Renormalization group and the Fermi surface in the Luttinger model.

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Abstract: the exactly soluble Luttinger model can be also analyzed from the point of view of the renormalization group. A perturbation theory of the beta function of the model is derived. We argue that the main terms of the beta function vanish identically if the anomalous dimension is properly treated and if suitable properties of the exact solution are taken into account. Our treatment is purely perturbative and we do not discuss the problems of convergence of the formal series defining the beta function: however the property that the series defining it is convergent has been recently established.

§*1 The Luttinger model*

The recent interest on the Luttinger model, [A], motivates our discussion of its properties in a formalism which admits extensions to higher dimensions, developed in [BG].

The model [L] describes two spinless fermions labeled by $\vec\omega = \pm1$, with a one dimensional hamiltonian:

$$\begin{aligned}H = [T_0] + \{H_I\} = \Big[\sum_{\vec\omega=\pm1}\int_0^L d\vec x\tilde\psi^+_{\vec x,\vec\omega}\beta_0(i\vec\omega\vec\partial - p_F)\tilde\psi^-_{\vec x,\vec\omega}\Big]+\\ + \Big\{\lambda\int_0^L d\vec x d\vec y v(\vec x-\vec y)(\sum_{\vec\omega=\pm1} q_{1,\vec\omega}\tilde\psi^+_{\vec x,\vec\omega}\tilde\psi^-_{\vec x,\vec\omega})(\sum_{\vec\omega=\pm1} q_{2,\vec\omega}\tilde\psi^+_{\vec y,\vec\omega}\tilde\psi^-_{\vec y,\vec\omega})+\\ + \nu\sum_{\vec\omega=\pm1}[q_{1,\vec\omega}+q_{2,\vec\omega}]\int_0^L d\vec x\tilde\psi^+_{\vec x,\vec\omega}\tilde\psi^-_{\vec x,\vec\omega} + \sigma\int_0^L d\vec x\Big\}\end{aligned} \tag{1.1}$$

where $\tilde\psi^\pm$ are creation and annihilation field operators, $\vec x$ and $\vec y$ are position variables in the interval $[0,L]$ considered with periodic boundary conditions, $p_F = \frac{2\pi}{L}(n_F+1/2)$ is the

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Fermi momentum (n_F is an integer depending on L so that p_F is independent of L up to terms of order $1/L$), and β_0 is the *velocity at the Fermi surface*; $\lambda v(\vec{r})$ is the interaction potential, which will be supposed with short range, equal to a fixed length p_0^{-1}, and even as a function of $\vec{r}$; the *charges* $q_{1,\vec{\omega}}$ and $q_{2,\vec{\omega}}$ are arbitrary constants. Finally ν and σ are *counterterms*, necessary to balance the ultraviolet divergences due to the unrealistic linear dispersion relation in the (kinetic energy − chemical potential) term T_0; in fact a fermion field of type $\vec{\omega}$ and momentum $\vec{k}$ has energy $\beta_0\vec{\omega}\vec{k}$. We fix units so that the Fermi velocity is $\beta_0 = 1$.

The case considered by Luttinger was $q_{1,+} = 1$, $q_{1,-} = q_{2,+} = 0$, $q_{2,-} = 1$. The model was solved in [ML], but the exact solution applies to the general choice of q_i; the case $q_{i,\pm} = 1/2$ is explicitly treated in [ML] and extended to a simple spinning model in [M].

The values of ν and σ have to be computed by introducing a ultraviolet cut-off in (1.1) (which otherwise does not have a well defined meaning) and, subsequently, by imposing that the Schwinger functions of the model are well defined uniformly in the cut-off. Their values depend upon the way the ultraviolet regularization is introduced and can be altered by an arbitrary finite constant (possibly affecting the physical value of the Fermi momentum or the Fermi velocity).

The regularization which is implicit in the exact theory of the ground state seems to be simply the suppression of the modes with $\vec{k} < -2^U p_0$ for the $\vec{\omega} = +1$ fermions and $\vec{k} > 2^U p_0$ for the $\vec{\omega} = -1$ fermions, where p_0 is an arbitrary (for the time being) momentum scale and U is a cut-off parameter to be let to ∞ eventually. It is natural and convenient to fix ${p_0}^{-1}$ equal to the range of the interaction potential, supposed of finite range.

Since the momenta $\pm p_F$ play a special role for the two fermions, it is convenient to measure the momenta of the $\vec{\omega}$-type fermions from $p_F\vec{\omega}$. If we call $\alpha^{\pm}_{\vec{k},\vec{\omega}}$ the creation and annihilation operators of the two fermions, we introduce the following field operators:

$$\begin{aligned}
\tilde{\psi}^{\pm}_{\vec{x},t,\vec{\omega}} &\equiv e^{tT_0}\tilde{\psi}^{\pm}_{\vec{x},\vec{\omega}}e^{-tT_0} = \frac{1}{\sqrt{L}}\sum_{\vec{k}} e^{\pm[i\vec{k}\vec{x}+(\vec{\omega}\vec{k}-p_F)t]}\alpha^{\pm}_{\vec{k},\vec{\omega}} = \\
&= e^{\pm ip_F\vec{\omega}\vec{x}}\psi^{\pm}_{\vec{x},t,\vec{\omega}} \\
\psi^{\pm}_{\vec{x},t,\vec{\omega}} &\equiv e^{tT_0}\psi^{\pm}_{\vec{x},\vec{\omega}}e^{-tT_0} = \frac{1}{\sqrt{L}}\sum_{\vec{k}} e^{\pm(i\vec{k}\vec{x}+t\vec{\omega}\vec{k})}a^{\pm}_{\vec{k},\vec{\omega}} \\
a^{\pm}_{\vec{k},\vec{\omega}} &\equiv \alpha^{\pm}_{\vec{k}+p_F\vec{\omega},\vec{\omega}}
\end{aligned} \tag{1.2}$$

The following hamiltonian operators, necessary to establish contact with the existing literature, will also be introduced, following [ML]:

$$T_0' = \sum_{\vec{\omega}}\sum_{\vec{k}>0}\vec{k}(a^{+}_{\vec{\omega}\vec{k},\vec{\omega}}a^{-}_{\vec{\omega}\vec{k},\vec{\omega}} + a^{-}_{-\vec{\omega}\vec{k},\vec{\omega}}a^{+}_{-\vec{\omega}\vec{k},\vec{\omega}}) \tag{1.3}$$

$$H'_I = L^{-1}\lambda \sum_{\vec p>0} \hat v(\vec p)[R_1(\vec p)R_2(-\vec p) + R_1(-\vec p)R_2(\vec p)]+$$
$$+ L^{-1}\lambda\hat v(0)\left[\sum_{\vec\omega} q_{1,\vec\omega}\mathcal{N}_{\vec\omega}\right]\left[\sum_{\vec\omega} q_{2,\vec\omega}\mathcal{N}_{\vec\omega}\right]$$

where:

$$\begin{aligned} R_i(\vec p) &= \sum_{\vec\omega} q_{i,\vec\omega}\rho_{\vec\omega}(\vec p), \qquad \rho_{\vec\omega}(\vec p) = \sum_{\vec k} a^+_{\vec k+\vec p,\vec\omega} a^-_{\vec k,\vec\omega} \\ \mathcal{N}_{\vec\omega} &= \sum_{\vec k>0}(a^+_{\vec\omega\vec k,\vec\omega}a^-_{\vec\omega\vec k,\vec\omega} - a^-_{-\vec\omega\vec k,\vec\omega}a^+_{-\vec\omega\vec k,\vec\omega}) \end{aligned} \tag{1.4}$$

note that T'_0 is equal to $\sum_{\vec\omega,\vec k}\vec\omega\vec k a^+_{\vec k,\vec\omega}a^-_{\vec k,\vec\omega}$ up to a constant; but the constant, see below, is infinite, hence this simpler form for T'_0 is not defined (although it can be very useful for heuristic purposes).

One can check that the operators (1.3), (1.4) can be regarded as operators acting on a Hilbert space $\mathcal{H}$ constructed as follows. Let:

$$|0\rangle = \prod_{\vec k\le 0} a^+_{\vec k,+1}a^+_{-\vec k,-1}|\text{vacuum}\rangle \tag{1.5}$$

be an abstract vector, formally in Fock space. Let $\mathcal{H}_0$ be the abstract linear span of the formal vectors obtained by applying finitely many creation and annihilation operators to $|0\rangle$. We get an abstract linear space on which we introduce the scalar product between two vectors by computing it in the obvious way *as if* they were Fock space vectors (no problem arises because we only deal with vectors obtained by acting finitely many times on $|0\rangle$ with the basic operators); then we define $\mathcal{H}$ to be the completion of $\mathcal{H}_0$ in the just introduced scalar product.

With such definitions it is easy to check the following basic commutation relations:

$$\begin{aligned} &[\rho_{\vec\omega}(\varepsilon\vec p), \rho_{\vec\omega'}(-\varepsilon\vec p\,'))] = -\varepsilon\vec\omega\vec p\delta_{\vec\omega,\vec\omega'}\delta_{\vec p,\vec p'}\frac{L}{2\pi}, \qquad [\rho_{\vec\omega}(\varepsilon\vec p), T'_0] = -\varepsilon\vec\omega\vec p\rho_{\vec\omega}(\varepsilon\vec p) \\ &[\rho_{\vec\omega}(\varepsilon\vec p), \sum_{\vec p>0,\vec\omega'}\rho_{\vec\omega'}(\vec\omega'\vec p)\rho_{\vec\omega'}(-\vec\omega'\vec p)] = -\varepsilon\vec\omega\vec p\frac{L}{2\pi}\rho_{\vec\omega}(\varepsilon\vec p), \qquad \rho_{\vec\omega}(-\vec\omega\vec p)|0\rangle \equiv 0 \end{aligned} \tag{1.6}$$

for $\vec p > 0, \varepsilon = \pm 1$.

Furthermore the operators (1.3), (1.4), regarded as operators on $\mathcal{H}$ with domain $\mathcal{H}_0$, are essentially selfadjoint. A simple calculation shows that, if (setting $q_\varepsilon = \sum_{\vec\omega} q_{\varepsilon\vec\omega}$):

$$\begin{aligned} \nu &= -\lambda\hat v(0)(2^U p_0 + p_F)/2\pi \\ \sigma &= q_+q_-\lambda\hat v(0)(2^U p_0 + p_F)^2/(4\pi^2) - L^{-1}\langle 0|T_0|0\rangle \end{aligned} \tag{1.7}$$

then one has $T_0' + H_I' = T_0 + H_I$, in a formal sense as the $T_0 + H_I$ is defined using a ultraviolet cut off $2^U p_0$. The latter relation becomes an identity in the limit $U \to \infty$.

