An Ordered Phase with Slow Decay of Correlations in Continuum $1/r^2$ Ising Models

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Abstract

For continuum $1/r^2$ Ising models, we prove that the critical value of the long range coupling constant (inverse temperature), above which an ordered phase occurs (for strong short range cutoff), is exactly one. This leads to a proof of the existence of an ordered phase with slow decay of correlations. Our arguments involve comparisons between continuum and discrete Ising models, including (quenched and annealed) site diluted models, which may be of independent interest.

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1 Introduction

The model to be discussed below is an Ising model in one dimension with long range, translation invariant, ferromagnetic pair interaction. However, unlike the usual case, its configurations are ± 1 valued functions *on the real line* $\mathbf{R}$ rather than on the discrete one dimensional lattice $\mathbf{Z}$. Thus, it is a continuous time stochastic process, to be more precisely defined in the next section.

Such a process, which is called *continuum Ising model*, arises in the study of a quantum mechanical model of the motion of a particle subjected to a field. The quantum mechanical energy operator, known as spin-boson Hamiltonian for this model, can be analysed by Feynman-Kac techniques, leading to the re-expression of quantities related to the quantum model in terms of continuum Ising quantities (see [1], [2], [3]).

Two parameters, α and ϵ, and a function, W, enter the model. The parameter α is the long range coupling constant which can also be interpreted as the inverse temperature; ϵ can be related to the inverse of the short range coupling strength. $W = W(r)$ is defined on $[0, \infty]$, nonnegative and decays at infinity as $1/r^2$.

In previous work ([1], [3]), the case of $1/r^2$ long range interactions (corresponding to the *ohmic* case of the quantum model) has been studied with, among other results, the following rigorous description of the phase diagram (Theorem 2 in [3]). For $\alpha \leq 1$ and any $\epsilon > 0$, the model shows no spontaneous magnetization, whereas if $\alpha > 2$, then for small ϵ (large short range coupling force), there is spontaneous magnetization.

The strategy applied to get these results is to use the FK representation of the Ising model, which in the continuum case leads to a continuum bond percolation model, and then adapt the results existing for the discrete FK model, obtained in [4], [5] and [6].

Here, we establish the existence of long range order (in the strong form known as *long long range order*, which implies spontaneous magnetization) for $\alpha > 1$, thus closing a gap in the phase picture.

Indeed, what we do is establish a comparison between the continuum model and the discrete $1/r^2$ Ising model at inverse temperature α_ϵ and nearest neighbor coupling J_ϵ, with α_ϵ close to α and J_ϵ large when ϵ is small. We then quote the results for the discrete model obtained in [7].

As in the discrete case, long long range order for $\alpha > 1$ leads to the existence of an intermediate phase (at least for $1 < \alpha < 2$) with slow decay of correlations. Here, we prove lower bounds for the decay of the truncated two-point function in the ordered phase. In the disordered phase, lower and upper bounds for the two-point function were obtained in [3].

Upper bounds in the ordered phase remain to be obtained for the continuum model, unlike for the discrete case, for which they were derived in [7].

Our results are stated and proved in the next 5 sections, one for the description of the model and statement of results, one for each of three steps of the comparison with the discrete model and the last one for the lower bounds on the truncated 2-point function.

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2 The Model

For T positive, let Ω_T be the space of functions σ_t defined in the interval $[-T, T]$ and taking values in $\{-1, +1\}$, which have only a finite number of flips (and are right continuous, say). Ω_T is the the set of configurations of the continuum system.

Let $P^f_{\epsilon,T}$ be the measure on Ω_T such that the flip points form a Poisson process with rate $\epsilon > 0$ and σ_{-T} equals $+1$ or -1 with equal probabilities (free boundary conditions). $P^+_{\epsilon,T}$ will denote the measure on Ω_T such that the flips form a Poisson process in $[-T, T]$ with rate ϵ, *conditioned to having only an even number of flips in* $[-T, T]$, and starting at $+1$ (ie, $\sigma_{-T} = +1$), which corresponds to plus boundary conditions.

Now, let $W(t)$ be a nonnegative bounded (piecewise) continuous function decaying like $1/t^2$ at infinity, i.e. $t^2 W(t) \to 1$, as $t \to \infty$. It defines the continuum ferromagnetic couplings and will be kept fixed throughout.

The finite volume continuum Ising measures with free and plus boundary conditions are defined as follows.

$$d \overset{\circ}{*}_T (\sigma) = \frac{1}{Z^*_T} dP^*_{\epsilon,T}(\sigma) \exp^{-\alpha H^*(\sigma)}, (2.1)$$

where $* = f$ or $+$, and

$$H^f(\sigma) = -\frac{1}{4}\int_{-T}^{T}\int_{-T}^{T} W(|t-s|)\sigma_t\sigma_s dtds \tag{2.2}$$

$$H^+(\sigma) = -\frac{1}{4}\int_{-\infty}^{\infty}\int_{-T}^{T} W(|t-s|)\sigma_t\sigma_s dtds \tag{2.3}$$

are the Hamiltonians and Z_T^* is a normalizing constant. (In (2.3), $\sigma \equiv +1$ in $\mathbf{R}\backslash[-T,T]$.)

We denote by $\mathring{*}$ the infinite volume limit ($T \to \infty$) of $\mathring{*}_T$ (which exists by standard arguments) and by $\langle\,\cdot\,\rangle^*$ or $\langle\,\cdot\,\rangle_T^*$ the expectations w.r.t. $\mathring{*}$ or $\mathring{*}_T$.

Definition 2.1 *For given α and ϵ* long long range order *is said to occur if there are positive constants ν and μ such that for all $T > 0$,*

$$\langle\sigma_0\sigma_t\rangle_T^f \geq \nu^2 \text{ for all } |t| \leq \mu T.$$

Let M denote the spontaneous magnetization of $\langle\,\cdot\,\rangle^+$, i.e.

$$M = \langle\sigma_0\rangle^+,$$

and $G^T(t)$ its truncated two-point function, i.e.

$$G^T(t) = \langle\sigma_0\sigma_t\rangle^+ - M^2.$$

We now state the main results.

Theorem 1 *If $\alpha > 1$, then long long range order occurs for ϵ small enough.*

Theorem 2 *If for given α and ϵ long long range order occurs, then, for any $\delta > 0$, there exists some $C > 0$ so that*

$$G^T(t) \geq C/|t|^{2\gamma}, \text{ for all } t \geq 1, \tag{2.4}$$

where $\gamma = \min(1, \alpha - 1 + \delta)$.

