag{83}$$

and

$$B(T) = \left[\frac{1}{4\pi R}2,673\left(\frac{\beta}{2\pi R}\right) + O\left(\left(\frac{\beta}{2\pi R}\right)^3\right)\right]. \tag{84}$$

References

[1] H. B. G. Casimir, Proc. K. Ned. Acad. Wet. **51**, 793 (1948).

[2] A. Chodos, R. L. Jaffe, C. B. Thorn and V. Weisskopf, Phys. Rev. **D9**, 3471 (1974).

[3] A. Chodos and C. B. Thorn, Phys. Rev. **D12**, 2733 (1975).

[4] H. Falomir and E. M. Santangelo, Phys. Rev. **D43**,539 (1991).

[5] R. Forman, Invent. Math. **88**, 447 (1987).

[6] H. Falomir, M. A. Muschietti and E. M. Santangelo, Jour. Math. Phys. **31** , 989 (1990).

[7] R. T. Seeley, Am. J. Math. **91**, 889 (1969); **91**, 963 (1969).

[8] O. Barraza, H. Falomir, R. E. Gamboa Saraví and E. M. Santangelo, "P-determinants and boundary values"; submitted to Jour. Math. Phys. , May 1991.

[9] I. Zahed, A. Wirzba and Ulf-G. Meissner, Ann. of Phys. **165**, 406 (1985).

[10] I. C. Gohberg and M. G. Krein, "Introduction to the theory of linear nonselfadjoint operators", Vol. 18 (Translation of Mathematical Monographs), Providence, Rhode Island, American Mathematical Society (1969).

[11] B. Simon, "Trace ideals and their applications", London Math. Soc. lecture note series 35, Cambridge University Press (1979).

[12] P. J. Mulders, Phys. Rev. **D30**, 1073 (1984).

[13] M. Jeżabek, Acta Physica Polonica **B** , (1988).

[14] I. S. Gradshtein and I. M. Ryzhik,"Table of integrals, series and products", Academic Press (1980).

[15] M. Abramowitz and I. A. Stegun (eds.), "Handbook of Mathematical Functions", Dover Publications, New York (1972).