Moreover one can also write:

$$
\begin{aligned}
T_0' + H_I' = & \sum_{\vec{\omega}} \int d\vec{x} \, : \psi^+_{\vec{x},\vec{\omega}} (i\vec{\omega}\vec{\partial}) \psi^-_{\vec{x},\vec{\omega}} : + \\
& + \lambda \int d\vec{x} d\vec{y} \, v(\vec{x}-\vec{y}) : (\sum_{\vec{\omega}} q_{1,\vec{\omega}} \psi^+_{\vec{x},\vec{\omega}} \psi^-_{\vec{x},\vec{\omega}}) :: (\sum_{\vec{\omega}} q_{2,\vec{\omega}} \psi^+_{\vec{y},\vec{\omega}} \psi^-_{\vec{y},\vec{\omega}}) :
\end{aligned}
\tag{1.8}
$$

where : : denotes the Wick ordering with respect to the vacuum $|0>$ of $\mathcal{H}$ (*i.e.* the Wick ordering of a product of creation and annihilation operators is obtained by rearranging the order so that $a^+_{-\vec{k},+}, a^-_{\vec{k},+}, a^+_{\vec{k},-}, a^-_{-\vec{k},-}, \vec{k} > 0$ are always to the right of the other operators, and the new product is multiplied by the parity sign of the permutation necessary to produce it).

We adopt the choice (1.7) of the counterterms because it allows the simple interpretation (1.8) of the hamiltonian in terms of Wick ordering. However the model thus obtained is not, strictly speaking, identical to the model of Luttinger as solved by Mattis-Lieb [ML]. They in fact add to (1.1) an extra term so that the operator H_I' in (1.3) is just given by the first line, *i.e.* $\hat{v}(0)$ is in some sense forced to vanish, without requiring $\hat{v}(\vec{p})$ to vanish continously when $\vec{p} \to 0$. But the second line in the definition (1.3) of H_I' is an operator C_I' commuting with T_0', as well as with all the operators $\rho_{\vec{\omega}}(\vec{p})$ and this, of course, implies that the model (1.1) with the choices (1.7) of ν, σ, *i.e.* (1.3) or (1.8), is exactly soluble in the same sense of the Luttinger model and the two hamiltonians are defined on the same Hilbert space and have the same eigenvectors. The only problem is that C_I' is not bounded below; however it is easy to see that $T_0'' \equiv T_0' + C_I'$ is still bounded below if $\lambda\hat{v}(0)$ satisfies the stability condition:

$$
\begin{aligned}
(\lambda\hat{v}(0)P)^2 &\leq (2\pi + 2\lambda\hat{v}(0)q_{1+}q_{2+})(2\pi + 2\lambda\hat{v}(0)q_{1-}q_{2-}) \\
P &= q_{1+}q_{2-} + q_{1-}q_{2+}
\end{aligned}
\tag{1.9}
$$

In fact, if we consider the action of T_0'' on the states with n_1 *excitations* (*i.e.* number of particles minus number of holes) of tipe $\vec{\omega} = +$ and n_2 of type $\vec{\omega} = -$, we have:

$$
T_0'' \geq \frac{\pi}{L}(n_1^2 + n_2^2) + \frac{\lambda\hat{v}(0)}{L}(q_{1+}q_{2+}n_1^2 + q_{1-}q_{2-}n_2^2) - |\frac{\lambda\hat{v}(0)}{L}P||n_1||n_2| \tag{1.10}
$$

and the r.h.s. is bounded below if and only if (1.9) is satisfied.

As we shall see below, the condition (1.9) is implied by the solubility condition of the model in the case considered in [ML] ($q_{i,\vec{\omega}} = 1/2$), but this is not true in general. However

(1.9) is always implied by the stability condition for the full hamiltonian, if $\hat{v}(p)$ is a continuous function, as we shall suppose.

Let us now define $H''_I \equiv H'_I - C'_I$, so that $T'_0 + H'_I = T''_0 + H''_I$. The basic remark of [ML] is that the commutation relations in (1.6) imply:

$$[\rho_{\vec{\omega}}(\pm\vec{p}), T - T''_0] \equiv 0, \qquad \text{for } \vec{p} > 0 \tag{1.11}$$

if $T \equiv (2\pi/L)\sum_{\vec{p}>0,\vec{\omega}} \rho_{\vec{\omega}}(\vec{\omega}\vec{p})\rho_{\vec{\omega}}(-\vec{\omega}\vec{p})$. Hence $T''_0 - T$ commutes with all the operators $\rho_{\vec{\omega}}(\vec{p})$, and, therefore, with $H''_I + T$. In this way we realize $T''_0 + H''_I$ as the sum of two commuting operators, the second of which is a sum of easily diagonalizable commuting operators and this leads to the exact solubility of the model, see [ML]. This is done by determining an even function $\varphi(p)$ such that setting $S = 2\pi L^{-1}\sum_{all\ p\neq 0}\varphi(p)p^{-1}\rho_+(p)\rho_-(-p)$ then the operator $e^{iS}(H''_I + T)e^{-iS}$ does not contain *mixed terms*, *i.e.* it can be written, if $E_0(\lambda)$ is a suitable constant, in the form:

$$\frac{2\pi}{L}\sum_{p>0}[\varepsilon_+(p)\rho_+(p)\rho_+(-p) + \varepsilon_-(p)\rho_-(-p)\rho_-(p)] + E_0(\lambda) \tag{1.12}$$

and one checks that this is achieved by taking:

$$\tanh 2\varphi(p) = -\frac{\lambda\hat{v}(p)P}{2\pi + \lambda\hat{v}(p)Q} \quad , \qquad \begin{cases} P = q_{1+}q_{2-} + q_{1-}p_{2+} \\ Q = q_{1+}q_{2+} + q_{1-}q_{2-} \end{cases} \tag{1.13}$$

and:

$$\begin{aligned} \varepsilon_+(p) &= c(p)^2\Big(1 + \frac{\lambda\hat{v}(p)}{\pi}q_{1+}q_{2+}\Big) + s(p)^2\Big(1 + \frac{\lambda\hat{v}(p)}{\pi}q_{1-}q_{2-}\Big) + \\ &+ \frac{\lambda\hat{v}(p)}{\pi}c(p)s(p)P \\ \varepsilon_-(p) &= s(p)^2\Big(1 + \frac{\lambda\hat{v}(p)}{\pi}q_{1+}q_{2+}\Big) + c(p)^2\Big(1 + \frac{\lambda\hat{v}(p)}{\pi}q_{1-}q_{2-}\Big) + \\ &+ \frac{\lambda\hat{v}(p)}{\pi}c(p)s(p)P \end{aligned} \tag{1.14}$$

where $c(p) = \cosh\varphi(p)$, $s(p) = \sinh\varphi(p)$. Of course one needs that the r.h.s. of the definition (1.13) of the hyperbolic tangent be < 1 in absolute value: we shall call this the "solubility condition". Moreover (1.12) and (1.14) imply that the hamiltonian is bounded below if and only if:

$$(\lambda\hat{v}(p)P)^2 \le (2\pi + 2\lambda\hat{v}(p)q_{1+}q_{2+})(2\pi + 2\lambda\hat{v}(p)q_{1-}q_{2-}) \tag{1.15}$$

This stability condition is a consequence of the solubility condition only if $q_{1+}q_{2+} = q_{1-}q_{2-}$, as is the case considered in [ML] or in the original Luttinger model. In general only the converse is true, *i.e.* the stability condition (1.15) implies that the r.h.s. of (1.13) is < 1 in absolute value, so that one should always assume the stability condition (1.15).

In the rest of this paper we shall consider, as in [ML], the case $q_{i\vec\omega} = 1/2$, $i = 1, 2$; then:

$$\varepsilon_+(p) = \varepsilon_-(p) = e^{-2\varphi(p)} = \big(1 + \frac{\lambda\hat v(p)}{2\pi}\big)^{1/2} \tag{1.16}$$

and the ground state energy is: $E_0(\lambda) = \sum_{p>0} p(e^{-2\varphi(p)} - 1)$.

Let us remark that the operator $T_0'' - T$ can be explicitly computed and it is a constant in every linear space containing a given number of excitations (this is non trivial and is implicit in [ML], as pointed out in [O]). The constant can be computed in a state with n_1 excitations of tipe $\vec\omega = +$ and n_2 of type $\vec\omega = -$, simply by evaluating the expectation value of $T_0'' - T$ on the ground state with the same number of excitations, namely the state with the first n_1 levels of type $\vec\omega = +$ occupied and the first n_2 of type $\vec\omega = -$ occupied (if $n_i < 0$ then one means, of course, holes created). And the problem is solved by the remark that the commutation rules (1.6) imply that $e^{iS}\rho_+(p)e^{-iS}$, $e^{iS}\rho_-(-p)e^{-iS}$ are bosonic creation operators while $e^{iS}\rho_+(-p)e^{-iS}$, $e^{iS}\rho_-(p)e^{-iS}$ are bosonic destruction operators annihilating the new ground state which is: $|\Omega\rangle = e^{iS}|0\rangle$ as well as all the similar ground states in the spaces with given numbers of excitations.