We prove Theorem 1 in three steps within the next three sections. The proof of Theorem 2 is presented in the last section.

3 Comparison to an Annealed Site-Diluted Model

We begin this section by representing the continuum Ising measure (2.1) introduced in the last section as a weak limit of discrete Ising measures. (This is a well known result, the arguments for which we sketch here for completeness.) This representation is then used to derive a comparison between the continuum model and a sort of annealed site diluted Ising model.

Consider a discrete Ising model on the lattice $\Lambda \equiv \delta\mathbf{Z}$, where $\mathbf{Z}$ is the set of integers and δ is a positive number, with interactions J_{ij} given by

$$\begin{aligned} J_{i,i+\delta} &= \frac{1}{2}|\log \epsilon\delta|, \ i \in \Lambda, \\ J_{ij} &= \alpha\delta^2 W(i-j), \ i,j \in \Lambda, \ |i-j| > \delta, \end{aligned}$$

and Hamiltonian

$$H_\delta(\sigma) = -\frac{1}{4}\sum_{i,j} J_{ij}\sigma_i\sigma_j. \tag{3.1}$$

The finite volume (in $[-T,T]$) Ising measure so defined is denoted by $\nu^*_{T,\delta}$ and its expectation by $\langle \cdot \rangle^*_{T,\delta}$, $* = f,+$, where the appropriate boundary conditions are used.

We make the discrete configurations into continuum ones by setting $\sigma_t = \sigma_i$ for $t \in [i, i+\delta)$, $i \in \Lambda$.

We can write H_δ as the sum $H^{(1)}_\delta + H^{(2)}_\delta$, where

$$\begin{aligned} H^{(1)}_\delta(\sigma) &= -\frac{1}{4}\sum_{|i-j|=\delta} J_{ij}\sigma_i\sigma_j \\ H^{(2)}_\delta(\sigma) &= -\frac{1}{4}\sum_{|i-j|>\delta} J_{ij}\sigma_i\sigma_j. \end{aligned}$$

Notice that $H^{(2)}_\delta(\sigma) \to H^*(\sigma)$ as $\delta \to 0$, with H^* the Hamiltonian of the continuum system given by (2.2) and (2.3). Also, the measure

$$\frac{e^{-H^{(1)}_\delta(\sigma)} \times \text{counting measure}}{\text{normalization}}$$

is that of a Markov chain which, as $\delta \to 0$, converges weakly to the Poisson measure $P^*_{\epsilon,T}$ entering the continuum Ising measure (2.1).

It follows that $\nu^*_{T,\delta}$ converges weakly to $\overset{\circ}{*}_T$, as $\delta \to 0$. In particular, $\langle\sigma_A\rangle^*_{T,\delta} \to \langle\sigma_A\rangle^*_T$ as $\delta \to 0$.

¿From now on, we write $H^f(\sigma)$ as $H(\sigma)$.

For K, N positive integers, let $L = T/K$ and $\delta = L/N$. Consider the discrete Ising model in Λ with interactions given by

$$J^\circ_{ij} = J_{ij},$$

for $|i-j| > \delta$ and for $|i-j| = \delta$, but $i \neq kL$, and given by

$$J^\circ_{kL,kL+\delta} = 0$$

for $k \in \{-K, \dots, K\}$. Denote it by $\nu^{\circ,f}_{T,\delta}$ and its expectations by $\langle\,\cdot\,\rangle^{\circ,f}_{T,\delta}$. Since $J_{ij} \geq J^\circ_{ij}$, $\forall\, i,j$, we have by the GKS inequalities (see [10] and [11]) that

$$\langle\sigma_A\rangle^f_{T,\delta} \geq \langle\sigma_A\rangle^{\circ,f}_{T,\delta}. \tag{3.2}$$

Now, as $N \to \infty$, the measures $\nu^{\circ,f}_{T,\delta}$ converge weakly to the measure

$$d\circ, f_T(\sigma) = \frac{1}{Z'} e^{-\alpha H(\sigma)} \overset{\circ}{\prod_{k=-K}^{K}} dP_k(\sigma_k), (3.3)$$

where H is given by (2.2), σ_k is a continuum configuration in the interval $I_k \equiv [kL, (k+1)L)$ and P_k is the measure in the set of those configurations such that the flips form a Poisson process of rate ϵ and the initial distribution assigns equal probabilities to ± 1. (By an abuse of notation, we will omit the subscript from P_k from now on.)

Denote expectations w.r.t. (3.3) by $\langle\cdot\rangle'$ (the dependence on T is omitted). From (3.2) we conclude that

$$\langle\sigma_A\rangle^f_T \geq \langle\sigma_A\rangle'. \tag{3.4}$$

(Concerning the above discussion, see also [1].)

Now, let N_i denote the number of jumps of σ_i in the interval I_i. We can write the measure P as

$$\begin{aligned} dP(\sigma_i) &= dP(\sigma_i|N_i = 0)\, P(N_i = 0) + dP(\sigma_i|N_i > 0)\, P(N_i > 0) \\ &= (1-\epsilon')dP_1(\sigma_i) + \epsilon' d\bar{P}(\sigma_i), \end{aligned}$$

where

$$\begin{aligned} dP_1(\sigma_i) &= dP(\sigma_i|N_i = 0) = \frac{1}{2}(\delta_{-1}(\sigma_i) + \delta_1(\sigma_i)), \\ d\bar{P}(\sigma_i) &= dP(\sigma_i|N_i > 0), \end{aligned}$$

$\epsilon' = 1 - e^{-\epsilon L}$ is approximately ϵL as $\epsilon \downarrow 0$ and $\delta_u(\cdot)$ is the Dirac delta at the constant function u.

We rewrite P further as

$$dP(\sigma_i) = (1 - 2\epsilon')dP_1(\sigma_i) + 2\epsilon' dP_0(\sigma_i),$$

where $P_0 = \frac{1}{2}(P_1 + \bar{P})$, and ϵ is so small that $2\epsilon' \leq 1$.

Now, we can write the correlations

$$\begin{aligned} \langle \sigma_A \rangle' &= \frac{1}{Z'} \int \sigma_A e^{-\alpha H(\sigma)} \prod_i dP(\sigma_i) \\ &= \frac{1}{Z'} \int \sigma_A e^{-\alpha H(\sigma)} \prod_i ((1 - \mathring{\ })dP_1(\sigma_i) + \mathring{\ }dP_0(\sigma_i)), \end{aligned}$$

where A is any set of (distinct) points $\{t_1, \ldots, t_k\}$ in $[-T, T]$ and $\sigma_A = \prod_{i=1}^{k} \sigma_{t_i}$. We have rewritten $2\epsilon'$ as $\mathring{\ }$.