For completeness we give the argument (see [H]) showing that $T_0'' - T$ is constant on the space with a fixed number of excitations. Since C_I' is clearly constant on this space, it is sufficient to consider the case $\lambda = 0$, so that $T_0'' = T_0'$. It is an immediate consequence of (1.11) that, if $E_j(n_1, n_2)$ are the eigenvalues of $T_0' - T$ in the space with excitations numbers n_1, n_2, then each of the corresponding eigenstates $|n_1, n_2, j\rangle$ generates a family of eigenvectors with the same eigenvalue simply by applying the operators $\rho_+(p)$ and $\rho_-(-p)$ an arbitrary number of times. Such vectors are all pairwise orthogonal and non zero. Furthermore the eigenvector $|n_1, n_2, j\rangle$ with eigenvalue $E_j(n_1, n_2)$ can be so chosen that $\rho_+(-p)$ and $\rho_-(p)$ annihilate it. Then we see that by applying the operators $\rho_+(p)$ and $\rho_-(-p)$ an arbitrary number of times to $|n_1, n_2, j\rangle$ one gets a family of vectors with the property that $(T_0' - T)$ has eigenvalue $E(n_1, n_2, j)$ on each of them while T has eigenvalue $\sum_{p>0} p(n_+(p) + n_-(p))$, where $n_+(p), n_-(p)$ are the number of times the operators $\rho_+(p), \rho_-(-p)$ are applied. Clearly the partition function for T_0' at positive temperature β^{-1} can be computed in two ways: one is by observing that it is the partition function of a free Fermi gas with two particles with dispersion relation $\vec\omega(k - p_F)$, which is obviously:

$$Z = [\prod_{n>0}(1 + z^{2n-1})]^4 \tag{1.17}$$

where $z = e^{-\beta\pi/L}$ (recall that $p_F = 2\pi/L(n_F + 1/2)$). Another way is to note that the above basis of vectors $|n_1, n_2, j, \{n_+(p)\}, \{n_-(p)\}\rangle$ is obviously complete and the operator

$T_0' \equiv (T_0'-T)+T$ has on it eigenvalues $E(n_1,n_2,j)+\sum_{p>0} p(n_+(p)+n_-(p))$, so that the partition function is:

$$Z = (\sum_{j,n_1} e^{-\beta E(n_1,0,j)})^2 \Big(\prod_{n>0}(1-z^{2n})^{-1}\Big)^2 \tag{1.18}$$

where the independence of the two species of fermions with $\vec{\omega}=\pm 1$ produces the squaring of the partition functions and the identity $E(j,n_1,n_2)=E(j,n_1,0)+E(j,0,n_2)$.

Note that, as remarked above, we know explicitly at least one among the eigenvectors $|j,n_1,n_2\rangle$ of $T_0'-T$, namely the one in which all the levels are filled up to the level n_1 (above $k=p_F$) with fermions of type $+$ and down to the level n_2 with fermions of type $-$. Furthermore on such states it is easy to see that T has eigenvalue 0 while T_0' has value $(n_1^2+n_2^2)\pi/L$. We see that if, *and only if*, such states were the only ones with n_1,n_2 excitations it would follow that the $(\sum_{j,n_1} e^{-\beta E(n_1,0,j)})^2$ would have to be the sum $(\sum_{k\in Z} z^{k^2})^2$. But $(\sum_{j,n_1} e^{-\beta E(n_1,0,j)})^2$ can be obviously ([ML]) computed by remarking that the two above methods of computing the partition function of the free gas must yield the same result (*i.e.* (1.17) equals (1.18)): so the property that there is only one eigenstate of $T_0'-T$ which has the quantum numbers n_1,n_2 and which is annihiilated by $\rho_+(-p),\rho_-(p)$ is equivalent to the validity of the following identity among power series:

$$\sum_{k=-\infty}^{+\infty} z^{k^2} = \prod_{k=1}^{\infty}(1+z^{2k-1})^2(1-z^{2k}) \tag{1.19}$$

which is a well known identity about theta functions (see [GR], 8.180, 8.181). Had we taken the Fermi momentum to be $p_F=2\pi n_F L^{-1}$ (instead of $p_F=2\pi(n_F+1/2)L^{-1}$) and performed consistently the above analysis, we would have found instead of (1.19) another remarkable identity:

$$\sum_{k=0}^{\infty} z^{k(k+1)/2} \equiv \prod_{k=1}^{\infty}(1+z^k)^2(1-z^k). \tag{1.20}$$

§2 Schwinger functions

By repeating the classical analysis of [LW], one finds expressions for the Schwinger functions of the Gibbs state at inverse temperature β for the system confined in a box $[0, L]$ with periodic boundary conditions.

If $x \equiv (\vec{x}, t)$, $\beta > t_i > 0$, $t_i \neq t_j$ if $i \neq j$, $\varepsilon_i = \pm 1$ and $\{\pi(1), \ldots, \pi(n)\}$ is the permutation of $\{1, \ldots, n\}$ (with parity σ_π) such that $\pi(1) > \pi(2) > \ldots > \pi(n)$, then:

$$\begin{aligned}
&S^{L,\beta}(x_1, \vec{\omega}_1, \varepsilon_1; \ldots; x_n, \vec{\omega}_n, \varepsilon_n) = (-1)^{\sigma_\pi} \left[\mathrm{Tr}\, e^{-\beta(H-E_0)} \right]^{-1} \cdot \\
&\cdot \mathrm{Tr}\, e^{-(\beta - t_{\pi(1)})(H-E_0)} \psi^{\varepsilon_{\pi(1)}}_{\vec{x}_{\pi(1)}, \vec{\omega}_{\pi(1)}} e^{-(t_{\pi(1)} - t_{\pi(2)})(H-E_0)} \ldots \psi^{\varepsilon_{\pi(n)}}_{\vec{x}_{\pi(n)}, \vec{\omega}_{\pi(n)}} e^{-t_{\pi(n)}(H-E_0)}
\end{aligned} \tag{2.1}$$

is the standard definition of the Schwinger functions, where E_0 is the ground state energy.

Therefore, if $|\Omega\rangle$ denotes the ground state of H, it is:

$$\begin{aligned}
&\lim_{\beta \to \infty} S^{L,\beta}(x_1, \vec{\omega}_1, \varepsilon_1; \ldots; x_n, \vec{\omega}_n, \varepsilon_n) \equiv S^L_n = \\
&\quad = (-1)^{\sigma_\pi} \langle \Omega | \psi^{\varepsilon_{\pi(1)}}_{\vec{x}_{\pi(1)}, \vec{\omega}_{\pi(1)}} e^{-(t_{\pi(1)} - t_{\pi(2)})(H-E_0)} \ldots \psi^{\varepsilon_{\pi(n)}}_{\vec{x}_{\pi(n)}, \vec{\omega}_{\pi(n)}} | \Omega \rangle
\end{aligned} \tag{2.2}$$

The mean number of particles with momentum $\vec{k} + p_F \vec{\omega}$ and type $\vec{\omega}$ can be, consequently, evaluated as:

$$n_{\vec{k}, \vec{\omega}} = [\lim_{L \to \infty} \lim_{\beta \to \infty} \frac{1}{L} \int d\vec{x} d\vec{y} e^{i\vec{k}(\vec{x}-\vec{y})} S^{L,\beta}((\vec{x}, 0^+), +, \vec{\omega}); (\vec{y}, 0), -, \vec{\omega})] \tag{2.3}$$

where 0^+ means that 0^+ should be replaced by $t > 0$ and then the limit of the square bracket as $t \to 0$ has to be considered.

The r.h.s. of (2.2) can be explicitely evaluated; for example, for the model with $\rho_i(\vec{\omega}) = 1/2$, $i = 1, 2$, one can show, [Ma], that:

$$S^L_n = e^{-Q^L_n} S^L_{0,n} \tag{2.4}$$

where $S^L_{0,n}$ are the free Schwinger functions and:

$$
\begin{aligned}
&Q^L_n(x_1,\vec\omega_1,\varepsilon_1;\ldots;x_n,\vec\omega_n,\varepsilon_n)=\\
&=\frac{2\pi}{L}\sum_{\vec p>0}\frac{1}{\vec p}\sum_{\vec\omega=\pm1}\Big\{s(\vec p)^2[\frac{n}{2}+2\sum_{\substack{i,j\in I_{\vec\omega}\\ i<j}}\varepsilon_i\varepsilon_j e^{-\vec p|t_i-t_j|/c_2(\vec p)}\cos\vec p(\vec x_i-\vec x_j)]+\\
&-\sum_{\substack{i,j\in I_{\vec\omega}\\ i<j}}\varepsilon_i\varepsilon_j[e^{-\vec p|t_i-t_j|}-e^{-\vec p|t_i-t_j|/c_2(\vec p)}]\cos\vec p(\vec x_i-\vec x_j)+\\
&-c(\vec p)s(\vec p)\sum_{\substack{i\in I_{\vec\omega}\\ j\in I_{-\vec\omega}}}\varepsilon_i\varepsilon_j e^{-\vec p|t_i-t_j|/c_2(\vec p)}\cos\vec p(\vec x_i-\vec x_j)+\\
&-i\vec\omega\sum_{\substack{i,j\in I_{\vec\omega}\\ i<j}}\frac{t_i-t_j}{|t_i-t_j|}[e^{-\vec p|t_i-t_j|}-e^{-\vec p|t_i-t_j|/c_2(\vec p)}]\sin\vec p(\vec x_i-\vec x_j)\Big\}
\end{aligned}
\tag{2.5}
$$

Here $I_{\vec\omega}$ denotes the set of indices such that $\vec\omega_i=\vec\omega$ and we have set:

$$
\begin{aligned}
&s(\vec p)=\sinh\varphi(\vec p),\qquad c(\vec p)=\cosh\varphi(\vec p),\qquad c_2(\vec p)=e^{2\varphi(\vec p)}\\
&\tanh 2\varphi(\vec p)=-\frac{\lambda\hat v(\vec p)}{\lambda\hat v(\vec p)+4\pi}
\end{aligned}
\tag{2.6}
$$

Of course the positions (2.6) are meaningless, unless:

$$\lambda\hat v(\vec p)>-2\pi \tag{2.7}$$

that we shall suppose satisfied in the following. We have seen that the physical meaning of (2.7) is simply that of the stability of the model, (*i.e.* boundedness from below of the energy spectrum, proportionally to the number of particles and holes).

Denoting $S(x,\vec\omega)\equiv\lim_{L\to\infty}\lim_{\beta\to\infty}S^{L,\beta}(x,\vec\omega,-;0,\vec\omega,+)$ and $S_0(x,\vec\omega)$ the corresponding free function, we find:

$$S(x,\vec\omega)=S_0(x,\vec\omega)e^{-Q(x)-R(x)-i\vec\omega\frac{t}{|t|}I(x)} \tag{2.8}$$

with, see [Ma] for details:

$$
\begin{aligned}
Q(x)&=\int_0^\infty d\vec p\frac{2s(\vec p)^2}{\vec p}(1-e^{-\vec p|t|/c_2(\vec p)}\cos\vec p\vec x)\\
R(x)&=\int_0^\infty d\vec p\frac{\cos\vec p\vec x}{\vec p}(e^{-\vec p|t|}-e^{-\vec p|t|/c_2(\vec p)})\\
I(x)&=-\int_0^\infty d\vec p\frac{\sin\vec p\vec x}{\vec p}(e^{-\vec p|t|}-e^{-\vec p|t|/c_2(\vec p)})
\end{aligned}
\tag{2.9}
$$

and:

$$S_0(x,\vec{\omega}) = \frac{1}{(2\pi)^2}\int dk_0 d\vec{k}\frac{e^{-i(k_0 t+\vec{k}\vec{x})}}{-ik_0+\vec{\omega}\vec{k}} = \frac{1}{2\pi}\frac{1}{i\vec{\omega}\vec{x}+t} \tag{2.10}$$

The (2.9) and (2.6) imply that $R(\vec{x},0) = I(\vec{x},0) = 0$ and that $Q(\vec{x},0) \to +\infty$ as $|\vec{x}| \to \infty$ like $2\eta\log|\vec{x}|$, with, if $\lambda_1 \equiv \lambda\hat{v}(0)$:

$$\begin{aligned} 2\eta &= 2[\sinh\varphi(0)]^2 = \\ &= [(1+\frac{\lambda_1}{2\pi})^{1/2} + (1+\frac{\lambda_1}{2\pi})^{-1/2} - 2]/2 = \frac{1}{8}(\frac{\lambda_1}{2\pi})^2 + \cdots \end{aligned} \tag{2.11}$$

This shows, using (2.3) and (2.8), that the occupation number $n_{\vec{k},\vec{\omega}}$ behaves, near $\vec{k}=0$, *i.e.* near the Fermi surface, as $a - \varepsilon(\vec{k})\vec{\omega}b|\vec{k}|^{\min\{2\eta,1\}}$, with $\varepsilon(\vec{k}) = \operatorname{sign}\vec{k}$ and a, b two suitable positive numbers; hence we have no discontinuity at the Fermi surface, if $\lambda_1 \neq 0$, but just a singularity in the derivatives of sufficiently high order, depending on the value of η (the first order if $2\eta < 1$). Note also that the stability condition enters naturally in the solubility restriction (2.7).

The (2.9) and (2.6) also imply that $R(\vec{x},t) + i\vec{\omega}(t/|t|)I(\vec{x},t)$ and $Q(\vec{x},t)$ behave respectively as $\log[(i\vec{\omega}\vec{x}+c_2(0)^{-1}t)/(i\vec{\omega}\vec{x}+t)]$ and $\eta\log(\vec{x}^2+c_2(0)^{-2}t^2)$, for $\vec{x}^2+t^2 \to \infty$, so that the asymptotic behavior of $S(x,\vec{\omega})$ is:

$$\frac{1}{2\pi}\frac{1}{i\vec{\omega}\vec{x}+c_2(0)^{-1}t}\frac{A(\lambda_1)}{(\vec{x}^2+c_2(0)^{-2}t^2)^\eta} \tag{2.12}$$

where $A(\lambda_1)$ is a constant such that $A(\lambda_1) \to 1$ as $\lambda_1 \to 0$.

Formula (2.12) can be written:

$$\hat{S}(p,\vec{\omega}) \simeq B(\lambda_1)\frac{|q|^{2\eta}}{-iq_0+\vec{\omega}\vec{q}} \qquad \textit{for } |p| \to 0 \tag{2.13}$$

where $q_0 = p_0$ and $\vec{q} = c_2(0)^{-1}\vec{p}$. This implies that there is no discontinuity at the Fermi surface and that the Fermi velocity is equal to $c_2(0)^{-1}$, which goes to 1 as $\lambda \to 0$. It is however possible to consider a variation of the model (1.1), such that the Fermi velocity stays equal to 1 for any λ. In fact, if we add a term $\bar{\delta}T_0$ to the Hamiltonian, the model is still exactly soluble and the Schwinger functions are obtained from (2.4),(2.5) by the replacements [Ma]:

$$\begin{aligned} &t \to (1+\bar{\delta})t \\ &c_2(\vec{p}) \to (1+\frac{\lambda\hat{v}(\vec{p})}{2\pi(1+\bar{\delta})})^{-1/2} \end{aligned} \tag{2.14}$$

It is possible to choose $\bar{\delta}$ so that:

$$\hat{S}(p,\vec{\omega}) \simeq B(\lambda_1)|p|^{2\eta}\hat{S}_0(p,\vec{\omega}) \qquad \textit{for } |p| \to 0 \tag{2.15}$$

and $\bar{\delta}$ is given by the condition [Ma]:

$$c_2(0)^{-1}(1+\bar{\delta}) = 1 \tag{2.16}$$

The exact solution (2.4) allows us to deduce all the properties of the Luttinger model. It is however interesting to investigate another approach to the theory of the ground state, which does not rest in principle on the solvability of the model and can then be extended to more realistic examples.

Starting from the expression (1.8) and going trough the well known pattern of deductions used to set up the theory of the ground state as a problem of the analysis of a suitable functional integral, one can easily find a functional integral formulation of the Luttinger model.

If we introduce a family of grassmanian fields $\psi^{\pm}_{x,\vec{\omega}}$, which we denote with the same symbols already used for the Fermi field operators (following a common practice, source of a lot of confusion), the (2.1) can be rewritten:

$$\begin{aligned} S^{L,\beta}(x_1,\vec{\omega}_1,\varepsilon_1;\ldots;x_n,\vec{\omega}_n,\varepsilon_n) &= \Xi^{-1}\int P_g^{L,\beta}(d\psi)\psi^{\varepsilon_1}_{x_1,\vec{\omega}_1}\ldots\psi^{\varepsilon_n}_{x_n,\vec{\omega}_n}e^{-V(\psi)} \\ \Xi &\equiv \int P_g^{L,\beta}(d\psi)e^{-V(\psi)} \end{aligned} \tag{2.17}$$

where $V(\psi)$ is:

$$\lambda\int d\vec{x}d\vec{y}dt\ v(\vec{x}-\vec{y}) : (\sum_{\vec{\omega}} q_{1,\vec{\omega}}\psi^+_{\vec{x},t,\vec{\omega}}\psi^-_{\vec{x},t,\vec{\omega}}) :: (\sum_{\vec{\omega}} q_{2,\vec{\omega}}\psi^+_{\vec{y},t,\vec{\omega}}\psi^-_{\vec{y},t,\vec{\omega}}) : \tag{2.18}$$

which is an element of the grassmanian algebra generated by $\psi^{\pm}_{x,\vec{\omega}}$ (hence it is not an operator), and the integrals over ψ in (2.17) are defined by expanding $\exp[-V(\psi)]$ in powers of $V(\psi)$, hence of ψ, and evaluating the integrals using the Wick rule with field propagator vanishing for all the pairings except for those between a ψ^- field and a ψ^+ field; in the latter case the propagator has the value:

$$\int P_g^{L,\beta}(d\psi)\psi^-_{x,\vec{\omega}}\psi^+_{0,\vec{\omega}} = \frac{\delta_{\vec{\omega}\vec{\omega}'}}{(2\pi)^2}\int^{(L,\beta)} dk_0 d\vec{k}\frac{e^{-i(k_0t+\vec{k}\vec{x})}}{-ik_0+\vec{\omega}\vec{k}} \equiv \delta_{\vec{\omega}\vec{\omega}'}S_0^{L,\beta}(x,\vec{\omega}) \tag{2.19}$$

where $(2\pi)^{-2}\int^{(L,\beta)}$ means $\sum_{\vec{k}}\sum_{k_0}(L\beta)^{-1}$, with the sums running over the values $\vec{k} = 2\pi L^{-1}n$, $k_0 = 2\pi\beta^{-1}(m+1/2)$, m and n integers. The latter structure of $k \equiv (\vec{k},k_0)$

means that one should regard the inverse temperature interval $[0,\beta]$ with antiperiodic boundary conditions: $\psi^{\pm}_{\vec x,0,\vec\omega}=-\psi^{\pm}_{\vec x,\beta,\vec\omega}$.

In the following sections we shall study, instead of the functions (2.17), the *truncated* Schwinger functions, which are simply related to them and can be derived by a well known procedure from the *generating function* $\mathcal{S}^T(\varphi)$ in the following way (we suppress the indices L,β):

$$
\begin{aligned}
&S^T_{2n}(x_1,\vec\omega_1;\ldots;x_n,\vec\omega_n;y_1,\vec\omega_1';\ldots;y_n,\vec\omega_n')=\frac{\delta^{2n}\mathcal{S}^T(\varphi)}{\delta\varphi^+_{x_1\vec\omega_1}\ldots\delta\varphi^-_{y_n\vec\omega_n'}}\Big|_{\varphi=0}\\
&\mathcal{S}^T(\varphi)\equiv\log\int P_g(d\psi)e^{-V(\psi)+(\varphi^+,\psi^-)+(\psi^+,\varphi^-)}
\end{aligned}
\tag{2.20}
$$

where $\varphi^{\pm}_{x\vec\omega}$ are auxiliary grassmanian variables, anticommuting also with the $\psi^{\pm}_{x\vec\omega}$ fields, δ denotes the formal functional derivative which, togheter with the logarithm and exponential, is defined in the sense of formal power series, and

$$
(\varphi^+,\psi^-)\equiv\sum_{\vec\omega}\int dx\varphi^+_{x\vec\omega}\psi^-_{x\vec\omega}
\tag{2.21}
$$

The truncated Schwinger functions are then constructed as power series in λ, whose terms are represented by suitable Feynman graphs. As long as $L,\beta<\infty$, the series can be shown to be convergent. One could then try to collect terms of the series so that the limits $L,\beta\to\infty$ can be taken, using renormalization group techniques. This is what we shall do, using however a different infrared cutoff, which has a meaning only with respect to the representation (2.20) of the Schwinger functions (it seems that our method does not work with the original cutoff). In order to solve the problem we use essential information from the fact that the model is exactly soluble. However the results that we obtain can be easily extended to the *more realistic* system of spinless electrons interacting with a symmetric potential (which is not soluble), see [BG]. Furthermore there is an intrinsic interest in the technique that we shall explain in the following sections, because of the *anomalous scaling* (2.12), which can be completely understood in this model from the point of view of the renormalization group.

§3 Anomalous scaling and running couplings.