Expanding the product, we see that the integral can be viewed as an expectation with respect to a family of i.i.d. random variables $\lambda = (\lambda_i)$, where λ_i has a Bernoulli distribution with parameter $1 - \mathring{\ }$, as follows.

$$\langle \sigma_A \rangle' = \frac{1}{Z'} E_\lambda (\int \sigma_A e^{-\alpha H(\sigma)} \prod_i dP_{\lambda_i}(\sigma_i)) \tag{3.5}$$

This model is a (sort of) annealed site diluted Ising model. In this case, dilution applies to configurations in an interval and means that there can be flips in the configuration in that interval.

We have thus shown that the correlations for $\langle \cdot \rangle^f$are bigger than the corresponding ones of an annealed model.

4 Comparison to a Quenched Site-Diluted Model

In this section we derive a comparison between the annealed site diluted model of the last section and a regular quenched one. ¿From now on, we restrict attention to sets $A = \{t_1, \ldots, t_n\}$ such that there are no two t_i's in the same I_j.

We write the expectation in (3.5) as

$$E_\lambda(\int \sigma_A e^{-\alpha H(\sigma)} \prod_i dP_{\lambda_i}(\sigma_i)) = E_\lambda(\frac{\int \sigma_A e^{-\alpha H(\sigma)} \prod_i dP_{\lambda_i}(\sigma_i)}{Z_\lambda(H)} Z_\lambda(H)),$$

where $Z_\lambda(H) = \int e^{-\alpha H(\sigma)} \prod_i dP_{\lambda_i}(\sigma_i)$. We denote the quotient inside the expectation sign above by $\langle \sigma_A \rangle_{\lambda,H}$.

Lemma 4.1

$$\int \sigma_A e^{-\alpha H(\sigma)} \prod_i dP_{\lambda_i}(\sigma_i) \geq 0.$$

Proposition 4.1 *$Z_\lambda(H)$ is increasing in λ (w.r.t. the usual partial ordering).*

Proposition 4.2

$$\langle \sigma_A \rangle_{\lambda,H} \geq \langle \sigma_A \rangle_{\lambda,\bar{H}},$$

where

$$\bar{H}(\sigma) = \sum_{i,j} \lambda_i \lambda_j \int_{I_i} \int_{I_j} W(|t-s|) \sigma_t \sigma_s dt ds. \tag{4.1}$$

Proof of Lemma 4.1: Expanding the exponential, we obtain

$$\int \sigma_A e^{-\alpha H(\sigma)} \prod_i dP_{\lambda_i}(\sigma_i) = \sum_n C_n \int \sigma_A H^n(\sigma) \prod_i dP_{\lambda_i}(\sigma_i),$$

where C_n are positive numbers.

Expanding the n-th power, the r.h.s. can be expressed as

$$\sum_n C_n \int \sigma_A \textstyle\int_{-T}^T \int_{-T}^T \cdots \int_{-T}^T \int_{-T}^T \displaystyle\prod_{j=1}^n (W(|t_j - s_j|) \sigma_{t_j} \sigma_{s_j} dt_j ds_j) \prod_i dP_{\lambda_i}(\sigma_i).$$

Moving the integral w.r.t. σ inside, we get

$$\sum_n C_n \textstyle\int \int \cdots \int \int \displaystyle\prod_j (W(|t_j - s_j|) dt_j ds_j)(\int \sigma_A \prod_j \sigma_{t_j} \sigma_{s_j} \prod_i dP_{\lambda_i}(\sigma_i)). \tag{4.2}$$

The expectation $\int \sigma_A \prod_j \sigma_{t_j}\sigma_{s_j} \prod_i dP_{\lambda_i}(\sigma_i)$ factors into $\prod_i \int \sigma_{A_i} dP_{\lambda_i}(\sigma_i)$, where A_i is a set of points in the interval I_i, for all i.

Now, if $|A_i|$ is odd ($|\cdot|$ denotes the cardinality), then $\int \sigma_{A_i} dP_{\lambda_i}(\sigma_i) = 0$, by the symmetry $\sigma_i \to -\sigma_i$ of P_0 and P_1. If $|A_i|$ is even and $\lambda_i = 1$, then $\int \sigma_{A_i} dP_{\lambda_i}(\sigma_i) = 1$. If $\lambda_i = 0$, we have

$$\begin{aligned}\int \sigma_{A_i} dP_0(\sigma) &= \frac{1}{2}\int \sigma_{A_i} dP_1(\sigma) + \frac{1}{2}\int \sigma_{A_i} d\bar{P} \\ &= \frac{1}{2}(1 + \int \sigma_{A_i} d\bar{P}).\end{aligned}$$

Now, $\int \sigma_{A_i} d\bar{P} \geq -1$. Therefore, $\int \sigma_{A_i} dP_0(\sigma) \geq 0$.

We conclude that $\int \sigma_{A_i} dP_{\lambda_i}(\sigma_i) \geq 0$, for all i. Thus, we see that (4.2) is nonnegative and the lemma is proved.

Proof of Proposition 4.1: Do the same steps as in the last proof and notice that

$$\int \sigma_{A_i} dP_0(\sigma) \leq \int \sigma_{A_i} dP_1(\sigma).$$

Indeed, both integrals are zero if $|A_i|$ is odd and, if $|A_i|$ is even, the r.h.s. is 1, which is the most the l.h.s. can be. This and positivity proves the proposition.

Proof of Proposition 4.2: We will use the following terminology. Let $\Lambda = \{i : \lambda_i = 1\}$ and $\Lambda^c = \{i : \lambda_i = 0\}$. We call an interval I_i either an 1-interval or a 0-interval depending on whether $i \in \Lambda$ or $i \in \Lambda^c$. Also, let J_{ij} denote the integral $\int_{I_i}\int_{I_j} W(|t-s|)\, dtds$.

First, notice that if any of the elements of A, say t_{j_o}, belongs to a 0-interval, say I_{i_0}, then $\langle \sigma_A \rangle_{\lambda,\bar{H}} = 0$. This is because under $\bar{H}$, all the 0-intervals get disconnected from the rest of the system, so that $\langle \sigma_A \rangle_{\lambda,\bar{H}}$ factors into terms, one of which is $\int \sigma_{t_{j_0}} dP_0(\sigma_{i_0})$ (it is here that the restriction on A made at the beginning of the section enters). This integral vanishes (by symmetry), making the product vanish. Thus, we need only consider A's all of whose elements belong to 1-intervals.

We change notation here and write S_t instead of σ_t for t's belonging to 1-intervals. Further, since S_t is constant in each 1-interval I_i, we write S_i instead. Notice that $\langle S_A \rangle_{\lambda,\bar{H}}$ is the correlation of a discrete Ising model (in $\mathbf{Z} \cap [-K, K]$) with interactions $\bar{J}_{ij} \equiv \lambda_i \lambda_j J_{ij}$. Now, the proof:

$$Z_\lambda(H)\, \langle S_A \rangle_{\lambda,H} = \int S_A\, e^{-\alpha(H_1(S)+H_2(\sigma)+H_{12}(S,\sigma))} \prod_{i\in\Lambda} dP_1(S_i) \prod_{i\in\Lambda^c} dP_0(\sigma_i)$$

$$= \int S_A \, e^{-\alpha H_{12}(S,\sigma)} dQ(S) d\tilde{Q}(\sigma),$$

where

$$\begin{aligned}
H_1(S) &= -\frac{1}{4} \sum_{i,j\in\Lambda} J_{ij} S_i S_j (= -\frac{1}{4} \sum_{i,j} \lambda_i \lambda_j J_{ij} S_i S_j), \\
H_2(\sigma) &= -\frac{1}{4} \sum_{i,j\in\Lambda^c} \int_{I_i} \int_{I_j} W(|t-s|) \sigma_t \sigma_s dt ds, \\
H_{12}(S,\sigma) &= -\frac{1}{4} \sum_{i\in\Lambda} S_i \psi_i(\sigma), \text{ with} \\
\psi_i(\sigma) &= \sum_{j\in\Lambda^c} \int_{I_i} \int_{I_j} W(|t-s|) \sigma_t dt ds, \\
dQ(S) &= e^{-\alpha H_1(S)} \prod_{i\in\lambda} dP_1(S_i), \\
d\tilde{Q}(\sigma) &= e^{-\alpha H_2(\sigma)} \prod_{i\in\Lambda^c} dP_0(\sigma_i).
\end{aligned}$$

Notice that Q is the unnormalized Ising measure with interactions $\bar{J}_{ij}$.

We expand the exponential.

$$\begin{aligned}
Z_\lambda(H) \langle S_A \rangle_{\lambda,\bar{H}} &= \int S_A \exp(\frac{\alpha}{4} \sum_{i\in\Lambda} S_i \psi_i(\sigma)) dQ(S) d\tilde{Q}(\sigma) \\
&= \sum_n C_n \int S_A (\sum_{i\in\Lambda} S_i \psi_i(\sigma))^n dQ(S) d\tilde{Q}(\sigma) \\
&= \sum_n C_n \sum_{(m)\in\Gamma_n} C_{(m)} \int S_A \prod_i S_i^{m_i} dQ(S) \int \prod_i \psi_i^{m_i}(\sigma) d\tilde{Q}(\sigma),
\end{aligned}$$

where $\Gamma_n = \{(m) \equiv (m_1, \ldots, m_n) | m_i \geq 0, \sum_i m_i = n\}$ and $C_{(m)}$ are positive numbers.

Now, by GKS inequality, the first integral is bigger than

$$\frac{1}{Z''} \int S_A \, dQ(S) \int \prod_i S_i^{m_i} dQ(S),$$

where $Z'' = \int dQ(S)$ is a normalizing factor.

Also, the second integral is positive (this is proved like Lemma 4.1 by expanding $\prod_i \psi_i^{m_i}(\sigma)$ as well as the exponential in $\tilde{Q}$ and checking that everything is positive, which is done as before).

Notice now that $\frac{1}{Z''}\int S_A dQ(S) = \langle S_A \rangle_{\lambda,\bar{H}}$ and that this quantity does not depend on (m) or n. We thus have

$$\begin{aligned} Z_\lambda(H)\, \langle S_A \rangle_{\lambda,H} &\geq \langle S_A \rangle_{\lambda,\bar{H}} \sum_n C_n \sum_{\Gamma_n} C_{(m)} \int \prod_i (S_i \psi_i(\sigma))^{m_i} dQ(S) d\tilde{Q}(\sigma) \\ &= \langle S_A \rangle_{\lambda,\bar{H}} \int e^{-\alpha H(\sigma)} \prod_i dP_{\lambda_i}(\sigma_i). \end{aligned}$$

And Proposition 4.2 follows by the fact that the last integral is $Z_\lambda(H)$.

Proposition 4.2 and Lemma 4.1 imply the inequality

$$E_\lambda(\langle \sigma_A \rangle_{\lambda,H} Z_\lambda(H)) \geq E_\lambda(\langle \sigma_A \rangle_{\lambda,\bar{H}} Z_\lambda(H)). \tag{4.3}$$

We need one last result in this section.

Proposition 4.3 $\langle \sigma_A \rangle_{\lambda,\bar{H}}$ *is increasing in* λ.

Proof: Since $\langle \sigma_A \rangle_{\lambda,\bar{H}}$ is always nonnegative and vanishes whenever there is an element of A in a 0-interval, we need only check the proposition for A's without elements in 0-intervals. But this case follows by monotonicity in the couplings for correlations of the ordinary discrete Ising ferromagnet.

Now, Propositions 4.1 and 4.3 imply, via Harris-FKG inequality (see [9]), the following inequality for (3.5):

$$\begin{aligned} \frac{1}{Z'} E_\lambda(\langle \sigma_A \rangle_{\lambda,H} Z_\lambda(H)) &\geq \frac{1}{Z'} E_\lambda(\langle \sigma_A \rangle_{\lambda,\bar{H}} Z_\lambda(H)) \\ &\overset{\text{H-FKG}}{\geq} \frac{1}{Z'} E_\lambda(\langle \sigma_A \rangle_{\lambda,\bar{H}}) E_\lambda(Z_\lambda(H)) \\ &= E_\lambda(\langle \sigma_A \rangle_{\lambda,\bar{H}}), \end{aligned}$$

since $E_\lambda(Z_\lambda(H))$ is by definition Z'.

We have thus obtained the following comparison.

$$\langle \sigma_A \rangle' \geq E_\lambda(\langle \sigma_A \rangle_{\lambda,\bar{H}}) \tag{4.4}$$

The latter expression is the correlation of a quenched site diluted Ising model.

Remark: For the case of regular (discrete) annealed site diluted models (i.e. those where the spins at the sites are independently randomly diluted), a similar inequality follows by the same arguments.