We consider the grassmanian integration with propagator:

$$\delta_{\vec\omega\vec\omega'}\int\frac{dk_0 d\vec k}{(2\pi)^2}\frac{e^{-i(k_0t+\vec k\vec x)}}{-ik_0+\vec\omega\vec k}\equiv\delta_{\vec\omega\vec\omega'}g(x,\vec\omega) \tag{3.1}$$

which is the limit of (2.19) when $L,\beta\to\infty$. The expression (3.1) must be handled with care; in fact one can see that the perturbative expansion of the Schwinger functions, expressed in terms of Feynman graphs, can agree with the exact expression (2.4) (in the limit $L\to\infty$) only if one calculates each contribution with an ultraviolet cutoff on the space momentum, $|\vec k|\le 2^U p_0$, and then takes the limit $U\to\infty$.

We now consider the *scaling decomposition*:

$$\begin{aligned}
g(x,\vec\omega)&=\sum_{n=-\infty}^{1}g^{(n)}(x,\vec\omega)\\
g^{(n)}(x,\vec\omega)&=\int_{p_0^{-2}2^{-2n-2}}^{p_0^{-2}2^{-2n}}d\alpha\int\frac{dk_0 d^d\vec k}{(2\pi)^2}e^{-i(k_0t+\vec k\vec x)-\alpha(k_0^2+\vec k^2)}(ik_0+\vec\omega\vec k),\qquad \text{if}\, n\le 0
\end{aligned} \tag{3.2}$$

while $g^{(1)}$ is given by the same integral over α with a different domain, namely $\alpha\in[0,p_0^{-2}/4]$. Of course the remark which follows (3.1) affects only $g^{(1)}$, which represents the ultraviolet part of the propagator.

We introduce, in correspondence with (3.2), a sequence of grassmanian fields $\psi^{(n)\pm}_{x,\vec\omega}$ with propagators $\delta_{n,n'}\delta_{\vec\omega\vec\omega'}g^{(n)}(x,\vec\omega)$. The reason for introducing them, as well as the related fields:

$$\psi^{(\le h)\pm}_{x,\vec\omega}=\sum_{n=-\infty}^{h}\psi^{(n)\pm}_{x,\vec\omega} \tag{3.3}$$

is to define a recursive method to study the functional integrals in (2.17).

The *normal scaling approach* would simply be to use that $P_g(d\psi)=\prod_{n=-\infty}^{1}P(d\psi^{(n)})$, in the sense that the integral of a function of ψ is the same regarding ψ as a field with propagator g or regarding it as $\psi=\psi^{(\le 1)}$ via (3.3) and integrating over the various fields $\psi^{(n)}$.The integration should be done recursively over $\psi^{(1)},\psi^{(0)},\psi^{(-1)},\ldots$, trying to find recursive estimates. It will be clear, however, that such approach is bound to fail. Therefore we set up an anomalous scaling approach, as it has been done in the theory of scalar fields in $4-\varepsilon$ dimensions, where a normal scaling approach could not have worked.

We write $\psi=\psi^{(1)}+\psi^{(\le 0)}$ and perform the integration over $\psi^{(1)}$ defining:

$$e^{-\bar V^{(0)}(\psi^{(\le 0)})}\equiv\int P(d\psi^{(1)})e^{-V(\psi^{(1)}+\psi^{(\le 0)})} \tag{3.4}$$

This is a preliminary step dealing with the ultraviolet part of the propagator; it is a step which has no relation with the long range slow decay of the propagator g, which is the main difficulty. Heuristically, one expects that $\bar V^{(0)}$ is not very different from the original V. This can be checked by studying the perturbation series for $\bar V^{(0)}$ in powers of λ.

¿From the point of view of field theory, the evaluation of $\bar V^{(0)}$ is a problem of renormalizable type, then it is not trivial. However one can easily show that the theory is divergence free (once ν, σ are properly chosen as in §2) and that, to all orders of perturbation theory, $V^{(0)}$ is an interaction containing terms of arbitrary degree in the fields, but with coefficients decaying exponentially fast on the scale p_0^{-1}. We think it possible to show that, for λ small enough, one can sum the terms of the same degree in the fields (there is some preliminary result in this direction [Ge]). We therefore proceed by assuming that $V^{(0)}$ has the form of a short range potential with many-body components (*i.e.* terms containing any number of $\psi^\pm$ fields), becoming very small as the number of bodies increases.

It is important to realize why it is inconvenient to break also $g^{(1)}$ into scales ranging from p_0^{-1} to 0 (in geometric progression with ratio 2); in fact at first sight this seems to provide the possibility of a symmetric treatment of the problem in its ultraviolet and infrared parts. But this would be illusory, for the simple reason that the interaction can be regarded as short ranged only on scales p_0^{-1} or larger. In the ultraviolet scales the interaction is very long ranged and we should rather treat it as a mean field. The only case in which it would seem reasonable not to distinguish between ultraviolet and infrared scales is the case of a delta function interaction (which has no scales intrinsic to it); this case is, however, well known to be pathological [ML] and in our formalism it is not even allowed because we suppose $p_0^{-1} < \infty$. In fact the model with the δ interaction is equivalent to the Thirring model for a quantum relativistic field theory and requires wave function renormalization to remove the ultraviolet divergences (absent if the range p_0^{-1} of the potential is positive), see [K, Ma].

To perform the integrals over $\psi^{(\le 0)}$ using an anomalous scaling method, we introduce a sequence $Z_0, Z_{-1}, \ldots$ of constants. While Z_0 is fixed to be $Z_0 = 1$, the others are left free to be determined inductively. The choice $Z_j = 1$ would give back the normal scaling procedure, but it will not be our choice, although most of what we do holds also for this choice (but the results are not useful, as it will appear).

In order to proceed we need:

1) the notion of relevant terms,
2) a more flexible notation for grassmanian integration.

The second point is an easy one; we denote $P_Z^{(h)}(d\psi)$ or $P_Z^{(\le h)}(d\psi)$ the grassmanian integrations with propagators:

$$Z^{-1} g^{(h)} \qquad \text{or} \qquad Z^{-1} g^{(\le h)} \tag{3.5}$$

If we introduce the convolution operator C_h with Fourier transform:

$$C_h(k) = e^{(k_0^2+\vec{k}^2)2^{-2h}p_0^{-2}/4} \tag{3.6}$$

we can write, formally:

$$P_Z^{(\le h)}(d\psi) \propto e^{-Z\sum_{\vec\omega}\int dx\psi^+_{x,\vec\omega}(\partial_t+i\vec\omega\partial_{\vec x})C_h(\partial)\psi^-_{x,\vec\omega}}d\psi \tag{3.7}$$

Coming to the notion of relevant operators, we consider a general element of the grassmanian algebra and we define the operation $\mathcal{L}$, the *localization operation*, as follows. $\mathcal{L}$ is a linear operator which annihilates all monomials in the field operators of degree > 4. Its definition on the monomials of degree 4 or 2 is simply:

$$\begin{aligned}
&\mathcal{L}\psi^+_{x_1\vec\omega_1}\psi^-_{x_2\vec\omega_2} = \psi^+_{x_1\vec\omega_1}(\psi^-_{x_1\vec\omega_2} + (x_2-x_1)\partial\psi^-_{x_1\vec\omega_2})\\
&\mathcal{L}\psi^+_{x_1\vec\omega_1}\psi^+_{x_2\vec\omega_2}\psi^-_{x_3\vec\omega_3}\psi^-_{x_4\vec\omega_4} = 2^{-1}\sum_{j=1,2}\psi^+_{x_j\vec\omega_1}\psi^+_{x_j\vec\omega_2}\psi^-_{x_j\vec\omega_3}\psi^-_{x_j\vec\omega_4}
\end{aligned} \tag{3.8}$$

This implies that the action of $\mathcal{L}$ on a V of the form:

$$\begin{aligned}
V(\psi) = &\sum_n \sum_{\substack{\vec\omega_1,\dots,\vec\omega_n\\ \vec\omega'_1,\dots,\vec\omega'_n}} \int W_n(x_1,\vec\omega_1;\dots;x_n,\vec\omega_n;y_1,\vec\omega'_1;\dots;y_n,\vec\omega'_n)\\
&\psi^+_{x_1,\vec\omega_1}\cdots\psi^+_{x_n,\vec\omega_n}\psi^-_{y_1,\vec\omega'_1}\cdots\psi^-_{y_n,\vec\omega'_n}\,dx_1\dots dy_n
\end{aligned} \tag{3.9}$$

gives a result which can be written, by collecting similar terms:

$$\begin{aligned}
\mathcal{L}V(\psi) = \lambda'\int dx\psi^+_{x,+}\psi^+_{x,-}\psi^-_{x,-}\psi^-_{x,+} + \nu'\sum_{\vec\omega}\int dx\psi^+_{x,\vec\omega}\psi^-_{x,\vec\omega}+\\
+\zeta'\sum_{\vec\omega}\int dx\psi^+_{x,\vec\omega}\partial_t\psi^-_{x,\vec\omega} + i\alpha'\sum_{\vec\omega}\int dx\psi^+_{x,\vec\omega}\vec\omega\partial_{\vec x}\psi^-_{x,\vec\omega}
\end{aligned} \tag{3.10}$$

provided the W's in (3.9) are not too singular distributions.

To make precise what we mean by *not too singular* we introduce the following fields:

$$\begin{aligned}
&\psi^\pm_{x,\vec\omega}\,,\quad \partial\psi^\pm_{x,\vec\omega}\\
&D^\pm_{x,y,\vec\omega} = \psi^\pm_{x,\vec\omega} - \psi^\pm_{y,\vec\omega}\,,\quad S^1_{x,y,\vec\omega} = \psi^-_{x,\vec\omega} - \psi^-_{y,\vec\omega} - (x-y)\partial\psi^-_{y,\vec\omega}\\
&S^2_{x,y,\vec\omega} = \partial\psi^-_{x,\vec\omega} - \partial\psi^-_{y,\vec\omega}\,,\quad S^3_{x_1,x_2,x_3,x_4,\vec\omega} = (x_3-x_4)S^1_{x_1,x_2,\vec\omega}\\
&K^{(h)}_{x,\vec\omega} = (\partial_t + i\vec\omega\partial_{\vec x})(1-C_h(\partial))\psi^-_{x,\vec\omega}
\end{aligned} \tag{3.11}$$

where C_h is the operator in (3.6).