5 Comparison to a Standard Discrete Ising Model

In this section, a general inequality relating quenched (site) diluted Ising model and an ordinary undiluted Ising model is derived. We then apply it to the quenched model at the end of last section to complete the comparison of the continuum to the discrete models and prove Theorem 1.

For an ordinary (discrete) Ising model in a finite volume Λ with ferromagnetic pair interactions J_{ij} and Hamiltonian

$$H(\sigma) = -\frac{1}{4}\sum_{i,j} J_{ij}\sigma_i\sigma_j,$$

let $\langle \cdot \rangle_{\alpha,J}$ denote the expectation w.r.t. the Ising measure (with free or + b.c.) at inverse temperature α.

Let $(\lambda_i)_{i\in\Lambda}$ be a sequence of independent nonnegative random variables which are less than one and have means (ξ_i). Let $\langle \cdot \rangle_{\alpha,\bar{J}}$ denote the (random) expectation w.r.t. the Ising measure with inverse temperature α and interactions $\bar{J}_{i,j} = \lambda_i\lambda_j J_{ij}$. We define the quenched site diluted Ising measure with expectation $\langle \cdot \rangle^q_{\alpha,J}$ to be the mean of the random measure w.r.t. the distribution of the λ's, i.e.

$$\langle \cdot \rangle^q_{\alpha,J} \equiv E_\lambda(\langle \cdot \rangle_{\alpha,\bar{J}}).$$

We prove the following result.

Proposition 5.1 *Let A be a set of points in Λ. Then*

$$\langle \sigma_A \rangle^q_{\alpha,J} \geq \langle \sigma_A \rangle_{\alpha,\tilde{J}}, \tag{5.1}$$

where $\tilde{J}_{ij} = \eta_i(\xi_i)\eta_j(\xi_j)J_{ij}$, and η_i is an increasing continuous function in $[0,1]$ which is 0 in 0 and 1 in 1, for each $i \in \Lambda$. (η_j does not depend on the distribution of the λ_i's, it does depend on α and the J_{ij}'s, especifically on $\frac{\alpha}{2}\sum_j J_{ij}$ — see (5.5).)

Proof:

For $i \in \Omega$, let $\eta_i(x)$ be a nonnegative function in $[0,1]$ such that $\eta_i(x) \leq x$. (The specific form of η_i will be chosen below.) By monotonicity of Ising correlations in the couplings, we have

$$\langle \sigma_A \rangle_{\alpha,\bar{J}} \geq \langle \sigma_A \rangle_{\alpha,J^*}, \tag{5.2}$$

where $J^*_{ij} = \eta_i(\lambda_i)\eta_j(\lambda_j)J_{ij}$.

Let $N = |\Omega|$ and write $\langle \sigma_A \rangle_{\alpha,J^*}$ as

$$\langle \sigma_A \rangle_{\alpha,J}(\eta_1(\lambda_1), \ldots, \eta_N(\lambda_N)).$$

Now suppose that

$$\frac{\partial^2}{\partial x^2} \langle \sigma_A \rangle_{\alpha,J}(\gamma_1, \ldots, \gamma_{i-1}, \eta_i(x), \gamma_{i+1}, \ldots, \gamma_N) \geq 0 \tag{5.3}$$

for all $0 \leq \gamma_i \leq 1$, $i \in \Lambda$ and all x in $(0,1)$. Then we get (5.1) by successive applying Jensen's inequality to $\langle \sigma_A \rangle^q_{\alpha,J}$.

By differentiating $\langle \sigma_A \rangle_{\alpha,J}$ as above, we obtain the following expression.

$$\begin{aligned}\{\eta_i'' - 2(\eta_i')^2 \langle \frac{\alpha}{4} \sum_{j\in\Lambda} \gamma_j J_{ij}\sigma_i\sigma_j \rangle\} \cdot [\langle \sigma_A (\frac{\alpha}{4} \sum_{j\in\Lambda} J_{ij}\gamma_j\sigma_i\sigma_j)\rangle - \langle \sigma_A \rangle \langle \frac{\alpha}{4} \sum_{j\in\Lambda} J_{ij}\gamma_j\sigma_i\sigma_j \rangle] \\ +(\eta_i')^2 [\langle \sigma_A (\frac{\alpha}{4} \sum_{j\in\Lambda} J_{ij}\gamma_j\sigma_i\sigma_j)^2 \rangle - \langle \sigma_A \rangle \langle (\frac{\alpha}{4} \sum_{j\in\Lambda} J_{ij}\gamma_j\sigma_i\sigma_j)^2 \rangle],\end{aligned}$$

where primes mean differentiation w.r.t. x. We have ommited the argument of η_i and the subscripts of the Ising expectation signs.

The expressions in square brackets are nonnegative, by GKS inequality, so that we only need the expression in braces to be positive. We use the boundedness of the σ's and γ's to bound it below by

$$\eta_i'' - \Gamma_i(\eta_i')^2, \tag{5.4}$$

where $\Gamma_i = \frac{\alpha}{2} \sum_{j\in\Lambda} J_{ij}$.

Setting (5.4) to zero and solving the differential equation with boundary conditions 0 in 0 and 1 in 1, we obtain

$$\eta_i = \zeta(\Gamma_i, x) \equiv \frac{1}{\Gamma_i} \log(1 - (1 - e^{-\Gamma_i})x)^{-1}, \tag{5.5}$$

which satisfies all the conditions above.

Remarks:

1. The above proposition holds for both free and + b.c..
2. A similar result is valid for a quenched *bond* model with a similar proof.
3. These results can be used to derive lower bounds for the critical temperature of diluted models in cases more general than, for example, those studied in [8]. For the cases studied in this reference, our bounds are weaker.

We are ready now for the

Proof of Theorem 1:

The results of this and previous sections give us the following comparison between the correlations of the continuum model (in $[-KL, KL]$, configurations denoted by the letter σ) and those of the discrete one (in $\{-K, \ldots, K\}$, configurations denoted by S).

$$\langle \sigma_A \rangle^f \geq \langle S_{A^*} \rangle^f_{\tilde{J}}, \tag{5.6}$$

where A is a set of points in $[-KL, KL]$ such that no two are in the same interval I_i $(= [iL, (i+1)L])$, $i \in \{-K, \ldots, K\}$, and A^* is the set of integers $i \in \{-K, \ldots, K\}$ such that there is a point of A in I_i. $\tilde{J}_{ij} = (\zeta(\Gamma, 1 - \mathring{)})^2 J_{ij}$, with $J_{ij} = \int_{I_i} \int_{I_j} W(|t-s|)\, dtds$ and $\Gamma = \frac{\alpha}{2} \sum_j J_{ij}$ (which does not depend on i, by translation invariance, and is finite, due to the decay of W — notice that by applying the proposition above directly, we obtain (5.6) but with the finite sum for Γ; we can then replace it by the infinite sum due to the monotonicity of both ζ and the Ising correlations).

Notice that the model in the r.h.s. of (5.6) is a (one dimensional) $1/r^2$ Ising model at inverse temperature $\alpha\zeta^2(\Gamma, 1 - \mathring{)}$. Notice also that $J_{ij} = J_{ij}(L)$ is such that, denoting

$$\int_i^{i+1} \int_j^{j+1} \frac{1}{|t-s|^2}\, dtds$$

by J'_{ij} for $|i-j| > 1$, we have that, as $L \to \infty$, J_{ij}/J'_{ij} converges to 1 uniformly in i, j such that $|i-j| \geq 2$, and $J_{i,i+1}$ (which does not depend on i) goes to ∞.

We want to use the result of [7] (Theorem 3.4) stating that, as $J'_{i,i+1} \to \infty$,

$$\langle S_0 S_x \rangle^f_{J'} \to 1$$

uniformly in the volume and in x inside the volume, provided $\alpha > 1$, to prove the following corresponding continuum result. As $\epsilon \to 0$,

$$\langle \sigma_0 \sigma_t \rangle_T^f \to 1$$

uniformly in the volume and in t, provided $\alpha > 1$.

We proceed as follows. Given $\alpha > 1$ and $\delta > 0$, let $\bar{\alpha}$ be such that $1 < \bar{\alpha} < \alpha$. By the discrete result just quoted, there exists J such that for the model with nearest neighbor interactions bigger than J, long range interactions given by J'_{ij} and inverse temperature $\bar{\alpha}$, we have

$$\langle S_0 S_x \rangle_{J'}^f \geq 1 - \delta$$

uniformly in the volume and in x. Now, let L be so big that $J_1 \equiv J_{i,i+1} > J$ and also $\alpha J_{ij} > \bar{\alpha} J'_{ij}$ for i, j with $|i - j| \geq 2$. Next, make ϵ so small that $^\circ$ is so small that ζ is so close to 1 that $\zeta^2 \alpha J_1 > \bar{\alpha} J$ and $\zeta^2 \alpha J_{ij} > \bar{\alpha} J'_{ij}$. By applying the comparison (5.6), we get

$$\langle \sigma_0 \sigma_t \rangle_T^f \geq 1 - \delta \tag{5.7}$$

uniformly in the volume and for $|t| > L$. If necessary we can take ϵ smaller so that (5.7) holds uniformly in t. The theorem is now proven.

6 Slow Decay of Correlations

In this last section, we use the FK representation of the continuum Ising model to derive lower bounds for the truncated two point function, proving Theorem 2. It is done almost exactly in the same way as has been done in [7] for discrete FK models, with a few modifications (to account for the extra randomness of the continuum case). For this reason we will be a bit sketchy, referring the reader to the discrete results for missing details (also to [3] for definitions and properties of continuum FK measures).

We start by defining the FK measures. Let θ_i, $i = 1, 2, \ldots$ be the points of a Poisson process of rate ϵ on the real line and $\omega_j = (s_j, t_j)$, $j = 1, 2, \ldots$ the points of a Poisson process in $R_0^2 = \{(s,t) : s \leq t\}$ with density $\Delta = \alpha W(t - s)$. Denote these (random) sets of points by θ and ω, respectively, and call them configurations. (We will alternatively use the terminology θ-points for θ.) The ω_j's will be given the meaning of occupied bonds linking s_j and t_j.

Consider the partition of R (resp., of the interval $[-T, T]$, for $T > 0$) into intervals, produced by the points θ_i in $[-T, T]$. Call those θ-intervals. Say that two disjoint intervals I and J are linked (denoted $I \frown J$) if there is an occupied bond linking two points, one in each interval. (If there are none, we denote $I \not\frown J$. Notice that the two infinite intervals of the partition of R are linked with probability 1.) Two θ-intervals I and J are connected if there is a sequence of θ-intervals $I_0, \ldots, I_n$ with $I_0 = I$ and $I_n = J$, so that $I_i \frown I_{i-1}$, $i = 1, \ldots, n$. Two points s and t are connected (denoted $s \leftrightarrow t$) if either they belong to the same θ-interval or belong to distinct connected θ-intervals. A *cluster* is a maximal union of connected θ-intervals.

Let $C_T^w(\theta, \omega)$ (resp., $C_T^f(\theta, \omega)$) be the number of distinct connected clusters obtained with the θ-intervals of the partition of R (resp., of $[-T, T]$). We define the finite volume,continuum FK measures with parameter q as follows

$$dP^*_{q,\Delta,T}(\theta, \omega) = \frac{1}{N} dP(\theta) dP'(\omega) q^{C_T^*(\theta,\omega)}, \tag{6.1}$$

where $* = w$ or f for the *wired* and *free* cases (see [3]), P and P' are the Poisson processes mentioned and N is the normalizing factor. (We will drop some subscripts sometimes).

The infinite volume measure exists (by standard arguments) and is denoted $P^*_{q,\Delta}$. Notice that for $q = 1$, $P^w_{1,\Delta} = P^f_{1,\Delta}$ is an independent (continuum) percolation model. We list the properties of the FK measures we will need.

1) $P^*_{q,\Delta}$ is a strong FKG measure, i.e., for any region A in $R \times R_0^2$ and f, g increasing functions in the configurations (w.r.t. the partial order $(\theta, \omega) \prec (\theta', \omega')$ whenever $\theta \supset \theta'$ and $\omega \subset \omega'$), we have

$$P^*_{q,\Delta}(fg|\mathcal{A}) \geq P^*_{q,\Delta}(f|\mathcal{A}) P^*_{q,\Delta}(g|\mathcal{A}),$$

where $\mathcal{A}$ is the σ-algebra generated by the configurations in A.

For the properties below, we use the notation $P(\cdot)$ for the expectation w.r.t. the measure P.

2) $P^*_{q,\Delta}(f) \geq P^*_{q',\Delta'}(f)$, for $q' \geq q, \Delta \geq \Delta'$ and f increasing.

3) $P^w_{q,\Delta}(f) \geq P^f_{q,\Delta}(f)$, for f increasing.