We shall only consider V's of the form (3.9), which can be rewritten as:

$$V(\psi) = \mathcal{L}V(\psi) + \sum_n \int d\xi_1 \ldots d\xi_n \tilde{W}_n(\xi_1, \ldots, \xi_n)\Phi_{\xi_1} \ldots \Phi_{\xi_n} \tag{3.12}$$

where Φ_ξ denotes one of the fields in (3.11) and ξ is $(x, \vec{\omega})$ or $(x, y, \vec{\omega})$ or $(x_1, x_2, x_3, x_4, \vec{\omega})$ and $d\xi$ means integration over the $x, y, \ldots$ coordinates and summation over the $\vec{\omega}$ coordinates; furthermare the $\tilde{W}$ are products of ordinary smooth kernels by suitable time delta functions.

We shall write the function $\bar{V}^{(0)}$ in (3.4) as:

$$\bar{V}^{(0)}(\psi) = \bar{\zeta}(\psi^+, (\partial_t + i\vec{\omega}\cdot\partial_{\vec{x}})\, C_0(\partial)\psi^-) + V^{(0)}(\sqrt{Z_0}\psi) \tag{3.13}$$

where $\bar{\zeta}$ is the coefficient of $(\psi^+, \partial_t\psi^-)$ in the expansion of $\bar{V}^{(0)}(\psi)$, and we set:

$$Z_0 \equiv 1 + \bar{\zeta} \tag{3.14}$$

We can now set up a recursive procedure for the analysis of the integral (which coincides with the Ξ in (2.17), because of (3.4) and the last two definitions):

$$\int P_{Z_0}^{(\le 0)}(d\psi) e^{-V^{(0)}(\sqrt{Z_0}\psi)} \tag{3.15}$$

by writing $P_{Z_0}^{(\le 0)}(d\psi) = P_{Z_0}^{(0)}(d\bar{\psi})P_{Z_0}^{(\le -1)}(d\tilde{\psi}), \psi = \bar{\psi} + \tilde{\psi}$. Integrating over $\bar{\psi}$ and using (3.7), we write (3.15) as:

$$\begin{aligned} &\int P_{Z_0}^{(\le -1)}(d\psi) e^{-\tilde{V}^{(-1)}(\sqrt{Z_0}\psi)} = \\ &= \text{const} \int d\psi e^{-Z_0 \sum_{\vec{\omega}} \int dx \psi^+_{x,\vec{\omega}}(\partial_t + i\vec{\omega}\partial_{\vec{x}})C_{-1}(\partial)\psi^-_{x,\vec{\omega}}} \cdot \\ &e^{-\mathcal{L}\tilde{V}^{(-1)}(\sqrt{Z_0}\psi) - (1-\mathcal{L})\tilde{V}^{(-1)}(\sqrt{Z_0}\psi)} \end{aligned} \tag{3.16}$$

which we rewrite, using (3.7), as:

$$\begin{aligned} &const \int P_{Z_{-1}}^{(\le -1)}(d\psi) e^{-(Z_0 - Z_{-1})\sum_{\vec{\omega}} \int dx \psi^+_{x,\vec{\omega}}(\partial_t + i\vec{\omega}\partial_{\vec{x}})C_{-1}(\partial)\psi^-_{x,\vec{\omega}}} \cdot \\ &\cdot e^{-\zeta' \sum_{\vec{\omega}} \int dx \psi^+_{x,\vec{\omega}}(\partial_t + i\vec{\omega}\partial_{\vec{x}})C_{-1}(\partial)\psi^-_{x,\vec{\omega}} + \text{other relevant terms}} \cdot \\ &\cdot e^{-(1-\mathcal{L})\tilde{V} - \zeta' \sum_{\vec{\omega}} \int dx \psi^+_{x,\vec{\omega}} K^{(0)}_{x,\vec{\omega}}} \end{aligned} \tag{3.17}$$

where *const* is a formally infinite but trivial constant, which we shall neglect in the following, together with similar ones.

In the anomalous scaling procedure one chooses Z_{-1} so that $Z_0 - Z_{-1} + \zeta' = 0$, *i.e.* the coefficient of $\int dx \psi^+_{x,\vec\omega}(\partial_t + i\vec\omega\partial_{\vec x})C_{-1}(\partial)\psi^-_{x,\vec\omega}$ vanishes and (3.17) becomes (thus defining $V^{(-1)}$):

$$\int P^{(\leq -1)}_{Z_{-1}}(d\psi)e^{-V^{(-1)}(\sqrt{Z_{-1}}\psi)} \tag{3.18}$$

where $V^{(-1)}$ can be expressed in terms of the fields (3.11) as in (3.12), with $h \geq -1$, if $V^{(0)}$ was expressible in terms of them as in (3.12). The latter property is seen to hold order by order of perturbation theory.

The iteration produces a sequence $Z_0, Z_{-1}, \ldots$, as well as a sequence of potentials $V^{(h)}$ such that, up to a trivial constant:

$$\int P^{(\leq 0)}_{Z_0}(d\psi)e^{-V^{(0)}(\sqrt{Z_0}\psi)} = \int P^{(\leq h)}_{Z_h}(d\psi)e^{-V^{(h)}(\sqrt{Z_h}\psi)} \tag{3.19}$$

and a sequence of coefficients $\vec r_h = (\nu_h, \delta_h, \lambda_h)$, called *running couplings*, which are defined by writing:

$$\begin{aligned}\mathcal{L}V^{(h)} = {} & Z_h^2\lambda_h \sum_{\vec\omega}\int dx \psi^+_{x,+}\psi^+_{x,-}\psi^-_{x,-}\psi^-_{x,+} + \\ & + Z_h i\delta_h \sum_{\vec\omega}\int dx \psi^+_{x,\vec\omega}\vec\omega\partial_{\vec x}\psi^-_{x,\vec\omega} + Z_h 2^h\nu_h \sum_{\vec\omega}\int dx \psi^+_{x,\vec\omega}\psi^-_{x,\vec\omega}\end{aligned} \tag{3.20}$$

Furthermore the $\tilde W_h$-functions, appearing in the expansion of $(1-\mathcal{L})V^{(h)}$ in powers of the fields, are also produced as formal power series in $\vec r_{h+1}, \ldots, \vec r_0$. No term proportional to $\int dx\, \psi^+_{x,\vec\omega}\partial_t\psi^-_{x,\vec\omega}$ appears in (3.20), because of our definition of the sequence Z_h. Finally, using the oddness of the propagator, it is easy to see that, for each h:

$$\nu_h = 0 \tag{3.21}$$

This property of the potential is related to the fact that, in the Luttinger model, the interaction does not modify the position of the Fermi surface, so that we have effectively only two running couplings.

Of course one could envisage other prescriptions to construct the sequence Z_j, but it will appear that only one of them has the possibility of being applicable to our problem, namely the just illustrated anomalous scaling choice.

On heuristic grounds we expect that an asymptotic behaviour of the running couplings like:

$$a)\begin{cases} Z_h = z2^{-2\eta h} \\ \nu_h \to 0 \\ \delta_h \to 0 \\ \lambda_h \to \lambda_{-\infty} \end{cases} \qquad b)\begin{cases} |\nu_h|, |\delta_h|, \,|\lambda_h| \,< C_0|\lambda_0| \\ e^{-q\varepsilon|\lambda_0|} < |\frac{Z_{h+q}}{Z_h}| < e^{q\varepsilon|\lambda_0|} \qquad q \ge 0 \end{cases} \tag{3.22}$$

for some $z, \eta, \lambda_{-\infty}, \varepsilon, C_0$, implies that the pair correlation function $S_2(x, \vec{\omega}) \equiv S_2^T(x, \vec{\omega}; 0, \vec{\omega})$ behaves like:

$$\hat{S}_2(k) \propto |k|^{2\eta}\hat{S}_0(k), \qquad k \to 0 \tag{3.23}$$

and that the four points truncated Schwinger function S_4^T, which we write as:

$$S_4^T(k_1+, k_2+, k_3-, k_4-) = -\delta(k_1 + k_2 - k_3 - k_4)W(k_1, k_2, k_3, k_4)\prod_{i=1}^{4}\hat{S}_0(k_i, \vec{\omega}_i) \tag{3.24}$$

verifies:

$$W(k, k, k, k) \propto |k|^{4\eta}[\lambda_{-\infty} + O(\lambda_{-\infty}^2)] \qquad k \to 0 \tag{3.25}$$

In section 5 we show that the relations between the running couplings:

$$\vec{r}_h = (\lambda_h, \delta_h, \nu_h) \quad \text{and the } \textit{scalings} \quad Z_h \tag{3.26}$$

of different h's is such that (3.22) can hold if and only if the *beta function* vanishes.

On the other hand we know from the exact solution that (3.23), (3.24) hold rigorously [Ma]. Hence we conclude that the beta function of the model ought to vanish.

In fact b) in (3.22) alone implies the vanishing of the beta function and is implied by it. Thus the vanishing of the beta function is implied by the boundedness assumption b) in (3.22). Once this is supposed to hold it follows that the model must have anomalous scaling properties, (3.23), (3.24), and in fact one can compute the expansion in powers of λ_0 of the anomalous exponent η.

§4 *Schwinger functions and running couplings.*

Let us now discuss the connection between the potentials $V^{(h)}$ and the truncated Schwinger functions (2.20).

We define the *effective potential* $V_{eff}(\varphi)$ by:

$$e^{-V_{eff}(\varphi)} = \int P_g(d\psi) e^{-V(\psi+\varphi)} \tag{4.1}$$

and we introduce, see (3.6):

$$\begin{aligned} &\hat{Q}_1(k,\vec{\omega}) = (-ik_0 + \vec{\omega}\vec{k})^{-1} \\ &\hat{Q}_h(k,\vec{\omega}) = \hat{Q}_1(k,\vec{\omega}) C_h(k)^{-1} \quad , \quad h \le 0 \end{aligned} \tag{4.2}$$

By using the formal relation $P_g(d\psi) \propto e^{-(\psi^+, Q_1^{-1}\psi)} d\psi$ and by doing in (4.1) the formal change of variable $\psi \to \psi - \varphi$, it is easy to see that:

$$S^T(\varphi) = (\varphi^+, Q_1 \varphi^-) - V_{eff}(Q_1\varphi) \tag{4.3}$$

The evaluation of the effective potential can be conveniently performed iteratively, starting from (3.13):

$$\begin{aligned} e^{-V_{eff}(\varphi)} &= \int P_1^{(\le 0)}(d\psi) e^{-\bar{V}^{(0)}(\psi+\varphi)} = \\ &= \int P_1^{(\le 0)}(d\psi) e^{-(Z_0-1)(\psi^+ + \varphi^+, Q_0^{-1}(\psi^- + \varphi^-)) - V^{(0)}(\sqrt{Z_0}(\psi+\varphi))} \\ &= \int P_{Z_0}^{(\le 0)}(d\psi) e^{-(Z_0-1)(\varphi^+, Q_0^{-1}\varphi^-) - (Z_0-1)[(\varphi^+, Q_0^{-1}\psi^-) + (\psi^+, Q_0^{-1}\varphi^-)] - V^{(0)}(\sqrt{Z_0}(\psi+\varphi))} \end{aligned} \tag{4.4}$$

and, by doing in (4.4) the formal change of variable $\psi \to \psi - \frac{Z_0-1}{Z_{-1}}\varphi$, we get, up to a trivial constant:

$$e^{-(1-\frac{1}{Z_0})(\varphi^+, Q_0^{-1}\varphi^-)} \int P_{Z_0}^{(\le 0)}(d\psi) e^{-V^{(0)}(\sqrt{Z_0}(\psi + \frac{\varphi}{Z_0}))} \tag{4.5}$$

Hence, by iteration, one can easily prove that, for any $p \le 0$:

$$e^{-V_{eff}(\varphi)} = e^{-\sum_{j=p+1}^{1} (\frac{1}{Z_j} - \frac{1}{Z_{j-1}})(\varphi^+, Q_{j-1}^{-1}\varphi^-)} \int P_{Z_p}^{(\le p)}(d\psi) e^{-V^{(p)}(\sqrt{Z_p}(\psi + \frac{\varphi}{Z_p}))} \tag{4.6}$$

having set $Z_1 \equiv 1$. So that, if Z_2 is defined to be equal to ∞:

$$\mathcal{S}^T(\varphi) = \sum_{j=p+1}^{2} (\frac{1}{Z_{j-1}} - \frac{1}{Z_j})(\varphi^+, Q_1 C_{j-1}\varphi^-) + \log \int P_{Z_p}^{(\le p)}(d\psi) e^{-V^{(p)}(\sqrt{Z_p}(\psi + \frac{Q_1\varphi}{Z_p}))} \tag{4.7}$$

Let us now remark that it is not possible to take the limit $p \to -\infty$ in (4.7); in fact the sum in the r.h.s. is divergent in this limit, because of the bad dependence on k of $C_j(k)$, see (3.6). However the analysis of the potentials $V^{(h)}$ [BGPS] shows that they all have the form of a short range potential on scale $(2^h p_0)^{-1}$, with many-body components which are convergent power series in the running couplings near zero. Therefore we see that the evaluation of the integral in (4.7), with φ having support in $2^p p_0 \le |k| \le 2^{p+1} p_0)$, is the same type of calculation, up to a trivial rescaling, that one would perform to evaluate the correlations on the scale of the potential. We can suppose that the latter problem is solvable, by the same techniques of [BG] suitably extended, and that the result is of order one (uniformly in h).

These considerations and (4.7) imply that the Fourier transform of the pair correlation function $S_2(x-y,\vec\omega)$ can be written, for momenta of order $2^h p_0$, in the following way:

$$\hat Q_1(k)^{-1}\hat S_2(k,\vec\omega) = \sum_{j=h+1}^{2} (\frac{1}{Z_{j-1}} - \frac{1}{Z_j}) C_{j-1}(k) + Z_h^{-1}\hat Q_1(k)[-\delta_h \vec\omega\vec k - 2^h \bar V^{(h)}(2^{-h}k)] \quad (4.8)$$

where $\bar V^{(h)}(k)$ is a smooth function of k, which has a smooth limit as $h \to -\infty$ and is of the second order in the running couplings.

The equation (4.8) implies that, if $k = 2^h \bar k$, with $|\bar k| > 0$ and independent of h, and if $\lim_{h\to-\infty} |h|^{-1} \log Z_h = \tilde\eta > 0$, then the asymptotic behaviour, for $h \to -\infty$, of $\hat S_2(k,\vec\omega)$ is of the type:

$$\hat S_2(k,\vec\omega) \simeq \frac{|k|^{\tilde\eta}}{(-ik_0 + \vec\omega\vec k)} \left\{ b(\bar k) + \frac{\vec\omega\vec k}{-ik_0 + \vec\omega\vec k}[\delta_h + a(\bar k)] \right\} \quad (4.9)$$

We can not compare (4.9) with the asymptotic behaviour of the pair correlation (2.13), because our renormalization procedure fixed the Fermi velocity to 1, which is not the value in the model (1.1). Instead of modifying the renormalization procedure, we choose to study the new model discussed after (2.13), obtained by adding a term $\bar\delta T$ to the Hamiltonian, so that the Fermi velocity is fixed to 1, independently of λ. Then, by comparing (2.15) and (4.9), we see that $b(\bar k)$ is a function of $|\bar k|$, $a(\bar k) = 0$ (it should be possible to derive directly these two results, but we did not do that) and

$$\delta_h \to 0 \qquad \text{as} \quad h \to -\infty \quad (4.10)$$

$$\tilde\eta = 2\eta = 2[\sinh\varphi(0)]^2 \quad (4.11)$$

A similar discussion for the four fields Schwinger function yields a similar result, that is:

$$\begin{aligned} &\hat S_4^T(p_1,+,p_2,-;p_3,+,p_4,-) = -\delta(p_1+p_2-p_3-p_4)\cdot \\ &\cdot g^{(>h)}(p_1,+) g^{(>h)}(p_2,-) g^{(>h)}(p_3,+) g^{(>h)}(p_4,-) \frac{1}{Z_h^2} \hat W_4^{(h)}(p_1,p_2,p_3) \end{aligned} \quad (4.12)$$

where, for momenta of order 2^h:

$$\hat{W}_4^{(h)}(p_1, p_2, p_3) = \lambda_h + \bar{W}_4^{(h)}(2^{-h}p_1, 2^{-h}p_2, 2^{-h}p_3) \tag{4.13}$$

with $\bar{W}_4^{(h)}$ having a smooth limit as $h \to -\infty$ and being of the second order in the running couplings.

The asymptotic behaviour of the l.h.s. in (4.12) can be calculated from the exact solution and one can see [Ma] that it is compatible with (4.12) and (4.13) only if λ_h has a finite limit as $h \to -\infty$:

$$\lambda_h \to \lambda_\infty(\lambda_1) \tag{4.14}$$

.

§5 *The beta function.*

The above analysis not only permits us to define the running couplings $\vec{r}_h$ and the scalings Z_h, (4.5), but also to find an expression of $\vec{r}_{h-1}, Z_{h-1}/Z_h$ in terms of $\vec{r}_{\ge h}, Z_{\ge h}/Z_h$. The latter can be studied from the explicit expressions of $\vec{r}_h$ in terms of the Feynman graphs of the model, which are constructed from the formal integration formula:

$$e^{\bar{V}(\psi)} \equiv \int P(d\bar{\psi}) e^{V(\psi+\bar{\psi})} = \exp \sum_{n=1}^{\infty} \frac{1}{n!} \mathcal{E}^T(V;\ldots;V) \tag{5.1}$$

where $\mathcal{E}^T$ denotes the truncated expectation with respect to the integration over $\bar{\psi}$. The latter is defined simply by imposing that, in the evaluation of the integrals $\mathcal{E}(V^n)$ with the Wick rule, only some terms are to be retained. Namely, if we think all the fields appearing in a monomial in one of the V factors as lines and we represent a Wick contraction by suitably joining together pairs of lines, then we only retain terms corresponding to Wick contractions generating a connected graph of lines.

The theory of the estimates of the series expansion of $\vec{r}_{h-1}, Z_{h-1}/Z_h$ in powers of $\vec{r}_{\ge h}, Z_{\ge h}/Z_h$ is technically involved, see [BG], where the main result is that the formal power series has coefficients of order n bounded uniformly in the scale parameter h, provided for some $\xi > 0$:

$$e^{-q\xi} < |Z_{h+q}/Z_h| < e^{q\xi}, \quad \text{for } q > 0 \tag{5.2}$$

by the bound:

$$D_\xi C_\xi^{n-1}(n-1)! \tag{5.3}$$

for some C_ξ, D_ξ and all $h \le 0$.

In fact the model is technically very similar to the one flavour Gross-Neveu model and it seems reasonable to us that one could improve (5.3) by taking out the $n!$ from the bounds.

It is in fact possible to prove [GS], [BGPS], that there is a convergent power series expansion:

$$\begin{aligned} \vec{r}_{h-1} =& \Lambda \vec{r}_h + B'_h(\vec{r}_h, Z_{h+1}/Z_h, \vec{r}_{h+1}, \ldots, Z_0/Z_h \vec{r}_0) \\ \frac{Z_{h-1}}{Z_h} =& (1 + B"_h(\vec{r}_h, Z_{h+1}/Z_h, \vec{r}_{h+1}, \ldots, Z_0/Z_h \vec{r}_0) \end{aligned} \tag{5.4}$$

where Λ is a 3×3 diagonal matrix, with elements $1,1,2$, see below, and with the functions B_h^σ being holomorphic when all their arguments $\vec{r}$-arguments are in a small enough disk with h-independent radius while the arguments Z_{h+q}/Z_h, $q \ge 0$ vary in an annulus like (5.2) for some ξ small enough. Furthermore the limit $\lim_{h\to\infty} B_h^s$ converging to a holomorphic function of infinitely many variables $B(\vec{z}^1, \vartheta_1, \vec{z}^2, \vartheta_2, \ldots)$, holomorphic in a disk of some radius $\rho > 0$ for the $\vec{z}$ variables and in an annulus like (5.2) for the ϑ_q variables with some $\xi > 0$. So that if $\lim \vec{r}^h = \vec{r}^{-\infty}$ and $\lim Z_{h+q}/Z_h = \vartheta_q$ exist then $\vec{r}^{-\infty}$ is a fixed point of the relation $\vec{r}^{-\infty} = \Lambda \vec{r}^{-\infty} + B_\infty(\vec{r}^{-\infty})$ where $B_\infty(\vec{r})$ is defined by setting $B_\infty(\vec{r}) = B'(\vec{r}, \vartheta_1, \vec{r}, \vartheta_2, \vec{r}, \ldots)$.