4) We have the following representation of continuum Ising correlations (where, the notation $0 \leftrightarrow \infty$ means that the cluster of the origin is infinite):

$$\begin{aligned} M &= \langle \sigma_0 \rangle^+ = P_2^w(0 \leftrightarrow \infty), \\ \langle \sigma_s \sigma_t \rangle^* &= P_2^*(s \leftrightarrow t). \end{aligned}$$

We conclude

$$\begin{aligned}\tau(t) &= \langle\sigma_0\sigma_t\rangle^+ - M^2 = P_2^f(0 \leftrightarrow t) - (P_2^w(0 \leftrightarrow \infty))^2 \\ &\geq P_2^w(0 \leftrightarrow t, 0 \not\leftrightarrow \infty, t \not\leftrightarrow \infty) \equiv \tau'(t).\end{aligned}$$

So, all we need to prove Theorem 2 is to derive the same bounds for τ'. We do that in the propositions below.

As in [7], we begin with an estimate for the *self similar* percolation case, i.e., the $q = 1$ case with

$$W(t) = \tilde{W}(t) \equiv \frac{1}{t^2} 1_{\{t>1\}}. \tag{6.2}$$

For ξ a real number, let

$$T_\xi = \inf\{\theta_i : \theta_i \geq \xi\}, \;\; S_\xi = \sup\{\theta_i : \theta_i \leq \xi\}.$$

Define

$$\begin{aligned}\mu_1^\xi &= T_\xi \\ \mu_{n+1}^\xi &= T_{\mu_n^\xi}, \; n \geq 1 \\ \nu_1^\xi &= S_\xi \\ \nu_{n+1}^\xi &= S_{\nu_n^\xi}, \; n \geq 1, \text{ i.e.}\end{aligned}$$

μ_n^ξ is the n-th θ-point after ξ and ν_n^ξ is the n-th θ-point before ξ.

Notice that $\mu_n^\xi = \xi + Y_n$ and $\nu_n^\xi = \xi - W_n$, where Y_n and W_n are random variables each having a Gamma distribution with parameters n and ϵ. In particular, $E\mu_n^\xi = \xi + n/\epsilon$ and $E\nu_n^\xi = \xi - n/\epsilon$.

We say that an interval $[\xi', \xi]$ is *dissociated* if there is no occupied bond from $[S_{\xi'}, T_\xi]$ to its complement. Below, we use the notation $\{I \not\leftrightarrow \infty\}$ for an interval I none of whose points are in an infinite cluster.

Proposition 6.1 *For L a positive integer, let $F_L = \{\exists$ an integer $k \in [1, L) | [0, \mu_k^L) \not\to (\mu_k^L, \infty)\}$.*

If $\alpha > 1$, then there exists constants C and C' so that in the self similar case (6.2)

$$\begin{aligned}P_1(F_L) &\geq C/L^{\alpha-1} \;\; \textit{for all } L, &(6.3)\\ P_1([0,L] \not\leftrightarrow \infty) &\geq C'/L^{2(\alpha-1)} \;\; \textit{for all } L. &(6.4)\end{aligned}$$

Proof: Define

$$\begin{aligned}
F_L^* &= \{\exists \text{ integer } k' \in [1,L) | [\nu_k^0, L] \not\to (-\infty, \nu_k^0)\},\\
H_L &= \{(\nu_L^0, 0) \not\to (L, \infty)\},\\
H_L^* &= \{(L, \mu_L^L) \not\to (-\infty, 0)\}.
\end{aligned}$$

Then

$$\begin{aligned}
P_1([0,L] \not\leftrightarrow \infty) &\geq P_1(F_L \cap F_L^* \cap H_L \cap H_L^*)\\
&\geq P_1^2(F_L)\, P_1^2(H_L),
\end{aligned}$$

with the second inequality due to the FKG property (all of the events are decreasing).

Now,

$$\begin{aligned}
P_1(H_L) &= E\{\exp(-\alpha \int_{\nu_L^0}^0 \int_L^\infty (t-s)^{-2}\, dtds)\}\\
&= E\{\exp(-\alpha \int_{-Y}^0 \int_1^\infty (t-s)^{-2}\, dtds)\}\\
&\geq \exp(-\alpha \int_{-EY}^0 \int_1^\infty (t-s)^{-2} dtds),
\end{aligned}$$

where the last inequality is Jensen's inequality and the expectation E is w.r.t. a Gamma random variable Y.

The last expression is positive and does not depend on L. It follows that (6.3) implies (6.4).

To derive (6.3), let

$$\mathcal{N} = \#\{k \in [1,L) \cap \mathbf{Z} : [0, \mu_k^L] \not\to (\mu_k^L, \infty)\}.$$

F_L is the event that $\mathcal{N} > 0$. We compute the expected value of $\mathcal{N}$.

$$\begin{aligned}
E_1(\mathcal{N}) &= \sum_{k=1}^{L-1} P_1([0,\mu_k^L] \not\to (\mu_k^L, \infty))\\
&= \sum_1^{L-1} E\{\exp[-\alpha(\int_0^{\mu_k^L} \int_{\mu_k^L+1}^\infty (t-s)^{-2}\, dtds\\
&\quad + \int_{\mu_k^L}^{\mu_k^L+1} \int_0^{t-1} (t-s)^{-2}\, dsdt)]\}
\end{aligned}$$

$$\begin{aligned}
&\geq const. \sum_{1}^{L-1} E(\frac{1}{(\mu_k^L)^\alpha}) \geq const. \sum_{1}^{L-1} \frac{1}{(E\mu_k^L)^\alpha} \\
&= const. \sum_{1}^{L-1} \frac{1}{(L+k/\epsilon)^\alpha} \geq const L^{1-\alpha},
\end{aligned}$$

where the second inequality follows by Jensen's inequality and E is expectation w.r.t. μ_k^L.

Now,

$$P_1(\mathcal{N} > 0) = \frac{E_1(\mathcal{N})}{E_1(\mathcal{N}|\mathcal{N} > 0)}.$$