For this reason we shall call *beta function* the function of three complex variables $\Lambda\vec{z} + B_\infty(\vec{z})$, while we call *beta functional* the functions B^s_h in (5.4), depending on h arguments. The beta function in the above sense is a function whose fixed points are the limit values of the running couplings $\vec{r}_h$ of our model.

In the literature one also considers often the function relating $\vec{r}_{h-1}$ to $\vec{r}_h$: it follows from the above that the latter also has a well defined expansion around $\vec{r}_h = \vec{0}$ but its coefficients grow as $n!$ with the order, hence it is not a priori well defined, and it seems to us that even if it is well defined it will be such because of non generic cancellations, absent in the case of spin non zero, for instance. The proof of the above convergence properties can be found in [GeS],[BGSP]. Hence we shall assume them and study their implications.

We stress, before continuing, that the above results would also hold if one used the normal scaling procedure. The bounds (5.3) and our convergence conjecture hold also in the normal scaling approach. The importance of the scaling does not come in at this point, yet.

If $\lambda_{\ge h}$ denotes the sequence $\lambda_h, \lambda_{h+1}, \ldots, \lambda_0$ and a similar notation is adopted for $\delta_{\ge h}, \nu_{\ge h}$, then the computation, via the Feynman graphs, of the running couplings leads to the following:

$$\begin{aligned}
\lambda_{h-1} =&(Z_h/Z_{h-1})^2\big[\lambda_h + \lambda_h^3 B_1(\lambda_{\ge h}) + \delta_h\lambda_h^3 B_2(\lambda_{\ge h}, \delta_{\ge h}) + \\
&+ \lambda_h^2\nu_h^2 B_3(\lambda_{\ge h}, \delta_{\ge h}, \nu_{\ge h}) + 2^h \bar{R}_1(\lambda_{\ge h}, \delta_{\ge h}, \nu_{\ge h}, 2^h)\big]\\
\delta_{h-1} =&(Z_h/Z_{h-1})[\delta_h + \lambda_h^2\delta_h B_4(\lambda_{\ge h}) + \nu_h^2 B_5(\lambda_{\ge h}, \delta_{\ge h}, \nu_{\ge h}) + \\
&2^h \bar{R}_2(\lambda_{\ge h}, \delta_{\ge h}, \nu_{\ge h}, 2^h)]\\
\nu_{h-1} =&2(Z_h/Z_{h-1})[\nu_h + \nu_h\lambda_h^2 B_6(\lambda_{\ge h}, \delta_{\ge h}, \nu_{\ge h}) + \\
&+ 2^h \bar{R}_3(\lambda_{\ge h}, \delta_{\ge h}, \nu_{\ge h}, 2^h)]\\
1 =&(Z_h/Z_{h-1})[1 + \lambda_h^2 B_8(\lambda_{\ge h}) + \\
&+ \delta_h\lambda_h^2 B_9(\lambda_{\ge h}, \delta_{\ge h}) + \lambda_h^2\nu_h^2 B_{10}(\lambda_{\ge h}, \delta_{\ge h}, \nu_{\ge h}) + 2^h \bar{R}_4(\lambda_{\ge h}, \delta_{\ge h}, \nu_{\ge h}, 2^h)]
\end{aligned} \tag{5.5}$$

where all the B_j functions do depend also on the ratios Z_{h+q}/Z_h, $q \ge 0$, as discussed above, but such dependence is noty explicitly indicated to simplify the notation. Furthermore we have computed a little more carefully the lowest terms to find out the minimal power to which each running constant is raised; in particular we have used the following facts:

a) the graphs containing two λ_h-vertices and any number of δ_h-vertices cancel out;
b) since the propagator is an odd function of x, in the equation for ν_{h-1} there is no contribution due to graphs containing only λ_h-vertices (and therefore an odd number

of innner lines) or containing only λ_h- and δ_h-vertices (a δ_h-vertex does not change the parity of the graph).

As we have stressed in the previous section, ν_h is exactly zero in the model (1.1), so we could cancel out the third equation in (5.5). However we prefer to study the complete set of equations (5.5), since they are valid also in the model with an ordinary kinetic energy, where ν_h is not zero, see [BG].

As a consequence of the discussion preceding (5.5), the functions $B_j, \bar{R}_j$ should be analytic in their arguments $\lambda_{\ge h}, \delta_{\ge h}, \nu_{\ge h}$ (with a suitably small radius M of convergence) and in Z_{h+q}/Z_h, $q > 0$ (in a suitably thin annulus around the unit circle, see (5.2)). Furthermore the B_j can be shown to have a limit as $j \to -\infty$ while the $\bar{R}_j$ terms disappear in this limit. The $\bar{R}_j$ vanish to second order in $\lambda_h, \delta_h, \nu_h$, see [BGPS].

Had we used the normal scaling approach, we would have found an equation like (5.5) with δ_h (or $\delta_{\ge h}$) replaced by a pair (α_h, ζ_h) of constants (or by $(\alpha_{\ge h}, \zeta_{\ge h})$, representing the coefficients of $\int dx \psi^+_{x,\vec{\omega}} i\partial_{\vec{x}} \psi^-_{x,\vec{\omega}}$ and $\int dx \psi^+_{x,\vec{\omega}} \partial_t \psi^-_{x,\vec{\omega}}$, and each of the two new relations would have had a non vanishing term proportional to λ_h^2. The reason why such term is missing in (5.5) is precisely due to our definition of anomalous scaling combined with the symmetry in the propagator between $\vec{x}$ and t, which makes identical the contributions to the variations of α_h and ζ_h due to graphs only involving λ_h-vertices, hence it makes idenically identically zero the contributions to δ_h of the same graphs.

It is convenient to eliminate completely the factors Z_h/Z_{h-1} from (5.5), using the last of (5.5) and expanding the denominators in power series:

$$\begin{aligned}
\lambda_{h-1} =&\lambda_h + \lambda_h^3 G_1(\lambda_{\ge h}) + \delta_h \lambda_h^3 G_2(\lambda_{\ge h}, \delta_{\ge h}) + \nu_h^2 \lambda_h^2 G_3(\lambda_{\ge h}, \delta_{\ge h}, \nu_{\ge h}) + \\
&+ t_h R_1(\lambda_{\ge h}, \delta_{\ge h}, \nu_{\ge h}, t_h) \\
\delta_{h-1} =&\delta_h + \lambda_h^2 \delta_h G_4(\lambda_{\ge h}, \delta_{\ge h}) + \nu_h^2 G_5(\lambda_{\ge h}, \delta_{\ge h}, \nu_{\ge h}) + t_h R_2(\lambda_{\ge h}, \delta_{\ge h}, \nu_{\ge h}, t_h) \\
\nu_{h-1} =&2\nu_h + \nu_h \lambda_h^2 G_6(\lambda_{\ge h}, \delta_{\ge h}, \nu_{\ge h}) + \\
&+ t_h R_3(\lambda_{\ge h}, \delta_{\ge h}, \nu_{\ge h}, t_h) \\
t_{h-1} =&2^{-1} t_h
\end{aligned} \tag{5.6}$$

having set $t_h = 2^h$, and not having once more written explicitly the dependence of the G_j on the variables $\vartheta_{q,h} = Z_{h+q}/Z_h$, $q > 0$.

The relation (5.6), defining the beta functional, does not permit us to infer much about the properties of the model; but we can derive extra information about the G_j functions from the fact that the model is exactly soluble.

Let us assume:

1) that the flow (5.6) admits, for each $\lambda_0 \neq 0$ small enough, initial values $\delta_0(\lambda_0)$, $\nu_0(\lambda_0)$

such that:

$$\begin{aligned}
&\delta_h, \nu_h \to 0 \quad , \quad \lambda_h \to \lambda_\infty(\lambda_0)\\
&Z_h \simeq 2^{-\eta h} \quad , \quad \eta = O(\lambda_0^2) > 0\\
&|\vec{r}_h| \le C_0|\lambda_0|
\end{aligned} \tag{5.7}$$

where $\simeq$ means that the logarithms of both sides, divided by $|h|$, have the same limit (see (4.3)), and $\lambda_{-\infty}(\lambda), \eta(\lambda)$ being analytic near $\lambda = 0$.

Call $\bar{G}_1(\lambda, \eta) \equiv \lim_{h\to-\infty} G_1(\lambda_{\ge h})$, with $\lambda_j \equiv \lambda$ and $\vartheta_{q,j} = 2^{-2\eta q}$, and let us suppose that:

2) the function G_1 in the first equation of (5.6) is analytic and not identically zero.

The assumption (5.7) immediately implies that $\bar{G}_1(\lambda_\infty, \eta) = 0$; then λ_∞ is independent of λ_0, as a consequence of the analyticity hypothesis. But the hypotheses (5.7) and imply that $|\lambda_\infty| \le C_0|\lambda_0|$ has to hold for all λ_0 small enough, so $\lambda_\infty = 0$ and the fourth of (5.5) tells us that $Z_h/Z_{h-1} \to 1$, which is incompatible with $\eta > 0$.

In conclusion, if the assumptions 1), 2) above are satisfied:

$$\bar{G}_1(\lambda) = 0 \tag{5.8}$$

This makes more precise the argument leading to the conjecture given in [BG]. A similar property has been proposed in [DM], supported by a symmetry argument.

We now observe that 1), *i.e.* (5.7), should be deducible from the exact solution of the model, using (2.11), (3.23), (3.25). Moreover $\lambda_\infty = \lambda_0 + O(\lambda_0^3)$, so that, if λ_0 is small, also λ_∞ is small. Also 2) should be provable by known techniques, as discussed above.

Then, by the previous discussion, our basic result is that the main term in the beta function not only is zero to second order, where it is easily calculated, but vanishes to all orders. We have checked by explicit calculation that (5.8) is verified also to third order [Ma].

It is remarkable that (5.8) holds: in fact as it can be used in other models which are not exactly soluble, but which can be shown to have the same G_i functions. One case, see [BG], is the model of one spinless species of fermions interacting via a short range interaction and with an ordinary kinetic energy (namely $(\vec{k}^2 - p_F^2)/2m$).

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