Let $X = \inf\{k' \in [1, L) \cap \mathbf{Z} : [0, \mu_{k'}^L] \not\to (\mu_{k'}^L, \infty)\}$. Then,

$$\begin{aligned}
E_1(\mathcal{N}|\mathcal{N} > 0) &= \sum_{k'=1}^{L-1} \int_L^\infty P_1(X = k', \mu_{k'}^L \in dt|\mathcal{N} > 0) E_1(\mathcal{N}|X = k', \mu_{k'}^L = t) \\
&= \sum_{k'=1}^{L-1} \int_L^\infty P_1(X = k', \mu_{k'}^L \in dt|\mathcal{N} > 0) \\
&\qquad \cdot (1 + \sum_{k=1}^{L-k'} P_1((t, \mu_k^t] \not\to (\mu_k^t, \infty)) \\
&\leq \sum_{k'=1}^{L-1} \int_L^\infty P_1(X = k', \mu_{k'}^L \in dt|\mathcal{N} > 0) \\
&\qquad \cdot (1 + \sum_{k=1}^{\infty} E\{\exp(-\alpha \int_t^{\mu_k^t} \int_{\mu_k^t+1}^\infty (r-s)^{-2} dr ds)\}) \\
&\leq \sum_{k'=1}^{L-1} \int_L^\infty P_1(X = k', \mu_{k'}^L \in dt|\mathcal{N} > 0)(1 + \sum_{k=1}^{\infty} E(\frac{1}{(\mu_k^t - t + 1)^\alpha})) \\
&= \sum_{k'=1}^{L-1} \int_L^\infty P_1(X = k', \mu_{k'}^L \in dt|\mathcal{N} > 0)(1 + \sum_{k=1}^{\infty} E(\frac{1}{(Y_k + 1)^\alpha})) \\
&\leq 1 + const \sum_{1}^{\infty} \frac{1}{k^\alpha} = const.
\end{aligned}$$

Combining all these inequalities, we get (6.3).

In the next proposition, we omit the subscript Δ in P^*.

Proposition 6.2 *For $|t| \leq L$,*

$$\tau'(t) \geq P_{2,L}^f(0 \leftrightarrow t)\, P_2^w([-L, L] \not\leftrightarrow \infty).$$

Proof: Let $\Lambda_L =$ bonds s, t with $|s|, |t| \leq L$. Then,

$$\tau'(t) \geq P_2^w(0 \leftrightarrow t \text{ by bonds in } \Lambda_L | [-L, L] \not\leftrightarrow \infty) P_2^w([-L, L] \not\leftrightarrow \infty).$$

The first probability on the r.h.s. can be estimated by first noticing that the conditioning event only depend on bonds in Λ_L^c and points of θ outside $[-L, L]$. Proceed now exactly as in the proof of Proposition 2.1 in [7], by conditioning further on such configurations, expressing the infinite volume measure as the proper limit of the finite volume ones, and then using the strong FKG property to conclude that

$$\begin{aligned}
& P_2^w(0 \leftrightarrow t \text{ by bonds in } \Lambda_L | [-L, L] \not\leftrightarrow \infty) \\
\geq \ & \lim_{L' \to \infty} P_{2,L'}^w(0 \leftrightarrow t \text{ by bonds in } \Lambda_L | \not\exists \text{ occupied bonds in } \Lambda_L^c) \\
= \ & P_2^f(0 \leftrightarrow t).
\end{aligned}$$

Proposition 6.3 *If $\alpha > 1$, then for any $\delta > 0$, there exists some $C' > 0$ so that*

$$P_{2,\Delta}^w([0, L] \not\leftrightarrow \infty) \geq C'/L^{2(\alpha-1)+\delta}, \text{ for all } L \geq 1.$$

Proof: Let $\hat{W} = W\chi(|t| < R) + \tilde{W}\chi(|t| > R)$, for $\tilde{W}$ given by (6.2), where χ is the indicator function of a set. Given δ, choose $\hat{\alpha} > \alpha$ so that $2(\hat{\alpha} - 1) = 2(\alpha - 1) + \delta$, and R so that $\hat{\alpha} W > \alpha \tilde{W}$, for $t > R$. Let $\hat{\alpha}\hat{W} = H_\delta$.

We then have

$$P_{2,\Delta}^w([0, L] \not\leftrightarrow \infty) \geq P_{1,\hat{\Delta}}([0, L] \not\leftrightarrow \infty),$$

by the monotonicity properties of the FK measures.

Exactly as in Proposition 6.1

$$P_{1,H_\delta}([0, L] \not\leftrightarrow \infty) \geq const P_{1,H_\delta}^2(F_L).$$

Define $\hat{F}_L = \{\exists\ \xi \in [L, 2L) \cap \mathbf{Z} |$ there is no occupied bond longer than R linking $[0, T_\xi]$ to $(T_\xi, \infty)\}$.

Since $\hat{F}_L$ does not involve the short bonds distinguishing between P_{1,H_δ} and $P_{1,\hat{\alpha}\tilde{W}}$, we have

$$P_{1,H_\delta}(\hat{F}_L) = P_{1,\hat{\alpha}\tilde{W}}(\hat{F}_L) \geq P_{1,\hat{\alpha}\tilde{W}}(F_L) \geq C/L^{\hat{\alpha}-1},$$

by Proposition 6.1.

Now, conditioning

$$
\begin{aligned}
P_{1,H_\delta}(F_L) &= P_{1,H_\delta}(\hat{F}_L)P_{1,H_\delta}(F_L|\hat{F}_L) \\
&\geq P_{1,H_\delta}(\hat{F}_L)E\{e^{-\hat{\alpha}\int_{T_\xi-R}^{T_\xi}\int_{T_\xi}^{T_\xi+R}W(t-s)\,dtds}\}, \\
&= const P_{1,H_\delta}(\hat{F}_L).
\end{aligned}
$$

To complete the proof of Theorem 2 we need the following result.

Proposition 6.4 *There is a constant $C > 0$ such that*

$$\tau'(t) \geq C/t^2, \text{ for } t \text{ large.}$$

Proof: We consider the event Λ_t that the θ-intervals I_0, I_t containing the origin and the point t respectively are connected to each other but to no other θ-interval. Condition on θ and observe that the resulting measure is a discrete FK measure. Follow the steps of [7] to find

$$
\begin{aligned}
P_2^w(\Lambda_t|\theta) &\geq (1 - e^{-\alpha\int_{I_0}\int_{I_t}W(|t-s|)\,dtds}) \\
&\quad \cdot e^{-\alpha\int_{I_0}\int_{I_0^c}W(|t-s|)\,dtds}e^{-\alpha\int_{I_t}\int_{I_t^c}W(|t-s|)\,dtds} \\
&\equiv f(I_0, I_t).
\end{aligned}
$$

Now, there exist constants $0 < a < b < \infty$, $0 < c < 1$, such that

$$P_2^w(a < |I_0| < b, a < |I_t| < b) \geq c,$$

where a, b, c do not depend on t.

We conclude that for t large enough

$$
\begin{aligned}
P_2^w(\Lambda_t) &\geq P_2^w(f(I_0, I_t)1(a < |I_0| < b, a < |I_t| < b)) \\
&\geq \frac{const}{t^2}P_2^w(a < |I_0| < b, a < |I_t| < b) \\
&= const/t^2.
\end{aligned}
$$

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